Advanced Control Systems: RPP manipulator

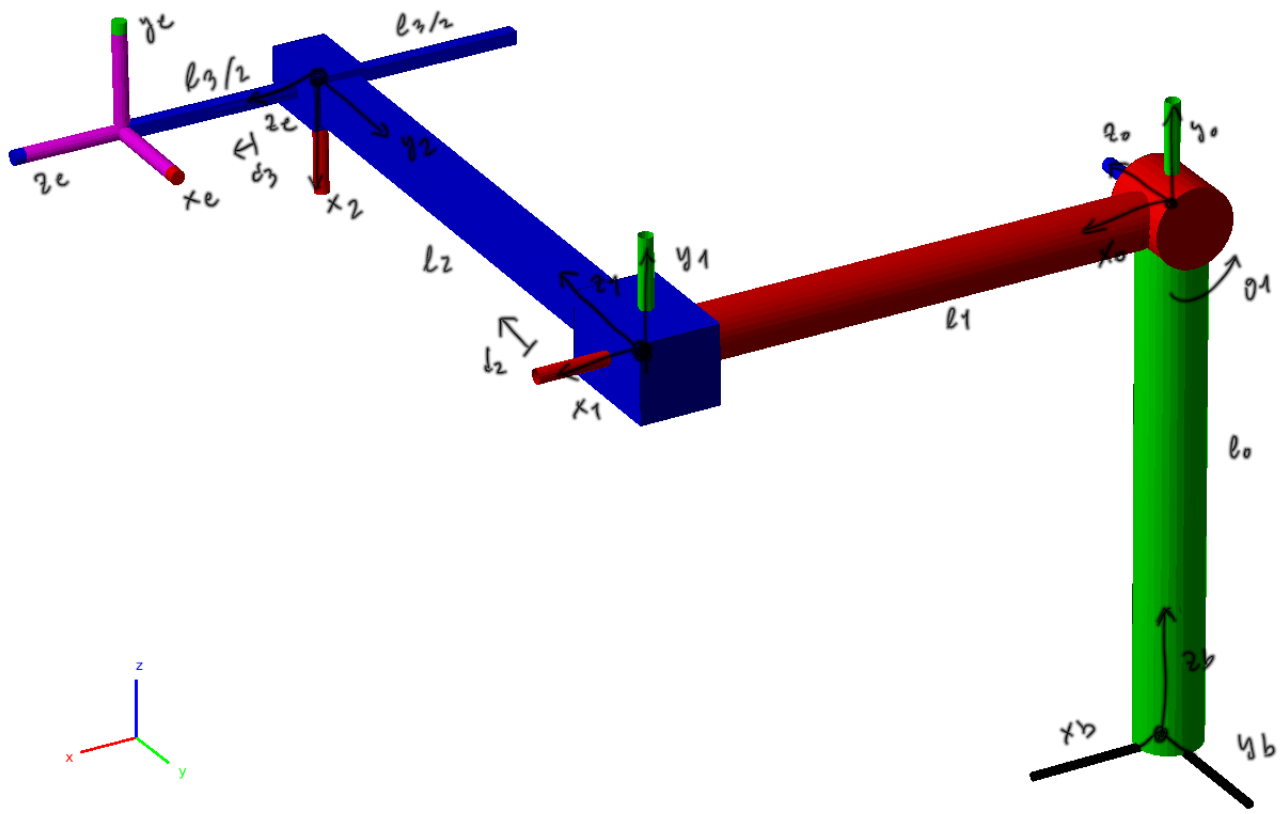
Filippo Grotto VR460638

November 20, 2021

Contents

1 Kinematics

1.1 Direct Kinematics



Lets define the DH table for our manipulator:

Σ_i	d_i	θ_i	a_i	α_i
$b-0$	ℓ_0	0	0	$\frac{\pi}{2}$
$0-1$	0	θ_1	ℓ_1	0
$1-2$	ℓ_2+d_2	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$
$2-3$	ℓ_3+d_3	$\frac{\pi}{2}$	0	0
$3-e$	0	0	0	0

The homogenous transformation is defined according to the following matrix and calculated for each row of the DH table. By multiplying $H_0^b H_1^0 H_2^1 H_3^2 H_e^3$ we obtain the final transformation

$$H_i^{i-1}(q_i) = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \ell_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_1^0(\theta_1) = \begin{bmatrix} c_1 & -s_1 & 0 & \ell_1 c_1 \\ s_1 & c_1 & 0 & \ell_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_2^1(d_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 + \ell_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2(d_3) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 + l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_e^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_e^b(\boldsymbol{q}) = \begin{bmatrix} 0 & -s_1 & c_1 & c_1(\ell_1 + \ell_3 + d_3) \\ -1 & 0 & 0 & -\ell_2 - d_2 \\ 0 & -c_1 & s_1 & s_1(\ell_1 + \ell_3 + d_3) + \ell_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2 Inverse Kinematics

Let's consider the position of ee with respect of the base frame to calculate the value of the joints.

$$p_e^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1(\ell_1 + \ell_3 + d_3) \\ -\ell_2 - d_2 \\ s_1(\ell_1 + \ell_3 + d_3) + \ell_0 \\ 1 \end{bmatrix}$$

It is easy to see that

$$d_2 = -\ell_2 - y$$

$$\theta_1 = Atan2(z - \ell_0, x)$$

For d_3 we can apply sum of squares and the result is:

$$d_3 = -\ell_1 \pm \sqrt{x^2 + (z - \ell_0)^2} - \ell_3$$

2 Differential Kinematics

2.1 Geometric Jacobians

The geometric jacobian is defined as follow with $\boldsymbol{q} = [\theta_1, d_2, d_3]^\top$. Note that the matlab robotic toolbox defines the angular velocities above the linear velocities:

$$\begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \left[\begin{array}{c|c|c} J_{P_1} & J_{P_2} & J_{P_3} \\ J_{O_1} & J_{O_2} & J_{O_3} \end{array}\right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$J_{P_1} = z_0 \times (d_e^0 - d_0^0) = \begin{bmatrix} -s_1(\ell_1 + \ell_3 + d_3) \\ 0 \\ c_1(\ell_1 + \ell_3 + d_3) \end{bmatrix} \qquad J_{O_1} = z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{P_2} = z_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad J_{O_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{P_3} = z_2 = \begin{bmatrix} c_1 \\ 0 \\ s_1 \end{bmatrix} \qquad J_{O_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can finally put all the pieces together and obtain the final geometric jacobian:

$$J(\boldsymbol{q}) = \begin{bmatrix} -s_1(\ell_1 + \ell_3 + d_3) & 0 & c1 \\ 0 & -1 & 0 \\ c_1(\ell_1 + \ell_3 + d_3) & 0 & s1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

2.2 Analytical Jacobian

The analytical jacobian can be easily calculated by using partial derivatives of p_e^b

$$p_e^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1(\ell_1 + \ell_3 + d_3) \\ -\ell_2 - d_2 \\ s_1(\ell_1 + \ell_3 + d_3) + \ell_0 \\ 1 \end{bmatrix}$$

Finally we end up with the analytical jacobian

$$Ja(\boldsymbol{q}) = \begin{bmatrix} -s_1(\ell_1 + \ell_3 + d_3) & 0 & c1 \\ 0 & -1 & 0 \\ c_1(\ell_1 + \ell_3 + d_3) & 0 & s1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Another possibility is to use the relation between the geometric and analytical jacobian as follow using ZYZ:

$$\omega_e = T(\phi_e)\dot{\phi}_e \qquad T(\phi_e) = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\theta \\ 0 & c_\varphi & s_\varphi s_\theta \\ 1 & 0 & c_\theta \end{bmatrix}$$

$$J(\boldsymbol{q}) = T_A(\phi_e) J_A(\boldsymbol{q})$$

$$T_A(\phi_e) = \begin{bmatrix} \mathbb{I}_3 & \varnothing_3 \\ \varnothing_3 & T(\phi_e) \end{bmatrix}$$

3 Lagrangian formulation

Let's calculate p_{ℓ_i} of the center of mass wrt of Σ_0. To get them let's calculate $p^i_{\ell_i}$ of the center of mass wrt of Σ_i

$$p^1_{\ell_1} = \begin{bmatrix} -\frac{\ell_1}{2} \\ 0 \\ 0 \end{bmatrix} \qquad p^2_{\ell_2} = \begin{bmatrix} 0 \\ \frac{\ell_2}{2} \\ 0 \end{bmatrix} \qquad p^3_{\ell_3} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\ell_3}{2} \end{bmatrix}$$

we can express the homogenous wrt of Σ_0 using the following formula:

$$p_{\ell_i} = R^0_i p^i_{\ell_i} + d^0_i$$

3.1 Potential Energy

The potential energy is calculated according to the formula:

$$U_i = -m_{l_i} g_0^T p_{l_i} \qquad g_0 = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

The total potential energy is the sum of the 3 contributions U_1 U_2 and U_3. The total expression is reported and was calculated using the MATLAB symbolic toolbox (L_{ih} is the length of i-th link and m_i is the mass)

$$U = \frac{gsin(\theta_1)(l_1 m_1 + 2l_1 m_2 + 2l_1 m_3 + l_3 m_3 + 2d_3 m_3)}{2}$$

3.2 Kinetic Energy

The kinetic energy is calculated using the following formula:

$$\mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} \dot{\boldsymbol{q}}^\top B(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

$$B(\boldsymbol{q}) = \sum_{i=1}^{n} B_i(\boldsymbol{q}) = \sum_{i=1}^{n} m_{\ell_i} \left({J_P^{\ell_i}}^\top J_P^{\ell_i} \right) + \left({R_i^0}^\top J_O^{\ell_i} \right)^\top I^i_{\ell_i} \left({R_i^0}^\top J_O^{\ell_i} \right)$$

It is necessary to calculate the inertia tensors $I^i_{\ell_i}$ and the partial jacobians $J_P^{\ell_i}$ and $J_O^{\ell_i}$. We will use the steiner theorem because all frames Σ_i are translated of $p^i_{\ell_i}$ w.r.t. of the center of mass (i.e inertia tensor w.r.t. of the axis of the joint that the link is attached).

$$I^1_{\ell_1} = I^{C_1}_{\ell_1} + m_{\ell_1} S^T(r) S(r) = I^{C_1}_{\ell_1} + m_{\ell_1} \left(r^\top r \mathbb{I}_{3,3} - r r^\top \right)$$

For the inertia tensors we can use the following formulas for the cylindrical and prismatic links considering that the prismatic links have a square base.

$$I^C_{cylinder} = \frac{1}{2} \begin{bmatrix} m(a^2 + b^2) & 0 & 0 \\ 0 & m(3(a^2 + b^2) + h^2) & 0 \\ 0 & 0 & m(3(a^2 + b^2) + h^2) \end{bmatrix}$$

$$I^C_{prismatic} = \frac{1}{12}\begin{bmatrix} m(b^2+c^2) & 0 & 0 \\ 0 & m(a^2+c^2) & 0 \\ 0 & 0 & m(a^2+b^2) \end{bmatrix}$$

Finally we need to compute the partial jacobians in order to calculate velocity of intermediate links.

$$J^{\ell_i}_{P_j} = \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_{l_i} - p_{j-1}) & \text{revolute joint} \end{cases} \qquad J^{\ell_i}_{O_j} = \begin{cases} 0 & \text{prismatic joint} \\ z_{j-1} & \text{revolute joint} \end{cases}$$

In our case the computer partial jacobians are:

$$J^{\ell_1}_P = \begin{bmatrix} -\ell_1 sin(\theta_1)/2 & 0 & 0 \\ \ell_1 cos(\theta_1)/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J^{\ell_2}_P = \begin{bmatrix} \ell_2 cos(\theta_1)/2 - \ell_1 sin(\theta_1) & 0 & 0 \\ \ell_1 cos(\theta_1) + (\ell_2 sin(\theta_1))/2 & 0 & 0 \\ 0 & 1 & 0; \end{bmatrix}$$

$$J^{\ell_3}_P = \begin{bmatrix} -sin(\theta_1)(\ell_1 + \ell_3/2 + d_3) & 0 & cos(\theta_1) \\ cos(\theta_1)(\ell_1 + \ell_3/2 + d_3) & 0 & sin(\theta_1) \\ 0 & 1 & 0 \end{bmatrix} \qquad J^{\ell_1}_O = J^{\ell_2}_O = J^{\ell_3}_O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B(\mathbf{q}) = B_1(\mathbf{q}) + B_2(\mathbf{q}) + B_3(\mathbf{q})$$

Finally we can recover the kinetic energy using the calculated $B(\mathbf{q})$ and $\dot{\mathbf{q}}$.

3.3 Dynamic Model of the manipulator

The aim is to find an expression that describes the dynamic model of the manipulator:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

The matrix $B(q)$ was previously calculated as a sum of the contributions of each link and $g(q)$ can be easily derived by differentiating U by the generalized positions $q = [\theta_1, d_2, d_3]$. In order to recover $C(q,\dot{q})$ some additional steps are required and described as follows:

$$\begin{aligned} \sum_{j=1}^{n} c_{ij}(q)\dot{q}_j &= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}\dot{q}_k\dot{q}_j \\ &= \sum_{j=1}^{n} c_{ij}\dot{q}_j \end{aligned} \tag{1}$$

A generalized formulation for the dynamic model of the manipulator is

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s sign(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

3.4 Recursive Newton Euler

The recursive newton euler has been implemented for the RPP robot. The calculations are lengthy and are not reported here. The results have been compared with the lagrangian model and using the following facts:

$$g(q) = NE(q,0,0,g_0)$$
$$C(q,\dot{q})\dot{q} = NE(q,\dot{q},0,0)$$
$$B_i(q) = NE(q,0,e_i,0) \qquad e_i = \text{i-th element equal to 1}$$

4 Control architectures

4.1 Joint Space PD Control with Gravity Compensation

The joint space PD Control with gravity compensation was implemented using an S-function to define the manipulator dynamics. In fact the symbolic B,C and G matrixes were used in this context. The values for Kp and Kd has been properly selected for our robot. In Fig 1 a plot of the positions with respect of the desired positions are reported as well as the related tau applied to each of the three joints.

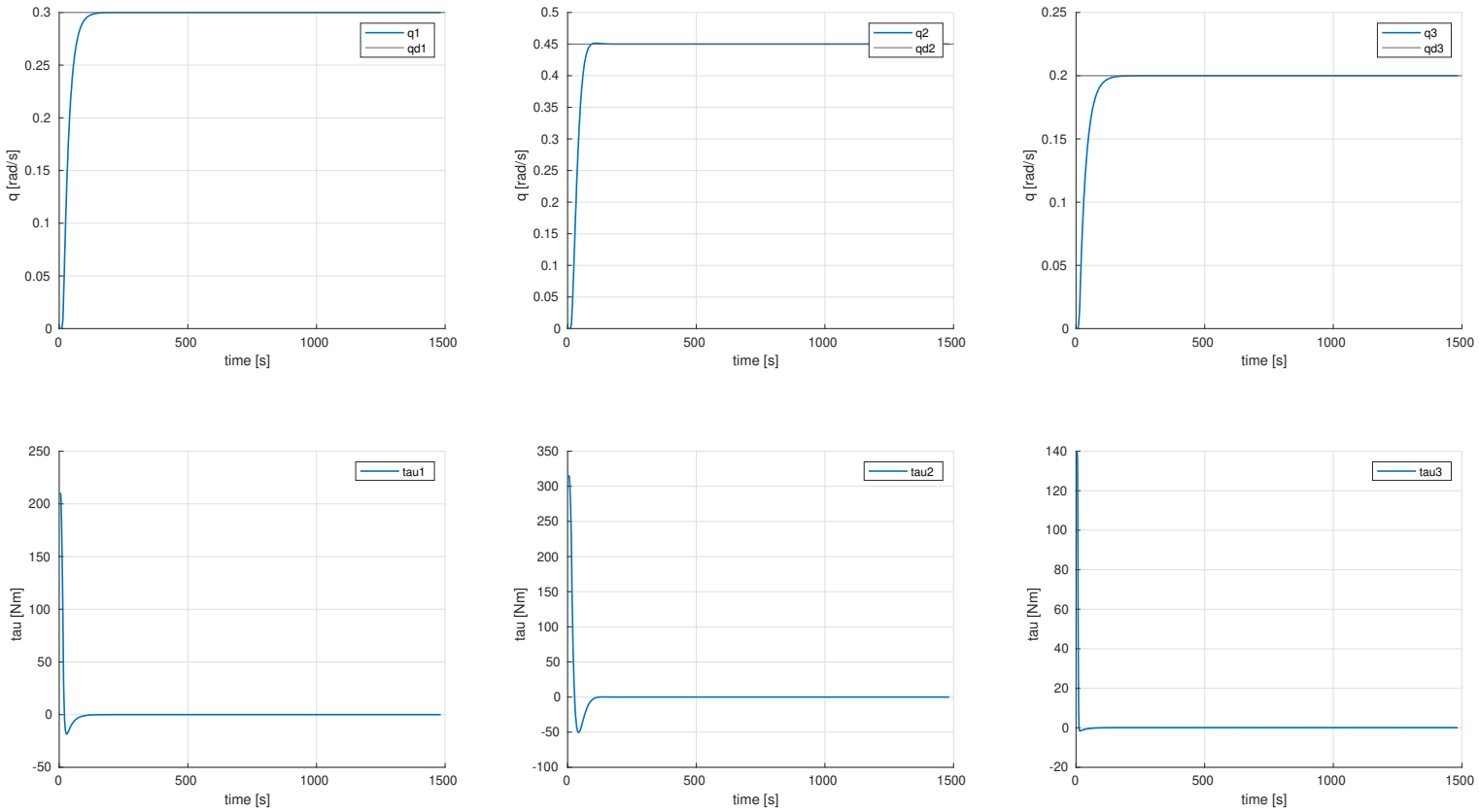


Figure 1: Joint Space PD Control with Gravity Compensation