

## Entrega: data

### Exercício

1. Define the  $\ell^1$ -**norm** on  $\mathbb{R}^n$  by

$$\|x\|_1 = \sum_{i=1}^n |x^i|,$$

and define the **sup-norm** on  $\mathbb{R}^n$  by

$$\|x\|_\infty = \sup \{|x^i|\}.$$

Show that these satisfy Theorem 1.

*Demonstração.*

□

2. Prove that  $\|x\| \leq \sum_{i=1}^n |x^i|$ . In other words, the usual norm is no greater than the  $\ell^1$ -norm.

*Demonstração.*

□

3. Prove that  $\|x - y\| \leq \|x\| + \|y\|$ . (Compare this with part (2) of Theorem 1.) When does equality hold?
4. Prove that  $\left| \|x\| - \|y\| \right| \leq \|x - y\|$ .
5. The quantity  $\|y - x\|$  is called the **distance** between  $x$  and  $y$ . Prove and interpret the “triangle inequality”:

$$\|z - x\| \leq \|z - y\| + \|y - x\|.$$

6. Let  $f$  and  $g$  be integrable on  $[a, b]$ .

(a) Prove the integral version of the Cauchy-Schwarz inequality:

$$\left| \int_a^b fg \right| \leq \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2}.$$

Hint: Consider separately the cases  $0 = \int_a^b (f - tg)^2$  for some  $t \in \mathbb{R}$ , and  $0 < \int_a^b (f - tg)^2$  for all  $t \in \mathbb{R}$ .

- (b) If equality holds, must  $f = tg$  for some  $t \in \mathbb{R}$ ? What if  $f$  and  $g$  are continuous?
  - (c) Show that the Cauchy-Schwarz inequality is a special case of (a).
7. A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is **norm preserving** if

$$\|T(x)\| = \|x\|,$$

for all  $x \in \mathbb{R}^n$ , and **inner product preserving** if

$$\langle Tx, Ty \rangle = \langle x, y \rangle,$$

for all  $x, y \in \mathbb{R}^n$ .

- (a) Prove that  $T$  is norm preserving if and only if it is inner product preserving.

- (b) Prove that such a linear transformation is 1-1, and  $T^{-1}$  is norm preserving (and inner product preserving).
8. If  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear transformation, show that there is a number  $M$  such that  $\|T(h)\| \leq M\|h\|$  for all  $h \in \mathbb{R}^m$ . Hint: Estimate  $\|T(h)\|$  in terms of  $\|h\|$  and the entries in the matrix for  $T$ .
9. If  $x, y \in \mathbb{R}^n$ , and  $z, w \in \mathbb{R}^m$ , show that  $\langle (x, z), (y, w) \rangle = \langle x, y \rangle + \langle z, w \rangle$ , and  $\|(x, z)\| = \sqrt{\|x\|^2 + \|z\|^2}$ . Note that  $(x, z)$  and  $(y, w)$  denote points in  $\mathbb{R}^{n+m}$ .
10. If  $x, y \in \mathbb{R}^n$ , then  $x$  and  $y$  are called **perpendicular** (or **orthogonal**), and we write  $x \perp y$ , if  $\langle x, y \rangle = 0$ . If  $x \perp y$ , prove that  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

## Exercises for Section 1.2: More Topology: Open and Closed Sets in $\mathbb{R}^n$

1. Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence of finitely many) open sets is open. Give a counterexample for the intersection of infinitely many open sets.
2. If  $A \subset B \subset \mathbb{R}^n$ , prove that
 
$$\text{cl}A \subset \text{cl}B, \quad \text{and} \quad \text{int}A \subset \text{int}B.$$
3. Prove that if  $B$  is an open subset of  $A$ , then  $B \subset \text{int}(A)$ . Note that this says that  $\text{int}(A)$  is the largest open subset of  $A$ .
4. Prove that the  $n$ -dimensional ball centered at  $a$  of radius  $r$ ,

$$B^n(a; r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

is open.

5. Find the interior, exterior, and boundary of the sets:

$$B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\},$$

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\},$$

$$\mathbb{Q}^n = \{x \in \mathbb{R}^n : \text{each } x^i \text{ is rational}\}.$$

**Solution.**

6. If  $A \subset [0, 1]$  is the union of open intervals  $(a_i, b_i)$  such that each rational number in  $(0, 1)$  is contained in some  $(a_i, b_i)$ , show that  $\partial A = [0, 1] - A$ .
7. If  $A$  is a closed set that contains every rational number  $r \in [0, 1]$ , show that  $[0, 1] \subset A$ .
8. Graph generic open balls in  $\mathbb{R}^2$  with respect to each of the “non-Euclidean” norms,  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$ . What shapes are they?

**Solution.**