Advanced Control Systems: RPP manipulator

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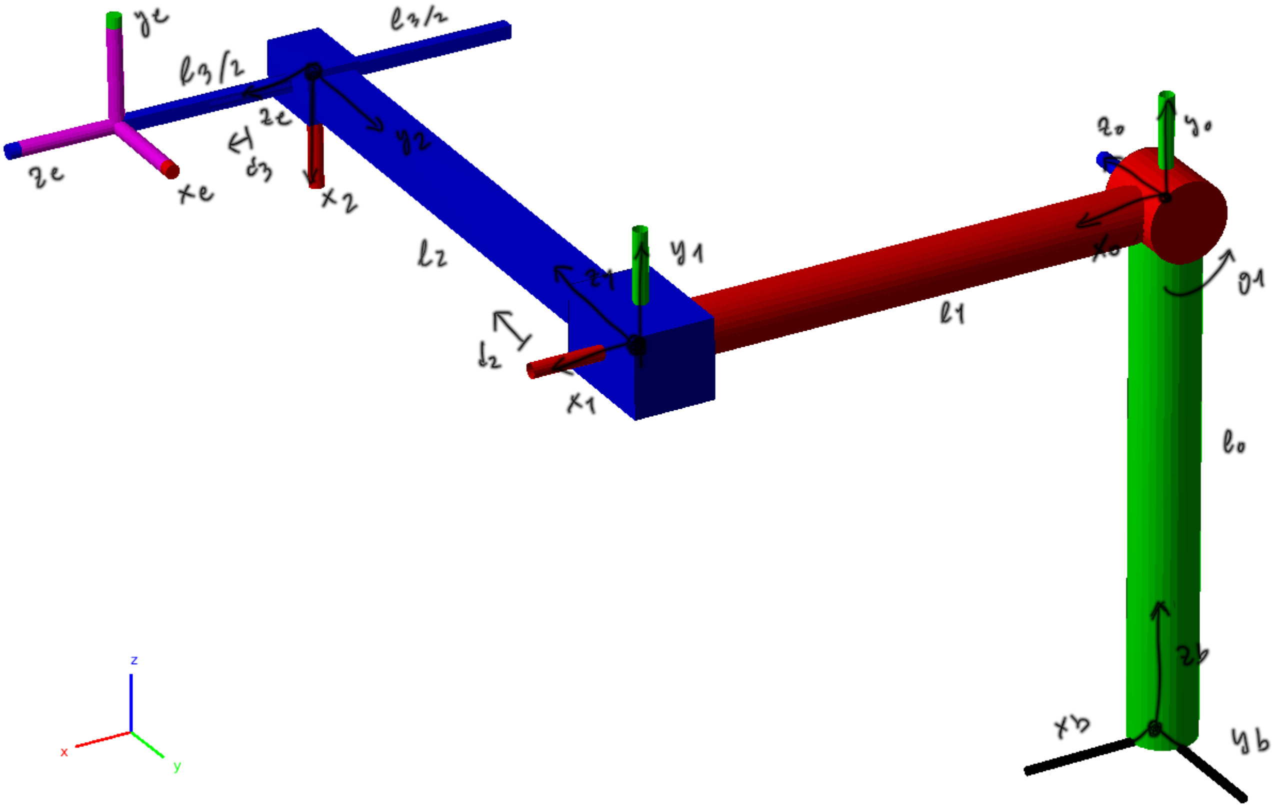
1 Kinematics

1.1 Direct Kinematics



Lets define the DH table for our manipulator:

Σ_i	d_i	θ_i	a_i	α_i
$b-0$	ℓ_0	0	0	$\frac{\pi}{2}$
$0-1$	0	θ_1	ℓ_1	0
$1-2$	ℓ_2+d_2	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$
$2-3$	ℓ_3+d_3	$\frac{\pi}{2}$	0	0
$3-e$	0	0	0	0

The homogenous transformation is defined according to the following matrix and calculated for each row of the DH table. By multiplying $H_0^b H_1^0 H_2^1 H_3^2 H_e^3$ we obtain the final transformation

$$H_i^{i-1}(q_i) = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \ell_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_1^0(\theta_1) = \begin{bmatrix} c_1 & -s_1 & 0 & \ell_1 c_1 \\ s_1 & c_1 & 0 & \ell_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_2^1(d_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 + \ell_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2(d_3) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 + l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_e^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_e^b(\boldsymbol{q}) = \begin{bmatrix} 0 & -s_1 & c_1 & c_1(\ell_1 + \ell_3 + d_3) \\ -1 & 0 & 0 & -\ell_2 - d_2 \\ 0 & -c_1 & s_1 & s_1(\ell_1 + \ell_3 + d_3) + \ell_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2 Inverse Kinematics

Let's consider the position of ee with respect of the base frame to calculate the value of the joints.

$$p_e^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1(\ell_1 + \ell_3 + d_3) \\ -\ell_2 - d_2 \\ s_1(\ell_1 + \ell_3 + d_3) + \ell_0 \\ 1 \end{bmatrix}$$

It is easy to see that

$$d_2 = -\ell_2 - y$$

$$\theta_1 = Atan2(z - \ell_0, x)$$

For d_3 we can apply sum of squares and the result is:

$$d_3 = -\ell_1 \pm \sqrt{x^2 + (z - \ell_0)^2} - \ell_3$$

2 Differential Kinematics

2.1 Geometric Jacobians

The geometric jacobian is defined as follow with $\boldsymbol{q} = [\theta_1, d_2, d_3]^\top$. Note that the matlab robotic toolbox defines the angular velocities above the linear velocities:

$$\begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \left[\begin{array}{c|c|c} J_{P_1} & J_{P_2} & J_{P_3} \\ J_{O_1} & J_{O_2} & J_{O_3} \end{array} \right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$J_{P_1} = z_0 \times (d_e^0 - d_0^0) = \begin{bmatrix} -s_1(\ell_1 + \ell_3 + d_3) \\ 0 \\ c_1(\ell_1 + \ell_3 + d_3) \end{bmatrix} \qquad J_{O_1} = z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{P_2} = z_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad J_{O_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{P_3} = z_2 = \begin{bmatrix} c_1 \\ 0 \\ s_1 \end{bmatrix} \qquad J_{O_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can finally put all the pieces together and obtain the final geometric jacobian:

$$J(\boldsymbol{q}) = \begin{bmatrix} -s_1(\ell_1 + \ell_3 + d_3) & 0 & c1 \\ 0 & -1 & 0 \\ c_1(\ell_1 + \ell_3 + d_3) & 0 & s1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

2.2 Analytical Jacobian

The analytical jacobian can be easily calculated by using partial derivatives of p_e^b

$$p_e^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1(\ell_1 + \ell_3 + d_3) \\ -\ell_2 - d_2 \\ s_1(\ell_1 + \ell_3 + d_3) + \ell_0 \\ 1 \end{bmatrix}$$

Finally we end up with the analytical jacobian

$$Ja(\boldsymbol{q}) = \begin{bmatrix} -s_1(\ell_1 + \ell_3 + d_3) & 0 & c1 \\ 0 & -1 & 0 \\ c_1(\ell_1 + \ell_3 + d_3) & 0 & s1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Another possibility is to use the relation between the geometric and analytical jacobian as follow using ZYZ:

$$\omega_e = T(\phi_e)\dot{\phi}_e \qquad T(\phi_e) = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\theta \\ 0 & c_\varphi & s_\varphi s_\theta \\ 1 & 0 & c_\theta \end{bmatrix}$$

$$J(\boldsymbol{q}) = T_A(\phi_e) J_A(\boldsymbol{q})$$

$$T_A(\phi_e) = \begin{bmatrix} \mathbb{I}_3 & \emptyset_3 \\ \emptyset_3 & T(\phi_e) \end{bmatrix}$$

3 Lagrangian formulation

Let's calculate p_{ℓ_i} of the center of mass wrt of Σ_0. To get them let's calculate $p^i_{\ell_i}$ of the center of mass wrt of Σ_i

$$p^1_{\ell_1} = \begin{bmatrix} -\frac{\ell_1}{2} \\ 0 \\ 0 \end{bmatrix} \qquad p^2_{\ell_2} = \begin{bmatrix} 0 \\ -\frac{\ell_2}{2} \\ 0 \end{bmatrix} \qquad p^3_{\ell_3} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\ell_3}{2} \end{bmatrix}$$

we can express the homogenous wrt of Σ_0 using the following formula:

$$p_{\ell_i} = R^0_i p^i_{\ell_i} + d^0_i$$

3.1 Potential Energy

The potential energy is calculated according to the formula:

$$U_i = -m_{l_i} g_0^T p_{l_i} \qquad g_0 = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

The total potential energy is the sum of the 3 contributions U_1 U_2 and U_3. The total expression is reported and was calculated using the MATLAB symbolic toolbox (L_{ih} is the length of i-th link and m_i is the mass)

$$U = \frac{-g sin(\theta_1)(l_1 m_1 + 2 l_1 m_2 + 2 l_1 m_3 + l_3 m_3 + 2 d_3 m_3)}{2}$$

3.2 Kinetic Energy

The kinetic energy is calculated using the following formula:

$$\mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} \dot{\boldsymbol{q}}^\top B(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

$$B(\boldsymbol{q}) = \sum_{i=1}^{n} B_i(\boldsymbol{q}) = \sum_{i=1}^{n} m_{\ell_i} (J_P^{\ell_i \top} J_P^{\ell_i}) + (R_i^{0\top} J_O^{\ell_i})^\top I^i_{\ell_i} (R_i^{0\top} J_O^{\ell_i})$$

It is necessary to calculate the inertia tensors $I^i_{\ell_i}$ and the partial jacobians $J_P^{\ell_i}$ and $J_O^{\ell_i}$. We will use the steiner theorem because all frames Σ_i are translated of $p^i_{\ell_i}$ w.r.t. of the center of mass (i.e inertia tensor w.r.t. of the axis of the joint that the link is attached).

$$I^1_{\ell_1} = I^{C_1}_{\ell_1} + m_{\ell_1} S^T(r) S(r) = I^{C_1}_{\ell_1} + m_{\ell_1} (r^\top r \mathbb{I}_{3,3} - r r^\top)$$

For the inertia tensors we can use the following formulas for the cylindrical and prismatic links considering that the prismatic links have a square base.

$$I^C_{cylinder} = \frac{1}{2} \begin{bmatrix} m(a^2 + b^2) & 0 & 0 \\ 0 & m(3(a^2+b^2)+h^2) & 0 \\ 0 & 0 & m(3(a^2+b^2)+h^2) \end{bmatrix}$$

$$I^C_{prismatic} = \frac{1}{12}\begin{bmatrix} m(b^2+c^2) & 0 & 0 \\ 0 & m(a^2+c^2) & 0 \\ 0 & 0 & m(a^2+b^2) \end{bmatrix}$$

Finally we need to compute the partial jacobians in order to calculate velocity of intermediate links.

$$J^{\ell_i}_{P_j} = \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_{l_i} - p_{j-1}) & \text{revolute joint} \end{cases} \qquad J^{\ell_i}_{O_j} = \begin{cases} 0 & \text{prismatic joint} \\ z_{j-1} & \text{revolute joint} \end{cases}$$

In our case the computer partial jacobians are:

$$J^{\ell_1}_P = \begin{bmatrix} -\ell_1 sin(\theta_1)/2 & 0 & 0 \\ \ell_1 cos(\theta_1)/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J^{\ell_2}_P = \begin{bmatrix} \ell_2 cos(\theta_1)/2 - \ell_1 sin(\theta_1) & 0 & 0 \\ \ell_1 cos(\theta_1) + (\ell_2 sin(\theta_1))/2 & 0 & 0 \\ 0 & 1 & 0; \end{bmatrix}$$

$$J^{\ell_3}_P = \begin{bmatrix} -sin(\theta_1)(\ell_1 + \ell_3/2 + d_3) & 0 & cos(\theta_1) \\ cos(\theta_1)(\ell_1 + \ell_3/2 + d_3) & 0 & sin(\theta_1) \\ 0 & 1 & 0 \end{bmatrix} \qquad J^{\ell_1}_O = J^{\ell_2}_O = J^{\ell_3}_O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B(\mathbf{q}) = B_1(\mathbf{q}) + B_2(\mathbf{q}) + B_3(\mathbf{q})$$

Finally we can recover the kinetic energy using the calculated $B(\mathbf{q})$ and $\dot{\mathbf{q}}$.

3.3 Dynamic Model of the manipulator

The aim is to find an expression that describes the dynamic model of the manipulator:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

The matrix $B(q)$ was previously calculated as a sum of the contributions of each link and $g(q)$ can be easily derived by differentiating U by the generalized positions $q = [\theta_1, d_2, d_3]$. In order to recover $C(q,\dot{q})$ some additional steps are required and described as follows:

$$\begin{aligned} \sum_{j=1}^{n} c_{ij}(q)\dot{q}_j &= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i})\dot{q}_k\dot{q}_j \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}\dot{q}_k\dot{q}_j \\ &= \sum_{j=1}^{n} c_{ij}\dot{q}_j \end{aligned} \tag{1}$$

A generalized formulation for the dynamic model of the manipulator is

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s sign(\dot{q}) + g(q) = \tau - J^T(q)h_e$$