

Gemini 2.5 Pro Capable of Winning Gold at IMO 2025*

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Abstract

The International Mathematical Olympiad (IMO) poses uniquely challenging problems requiring deep insight, creativity, and formal reasoning. While Large Language Models (LLMs) perform well on mathematical benchmarks like AIME, they struggle with Olympiad-level tasks. We use Google’s Gemini 2.5 Pro on the newly released IMO 2025 problems, avoiding data contamination. Using a self-verification pipeline with careful prompt design, 5 (out of 6) problems are solved correctly. This result underscores the importance of developing optimal strategies to harness the full potential of powerful LLMs for complex reasoning tasks.

1 Introduction

The International Mathematical Olympiad (IMO) [1] is an esteemed annual competition that convenes the world’s most talented pre-university mathematicians. Established in Romania in 1959 with just seven participating countries, it has since expanded to include over 100 nations, each represented by a team of up to six contestants. Held annually, with the sole exception of 1980, the IMO challenges participants with exceptionally difficult problems in fields like algebra, geometry, number theory, and combinatorics. Contestants are given two 4.5-hour sessions over two days to solve three problems per session, each graded out of seven points. Unlike typical mathematical exercises, IMO problems demand profound insight, originality, and the ability to synthesize diverse mathematical concepts. This emphasis on creative, proof-based reasoning makes the IMO a hallmark of mathematical excellence and a vital platform for identifying future leaders in the field.

Consequently, the IMO has also become a grand challenge and a formidable benchmark for evaluating the advanced reasoning capabilities of Artificial Intelligence, particularly Large Language Models (LLMs), providing a rigorous test of their ability to perform complex, multi-step logical deduction rather than rote calculation [9, 15, 3]. Traditional benchmarks

*Code available at: <https://github.com/lyang36/IMO25>

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like GSM8K and MATH focus on grade-school and high-school level problems, respectively, where LLMs have achieved high performance through pattern recognition and retrieval from training data [6, 10]. However, IMO problems surpass these in complexity, requiring multi-step reasoning, abstraction, and innovation akin to human expert-level cognition, thereby exposing limitations in LLMs’ generalization and vulnerability to hallucinations or superficial heuristics [7]. This positions the IMO as an ideal probe for assessing whether LLMs can truly “reason” rather than merely replicate memorized solutions, addressing concerns about their reliability in high-stakes domains like scientific discovery and formal verification [19].

The pursuit of automated mathematical reasoning has seen remarkable progress with the advent of LLMs [4, 16]. Early successes on foundational benchmarks have rapidly escalated to tackling complex, competition-level mathematics. This progress has been significantly propelled by innovations such as Chain-of-Thought (CoT) prompting, which enables models to generate intermediate reasoning steps, thereby improving performance on tasks requiring complex logic and calculation [18]. Nevertheless, even state-of-the-art models have demonstrated significant limitations when confronted with Olympiad-level problems. For example, recent evaluations on problems from the USA Mathematical Olympiad (USAMO) 2025 and IMO 2025 showed that top-tier public models still struggle to produce sound, rigorous proofs and fail to achieve scores comparable to human medalists, often succumbing to logical fallacies and a lack of creative insight [13, 3]. This highlights a critical gap between generating numerically correct answers and constructing logically sound arguments [12].

In this paper, we construct a self-verification pipeline with careful prompt design and implemented using the Gemini 2.5 Pro model, a strong base model released by Google [?]. We solved 5 out of the 6 problems of IMO 2025. A persistent and critical challenge in the evaluation of LLMs is the issue of data contamination, where test data from public benchmarks is inadvertently included in the vast pre-training corpora, leading to inflated and unreliable performance metrics [5]. To ensure a rigorous and uncontaminated assessment of the model’s genuine problem-solving capabilities, this work exclusively utilizes the problems from the most recent IMO 2025 competition. As these problems were released only days before our evaluation, they serve as a pristine testbed, mitigating the risk of data leakage and providing a robust measure of the model’s ability to generalize and reason on genuinely unseen challenges. Our approach shows that strong existing models are already capable of solving difficult math reasoning problems, but directly using them can result in poor results as shown in [3]. Our results demonstrate a significant advance in automated mathematical reasoning.

Concurrent with our work, other teams also reported high-level performance on the IMO 2025 problems. These include OpenAI [17], Google DeepMind [11], and ByteDance [2].

2 Methods

2.1 Pipeline

At a high level, our pipeline proceeds as follows (illustrated in Figure 1):

- Step 1: Initial solution generation with the prompt in Section 3.1;
- Step 2: Self-improvement;
- Step 3: Verifying the solution with the prompt in Section 3.2 and generating a bug report; go to Step 4 or Step 6 (see below for explanations);
- Step 4: Review of the bug report (optional);
- Step 5: Correcting or improving the solution based on the bug report; go to Step 3;
- Step 6: Accept or Reject.

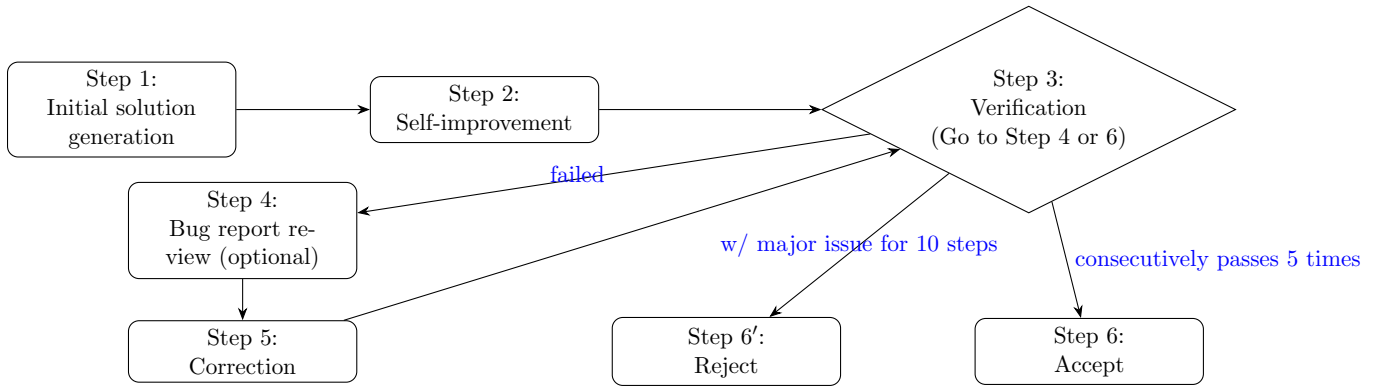


Figure 1: Flow diagram of our pipeline. See the main text for detailed explanations of each step.

We run the procedure some number of times (in parallel or in serial, independently) to obtain a correct solution. We hope the model either outputs the correct solution, or claims that it fails to identify the solution.

2.2 Solver

The solver prompt in Section 3.1 for Step 1 is designed to emphasize rigor rather than focus on finding the final answer and thus matches the theme of IMO. We have randomly selected some outputs of this step and found that the overall quality of the solutions are pretty low. This is consistent with very recent findings of Ref. [3].

In Step 2, the model is prompted to review and try to improve its work. While Gemini 2.5 Pro is good at mathematics, as a general-purpose LLM, it is not tailored to solving especially challenging mathematical problems. One significant constraint is the thinking budget. Note that thinking is quite token consuming: Even a trivial fact might take a few thousand tokens for the model to prove. The maximum number of thinking tokens of Gemini 2.5 Pro is 32768, which is not enough for solving a typical IMO problem. We observe that in Step 1, the model almost always uses up its thinking budget. Thus, the model does not even have the capacity

to fully solve the problem. This is why we choose to break down the problem solving process into steps. Step 2 effectively injects another budget of 32768 thinking tokens to allow the model review and continue its work. We keep monitoring the entire process and do observe that the outputs have been noticeably improved during Step 2.

Next we will use the verifier to make iterative improvement and decide whether to accept an improved solution.

2.3 Verifier

The verifier plays an important role in our pipeline. Its functionality is to carefully review a solution step by step and find out issues (if any). We emphasize mathematical rigor and classify issues into critical errors and justification gaps. Critical errors are something that is demonstratively false or with clear logical fallacies, while justification gaps can be major or minor. A major justification gap that cannot be repaired would crash an entire proof, while minor justification gaps may not even be well defined: A minor gap could sometimes be viewed as concise argument.

In Step 3, we use the verifier to generate a bug report for each solution outputted in Step 2. The bug report contains a list of issues classified as critical errors or justification gaps. For each issue, an explanation is required. The bug report will serve as useful information for the model to improve the solution, either fixing errors or filling gaps. Step 4 (optional) is to carefully review each issue in the bug report. If the verifier makes a mistake and reports an issue which is not really an issue, the issue would be deleted from the bug report. Thus, Step 4 increases the reliability of the bug report. In Step 5, the model tries to improve the solution based on the bug report. We iterate Steps 3-5 a sufficient number of times until we decide to accept or decline a solution. We accept a solution if it robustly passes the verification process and decline a solution if there are always critical errors or major justification gaps during the iterations.

We observe that the verifier is quite reliable but can make mistakes. Since our major goal is not to benchmark the verifier, we do not have quantitative results on its effectiveness. However, we have used this verifier for quite a while (starting from well before IMO 2025). We have been keeping an eye on its performance and below is our qualitative observation:

- Critical errors are seldom missed by the verifier. This is consistent with the observations in Refs. [14, 8]. In the unlikely event such errors are not caught, simply running the verifier a few more times would very likely catch it. This is good because we do not wish to miss critical errors.
- If the verifier reports a critical error, it may not always be critical, but it almost always needs some revision.
- The verifier may report some justification gaps which are only slightly beyond trivial statements and thus are not really gaps for mathematicians.

Indeed, our system is quite robust to errors made by the verifier. We iteratively use the verifier a sufficiently number of times. If it misses an error in one iteration, it still has some

probability to catch it in the next iteration. Also, if it claims an error which is actually not an error, such a false negative may not go through the bug report review step (Step 4). Furthermore, we instruct the model (who generates the solution) to review each item in the bug report. If the model does not agree with a particular item, it is encouraged to revise its solution to minimize misunderstanding. This is analogous to the peer review process. If a referee makes a wrong judgment, the authors are encouraged to revise the paper. Ultimately, the presentation is improved.

At the time we plan to accept a solution, we do not wish the verifier misses any issue; we run the verifier five times and accept a solution only if it passes every time.

3 Experiment Setup

We choose low temperature: 0.1. High temperature may lead to more random errors, which may be harmful. We use the maximum thinking budget (32768 reasoning tokens) of Gemini 2.5 Pro. We do not use web search (of course), code, or any other tools. We share the most important prompts below.

3.1 Step 1 Prompt

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    ### Core Instructions ###

*   **Rigor is Paramount:** Your primary goal is to produce a
    complete and rigorously justified solution. Every step in
    your solution must be logically sound and clearly
    explained. A correct final answer derived from flawed or
    incomplete reasoning is considered a failure.
*   **Honesty About Completeness:** If you cannot find a
    complete solution, you must not guess or create a
    solution that appears correct but contains hidden flaws or
    justification gaps. Instead, you should present only
    significant partial results that you can rigorously prove.
    A partial result is considered significant if it represents
    a substantial advancement toward a full solution. Examples
    include:
    *   Proving a key lemma.
    *   Fully resolving one or more cases within a logically
        sound case-based proof.
    *   Establishing a critical property of the mathematical
        objects in the problem.
    *   For an optimization problem, proving an upper or lower
        bound without proving that this bound is achievable.

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* ****Use TeX for All Mathematics:**** All mathematical variables, expressions, and relations must be enclosed in TeX delimiters (e.g., 'Let n be an integer.').

Output Format

Your response MUST be structured into the following sections, in this exact order.

****1. Summary****

Provide a concise overview of your findings. This section must contain two parts:

- * ****a. Verdict:**** State clearly whether you have found a complete solution or a partial solution.
 - * ****For a complete solution:**** State the final answer, e.g., "I have successfully solved the problem. The final answer is..."
 - * ****For a partial solution:**** State the main rigorous conclusion(s) you were able to prove, e.g., "I have not found a complete solution, but I have rigorously proven that..."
- * ****b. Method Sketch:**** Present a high-level, conceptual outline of your solution. This sketch should allow an expert to understand the logical flow of your argument without reading the full detail. It should include:
 - * A narrative of your overall strategy.
 - * The full and precise mathematical statements of any key lemmas or major intermediate results.
 - * If applicable, describe any key constructions or case splits that form the backbone of your argument.

****2. Detailed Solution****

Present the full, step-by-step mathematical proof. Each step must be logically justified and clearly explained. The level of detail should be sufficient for an expert to verify the correctness of your reasoning without needing to fill in any gaps. This section must contain ONLY the complete, rigorous proof, free of any internal commentary, alternative approaches, or failed attempts.

Self-Correction Instruction

Before finalizing your output, carefully review your "Method Sketch" and "Detailed Solution" to ensure they are clean, rigorous, and strictly adhere to all instructions provided above. Verify that every statement contributes directly to the final, coherent mathematical argument.

3.2 Verification Prompt

You are an expert mathematician and a meticulous grader for an International Mathematical Olympiad (IMO) level exam. Your primary task is to rigorously verify the provided mathematical solution. A solution is to be judged correct **only if every step is rigorously justified.** A solution that arrives at a correct final answer through flawed reasoning, educated guesses, or with gaps in its arguments must be flagged as incorrect or incomplete.

Instructions

****1. Core Instructions****

- * Your sole task is to find and report all issues in the provided solution. You must act as a ****verifier****, NOT a solver. ****Do NOT attempt to correct the errors or fill the gaps you find.****
- * You must perform a ****step-by-step**** check of the entire solution. This analysis will be presented in a ****Detailed Verification Log****, where you justify your assessment of each step: for correct steps, a brief justification suffices; for steps with errors or gaps, you must provide a detailed explanation.

****2. How to Handle Issues in the Solution****

When you identify an issue in a step, you **MUST** first classify it into one of the following two categories and then follow the specified procedure.

- * ****a. Critical Error:****

This is any error that breaks the logical chain of the proof. This includes both **logical fallacies** (e.g., claiming that 'A>B, C>D' implies 'A-C>B-D') and **factual errors** (e.g., a calculation error like '2+3=6').

* **Procedure:**

- * Explain the specific error and state that it **invalidates the current line of reasoning**.
- * Do NOT check any further steps that rely on this error.
- * You MUST, however, scan the rest of the solution to identify and verify any fully independent parts. For example, if a proof is split into multiple cases, an error in one case does not prevent you from checking the other cases.

* **b. Justification Gap:**

This is for steps where the conclusion may be correct, but the provided argument is incomplete, hand-wavy, or lacks sufficient rigor.

* **Procedure:**

- * Explain the gap in the justification.
- * State that you will **assume the step's conclusion is true** for the sake of argument.
- * Then, proceed to verify all subsequent steps to check if the remainder of the argument is sound.

3. Output Format

Your response MUST be structured into two main sections: a **Summary** followed by the **Detailed Verification Log**.

* **a. Summary**

This section MUST be at the very beginning of your response. It must contain two components:

- * **Final Verdict:** A single, clear sentence declaring the overall validity of the solution. For example: "The solution is correct," "The solution contains a Critical Error and is therefore invalid," or "The solution's approach is viable but contains several Justification Gaps."
- * **List of Findings:** A bulleted list that summarizes **every** issue you discovered. For each finding, you


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    must provide:
    *    **Location:** A direct quote of the key phrase or
        equation where the issue occurs.
    *    **Issue:** A brief description of the problem and
        its classification (**Critical Error** or
        **Justification Gap**).

*    **b. Detailed Verification Log**
    Following the summary, provide the full, step-by-step
    verification log as defined in the Core Instructions.
    When you refer to a specific part of the solution,
    **quote the relevant text** to make your reference
    clear before providing your detailed analysis of that
    part.

**Example of the Required Summary Format**
*This is a generic example to illustrate the required format.
  Your findings must be based on the actual solution provided
  below.*

**Final Verdict:** The solution is **invalid** because it
  contains a Critical Error.

**List of Findings:**
*    **Location:** "By interchanging the limit and the
    integral, we get..."
    *    **Issue:** Justification Gap - The solution
        interchanges a limit and an integral without providing
        justification, such as proving uniform convergence.
*    **Location:** "From  $A > B$  and  $C > D$ , it follows that
     $A - C > B - D$ "
    *    **Issue:** Critical Error - This step is a logical
        fallacy. Subtracting inequalities in this manner is not
        a valid mathematical operation.

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### Problem ###

[Paste the TeX for the problem statement here]

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### Solution ###

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[Paste the TeX for the solution to be verified here]
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### Verification Task Reminder ###
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Your task is to act as an IMO grader. Now, generate the  
  **summary** and the **step-by-step verification log** for  
  the solution above. In your log, justify each correct step  
  and explain in detail any errors or justification gaps you  
  find, as specified in the instructions above.
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4 Results and Discussion

Our pipeline successfully generated complete and mathematically rigorous solutions for 5 out of the 6 problems from the IMO 2025. The full, verbatim proofs, which constitute the primary evidence for this claim, are provided in Appendix A.

The solutions for Problems 3, 4, and 5 were generated without any problem-specific hints. For Problems 1 and 2, we present two distinct solutions each: one generated without a hint, and one generated with a hint. The inclusion of both solution types for Problems 1 and 2 allows for a direct analysis of the trade-off between computational sampling and explicit strategic guidance. For Problem 1, the hint was to use mathematical induction; for Problem 2, it was to use analytic geometry. In arXiv versions 1 and 2 of this paper, we presented only the hint-based solutions and hypothesized that these general hints do not provide problem-specific insight but rather serve to reduce the computational search space. We argued that a comprehensive, multi-agent system would naturally dedicate resources to exploring such standard methods. The hint-free solutions, newly presented in this version, provide strong empirical validation for this hypothesis. Our success in finding these solutions, albeit with increased sampling, confirms that the hints primarily improve efficiency rather than enabling a fundamentally new capability.

Despite the high success rate, the pipeline failed to solve Problem 6. The model’s output for this problem, provided in Appendix A.6, correctly identifies the trivial upper bound of 4048. However, its attempt to prove a matching lower bound is built on a flawed premise. The core error is the assertion that any tile must lie entirely in one of two disjoint regions: the cells to the left of the uncovered squares (C_L) or the cells to the right (C_R). This is incorrect, as a single tile can span columns that are to the left of one row’s uncovered square and to the right of another’s. This invalidates the subsequent proof.

The high success rate on these diverse problems validates the central thesis of this work: that a structured, iterative pipeline is essential for converting the latent capabilities of large models into rigorous mathematical proofs. This approach directly addresses the challenge, highlighted in recent findings by [3], that raw model capability alone is often insufficient

for such tasks. Our results demonstrate that the iterative refinement process systematically overcomes the limitations of single-pass generation, such as finite reasoning budgets and the critical errors and justification gaps that often appear in initial drafts. The verifier-guided loop, in particular, proved essential for eliciting rigorous, trustworthy, and complete mathematical arguments from the model.

5 Outlook

All results reported in this work are obtained by using Gemini 2.5 Pro only. We expect that using a diverse set of leading models (e.g, Grok 4, OpenAI-o series) would yield stronger mathematical capability.

In our pipeline, we sample solutions and then try to individually improve each solution. It is expected that a multi-agent system (analogues to Grok 4 heavy) where the strengths of different solutions can be combined would lead to stronger mathematical capability.

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A Verbatim Model Outputs for IMO 2025 Problems

This appendix contains the verbatim solutions produced by the model. To preserve authenticity, they are rendered in their original format, which is a mixture of Markdown and TeX.

A.1 Problem 1

Problem 1. A line in the plane is called *sunny* if it is not parallel to any of the x -axis, the y -axis, and the line $x + y = 0$.

Let $n \geq 3$ be a given integer. Determine all nonnegative integers k such that there exist n distinct lines in the plane satisfying both the following:

- For all positive integers a and b with $a + b \leq n + 1$, the point (a, b) is on at least one of the lines; and
- Exactly k of the lines are sunny.

A.1.1 Solution without hint

Let $P_n = \{(a, b) \in \mathbb{Z}^2 \mid a > 0, b > 0, a + b \leq n + 1\}$ be the set of points to be covered. A line is called **sunny** if it is not parallel to the x -axis (slope 0), the y -axis (undefined slope), or the line $x + y = 0$ (slope -1). Let k be the number of sunny lines in a set of n distinct lines covering P_n . Let K_n be the set of all possible values of k for a given integer $n \geq 3$.

****Part 1: The case $n = 3$ ****

For $n = 3$, the set of points is $P_3 = \{(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1)\}$. We show that $K_3 = \{0, 1, 3\}$.

* **** $k = 0$ is possible:**** The set of non-sunny lines $\mathcal{L} = \{x = 1, x = 2, x = 3\}$ covers all points in P_3 . For any $(a, b) \in P_3$, $a \in \{1, 2, 3\}$, so (a, b) lies on the line $x = a$. Thus, $0 \in K_3$.

* **** $k = 1$ is possible:**** Consider $\mathcal{L} = \{x = 1, y = 1, y = x\}$. The lines $x = 1$ and $y = 1$ are not sunny. The line $y = x$ (slope 1) is sunny. This set of lines covers all points in P_3 . Thus, $1 \in K_3$.

* **** $k = 3$ is possible:**** Consider $\mathcal{L} = \{y = x, x + 2y = 5, 2x + y = 5\}$. The slopes are $1, -1/2, -2$. All three lines are sunny. The line $y = x$ covers $(1, 1)$ and $(2, 2)$. The line $x + 2y = 5$ covers $(1, 2)$ and $(3, 1)$. The line $2x + y = 5$ covers $(1, 3)$ and $(2, 1)$. Together, these three lines cover all six points of P_3 . Thus, $3 \in K_3$.

* **** $k = 2$ is impossible:**** We prove this by contradiction. Assume there exists a set of 3 lines with 2 sunny lines (L_1, L_2) and 1 non-sunny line (L_3) that covers P_3 .

****Lemma:**** A sunny line contains at most two points of P_3 . ****Proof:**** The sets of three or more collinear points in P_3 are $\{(1, 1), (1, 2), (1, 3)\}$ (on line $x = 1$), $\{(1, 1), (2, 1), (3, 1)\}$

(on line $y = 1$), and $\{(1, 3), (2, 2), (3, 1)\}$ (on line $x + y = 4$). All three of these lines are non-sunny. Any other line can intersect P_3 in at most two points. Thus, a sunny line contains at most two points of P_3 .

Now we proceed with a case analysis on the non-sunny line L_3 . 1. L_3 covers 3 points of P_3 : L_3 must be one of $x = 1, y = 1$, or $x + y = 4$. * If $L_3 = x = 1$, it covers $\{(1, 1), (1, 2), (1, 3)\}$. The remaining points are $R_1 = \{(2, 1), (2, 2), (3, 1)\}$. These three points are not collinear. To cover them with two lines L_1, L_2 , one line must cover two points. The lines defined by pairs of points in R_1 are $x = 2, y = 1$, and $x + y = 4$. All are non-sunny, contradicting that L_1, L_2 are sunny. * If $L_3 = y = 1$, it covers $\{(1, 1), (2, 1), (3, 1)\}$. The remaining points are $R_2 = \{(1, 2), (1, 3), (2, 2)\}$. The lines defined by pairs of points in R_2 are $x = 1, y = 2$, and $x + y = 4$. All are non-sunny. Contradiction. * If $L_3 = x + y = 4$, it covers $\{(1, 3), (2, 2), (3, 1)\}$. The remaining points are $R_3 = \{(1, 1), (1, 2), (2, 1)\}$. The lines defined by pairs of points in R_3 are $x = 1, y = 1$, and $x + y = 3$. All are non-sunny. Contradiction. 2. L_3 covers 2 points of P_3 : The remaining 4 points must be covered by two sunny lines L_1, L_2 . By the lemma, each must cover at most 2 points. Thus, each must cover exactly 2 points. The set of 4 remaining points must be partitioned into two pairs, each defining a sunny line. The non-sunny lines covering exactly 2 points of P_3 are $x = 2, y = 2, x + y = 3$. * If $L_3 = x = 2$, it covers $\{(2, 1), (2, 2)\}$. The remaining points are $R_4 = \{(1, 1), (1, 2), (1, 3), (3, 1)\}$. The partitions of R_4 into two pairs are: - $\{(1, 1), (1, 2)\}$ (on non-sunny $x = 1$) and $\{(1, 3), (3, 1)\}$ (on non-sunny $x + y = 4$). - $\{(1, 1), (1, 3)\}$ (on non-sunny $x = 1$) and $\{(1, 2), (3, 1)\}$ (on sunny $x + 2y = 5$). - $\{(1, 1), (3, 1)\}$ (on non-sunny $y = 1$) and $\{(1, 2), (1, 3)\}$ (on non-sunny $x = 1$). In no case do we get two sunny lines. Contradiction. * If $L_3 = y = 2$, it covers $\{(1, 2), (2, 2)\}$. The remaining points are $R_5 = \{(1, 1), (2, 1), (1, 3), (3, 1)\}$. This case is symmetric to $L_3 = x = 2$ by reflection across $y = x$, which preserves sunniness. No partition yields two sunny lines. Contradiction. * If $L_3 = x + y = 3$, it covers $\{(1, 2), (2, 1)\}$. The remaining points are $R_6 = \{(1, 1), (1, 3), (2, 2), (3, 1)\}$. The partitions of R_6 into two pairs are: - $\{(1, 1), (2, 2)\}$ (on sunny $y = x$) and $\{(1, 3), (3, 1)\}$ (on non-sunny $x + y = 4$). - $\{(1, 1), (1, 3)\}$ (on non-sunny $x = 1$) and $\{(2, 2), (3, 1)\}$ (on non-sunny $x + y = 4$). - $\{(1, 1), (3, 1)\}$ (on non-sunny $y = 1$) and $\{(1, 3), (2, 2)\}$ (on non-sunny $x + y = 4$). In no case do we get two sunny lines. Contradiction. 3. L_3 covers 1 or 0 points of P_3 : The remaining ≥ 5 points must be covered by two sunny lines L_1, L_2 . By the lemma, they can cover at most $2 + 2 = 4$ points. This is not enough. Contradiction.

Since all cases lead to a contradiction, $k = 2$ is impossible for $n = 3$. Thus, $K_3 = \{0, 1, 3\}$.

****Part 2: A Preliminary Lemma****

****Lemma:**** For any integer $m \geq 2$, the set of points P_m cannot be covered by $m - 1$ lines. ****Proof:**** We proceed by induction on m . * ****Base Case ($m = 2$):**** $P_2 = \{(1, 1), (1, 2), (2, 1)\}$. These three points are not collinear, so they cannot be covered by $2 - 1 = 1$ line. * ****Inductive Step:**** Assume for some $m \geq 3$ that P_{m-1} cannot be covered by $m - 2$ lines. Suppose for contradiction that P_m can be covered by a set \mathcal{L} of $m - 1$ lines. The set P_m contains the m points $S = \{(1, 1), (1, 2), \dots, (1, m)\}$, which lie on the line $x = 1$. To cover these m points with $m - 1$ lines from \mathcal{L} , by the Pigeonhole Principle, at least one line in \mathcal{L} must contain at least two points from S . Such a line must be the line $x = 1$. Let

$L_1 = (x = 1) \in \mathcal{L}$. The remaining $m - 2$ lines in $\mathcal{L} \setminus \{L_1\}$ must cover the remaining points $P_m \setminus S = \{(a, b) \in P_m \mid a \geq 2\}$. The affine transformation $T(x, y) = (x - 1, y)$ maps the set $P_m \setminus S$ bijectively onto P_{m-1} , since $T(P_m \setminus S) = \{(a - 1, b) \mid a \geq 2, b > 0, a + b \leq m + 1\} = \{(a', b) \mid a' \geq 1, b > 0, a' + 1 + b \leq m + 1\} = \{(a', b) \mid a' > 0, b > 0, a' + b \leq m\} = P_{m-1}$. The set of transformed lines $\{T(L) \mid L \in \mathcal{L} \setminus \{L_1\}\}$ consists of $m - 2$ lines that cover P_{m-1} . This contradicts our induction hypothesis. Therefore, P_m cannot be covered by $m - 1$ lines for any $m \geq 2$.

****Part 3: Key Lemma for $n \geq 4$ ****

****Lemma:**** For $n \geq 4$, any set \mathcal{L} of n lines covering P_n must contain at least one of the lines $x = 1$, $y = 1$, or $x + y = n + 1$. ****Proof:**** Assume for contradiction that \mathcal{L} covers P_n but contains none of these three lines. Let $S_V = \{(1, j) \mid 1 \leq j \leq n\}$, $S_H = \{(i, 1) \mid 1 \leq i \leq n\}$, and $S_D = \{(l, n + 1 - l) \mid 1 \leq l \leq n\}$. Since $x = 1 \notin \mathcal{L}$, each of the n lines in \mathcal{L} must intersect the line $x = 1$ at exactly one point, so each line covers exactly one point from S_V . Similarly, each line covers exactly one point from S_H and one from S_D . This establishes a bijection between the lines in \mathcal{L} and the points in each of the sets S_V, S_H, S_D .

Let L_j be the line covering $(1, j)$ for $j \in \{1, \dots, n\}$. Let L_j intersect S_H at $(\sigma(j), 1)$ and S_D at $(\tau(j), n + 1 - \tau(j))$. Since each line covers one point from each set, σ and τ must be permutations of $\{1, \dots, n\}$. For $j = 1$, L_1 covers $(1, 1) \in S_V \cap S_H$, so $\sigma(1) = 1$. For $j = n$, L_n covers $(1, n) \in S_V \cap S_D$, so $\tau(n) = 1$. For $j \in \{2, \dots, n - 1\}$, the points $(1, j)$, $(\sigma(j), 1)$, and $(\tau(j), n + 1 - \tau(j))$ are distinct and collinear. The collinearity condition implies $\tau(j) = n + 1 - j + \frac{(n-j)(j-1)}{\sigma(j)-j}$.

1. ****Case $j = 2$ (since $n \geq 4$):**** Let $i_2 = \sigma(2)$ and $l_2 = \tau(2)$. $l_2 = n - 1 + \frac{n-2}{i_2-2}$. Since σ is a permutation and $\sigma(1) = 1$, $i_2 \in \{2, \dots, n\}$. If $i_2 = 2$, the line is $x + y = 3$. For this line to pass through a point in S_D , we need $n + 1 = 3 \implies n = 2$, which contradicts $n \geq 4$. So $i_2 \neq 2$. Let $d = i_2 - 2 \in \{1, \dots, n - 2\}$. For l_2 to be an integer, d must divide $n - 2$. If $d < n - 2$, then $\frac{n-2}{d} > 1$, so $l_2 = n - 1 + \frac{n-2}{d} > n$, which contradicts $l_2 \in \{1, \dots, n\}$. Thus, $d = n - 2$, which implies $i_2 = n$. This gives $l_2 = n - 1 + 1 = n$. So we must have $\sigma(2) = n$ and $\tau(2) = n$.

2. ****Case $j = 3$ (since $n \geq 4$):**** Let $i_3 = \sigma(3)$ and $l_3 = \tau(3)$. $l_3 = n - 2 + \frac{2(n-3)}{i_3-3}$. Since σ is a permutation with $\sigma(1) = 1, \sigma(2) = n$, we have $i_3 \in \{2, \dots, n - 1\}$. If $i_3 = 3$, the line is $x + y = 4$. For this to pass through a point in S_D , we need $n + 1 = 4 \implies n = 3$, which contradicts $n \geq 4$. So $i_3 \neq 3$. Let $d = i_3 - 3$. Then $i_3 \in \{2, 4, \dots, n - 1\}$, so $d \in \{-1, 1, 2, \dots, n - 4\}$. If $d > 0$, then $d \leq n - 4$. For $l_3 \leq n$, we need $n - 2 + \frac{2(n-3)}{d} \leq n \implies \frac{2(n-3)}{d} \leq 2 \implies n - 3 \leq d$. This requires $n - 3 \leq d \leq n - 4$, which is impossible. If $d < 0$, the only possibility is $d = -1$, which means $i_3 = 2$. Then $l_3 = n - 2 + \frac{2(n-3)}{-1} = 4 - n$. For $l_3 \in \{1, \dots, n\}$, we need $1 \leq 4 - n$, which implies $n \leq 3$. This contradicts $n \geq 4$.

Since all possibilities for $\sigma(3)$ lead to a contradiction, our initial assumption must be false. Thus, for $n \geq 4$, \mathcal{L} must contain one of the lines $x = 1, y = 1, x + y = n + 1$.

****Part 4: Recurrence Relation for K_n ****

**** $K_n \subseteq K_{n-1}$ for $n \geq 4$:**** Let \mathcal{L} be a valid configuration of n lines for P_n with k sunny lines. By the Key Lemma, \mathcal{L} must contain a line $L \in \{x = 1, y = 1, x + y = n + 1\}$. All these lines are non-sunny. The number of sunny lines in $\mathcal{L} \setminus \{L\}$ is k . If $L = x + y = n + 1$, it

covers the points $\{(i, n+1-i)\}_{i=1}^n$. The remaining $n-1$ lines must cover $P_n \setminus L = P_{n-1}$. So $\mathcal{L} \setminus \{L\}$ is a valid configuration for P_{n-1} with k sunny lines. Thus $k \in K_{n-1}$. If $L = x = 1$, the remaining $n-1$ lines cover $P_n \setminus \{(1, j)\}_{j=1}^n$. The transformation $T(x, y) = (x-1, y)$ maps these points to P_{n-1} . The set of transformed lines is a valid configuration for P_{n-1} . T preserves slopes, so sunniness is preserved. The number of sunny lines is k , so $k \in K_{n-1}$. The case $L = y = 1$ is symmetric. In all cases, if $k \in K_n$, then $k \in K_{n-1}$.

**** $K_{n-1} \subseteq K_n$ for $n \geq 4$ **** Let \mathcal{L}' be a valid configuration for P_{n-1} with k sunny lines. Let $T^{-1}(x', y') = (x'+1, y')$. Let $\mathcal{L}'' = \{T^{-1}(L') \mid L' \in \mathcal{L}'\}$. \mathcal{L}'' is a set of $n-1$ lines covering $T^{-1}(P_{n-1}) = \{(a, b) \in P_n \mid a \geq 2\}$. The number of sunny lines in \mathcal{L}'' is k . Consider the set of n lines $\mathcal{L} = \mathcal{L}'' \cup \{x = 1\}$. The line $x = 1$ is not sunny and covers all points in P_n with x -coordinate 1. Thus \mathcal{L} covers P_n . We must check that the lines in \mathcal{L} are distinct. This requires $x = 1 \notin \mathcal{L}''$, which is equivalent to showing that the line $x' = 0$ is not in \mathcal{L}' . Suppose $x' = 0$ is in \mathcal{L}' . Since $x' = 0$ contains no points of P_{n-1} (where $a > 0$), the remaining $(n-1) - 1 = n-2$ lines in \mathcal{L}' must cover all of P_{n-1} . But by our Preliminary Lemma, for $n \geq 4$, P_{n-1} cannot be covered by $n-2$ lines. This is a contradiction. Thus, $x = 1 \notin \mathcal{L}''$, and \mathcal{L} is a valid configuration for P_n with k sunny lines. So if $k \in K_{n-1}$, then $k \in K_n$.

****Part 5: Final Conclusion****

From Part 4, we have $K_n = K_{n-1}$ for all $n \geq 4$. By induction, this means $K_n = K_3$ for all $n \geq 3$. In Part 1, we established that $K_3 = \{0, 1, 3\}$. Therefore, for any integer $n \geq 3$, the set of all possible values for k is $\{0, 1, 3\}$.

A.1.2 Solution with hint

The following solution was generated by providing the model with the additional prompt: "Let us try to solve the problem by induction."

Let $S_n = \{(a, b) \in \mathbb{Z}^2 \mid a \geq 1, b \geq 1, a+b \leq n+1\}$. A line is **sunny** if its slope is not $0, \infty$, or -1 . We seek the set of non-negative integers k such that for a given integer $n \geq 3$, there exist n distinct lines covering S_n , exactly k of which are sunny.

****Part 1: Reduction to a Core Problem****

Let \mathcal{L} be a set of n distinct lines covering S_n . The **principal boundary lines** of S_n are the lines $x = 1$, $y = 1$, and $x + y = n + 1$. These are all non-sunny.

****Lemma**** If a principal boundary line L of S_n is in \mathcal{L} , then there exists a set of $n-1$ distinct lines \mathcal{L}' that covers S_{n-1} and has the same number of sunny lines as \mathcal{L} .

Proof. Let k be the number of sunny lines in \mathcal{L} . The line L is not sunny. 1. Suppose L is the line $x = 1$. This line covers the points $\{(1, b) \mid 1 \leq b \leq n\}$. The remaining $n-1$ lines in $\mathcal{L} \setminus \{L\}$ must cover the set $S'_n = \{(a, b) \in S_n \mid a \geq 2\}$. The affine transformation $\phi(a, b) = (a-1, b)$ is a bijection from S'_n to S_{n-1} . Let \mathcal{L}' be the set of lines obtained by transforming the lines in $\mathcal{L} \setminus \{L\}$. A point (x', y') is on a transformed line l' if its pre-image under ϕ , which is $(x'+1, y')$, is on the original line l . If l has equation $Ax + By + C = 0$, the transformed line l' has equation $A(x+1) + By + C = 0$. This transformation preserves the slope of any line, so the number of sunny lines in \mathcal{L}' is k . The lines in \mathcal{L}' are distinct: if $l_1 : A_1x + B_1y + C_1 = 0$ and $l_2 : A_2x + B_2y + C_2 = 0$ are distinct lines in $\mathcal{L} \setminus \{L\}$, their images are $l'_1 : A_1x + B_1y + (A_1 + C_1) = 0$ and $l'_2 : A_2x + B_2y + (A_2 + C_2) = 0$. If $l'_1 = l'_2$,

their coefficients are proportional, so $(A_1, B_1, A_1 + C_1) = \lambda(A_2, B_2, A_2 + C_2)$ for some $\lambda \neq 0$. This implies $A_1 = \lambda A_2$, $B_1 = \lambda B_2$, and $A_1 + C_1 = \lambda(A_2 + C_2)$. Substituting the first into the third gives $\lambda A_2 + C_1 = \lambda A_2 + \lambda C_2$, so $C_1 = \lambda C_2$. Thus $(A_1, B_1, C_1) = \lambda(A_2, B_2, C_2)$, contradicting the distinctness of l_1, l_2 . 2. If L is $y = 1$, a symmetric argument with the transformation $(a, b) \mapsto (a, b - 1)$ applies. 3. If L is $x + y = n + 1$, it covers the points $\{(a, b) \in S_n \mid a + b = n + 1\}$. The other $n - 1$ lines must cover the remaining points, which form the set $\{(a, b) \in S_n \mid a + b \leq n\} = S_{n-1}$. Thus, $\mathcal{L}' = \mathcal{L} \setminus \{L\}$ is a set of $n - 1$ distinct lines covering S_{n-1} with k sunny lines.

By repeatedly applying this lemma, any configuration for S_n can be reduced. This process terminates when we obtain a set of m lines covering S_m (for some $m \leq n$) where none of the lines are principal boundary lines of S_m . The number of sunny lines in this new configuration is still k .

Let \mathcal{L}_m be such a set of m lines covering S_m . Let $C_{m,x} = \{(1, b) \mid 1 \leq b \leq m\}$. These m points must be covered by \mathcal{L}_m . Since $x = 1 \notin \mathcal{L}_m$, any vertical line in \mathcal{L}_m (of the form $x = c, c \neq 1$) misses all points in $C_{m,x}$. Each of the non-vertical lines in \mathcal{L}_m can cover at most one point of $C_{m,x}$. If there are v vertical lines in \mathcal{L}_m , then the remaining $m - v$ lines must cover all m points, so $m - v \geq m$, which implies $v = 0$. Thus, \mathcal{L}_m contains no vertical lines (slope ∞). By a symmetric argument considering $C_{m,y} = \{(a, 1) \mid 1 \leq a \leq m\}$, we deduce \mathcal{L}_m contains no horizontal lines (slope 0). By considering $C_{m,sum} = \{(a, b) \in S_m \mid a + b = m + 1\}$, we deduce \mathcal{L}_m contains no lines of slope -1 . Therefore, all m lines in \mathcal{L}_m must be sunny. This implies $k = m$. The problem reduces to finding for which non-negative integers k it is possible to cover S_k with k distinct sunny lines, none of which are principal boundary lines of S_k .

****Part 2: Analysis of the Core Problem****

* ** $k = 0$:** $S_0 = \emptyset$. This is covered by 0 lines. So $k = 0$ is possible. * ** $k = 1$:** $S_1 = \{(1, 1)\}$. The line $y = x$ is sunny, covers $(1, 1)$, and is not a principal boundary line of S_1 (which are $x = 1, y = 1, x + y = 2$). So $k = 1$ is possible. * ** $k = 2$:** $S_2 = \{(1, 1), (1, 2), (2, 1)\}$. We need to cover these three points with two distinct sunny lines, neither of which can be a principal boundary line of S_2 ($x = 1, y = 1, x + y = 3$). A line passing through any two of the points in S_2 must be one of these three principal boundary lines, which are non-sunny. Therefore, a sunny line can pass through at most one point of S_2 . Consequently, two sunny lines can cover at most two points, leaving at least one point of S_2 uncovered. So $k = 2$ is impossible. * ** $k = 3$:** $S_3 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$. The following three lines are sunny, distinct, not principal boundary lines of S_3 , and cover S_3 : $L_1 : y = -2x + 5$ (covers $(1, 3), (2, 1)$) $L_2 : y = x$ (covers $(1, 1), (2, 2)$) $L_3 : x + 2y = 5$ (covers $(1, 2), (3, 1)$) So $k = 3$ is possible. * ** $k \geq 4$:** Assume there exists a set \mathcal{L} of k distinct sunny lines covering S_k , with no principal boundary lines. As argued, each line in \mathcal{L} must intersect each of the sets $C_y = \{(i, 1)\}_{i=1}^k$, $C_x = \{(1, j)\}_{j=1}^k$, and $C_s = \{(p, k + 1 - p)\}_{p=1}^k$ at exactly one point. We can label the lines L_i for $i \in \{1, \dots, k\}$ such that L_i is the unique line in \mathcal{L} passing through $(i, 1)$. For each i , L_i must also pass through a unique point $(1, \sigma(i)) \in C_x$ and a unique point $(\pi(i), k + 1 - \pi(i)) \in C_s$. The maps $\sigma, \pi : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ must be permutations.

These permutations must satisfy several properties: 1. L_1 passes through $(1, 1)$, so $\sigma(1) = 1$. 2. L_k passes through $(k, 1)$, which is also in C_s . Thus, $(\pi(k), k+1-\pi(k)) = (k, 1)$, which implies $\pi(k) = k$. 3. There is a unique line L_j passing through $(1, k) \in C_x$, so $\sigma(j) = k$. The point $(1, k)$ is also in C_s , so for L_j , $(\pi(j), k+1-\pi(j)) = (1, k)$, which implies $\pi(j) = 1$. Since $\sigma(1) = 1 \neq k$ and $\pi(k) = k \neq 1$, we have $j \neq 1, k$, so $j \in \{2, \dots, k-1\}$. 4. For $i \in \{2, \dots, k\}$, if $\sigma(i) = i$, L_i would pass through $(i, 1)$ and $(1, i)$, giving it slope -1 , which is not allowed. So $\sigma(i) \neq i$ for $i \geq 2$. 5. For $i \in \{1, \dots, k-1\}$, if $\pi(i) = i$, L_i would pass through $(i, 1)$ and $(i, k+1-i)$, making it a vertical line, which is not allowed. So $\pi(i) \neq i$ for $i \leq k-1$.

For any $i \in \{1, \dots, k\}$, the three points $P_1 = (i, 1)$, $P_2 = (1, \sigma(i))$, and $P_3 = (\pi(i), k+1-\pi(i))$ must be collinear. For $i \in \{2, \dots, k-1\} \setminus \{j\}$, these three points are distinct. To prove this: - $P_1 = P_2 \implies i = 1$, but $i \geq 2$. - $P_1 = P_3 \implies \pi(i) = i$, but $\pi(i) \neq i$ for $i \leq k-1$. - $P_2 = P_3 \implies \pi(i) = 1$ and $\sigma(i) = k$. By definition of j , this means $i = j$. But we consider $i \neq j$. Thus, for $i \in \{2, \dots, k-1\} \setminus \{j\}$, the points are distinct. Collinearity implies their slopes are equal: $\frac{\sigma(i)-1}{1-i} = \frac{k+1-\pi(i)-1}{\pi(i)-i}$. This gives $\sigma(i) = 1 + (i-1)\frac{k-\pi(i)}{i-\pi(i)}$.

****Step 1: Show $j = k-1$.**** Assume for contradiction that $j \neq k-1$. Then the formula for $\sigma(i)$ is valid for $i = k-1$. $\sigma(k-1) = 1 + (k-2)\frac{k-\pi(k-1)}{k-1-\pi(k-1)}$. From the properties of π : $\pi(k-1) \in \{1, \dots, k\}$, $\pi(k-1) \neq \pi(j) = 1$, $\pi(k-1) \neq \pi(k) = k$, and $\pi(k-1) \neq k-1$. So $\pi(k-1) \in \{2, \dots, k-2\}$. Let $d = k-1-\pi(k-1)$. Then $d \in \{1, \dots, k-3\}$. $\sigma(k-1) = 1 + (k-2)\frac{k-(k-1-d)}{d} = 1 + (k-2)\frac{d+1}{d} = 1 + (k-2)(1 + \frac{1}{d}) = k-1 + \frac{k-2}{d}$. Since $k \geq 4$, we have $d \leq k-3$, so $\frac{k-2}{d} \geq \frac{k-2}{k-3} = 1 + \frac{1}{k-3} > 1$. So $\sigma(k-1) > k-1+1 = k$. This contradicts $\sigma(k-1) \in \{1, \dots, k\}$. Thus, our assumption was false. We must have $j = k-1$.

****Step 2: Derive a final contradiction.**** We have established that for $k \geq 4$, it must be that $j = k-1$, which means $\sigma(k-1) = k$ and $\pi(k-1) = 1$. Consider $i = 2$. Since $k \geq 4$, $j = k-1 \geq 3$, so $j \neq 2$. The formula for $\sigma(i)$ is valid for $i = 2$: $\sigma(2) = 1 + \frac{k-\pi(2)}{2-\pi(2)}$. Since π is a permutation, its image on $\{1, \dots, k-2\}$ is $\{1, \dots, k\} \setminus \{\pi(k-1), \pi(k)\}$. With $\pi(k-1) = 1$ and $\pi(k) = k$, we have $\pi(\{1, \dots, k-2\}) = \{2, \dots, k-1\}$. So $\pi(2) \in \{2, \dots, k-1\}$. By property 5, $\pi(2) \neq 2$. Thus $\pi(2) \in \{3, \dots, k-1\}$. Let $d' = \pi(2) - 2$. Then $d' \in \{1, \dots, k-3\}$. The denominator is $2 - \pi(2) = -d'$. $\sigma(2) = 1 + \frac{k-(d'+2)}{-d'} = 1 - \frac{k-d'-2}{d'} = 1 - (\frac{k-2}{d'} - 1) = 2 - \frac{k-2}{d'}$. Since $d' \leq k-3$, we have $\frac{k-2}{d'} \geq \frac{k-2}{k-3} = 1 + \frac{1}{k-3} > 1$ for $k \geq 4$. So $\sigma(2) = 2 - \frac{k-2}{d'} < 2 - 1 = 1$. This contradicts $\sigma(2) \in \{1, \dots, k\}$. This final contradiction shows that no such configuration is possible for $k \geq 4$.

****Part 3: Constructions for general $n \geq 3$ ****

The set of possible values for k is $\{0, 1, 3\}$. We now show these are all possible for any $n \geq 3$.

*** ** $k = 0$:**** Let $\mathcal{L} = \{x = i \mid i = 1, 2, \dots, n\}$. These are n distinct non-sunny lines. Any point $(a, b) \in S_n$ has $1 \leq a \leq n$, so it lies on the line $x = a \in \mathcal{L}$. *** ** $k = 1$:**** Let $\mathcal{L} = \{x = i \mid i = 1, \dots, n-1\} \cup \{y = x - (n-1)\}$. The first $n-1$ lines are non-sunny and cover all points $(a, b) \in S_n$ with $a \leq n-1$. The only points in S_n not covered are those with $a \geq n$. If $(a, b) \in S_n$ and $a \geq n$, then $n+b \leq a+b \leq n+1$, which implies $b \leq 1$. Since $b \geq 1$, we must have $b = 1$. Then $a+1 \leq n+1 \implies a \leq n$. The only uncovered point is $(n, 1)$. The line $y = x - (n-1)$ has slope 1 (sunny), is distinct

from the other lines, and passes through $(n, 1)$ since $1 = n - (n - 1)$. * ** $k = 3$:** Let $\mathcal{L}_{ns} = \{x + y = c \mid c = 5, 6, \dots, n + 1\}$. This is a set of $n - 3$ distinct non-sunny lines (if $n = 3$, this set is empty). These lines cover all points $(a, b) \in S_n$ with $a + b \geq 5$. The remaining points to be covered are $U = \{(a, b) \in S_n \mid a + b \leq 4\}$. This set is precisely S_3 . As shown in Part 2, S_3 can be covered by 3 distinct sunny lines. Let \mathcal{L}_s be this set of 3 lines. Then $\mathcal{L} = \mathcal{L}_{ns} \cup \mathcal{L}_s$ is a set of $(n - 3) + 3 = n$ lines covering S_n . The lines in \mathcal{L}_s have slopes different from -1 , so they are distinct from lines in \mathcal{L}_{ns} . This construction provides exactly 3 sunny lines.

Thus, for any integer $n \geq 3$, configurations with $k \in \{0, 1, 3\}$ sunny lines are possible.

A.2 Problem 2

Problem 2. Let Ω and Γ be circles with centers M and N , respectively, such that the radius of Ω is less than the radius of Γ . Suppose circles Ω and Γ intersect at two distinct points A and B . Let MN intersect Ω at C and Γ at D , such that points C, M, N , and D lie on the line in that order. Let P be the circumcenter of triangle ACD . Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocenter of triangle PMN .

Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF .

(The orthocenter of a triangle is the point of intersection of its altitudes.)

A.2.1 Proof without hint

Let the line containing the centers M and N be the x-axis. The radical axis of circles Ω and Γ is the line AB , which is perpendicular to the line of centers. We place the origin $O(0, 0)$ at the intersection of MN and AB .

Let $M = (m, 0)$ and $N = (n, 0)$. The points A and B have coordinates $(0, a)$ and $(0, -a)$ for some $a > 0$. Since A lies on both circles: $MA^2 = (0 - m)^2 + (a - 0)^2 = r^2 \implies m^2 + a^2 = r^2$. $NA^2 = (0 - n)^2 + (a - 0)^2 = R^2 \implies n^2 + a^2 = R^2$. This gives the key relation $a^2 = r^2 - m^2 = R^2 - n^2$, which implies $n^2 - m^2 = R^2 - r^2$. The problem states that C, M, N, D lie on the line in that order, so $x_C < x_M < x_N < x_D$. C is on Ω and the x-axis, so its x-coordinate is $m - r$. Thus $C = (m - r, 0)$. D is on Γ and the x-axis, so its x-coordinate is $n + R$. Thus $D = (n + R, 0)$.

**1. Coordinates of P and H ** $P(x_P, y_P)$ is the circumcenter of $\triangle ACD$. It lies on the perpendicular bisector of CD , so its x-coordinate is $x_P = \frac{(m-r)+(n+R)}{2}$. The condition $PA^2 = PC^2$ gives $x_P^2 + (y_P - a)^2 = (x_P - (m - r))^2 + y_P^2$, which simplifies to $x_P^2 + y_P^2 - 2ay_P + a^2 = x_P^2 - 2x_P(m - r) + (m - r)^2 + y_P^2$. This yields $2ay_P = 2x_P(m - r) - (m - r)^2 + a^2$. Substituting the expression for x_P : $2ay_P = (m - r + n + R)(m - r) - (m - r)^2 + a^2 = (m - r)(n + R) + a^2$.

$H(x_H, y_H)$ is the orthocenter of $\triangle PMN$. The altitude from P to MN (the x-axis) is the vertical line $x = x_P$, so $x_H = x_P$. The altitude from M in $\triangle PMN$ is perpendicular to the line PN .

* **Lemma:** The line PN is perpendicular to the line AD . * **Proof:** P is the circumcenter of $\triangle ACD$, so by definition, P is equidistant from its vertices. Thus $PA = PD$. This means P lies on the perpendicular bisector of segment AD . N is the center of circle Γ . Points $A = (0, a)$ and $D = (n + R, 0)$ are on Γ . The radius of Γ is R . We have $NA^2 = (0 - n)^2 + (a - 0)^2 = n^2 + a^2 = R^2$, so $NA = R$. The distance ND is $|(n + R) - n| = R$. Thus $NA = ND = R$. This means N also lies on the perpendicular bisector of segment AD . Since both P and N lie on the perpendicular bisector of AD , the line PN is the perpendicular bisector of AD . Thus, $PN \perp AD$.

Since the altitude from M in $\triangle PMN$ is perpendicular to PN , and we have shown $PN \perp AD$, this altitude must be parallel to AD . The slope of AD is $k_{AD} = \frac{a-0}{0-(n+R)} = \frac{-a}{n+R}$. The slope of the line MH is $k_{MH} = \frac{y_H-0}{x_H-m} = \frac{y_H}{x_P-m}$. Equating the slopes gives $y_H = k_{AD}(x_P - m) = -\frac{a(x_P-m)}{n+R}$.

2. The Tangency Condition Let ω be the circumcircle of $\triangle BEF$ with center $O_\omega(x_c, y_c)$ and radius R_ω . Let k_{AP} be the slope of line AP . The line ℓ through H parallel to AP has equation $y - y_H = k_{AP}(x - x_P)$. The condition for ℓ to be tangent to ω is that the square of the distance from O_ω to ℓ equals R_ω^2 :

$$\frac{(k_{AP}(x_c - x_P) - (y_c - y_H))^2}{1 + k_{AP}^2} = R_\omega^2$$

Since $B(0, -a)$ is on ω , $R_\omega^2 = (x_c - 0)^2 + (y_c - (-a))^2 = x_c^2 + (y_c + a)^2$. Let $T_1 = k_{AP}(x_c - x_P) - (y_c - y_H)$ and $T_2 = (1 + k_{AP}^2)(x_c^2 + (y_c + a)^2)$. We need to prove $T_1^2 = T_2$.

**3. Coordinates of E, F and O_ω ** The line AP passes through $A(0, a)$ and has slope k_{AP} . Its equation is $y = k_{AP}x + a$. E is the second intersection of $y = k_{AP}x + a$ with $\Omega : (x - m)^2 + y^2 = r^2$. Substituting y gives $(x - m)^2 + (k_{AP}x + a)^2 = r^2$, which simplifies to $(1 + k_{AP}^2)x^2 + 2(ak_{AP} - m)x + (m^2 + a^2 - r^2) = 0$. Since $m^2 + a^2 = r^2$, this becomes $x((1 + k_{AP}^2)x + 2(ak_{AP} - m)) = 0$. The solutions are $x = 0$ (for point A) and $x_E = -\frac{2(ak_{AP} - m)}{1 + k_{AP}^2}$. Similarly, for F on $\Gamma : (x - n)^2 + y^2 = R^2$, we get $x_F = -\frac{2(ak_{AP} - n)}{1 + k_{AP}^2}$.

* **Lemma:** The coordinates of $O_\omega(x_c, y_c)$ satisfy: a) $x_c + k_{AP}y_c = m + n - ak_{AP}$ b) $y_c = -\frac{(ak_{AP} - m)(ak_{AP} - n)}{a(1 + k_{AP}^2)}$ * **Proof:** a) O_ω lies on the perpendicular bisector of EF . The midpoint of EF is $(\frac{x_E + x_F}{2}, \frac{y_E + y_F}{2})$. The slope of EF (line AP) is k_{AP} . The equation of the perpendicular bisector is $y - \frac{y_E + y_F}{2} = -\frac{1}{k_{AP}}(x - \frac{x_E + x_F}{2})$. Since $O_\omega(x_c, y_c)$ is on this line, we have $x_c + k_{AP}y_c = \frac{x_E + x_F}{2} + k_{AP}\frac{y_E + y_F}{2}$. Using $y_E = k_{AP}x_E + a$ and $y_F = k_{AP}x_F + a$, the right side is $\frac{1 + k_{AP}^2}{2}(x_E + x_F) + ak_{AP}$. Substituting the expressions for x_E and x_F : $\frac{1 + k_{AP}^2}{2} \left(-\frac{2(ak_{AP} - m) + 2(ak_{AP} - n)}{1 + k_{AP}^2} \right) + ak_{AP} = -(2ak_{AP} - m - n) + ak_{AP} = m + n - ak_{AP}$. So, $x_c + k_{AP}y_c = m + n - ak_{AP}$.

b) O_ω is equidistant from $B(0, -a)$ and $E(x_E, y_E)$. So $O_\omega B^2 = O_\omega E^2$. $x_c^2 + (y_c + a)^2 = (x_c - x_E)^2 + (y_c - y_E)^2$. $2ay_c + a^2 = -2x_c x_E + x_E^2 - 2y_c y_E + y_E^2$. Using $y_E = k_{AP}x_E + a$: $2ay_c + a^2 = -2x_c x_E + x_E^2 - 2y_c(k_{AP}x_E + a) + (k_{AP}x_E + a)^2$. $4ay_c = -2x_E(x_c + k_{AP}y_c) + (1 + k_{AP}^2)x_E^2 + 2ak_{AP}x_E$. Dividing by $x_E \neq 0$: $\frac{4ay_c}{x_E} = -2(x_c + k_{AP}y_c) + (1 + k_{AP}^2)x_E + 2ak_{AP}$. Using part (a) and the expression for x_E : $\frac{4ay_c}{x_E} = -2(m + n - ak_{AP}) - 2(ak_{AP} - m) + 2ak_{AP} =$

$-2m - 2n + 2ak_{AP} - 2ak_{AP} + 2m + 2ak_{AP} = 2(ak_{AP} - n)$. $2ay_c = x_E(ak_{AP} - n) = \left(-\frac{2(ak_{AP}-m)}{1+k_{AP}^2}\right)(ak_{AP} - n)$. $y_c = -\frac{(ak_{AP}-m)(ak_{AP}-n)}{a(1+k_{AP}^2)}$. This completes the proof of the lemma.

****4. Algebraic Verification****

****Lemma.**** $T_1 = k_{AP}(x_c - x_P) - (y_c - y_H) = \frac{rR}{a}$. ****Proof.**** $T_1 = (k_{AP}x_c - y_c) - (k_{AP}x_P - y_H)$. From part (a) of the previous lemma, $x_c = m+n-ak_{AP}-k_{AP}y_c$. $k_{AP}x_c - y_c = k_{AP}(m+n-ak_{AP}) - (1+k_{AP}^2)y_c$. Substituting y_c : $k_{AP}(m+n-ak_{AP}) + \frac{(ak_{AP}-m)(ak_{AP}-n)}{a} = \frac{ak_{AP}(m+n)-a^2k_{AP}^2+a^2k_{AP}^2-ak_{AP}(m+n)+mn}{a} = \frac{mn}{a}$. The slope of AP is $k_{AP} = \frac{y_P-a}{x_P}$, so $k_{AP}x_P = y_P - a$. Thus $k_{AP}x_P - y_H = y_P - a - y_H$. $T_1 = \frac{mn}{a} - (y_P - a - y_H) = \frac{mn-a(y_P-a-y_H)}{a}$. $2aT_1 = 2mn - 2ay_P + 2a^2 + 2ay_H$. Using $2ay_P = (m-r)(n+R) + a^2$ and $2ay_H = -2a^2\frac{x_P-m}{n+R} = -a^2\frac{n-m-r+R}{n+R}$: $2aT_1 = 2mn - (mn+mR-rn-rR+a^2) + 2a^2 - a^2\frac{n-m-r+R}{n+R}$. $2aT_1 = mn - mR + rn + rR + a^2 - (R^2 - n^2)\frac{n-m-r+R}{n+R}$ since $a^2 = R^2 - n^2$. $2aT_1 = mn - mR + rn + rR + (R-n)(R+n) - (R-n)(n-m-r+R)$. $2aT_1 = mn - mR + rn + rR + (R-n)[(R+n) - (n-m-r+R)]$. The term in brackets is $R+n-n+m+r-R = m+r$. $2aT_1 = mn - mR + rn + rR + (R-n)(m+r) = mn - mR + rn + rR + Rm + Rr - nm - nr = 2rR$. Thus, $T_1 = \frac{rR}{a}$.

****Lemma.**** $T_2 = (1+k_{AP}^2)(x_c^2 + (y_c+a)^2) = \frac{r^2R^2}{a^2}$. ****Proof.**** From the relations for x_c, y_c , we have $x_c = \frac{a(m+n)+k_{AP}(mn-a^2)}{a(1+k_{AP}^2)}$ and $y_c+a = \frac{a^2-mn+ak_{AP}(m+n)}{a(1+k_{AP}^2)}$. Let $U = a(m+n)$ and $V = mn - a^2$. Then $a(1+k_{AP}^2)x_c = U + k_{AP}V$ and $a(1+k_{AP}^2)(y_c+a) = -V + k_{AP}U$. $a^2(1+k_{AP}^2)^2(x_c^2 + (y_c+a)^2) = (U+k_{AP}V)^2 + (-V+k_{AP}U)^2 = (U^2+V^2)(1+k_{AP}^2)$. So, $T_2 = (1+k_{AP}^2)(x_c^2 + (y_c+a)^2) = \frac{U^2+V^2}{a^2} = \frac{(a(m+n))^2+(mn-a^2)^2}{a^2}$. We verify that the numerator is r^2R^2 : $r^2R^2 = (m^2+a^2)(n^2+a^2) = m^2n^2 + a^2(m^2+n^2) + a^4$. $(a(m+n))^2 + (mn-a^2)^2 = a^2(m^2+2mn+n^2) + (m^2n^2 - 2mna^2 + a^4) = m^2n^2 + a^2(m^2+n^2) + a^4$. The expressions are identical. So, $T_2 = \frac{r^2R^2}{a^2}$.

****Conclusion**** From our lemmas, $T_1^2 = \left(\frac{rR}{a}\right)^2 = \frac{r^2R^2}{a^2}$ and $T_2 = \frac{r^2R^2}{a^2}$. Since $T_1^2 = T_2$, the tangency condition is satisfied. The line through H parallel to AP is tangent to the circumcircle of $\triangle BEF$.

A.2.2 Proof with hint

The following proof was generated by providing the model with the additional prompt: "Let us try to solve the problem by analytic geometry."

****1. Coordinate System and Geometric Setup****

We establish a Cartesian coordinate system. Let the line of centers MN be the x -axis, and the common chord AB be the y -axis. Let their intersection be the origin $O(0,0)$.

* The points of intersection of the circles are $A(0,a)$ and $B(0,-a)$ for some $a > 0$. * The centers of the circles are $M(m,0)$ and $N(n,0)$. * Circle Ω has center $M(m,0)$ and passes through $A(0,a)$. Its radius is $R_\Omega = \sqrt{m^2+a^2}$. The equation of Ω is $(x-m)^2 + y^2 = m^2+a^2$, which simplifies to $x^2 - 2mx + y^2 = a^2$. * Circle Γ has center $N(n,0)$ and passes through $A(0,a)$. Its radius is $R_\Gamma = \sqrt{n^2+a^2}$. The equation of Γ is $(x-n)^2 + y^2 = n^2+a^2$, which simplifies to $x^2 - 2nx + y^2 = a^2$. * The line MN is the x -axis. The points C and D lie on the x -axis. Let their coordinates be $C(c,0)$ and $D(d,0)$. * The condition that C, M, N, D

lie on the line in that order implies their x -coordinates are ordered: $c < m < n < d$. * C is an intersection of Ω with the x -axis. Its coordinate c is a root of $x^2 - 2mx - a^2 = 0$. The roots are $m \pm \sqrt{m^2 + a^2}$. Since $c < m$, we must have $c = m - \sqrt{m^2 + a^2}$. * D is an intersection of Γ with the x -axis. Its coordinate d is a root of $x^2 - 2nx - a^2 = 0$. The roots are $n \pm \sqrt{n^2 + a^2}$. Since $d > n$, we must have $d = n + \sqrt{n^2 + a^2}$. * From the equations for c and d , we have $c^2 - 2mc - a^2 = 0$ and $d^2 - 2nd - a^2 = 0$. Since $c < 0$ and $d > 0$, $c, d \neq 0$. We can express m and n as: $m = \frac{c^2 - a^2}{2c}$ and $n = \frac{d^2 - a^2}{2d}$.

2. Preliminary Geometric and Algebraic Properties

We establish two key properties derived from the problem's conditions.

* **Lemma 1:** $c + d \neq 0$. * **Proof:** The condition $R_\Omega < R_\Gamma$ implies $m^2 + a^2 < n^2 + a^2$, so $m^2 < n^2$. * The ordering C, M, N, D implies $m < n$. * From $m < n$ and $m^2 < n^2$, we have $n^2 - m^2 > 0$, so $(n - m)(n + m) > 0$. Since $n - m > 0$, we must have $n + m > 0$. * Also, $m^2 < n^2 \implies m^2 + a^2 < n^2 + a^2 \implies \sqrt{m^2 + a^2} < \sqrt{n^2 + a^2}$. * Assume for contradiction that $c + d = 0$. Using the expressions for c and d : $(m - \sqrt{m^2 + a^2}) + (n + \sqrt{n^2 + a^2}) = 0 \implies m + n = \sqrt{m^2 + a^2} - \sqrt{n^2 + a^2}$. * The left side, $m + n$, is positive. The right side, $\sqrt{m^2 + a^2} - \sqrt{n^2 + a^2}$, is negative. This is a contradiction. * Thus, our assumption is false, and $c + d \neq 0$.

* **Lemma 2:** $a^2 + cd < 0$. * **Proof:** The condition $m < n$ implies $\frac{c^2 - a^2}{2c} < \frac{d^2 - a^2}{2d}$. * From their definitions, $c < 0$ and $d > 0$, so $cd < 0$. Multiplying the inequality by $2cd$ (which is negative) reverses the inequality sign: $d(c^2 - a^2) > c(d^2 - a^2) \implies c^2d - a^2d > cd^2 - a^2c \implies cd(c - d) > -a^2(c - d)$. * Since $c < d$, we have $c - d < 0$. Dividing by $c - d$ reverses the inequality sign again: $cd < -a^2 \implies a^2 + cd < 0$.

3. Coordinates of P and H

* Let $P(x_P, y_P)$ be the circumcenter of $\triangle ACD$ with vertices $A(0, a)$, $C(c, 0)$, $D(d, 0)$. * P lies on the perpendicular bisector of segment CD , which is the line $x = \frac{c+d}{2}$. So, $x_P = \frac{c+d}{2}$. * The condition $PA^2 = PC^2$ gives $(x_P - 0)^2 + (y_P - a)^2 = (x_P - c)^2 + (y_P - 0)^2$. $x_P^2 + y_P^2 - 2ay_P + a^2 = x_P^2 - 2cx_P + c^2 + y_P^2 \implies -2ay_P + a^2 = -2cx_P + c^2$. Substituting $x_P = \frac{c+d}{2}$: $-2ay_P + a^2 = -c(c + d) + c^2 = -cd \implies y_P = \frac{a^2 + cd}{2a}$. So, $P\left(\frac{c+d}{2}, \frac{a^2 + cd}{2a}\right)$.

* Let $H(x_H, y_H)$ be the orthocenter of $\triangle PMN$ with vertices $P(x_P, y_P)$, $M(m, 0)$, $N(n, 0)$. * The altitude from P to MN (on the x -axis) is the line $x = x_P$. Thus, $x_H = x_P = \frac{c+d}{2}$. * The altitude from M is perpendicular to PN . The slope of this altitude is $-\frac{x_P - n}{y_P}$. The line is $y - 0 = -\frac{x_P - n}{y_P}(x - m)$. * H lies on this line, so $y_H = -\frac{x_P - n}{y_P}(x_H - m) = -\frac{(x_P - m)(x_P - n)}{y_P}$. * We express the numerator in terms of a, c, d : $x_P - m = \frac{c+d}{2} - \frac{c^2 - a^2}{2c} = \frac{c(c+d) - (c^2 - a^2)}{2c} = \frac{cd + a^2}{2c}$. $x_P - n = \frac{c+d}{2} - \frac{d^2 - a^2}{2d} = \frac{d(c+d) - (d^2 - a^2)}{2d} = \frac{cd + a^2}{2d}$. * Substituting these into the expression for y_H : $y_H = -\frac{1}{y_P} \left(\frac{cd + a^2}{2c}\right) \left(\frac{cd + a^2}{2d}\right) = -\frac{(cd + a^2)^2}{4cdy_P}$. Using $y_P = \frac{a^2 + cd}{2a}$ and $a^2 + cd \neq 0$ (from Lemma 2), we get: $y_H = -\frac{(cd + a^2)^2}{4cd} \frac{2a}{a^2 + cd} = -\frac{a(a^2 + cd)}{2cd}$. * So, the coordinates of the orthocenter are $H\left(\frac{c+d}{2}, -\frac{a(a^2 + cd)}{2cd}\right)$.

4. The Line AP and its Intersections

* Since $a \neq 0$ and $c + d \neq 0$ (Lemma 1), the slope of line AP , denoted k_{AP} , is well-

defined: $k_{AP} = \frac{y_P - a}{x_P - 0} = \frac{\frac{a^2 + cd}{2a} - a}{\frac{c+d}{2}} = \frac{a^2 + cd - 2a^2}{a(c+d)} = \frac{cd - a^2}{a(c+d)}$. * The line AP has equation $y = k_{AP}x + a$. $E(x_E, y_E)$ and $F(x_F, y_F)$ are the other intersection points of this line with Ω and Γ respectively. * To find E , substitute $y = k_{AP}x + a$ into the equation of Ω , $x^2 - 2mx + y^2 = a^2$: $x^2 - 2mx + (k_{AP}x + a)^2 = a^2 \implies x((1 + k_{AP}^2)x - 2m + 2ak_{AP}) = 0$. The roots are $x = 0$ (for point A) and $x_E = \frac{2(m - ak_{AP})}{1 + k_{AP}^2}$. * Similarly, for F on Γ ($x^2 - 2nx + y^2 = a^2$): $x_F = \frac{2(n - ak_{AP})}{1 + k_{AP}^2}$.

5. Auxiliary Algebraic Identities

We derive identities that will simplify subsequent calculations. * **Identity 1:** $m - ak_{AP} = \frac{c^2 - a^2}{2c} - a \left(\frac{cd - a^2}{a(c+d)} \right) = \frac{(c^2 - a^2)(c+d) - 2c(cd - a^2)}{2c(c+d)} = \frac{(c^2 + a^2)(c-d)}{2c(c+d)}$. * **Identity 2:** $n - ak_{AP} = \frac{d^2 - a^2}{2d} - \frac{cd - a^2}{c+d} = \frac{(d^2 - a^2)(c+d) - 2d(cd - a^2)}{2d(c+d)} = \frac{(d^2 + a^2)(d-c)}{2d(c+d)}$. * **Identity 3:** $1 + k_{AP}^2 = 1 + \left(\frac{cd - a^2}{a(c+d)} \right)^2 = \frac{a^2(c+d)^2 + (cd - a^2)^2}{a^2(c+d)^2} = \frac{(c^2 + a^2)(d^2 + a^2)}{a^2(c+d)^2}$. * **Identity 4:** $k_{AP} + \frac{2a}{x_E} = \frac{cd - a^2}{a(c+d)} + \frac{a(1 + k_{AP}^2)}{m - ak_{AP}} = \frac{cd - a^2}{a(c+d)} + a \frac{(c^2 + a^2)(d^2 + a^2)}{a^2(c+d)^2} \bigg/ \frac{(c^2 + a^2)(c-d)}{2c(c+d)} = \frac{cd - a^2}{a(c+d)} + \frac{2c(d^2 + a^2)}{a(c+d)(c-d)} = \frac{(cd - a^2)(c-d) + 2c(d^2 + a^2)}{a(c+d)(c-d)} = \frac{cd(c+d) + a^2(c+d)}{a(c+d)(c-d)} = \frac{cd + a^2}{a(c-d)}$.

**6. The Circumcircle of $\triangle BEF$ **

Let $K(x_K, y_K)$ be the circumcenter of $\triangle BEF$. K lies on the perpendicular bisectors of BE and BF . The perpendicular bisector of BE is given by $KB^2 = KE^2$, which simplifies to $2x_Kx_E + 2y_K(y_E + a) = x_E^2 + y_E^2 - a^2 + 2a^2 = 2mx_E + 2a^2$. Using $y_E = k_{AP}x_E + a$, we get $x_Kx_E + y_K(k_{AP}x_E + 2a) = mx_E + a^2$. This is incorrect. Let's re-derive. $KB^2 = KE^2 \implies x_K^2 + (y_K + a)^2 = (x_K - x_E)^2 + (y_K - y_E)^2$. $x_K^2 + y_K^2 + 2ay_K + a^2 = x_K^2 - 2x_Kx_E + x_E^2 + y_K^2 - 2y_Ky_E + y_E^2$. $2ay_K + a^2 = -2x_Kx_E + x_E^2 - 2y_Ky_E + y_E^2$. Since E is on Ω , $x_E^2 + y_E^2 = 2mx_E + a^2$. $2ay_K + a^2 = -2x_Kx_E + 2mx_E + a^2 - 2y_Ky_E$. $2ay_K = -2x_Kx_E + 2mx_E - 2y_Ky_E$. $ay_K = -x_Kx_E + mx_E - y_Ky_E$. $ay_K = -x_Kx_E + mx_E - y_K(k_{AP}x_E + a) \implies 2ay_K = -x_Kx_E + mx_E - y_Kk_{AP}x_E$. $x_E(x_K + y_Kk_{AP}) + 2ay_K = mx_E$. Dividing by $x_E \neq 0$: (1) $x_K + y_K(k_{AP} + \frac{2a}{x_E}) = m$. A similar derivation for the perpendicular bisector of BF (using point F on Γ) yields: (2) $x_K + y_K(k_{AP} + \frac{2a}{x_F}) = n$. Subtracting (1) from (2): $y_K(\frac{2a}{x_F} - \frac{2a}{x_E}) = n - m \implies 2ay_K \frac{x_E - x_F}{x_Ex_F} = n - m$. Using $x_E - x_F = \frac{2(m-n)}{1 + k_{AP}^2}$, we get $2ay_K \frac{2(m-n)/(1 + k_{AP}^2)}{x_Ex_F} = n - m$. Since $m \neq n$, we find $y_K = -\frac{x_Ex_F(1 + k_{AP}^2)}{4a} = -\frac{(m - ak_{AP})(n - ak_{AP})}{a(1 + k_{AP}^2)}$. Using Identities 1, 2, 3: $y_K = -\frac{1}{a} \left(\frac{(c^2 + a^2)(c-d)}{2c(c+d)} \right) \left(\frac{(d^2 + a^2)(d-c)}{2d(c+d)} \right) \bigg/ \frac{(c^2 + a^2)(d^2 + a^2)}{a^2(c+d)^2} = \frac{a(c-d)^2}{4cd}$. From (1) and Identity 4: $x_K = m - y_K(k_{AP} + \frac{2a}{x_E}) = m - y_K \frac{cd + a^2}{a(c-d)}$. $x_K = \frac{c^2 - a^2}{2c} - \frac{a(c-d)^2}{4cd} \frac{cd + a^2}{a(c-d)} = \frac{c^2 - a^2}{2c} - \frac{(c-d)(cd + a^2)}{4cd} = \frac{2d(c^2 - a^2) - (c-d)(cd + a^2)}{4cd} = \frac{cd(c+d) - a^2(c+d)}{4cd} = \frac{(cd - a^2)(c+d)}{4cd}$. The radius squared of the circumcircle of $\triangle BEF$ is $R_K^2 = KB^2 = x_K^2 + (y_K + a)^2$. $y_K + a = \frac{a(c-d)^2}{4cd} + a = \frac{a(c+d)^2}{4cd}$. $R_K^2 = \left(\frac{(cd - a^2)(c+d)}{4cd} \right)^2 + \left(\frac{a(c+d)^2}{4cd} \right)^2 = \frac{(c+d)^2}{(4cd)^2} ((cd - a^2)^2 + a^2(c+d)^2) = \frac{(c+d)^2(c^2 + a^2)(d^2 + a^2)}{16c^2d^2}$.

7. The Tangency Proof

The line ℓ_H passes through $H(x_H, y_H)$ and is parallel to AP . Its equation is $y - y_H = k_{AP}(x - x_H)$, which can be written as $k_{AP}x - y - (k_{AP}x_H - y_H) = 0$. This line is tangent to the circumcircle of $\triangle BEF$ (center K , radius R_K) if the square of the distance from K to ℓ_H is R_K^2 . The squared distance is $\frac{(k_{AP}x_K - y_K - (k_{AP}x_H - y_H))^2}{k_{AP}^2 + 1}$. The condition

for tangency is $(k_{AP}(x_K - x_H) - (y_K - y_H))^2 = R_K^2(1 + k_{AP}^2)$. Let's compute the terms on the left side (LHS). $x_K - x_H = \frac{(cd-a^2)(c+d)}{4cd} - \frac{c+d}{2} = \frac{(c+d)(cd-a^2-2cd)}{4cd} = -\frac{(c+d)(cd+a^2)}{4cd}$. $y_K - y_H = \frac{a(c-d)^2}{4cd} - \left(-\frac{a(a^2+cd)}{2cd}\right) = \frac{a(c^2-2cd+d^2)+2a(a^2+cd)}{4cd} = \frac{a(c^2+d^2+2a^2)}{4cd}$. $k_{AP}(x_K - x_H) = \frac{cd-a^2}{a(c+d)} \left(-\frac{(c+d)(cd+a^2)}{4cd}\right) = -\frac{(cd-a^2)(cd+a^2)}{4acd} = -\frac{c^2d^2-a^4}{4acd}$. $k_{AP}(x_K - x_H) - (y_K - y_H) = -\frac{c^2d^2-a^4}{4acd} - \frac{a^2(c^2+d^2+2a^2)}{4acd} = -\frac{c^2d^2-a^4+a^2c^2+a^2d^2+2a^4}{4acd} = -\frac{(c^2+a^2)(d^2+a^2)}{4acd}$. LHS = $\left(-\frac{(c^2+a^2)(d^2+a^2)}{4acd}\right)^2 = \frac{(c^2+a^2)^2(d^2+a^2)^2}{16a^2c^2d^2}$. Now we compute the right side (RHS) using R_K^2 and Identity 3: RHS = $R_K^2(1 + k_{AP}^2) = \left(\frac{(c+d)^2(c^2+a^2)(d^2+a^2)}{16c^2d^2}\right) \left(\frac{(c^2+a^2)(d^2+a^2)}{a^2(c+d)^2}\right) = \frac{(c^2+a^2)^2(d^2+a^2)^2}{16a^2c^2d^2}$. Since LHS = RHS, the tangency condition is satisfied. This completes the proof.

A.3 Problem 3

Problem 3. Let \mathbb{N} denote the set of positive integers. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be *bonza* if

$$f(a) \mid b^a - f(b)^{f(a)}$$

for all positive integers a and b .

Determine the smallest real constant c such that $f(n) \leq cn$ for all bonza functions f and all positive integers n .

Solution without hint. Let $P(a, b)$ be the assertion $f(a) \mid b^a - f(b)^{f(a)}$.

****Step 1: Prove that $f(1) = 1$ **** Let $C = f(1)$. Assume for contradiction that $C > 1$. Let p be a prime divisor of C . The assertion $P(p, 1)$ states $f(p) \mid 1^p - f(1)^{f(p)}$, which is $f(p) \mid 1 - C^{f(p)}$. Since $p \mid C$, we have $C \equiv 0 \pmod{p}$, so $1 - C^{f(p)} \equiv 1 \pmod{p}$. If p were a divisor of $f(p)$, then p would have to divide $1 - C^{f(p)}$, which is impossible as $1 - C^{f(p)} \equiv 1 \pmod{p}$. Thus, $p \nmid f(p)$.

Now consider the assertion $P(1, p)$, which states $f(1) \mid p - f(p)^{f(1)}$, i.e., $C \mid p - f(p)^C$. Since $p \mid C$, this implies $p \mid p - f(p)^C$, which in turn implies $p \mid f(p)^C$. As p is prime, this means $p \mid f(p)$. We have derived a contradiction: $p \nmid f(p)$ and $p \mid f(p)$. Therefore, the assumption $C > 1$ must be false. Since $f(1) \in \mathbb{N}$, we must have $f(1) = 1$.

****Step 2: Properties of bonza functions**** Let $S_f = \{p \text{ prime} \mid \exists n \in \mathbb{N}, p \mid f(n)\}$.

****Property 1:**** For any $p \in S_f$ and $b \in \mathbb{N}$, if $p \mid f(b)$, then $p \mid b$. ***Proof:*** Let $p \in S_f$. By definition, there exists an $a_0 \in \mathbb{N}$ such that $p \mid f(a_0)$. The bonza condition $P(a_0, b)$ is $f(a_0) \mid b^{a_0} - f(b)^{f(a_0)}$. As $p \mid f(a_0)$, we have $p \mid b^{a_0} - f(b)^{f(a_0)}$. If we assume $p \nmid f(b)$, this implies $p \mid b^{a_0}$, and since p is prime, $p \mid b$.

****Lemma A:**** For any prime $p \in S_f$, we have $f(b) \equiv b \pmod{p}$ for all $b \in \mathbb{N}$. ***Proof:*** Let $p \in S_f$. First, we show $f(p)$ is a power of p . If $f(p) = 1$, then for any a with $p \mid f(a)$, $P(a, p) \implies f(a) \mid p^a - f(p)^{f(a)} = p^a - 1$, which contradicts $p \mid f(a)$. So $f(p) > 1$. Let q be a prime divisor of $f(p)$. Then $q \in S_f$. By Property 1, since $q \mid f(p)$, we must have $q \mid p$. As p, q are primes, $q = p$. Thus $f(p) = p^k$ for some $k \geq 1$. Now, $P(p, b) \implies f(p) \mid b^p - f(b)^{f(p)}$, so $p^k \mid b^p - f(b)^{p^k}$. This implies $b^p \equiv f(b)^{p^k} \pmod{p}$. By Fermat's Little Theorem, $b^p \equiv b \pmod{p}$ and $f(b)^{p^k} \equiv f(b) \pmod{p}$. Thus, $b \equiv f(b) \pmod{p}$.

****Lemma B:**** For any prime $q \notin S_f$, we have $f(q) = 1$. ***Proof:*** Let q be a prime with $q \notin S_f$. Suppose $f(q) > 1$. Let p be a prime divisor of $f(q)$. Then $p \in S_f$. By Property 1, since $p \mid f(q)$, we must have $p \mid q$. As p, q are primes, $p = q$. This contradicts the fact that $p \in S_f$ and $q \notin S_f$. Thus, $f(q) = 1$.

****Step 3: Classification of bonza functions**** For any $p \in S_f$ and any prime $q \notin S_f$, by Lemma A, $f(q) \equiv q \pmod{p}$. By Lemma B, $f(q) = 1$. Thus, we have the crucial condition:

$$\forall p \in S_f, \forall q \notin S_f \text{ (prime)}, \quad q \equiv 1 \pmod{p} \quad (*)$$

We analyze the possible structures for the set of primes S_f . 1. $S_f = \emptyset$: Condition (*) is vacuously true. This is a possible case. 2. S_f is the set of all primes: Condition (*) is vacuously true as there are no primes $q \notin S_f$. This is a possible case. 3. S_f is a non-empty proper subset of the set of all primes. a) Suppose S_f is an infinite proper subset. Let q be a prime such that $q \notin S_f$. By (*), $q \equiv 1 \pmod{p}$ for all $p \in S_f$. This means every prime $p \in S_f$ is a divisor of $q - 1$. Since S_f is infinite, this implies the integer $q - 1$ has infinitely many distinct prime divisors, which is impossible. Thus, S_f cannot be an infinite proper subset. b) Suppose S_f is a finite, non-empty proper subset. Let $S_f = \{p_1, \dots, p_r\}$. Let $P = p_1 p_2 \dots p_r$. Condition (*) implies that any prime $q \notin S_f$ must satisfy $q \equiv 1 \pmod{P}$. If $P > 2$, then Euler's totient function $\phi(P) > 1$. This means there exists an integer a with $1 < a < P$ and $\gcd(a, P) = 1$. By Dirichlet's theorem on arithmetic progressions, there are infinitely many primes q of the form $kP + a$. We can choose such a prime q that is not in the finite set S_f . For this prime q , we have $q \notin S_f$, so by our condition, it must satisfy $q \equiv 1 \pmod{P}$. But we chose q such that $q \equiv a \pmod{P}$. This leads to $a \equiv 1 \pmod{P}$, which contradicts our choice of a . Therefore, the assumption $P > 2$ must be false. We must have $P \leq 2$. Since S_f is non-empty, $P \geq 2$. Thus, $P = 2$, which implies $S_f = \{2\}$. Thus, the only possibilities for S_f are \emptyset , $\{2\}$, or the set of all primes.

****Step 4: Analysis of $f(n)/n$ for each case**** ****Case 1: $S_f = \emptyset$.**** This implies that for all n , $f(n)$ has no prime factors, so $f(n) = 1$. The function $f(n) = 1$ for all n is bonza, since $1 \mid b^a - 1^1$ is always true. For this function, $f(n)/n = 1/n \leq 1$. ****Case 2: S_f is the set of all primes.**** For any prime p , Lemma A gives $f(n) \equiv n \pmod{p}$. This means $p \mid (f(n) - n)$ for all primes p . If $f(n) \neq n$, then $f(n) - n$ is a non-zero integer. A non-zero integer can only have a finite number of prime divisors. This forces $f(n) - n = 0$, so $f(n) = n$. The function $f(n) = n$ is bonza, since $a \mid b^a - b^a$ is true. For this function, $f(n)/n = 1$. ****Case 3: $S_f = \{2\}$.**** The range of f is a subset of $\{1\} \cup \{2^k \mid k \in \mathbb{N}\}$. By Lemma A, $f(n) \equiv n \pmod{2}$. This implies $f(n) = 1$ for odd n , and $f(n) = 2^{k_n}$ for some $k_n \geq 1$ for even n . Let n be an even integer. Let $v_2(m)$ be the exponent of 2 in the prime factorization of m . 1. For any odd b , $P(n, b) \implies f(n) \mid b^n - f(b)^{f(n)}$. Since b is odd, $f(b) = 1$. So $2^{k_n} \mid b^n - 1$. This must hold for all odd b . Thus, $k_n \leq \min_{b \text{ odd}} v_2(b^n - 1)$. Let $n = 2^s t$ with t odd, $s \geq 1$. For any odd b , let $x = b^t$, which is also odd. We have $v_2(b^n - 1) = v_2(x^{2^s} - 1)$. It is a known property that for an odd integer x and $k \geq 1$, $v_2(x^{2^k} - 1) = v_2(x^2 - 1) + k - 1$. Applying this with $k = s$, we get $v_2(b^n - 1) = v_2((b^t)^2 - 1) + s - 1$. To find the minimum value, we must minimize $v_2((b^t)^2 - 1)$ over odd b . The map $\phi_t : (\mathbb{Z}/8\mathbb{Z})^\times \rightarrow (\mathbb{Z}/8\mathbb{Z})^\times$ given by $\phi_t(y) = y^t$ is a permutation for any odd t . This is because for any $y \in (\mathbb{Z}/8\mathbb{Z})^\times$, $y^2 \equiv 1 \pmod{8}$, so

for $t = 2k + 1$, $y^t = y^{2k+1} = (y^2)^k y \equiv y \pmod{8}$. Thus, as b runs through odd integers, $b^t \pmod{8}$ also takes on all values in $\{1, 3, 5, 7\}$. We can choose b such that $b^t \equiv 3 \pmod{8}$. For such b , $v_2((b^t)^2 - 1) = v_2(3^2 - 1) = v_2(8) = 3$. The minimum value of $v_2((b^t)^2 - 1)$ for $b^t \not\equiv 1 \pmod{8}$ is 3. So, $\min_{b \text{ odd}} v_2(b^n - 1) = 3 + s - 1 = s + 2 = v_2(n) + 2$. Thus, $k_n \leq v_2(n) + 2$. 2. For $b = 2$, $P(n, 2) \implies f(n) \mid 2^n - f(2)^{f(n)}$. Let $f(2) = 2^{k_2}$ for some integer $k_2 \geq 1$. The condition is $2^{k_n} \mid 2^n - 2^{k_2 2^{k_n}}$. This requires $k_n \leq v_2(2^n - 2^{k_2 2^{k_n}})$. If $n \neq k_2 2^{k_n}$, then $v_2(2^n - 2^{k_2 2^{k_n}}) = \min(n, k_2 2^{k_n})$. The condition becomes $k_n \leq \min(n, k_2 2^{k_n})$, which implies $k_n \leq n$. If $n = k_2 2^{k_n}$, the divisibility is on 0, which holds for any k_n . However, $n = k_2 2^{k_n}$ with $k_2 \geq 1$ implies $n \geq 2^{k_n}$. Since $2^x > x$ for all $x \geq 1$, we have $n > k_n$. In all cases, we must have $k_n \leq n$. Combining these constraints, for any bonza function in this class, $f(n) \leq 2^{\min(n, v_2(n)+2)}$ for even n .

****Step 5: Construction of the maximal function and determination of c^{**}** Let's define a function f_0 based on the derived upper bound:

$$f_0(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2^{\min(n, v_2(n)+2)} & \text{if } n \text{ is even} \end{cases}$$

Let $k_m = \min(m, v_2(m) + 2)$ for any even m . We verify that f_0 is a bonza function. - If a is odd, $f_0(a) = 1$. The condition is $1 \mid b^a - f_0(b)^1$, which is always true. - If a is even and b is odd: $f_0(a) = 2^{k_a}$, $f_0(b) = 1$. We need $2^{k_a} \mid b^a - 1$. By definition, $k_a \leq v_2(a) + 2$. As established in Step 4, for any even a and odd b , $v_2(b^a - 1) \geq v_2(a) + 2$. Thus, $k_a \leq v_2(a) + 2 \leq v_2(b^a - 1)$, so the condition holds. - If a, b are even: $f_0(a) = 2^{k_a}$, $f_0(b) = 2^{k_b}$. We need $2^{k_a} \mid b^a - (2^{k_b})^{2^{k_a}}$. Let $E = b^a - 2^{k_b 2^{k_a}}$. We need to show $v_2(E) \geq k_a$. Let $v_2(b^a) = av_2(b)$ and the exponent of 2 in the second term is $k_b 2^{k_a}$. If $v_2(b^a) \neq k_b 2^{k_a}$, then $v_2(E) = \min(v_2(b^a), k_b 2^{k_a})$. We have $k_a \leq a \leq av_2(b) = v_2(b^a)$ since $v_2(b) \geq 1$. Also $k_b \geq 1$, so $k_b 2^{k_a} \geq 2^{k_a} \geq k_a$ for $k_a \geq 1$. Thus $k_a \leq v_2(E)$. If $v_2(b^a) = k_b 2^{k_a}$, let $v_b = v_2(b)$ and $m = b/2^{v_b}$. The condition is $av_b = k_b 2^{k_a}$. $E = b^a - 2^{k_b 2^{k_a}} = (m 2^{v_b})^a - 2^{av_b} = m^a 2^{av_b} - 2^{av_b} = 2^{av_b}(m^a - 1)$. If $m = 1$, b is a power of 2, then $E = 0$ and $2^{k_a} \mid 0$ holds. If $m > 1$, $v_2(E) = av_b + v_2(m^a - 1)$. Since $k_a \leq a \leq av_b$, and $v_2(m^a - 1) \geq 1$ (as a is even, m is odd), we have $k_a \leq v_2(E)$. Thus, f_0 is a bonza function.

Now we find $\sup_n f_0(n)/n$. For odd n , $f_0(n)/n = 1/n \leq 1$. For even $n = 2^{st}$ with t odd, $s \geq 1$: $\frac{f_0(n)}{n} = \frac{2^{\min(2^s t, s+2)}}{2^{st}}$. - If $t \geq 3$, then $2^{st} > s + 2$ for $s \geq 1$. (Proof: $g(s) = 3 \cdot 2^s - s - 2$ is increasing for $s \geq 1$ and $g(1) = 3 > 0$. So $3 \cdot 2^s > s + 2$. Then $t \cdot 2^s \geq 3 \cdot 2^s > s + 2$.) So $\frac{f_0(n)}{n} = \frac{2^{s+2}}{2^{st}} = \frac{4}{t} \leq \frac{4}{3}$. - If $t = 1$, $n = 2^s$. $\frac{f_0(2^s)}{2^s} = \frac{2^{\min(2^s, s+2)}}{2^s}$. The inequality $2^s \geq s + 2$ holds for all integers $s \geq 2$. (Proof by induction: for $s = 2, 4 \geq 4$. Assume $2^k \geq k + 2$ for $k \geq 2$, then $2^{k+1} = 2 \cdot 2^k \geq 2(k + 2) = 2k + 4 = (k + 3) + (k + 1) > k + 3$.) For $s = 1$, $n = 2$: $\frac{f_0(2)}{2} = \frac{2^{\min(2, 3)}}{2} = \frac{2^2}{2} = 2$. For $s \geq 2$, since $2^s \geq s + 2$, we have $\min(2^s, s + 2) = s + 2$. So $\frac{f_0(2^s)}{2^s} = \frac{2^{s+2}}{2^s} = 4$. This value of 4 is achieved for $n = 4, 8, 16, \dots$. For example, $f_0(4) = 2^{\min(4, v_2(4)+2)} = 2^{\min(4, 4)} = 2^4 = 16$, so $f_0(4)/4 = 4$.

The supremum of $f(n)/n$ over all bonza functions is 4. For any constant $c < 4$, the function f_0 and $n = 4$ gives $f_0(4) = 16 > 4c$, so $f_0(4) > cn$. Thus, the smallest constant is $c = 4$. \square

A.4 Problem 4

Problem 4. A proper divisor of a positive integer N is a positive divisor of N other than N itself.

The infinite sequence a_1, a_2, \dots consists of positive integers, each of which has at least three proper divisors. For each $n \geq 1$, the integer a_{n+1} is the sum of the three largest proper divisors of a_n .

Determine all possible values of a_1 .

Solution without hint. Let $f(N)$ be the sum of the three largest proper divisors of a positive integer N . The sequence is defined by $a_{n+1} = f(a_n)$ for $n \geq 1$. The problem requires that each a_n has at least three proper divisors. A proper divisor of N is any positive divisor of N except N itself. The condition that N has at least three proper divisors means that the number of divisors of N , denoted by $d(N)$, must satisfy $d(N) - 1 \geq 3$, so $d(N) \geq 4$.

Let the divisors of N in increasing order be $1 = d_1 < d_2 < d_3 < \dots < d_k = N$, where $k = d(N) \geq 4$. The proper divisors of N are $\{d_1, d_2, \dots, d_{k-1}\}$. The three largest proper divisors are $d_{k-1}, d_{k-2}, d_{k-3}$. The map $d \mapsto N/d$ is a bijection on the set of divisors of N that reverses the order. Therefore, the largest proper divisor is $d_{k-1} = N/d_2$, the second largest is $d_{k-2} = N/d_3$, and the third largest is $d_{k-3} = N/d_4$. This holds for any N with $d(N) \geq 4$. Thus, the sum of the three largest proper divisors is given by: $f(N) = d_{k-1} + d_{k-2} + d_{k-3} = \frac{N}{d_2} + \frac{N}{d_3} + \frac{N}{d_4} = N \left(\frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4} \right)$.

****Lemma 1:**** For any valid sequence, a_n must be even for all $n \geq 1$. *** **Proof:**** Suppose a_k is an odd integer for some $k \geq 1$. All its divisors are odd, so its three largest proper divisors are odd. Their sum, $a_{k+1} = f(a_k)$, is also odd. Therefore, if any term is odd, all subsequent terms are odd. Let N be an odd integer with $d(N) \geq 4$. Its smallest divisors greater than 1 are d_2, d_3, d_4 . Since N is odd, all its divisors are odd. The smallest possible value for d_2 is 3. Thus, $d_2 \geq 3$. Since $d_3 > d_2$, $d_3 \geq 5$. Similarly, $d_4 > d_3$, so $d_4 \geq 7$. The sum $\frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4}$ is maximized when d_2, d_3, d_4 are minimized. Thus, $\frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4} \leq \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{71}{105}$. Therefore, $f(N) \leq \frac{71}{105}N < N$. So, if a_k is odd, the sequence $(a_n)_{n \geq k}$ is a strictly decreasing sequence of positive integers. Such a sequence must terminate, meaning it must produce a term which does not satisfy $d(a_n) \geq 4$. Therefore, an infinite sequence is not possible if any term is odd.

****Lemma 2:**** For any valid sequence, a_n must be divisible by 3 for all $n \geq 1$. *** **Proof:**** Suppose there is a term a_k not divisible by 3. By Lemma 1, a_k is even. Let a_m be any term with $v_3(a_m) = 0$. Its smallest divisor is $d_2 = 2$. Its divisors d_3, d_4 are not multiples of 3. Thus $d_3 \geq 4$ and $d_4 \geq 5$. $a_{m+1} = a_m \left(\frac{1}{2} + \frac{1}{d_3} + \frac{1}{d_4} \right) \leq a_m \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{5} \right) = \frac{19}{20}a_m < a_m$. So, if a term is not divisible by 3, the next term is strictly smaller.

Now, consider the sequence $(a_n)_{n \geq k}$. If for all $n \geq k$, $v_3(a_n) = 0$, then the sequence is strictly decreasing. A strictly decreasing sequence of positive integers must terminate. This is a failure. Therefore, for the sequence to be infinite, there must be a first term a_m (with $m \geq k$) such that $v_3(a_m) = 0$ and $v_3(a_{m+1}) > 0$. For a_{m+1} to gain a factor of 3, the numerator of the fraction in $a_{m+1} = a_m \left(\frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4} \right)$ must be divisible by 3. With $d_2 = 2$, this means $d_3d_4 + 2d_4 + 2d_3 \equiv 0 \pmod{3}$, which implies $(d_3 - 1)(d_4 - 1) \equiv 1 \pmod{3}$. This

holds if and only if $d_3 \equiv 2 \pmod{3}$ and $d_4 \equiv 2 \pmod{3}$. If $v_2(a_m) \geq 2$, then $d_3 = 4 \equiv 1 \pmod{3}$. The condition is not met. So, for the transition to happen, the term a_m must have $v_2(a_m) = 1$. In this case, $d_3 = p$, the smallest odd prime factor of a_m . We need $p \equiv 2 \pmod{3}$. d_4 is the smallest divisor of a_m greater than p . We also need $d_4 \equiv 2 \pmod{3}$. Let the prime factorization of a_m be $2 \cdot p^{e_p} \cdot q^{e_q} \cdots$, where $p < q < \dots$ are odd primes. The candidates for d_4 are the smallest divisors of a_m greater than p . These are p^2 (if $e_p \geq 2$), $2p$, and q (if a_m has a second odd prime factor q). Any other divisor of a_m greater than p is larger than one of these three. We check the congruences modulo 3, given $p \equiv 2 \pmod{3}$:
- $p^2 \equiv 2^2 = 4 \equiv 1 \pmod{3}$.
- $2p \equiv 2(2) = 4 \equiv 1 \pmod{3}$.
So neither p^2 nor $2p$ can be d_4 . This implies that d_4 must be q , the second smallest odd prime factor of a_m . For this to be the case, we must have $q < p^2$ and $q < 2p$. And for the condition to be met, we must have $q \equiv 2 \pmod{3}$. So, if a term a_m with $v_2 = 1, v_3 = 0$ gains a factor of 3, its smallest two odd prime factors, p and q , must both be congruent to 2 (mod 3). In this case, $d_3 = p$ and $d_4 = q$. Both are odd primes. Let's check $v_2(a_{m+1}) = v_2(a_m) + v_2(pq + 2(p + q)) - v_2(2pq)$. We have $v_2(a_m) = 1$. Since p, q are odd, pq is odd. $p + q$ is a sum of two odd numbers, so it's even. $2(p + q)$ is a multiple of 4. So $pq + 2(p + q)$ is odd + (multiple of 4), which is odd. Thus $v_2(pq + 2(p + q)) = 0$. Also, $v_2(2pq) = 1$ since p, q are odd. $v_2(a_{m+1}) = 1 + 0 - 1 = 0$. So a_{m+1} is odd. By Lemma 1, this leads to failure. In summary, if any term is not divisible by 3, the sequence must fail.

****Lemma 3:**** For any valid sequence, no term a_n can be divisible by 5. *** **Proof:****
By Lemmas 1 and 2, any term a_n must be divisible by 2 and 3. First, suppose $v_5(a_k) = 0$ for some $k \geq 1$. We show that $v_5(a_{k+1}) = 0$. The smallest divisors of a_k are $d_2 = 2, d_3 = 3$. d_4 is the smallest divisor of a_k greater than 3. Let p be the smallest prime factor of a_k other than 2 or 3. Since $v_5(a_k) = 0$, $p \geq 7$. The candidates for d_4 are divisors of a_k smaller than p , which can only be composed of primes 2 and 3. The smallest such divisor greater than 3 is 4. - If $v_2(a_k) \geq 2$, then 4 is a divisor of a_k . Since $3 < 4 < p$, we have $d_4 = 4$. - If $v_2(a_k) = 1$, then 4 is not a divisor of a_k . The smallest divisor greater than 3 must be $2 \cdot 3 = 6$ or $3^2 = 9$. Since $6 < 9$ and $6 < p$, $d_4 = 6$. In either case, d_4 is not a multiple of 5. If $d_4 = 4$, $a_{k+1} = \frac{13}{12}a_k$. If $d_4 = 6$, $a_{k+1} = a_k$. Neither operation introduces a factor of 5. So if $v_5(a_k) = 0$, then $v_5(a_{k+1}) = 0$.

Now, assume for contradiction that some term is divisible by 5. Let a_n be such a term. If $v_2(a_n) = 1$, then $d_2 = 2, d_3 = 3$. Since $v_5(a_n) \geq 1$, 5 is a divisor. $d_4 = \min(6, 5) = 5$. Then $a_{n+1} = a_n \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} \right) = \frac{31}{30}a_n$. $v_2(a_{n+1}) = v_2(a_n) - v_2(30) = 1 - 1 = 0$. So a_{n+1} is odd, which fails by Lemma 1. Therefore, for a sequence with a term divisible by 5 to be valid, every term a_n must satisfy $v_2(a_n) \geq 2$. This implies $d_4 = 4$ for all n , so $a_{n+1} = \frac{13}{12}a_n$. This means $v_2(a_{n+1}) = v_2(a_n) - 2$ and $v_3(a_{n+1}) = v_3(a_n) - 1$. This cannot continue indefinitely. The sequence must eventually produce a term a_m with $v_2(a_m) < 2$ or $v_3(a_m) < 1$. If $v_2(a_m) = 1$, the next term is odd. If $v_2(a_m) = 0$, the term is odd. If $v_3(a_m) = 0$, the term is not divisible by 3. All these cases lead to failure.

****Main Analysis**** From the lemmas, any term a_n in a valid sequence must be of the form $N = 2^a 3^b M$, where $a, b \geq 1$ and all prime factors of M are ≥ 7 . We analyze the sequence based on the value of $a = v_2(N)$.

****Case 1: $v_2(N) = 1$ **** Let $N = 2^1 3^b M$ with $b \geq 1$ and prime factors of M being at least 7. The smallest divisors of N are $d_1 = 1, d_2 = 2, d_3 = 3$. The next smallest divisor is $d_4 = \min(2^2, 2 \cdot 3, 3^2, p)$, where p is the smallest prime factor of M . Since $v_2(N) = 1$, 4 is not a divisor. $d_4 = \min(6, 9, p)$. As $p \geq 7$, $d_4 = 6$. Then $f(N) = N \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) = N(1) = N$. Such numbers are fixed points. For a_1 to be one of these values, we must check that $d(a_1) \geq 4$. $d(a_1) = d(2^1 3^b M) = 2(b+1)d(M)$. Since $b \geq 1, b+1 \geq 2, d(M) \geq 1$. So $d(a_1) \geq 2(2)(1) = 4$. The condition is satisfied. Thus, any integer $N = 2^1 3^b M$ with $b \geq 1$ and prime factors of M being at least 7 is a possible value for a_1 .

****Case 2: $v_2(N) \geq 2$ **** Let $N = 2^a 3^b M$ with $a \geq 2, b \geq 1$ and prime factors of M being at least 7. The smallest divisors are $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 4$. Then $f(N) = N \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = \frac{13}{12}N$. Let a_1 be such a number. The sequence starts with $a_{s+1} = \frac{13}{12}a_s$ as long as $v_2(a_s) \geq 2$. This recurrence implies $v_2(a_{s+1}) = v_2(a_s) - 2$ and $v_3(a_{s+1}) = v_3(a_s) - 1$. For the sequence to be infinite, it must transition to a fixed point, which requires an iterate a_k to have $v_2(a_k) = 1$. Let $v_2(a_1) = a$. The sequence of 2-adic valuations is $a, a-2, a-4, \dots$. For this sequence to attain the value 1, a must be odd. If a were even, it would eventually become 0, making the term odd and causing failure. So, a must be an odd integer, $a \geq 3$. The transition to $v_2 = 1$ occurs at step $s_0 = (a-1)/2$. The term is $a_{s_0+1} = a_{(a+1)/2}$. $a_{s+1} = \left(\frac{13}{12} \right)^s a_1 = 2^{a-2s} 3^{b-s} 13^s M$. For $s = s_0 = (a-1)/2$, we get $v_2(a_{s_0+1}) = a - 2 \frac{a-1}{2} = 1$. For this term to be a fixed point, its 3-adic valuation must be at least 1. $v_3(a_{s_0+1}) = b - s_0 = b - \frac{a-1}{2}$. We need $b - \frac{a-1}{2} \geq 1 \iff b \geq 1 + \frac{a-1}{2} = \frac{a+1}{2}$. This condition also ensures $v_3(a_s) \geq 1$ for all $s \leq s_0$.

Finally, we verify that all terms in such a sequence satisfy $d(a_n) \geq 4$. For $s \in \{0, 1, \dots, \frac{a-3}{2}\}$, the term is $a_{s+1} = 2^{a-2s} 3^{b-s} 13^s M$. $v_2(a_{s+1}) = a - 2s \geq a - 2 \left(\frac{a-3}{2} \right) = 3$. $v_3(a_{s+1}) = b - s \geq b - \frac{a-3}{2} \geq \frac{a+1}{2} - \frac{a-3}{2} = 2$. So $d(a_{s+1}) = (v_2(a_{s+1}) + 1)(v_3(a_{s+1}) + 1)d(13^s M) \geq (3+1)(2+1)(1) = 12 \geq 4$. The term $a_{(a+1)/2}$ is the fixed point $2^1 \cdot 3^{b-(a-1)/2} \cdot 13^{(a-1)/2} M$. $v_2 = 1, v_3 = b - \frac{a-1}{2} \geq 1$. $d(a_{(a+1)/2}) = (1+1)(b - \frac{a-1}{2} + 1)d(13^{(a-1)/2} M) \geq 2(1+1)(1) = 4$. All terms satisfy the condition.

****Conclusion**** The possible values for a_1 are integers $N = 2^a 3^b M$ where $a, b \geq 1$ and prime factors of M are ≥ 7 , satisfying one of: 1. $a = 1, b \geq 1$ (fixed points). 2. a is an odd integer, $a \geq 3$, and $b \geq \frac{a+1}{2}$ (converging to a fixed point). \square

A.5 Problem 5

Problem 5. Alice and Bazza are playing the *inekoalaty* game, a two-player game whose rules depend on a positive real number λ which is known to both players. On the n th turn of the game (starting with $n = 1$) the following happens:

- If n is odd, Alice chooses a nonnegative real number x_n such that

$$x_1 + x_2 + \dots + x_n \leq \lambda n.$$

- If n is even, Bazza chooses a nonnegative real number x_n such that

$$x_1^2 + x_2^2 + \dots + x_n^2 \leq n.$$

If a player cannot choose a suitable number x_n , the game ends and the other player wins. If the game goes forever, neither player wins. All chosen numbers are known to both players.

Determine all values of λ for which Alice has a winning strategy and all those for which Bazza has a winning strategy.

Solution without hint. Let $S_n = \sum_{i=1}^n x_i$ and $Q_n = \sum_{i=1}^n x_i^2$. Alice plays on odd turns n , Bazza on even turns n . Alice's move x_n is valid if $S_{n-1} + x_n \leq \lambda n$. If $S_{n-1} > \lambda n$, she loses. Bazza's move x_n is valid if $Q_{n-1} + x_n^2 \leq n$. If $Q_{n-1} > n$, he loses.

****Case 1: $\lambda > \frac{\sqrt{2}}{2}$ (Alice has a winning strategy)****

Alice can devise a plan to win on a predetermined turn $2m - 1$. ****Alice's Plan:**** 1. Alice chooses an integer m large enough such that $\lambda > \frac{m\sqrt{2}}{2m-1}$. Such an m exists because the function $g(m) = \frac{m\sqrt{2}}{2m-1}$ is strictly decreasing for $m \geq 1$ and $\lim_{m \rightarrow \infty} g(m) = \frac{\sqrt{2}}{2}$. 2. For her turns $2k - 1$ where $k = 1, \dots, m - 1$, Alice chooses $x_{2k-1} = 0$. 3. On turn $2m - 1$, Alice will choose a value x_{2m-1} that makes Bazza's next move impossible.

****Analysis of Alice's Plan:**** Alice wins on turn $2m - 1$ if she can choose $x_{2m-1} \geq 0$ such that her move is valid and Bazza's next move is not. This is possible if and only if the interval of winning moves for x_{2m-1} , which is $(\sqrt{2m - Q_{2m-2}}, \lambda(2m - 1) - S_{2m-2}]$, is non-empty. This requires the condition:

$$S_{2m-2} + \sqrt{2m - Q_{2m-2}} < \lambda(2m - 1)$$

Bazza's goal is to prevent this. Given Alice's plan ($x_{2k-1} = 0$ for $k < m$), Bazza controls the values $y_k = x_{2k}$ for $k = 1, \dots, m - 1$. These choices determine $S_{2m-2} = \sum_{k=1}^{m-1} y_k$ and $Q_{2m-2} = \sum_{k=1}^{m-1} y_k^2$. Bazza's best defense is to choose his moves to maximize the function $F = S_{2m-2} + \sqrt{2m - Q_{2m-2}}$.

****Lemma:**** The maximum value of F that Bazza can achieve is $m\sqrt{2}$. ****Proof:**** Let $S = S_{2m-2}$ and $Q = Q_{2m-2}$. Bazza's moves are constrained by $\sum_{i=1}^j y_i^2 \leq 2j$ for $j = 1, \dots, m - 1$. This implies $Q = \sum_{k=1}^{m-1} y_k^2 \leq 2(m - 1)$. By the Cauchy-Schwarz inequality, $S^2 = (\sum_{k=1}^{m-1} y_k)^2 \leq (m - 1) \sum_{k=1}^{m-1} y_k^2 = (m - 1)Q$. Thus, $S \leq \sqrt{(m - 1)Q}$. So, $F \leq \sqrt{(m - 1)Q} + \sqrt{2m - Q}$. Let this upper bound be $h(Q)$. We maximize $h(Q)$ for $Q \in [0, 2(m - 1)]$. The derivative $h'(Q) = \frac{\sqrt{m-1}}{2\sqrt{Q}} - \frac{1}{2\sqrt{2m-Q}}$ is positive for $Q < 2(m - 1)$. So $h(Q)$ is strictly increasing on its domain. The maximum is at $Q = 2(m - 1)$. The maximum value of $h(Q)$ is $h(2(m - 1)) = \sqrt{(m - 1)2(m - 1)} + \sqrt{2m - 2(m - 1)} = \sqrt{2}(m - 1) + \sqrt{2} = m\sqrt{2}$. This maximum is achieved when $Q = 2(m - 1)$ and the Cauchy-Schwarz inequality is an equality, which means all y_k are equal. Let $y_k = c$. Then $Q = (m - 1)c^2 = 2(m - 1) \implies c = \sqrt{2}$. The sequence of moves $x_{2k} = \sqrt{2}$ for $k = 1, \dots, m - 1$ is valid for Bazza and it maximizes the defensive function F .

****Alice's Victory:**** Alice's strategy is guaranteed to work if her winning condition holds even against Bazza's best defense. This requires $\max(F) < \lambda(2m - 1)$, which is $m\sqrt{2} < \lambda(2m - 1)$, or $\lambda > \frac{m\sqrt{2}}{2m-1}$. By her initial choice of m , this condition is met. We must also check that Alice's moves $x_{2k-1} = 0$ for $k < m$ are valid. This requires $S_{2k-2} \leq \lambda(2k - 1)$. Bazza's best defense maximizes S_{2k-2} to $(k - 1)\sqrt{2}$. The condition is $(k - 1)\sqrt{2} \leq \lambda(2k - 1)$,

or $\lambda \geq \frac{(k-1)\sqrt{2}}{2k-1}$. Since $\lambda > \frac{\sqrt{2}}{2}$ and $\frac{(k-1)\sqrt{2}}{2k-1}$ is an increasing function of k with limit $\frac{\sqrt{2}}{2}$, this condition holds for all k . Thus, Alice has a winning strategy.

****Case 2: $\lambda < \frac{\sqrt{2}}{2}$ (Bazza has a winning strategy)****

Bazza's strategy is to always play $x_{2k} = \sqrt{2k - Q_{2k-1}}$ if possible. This sets $Q_{2k} = 2k$. As shown in Case 1, Alice cannot win against this strategy because her winning condition $\lambda > \frac{m\sqrt{2}}{2m-1}$ can never be met if $\lambda < \frac{\sqrt{2}}{2}$. We now show that Bazza will win.

Alice loses on turn $2m-1$ if $S_{2m-2} > \lambda(2m-1)$. To survive, Alice must choose her moves to keep the sequence of sums S_{2k-2} as small as possible for as long as possible. With Bazza's strategy, Alice's choice of x_{2k-1} (provided $x_{2k-1} \leq \sqrt{2}$) determines Bazza's response $x_{2k} = \sqrt{2 - x_{2k-1}^2}$. The sum grows by $C_k = x_{2k-1} + \sqrt{2 - x_{2k-1}^2}$ over turns $2k-1$ and $2k$. To minimize the sum $S_{2m-2} = \sum_{k=1}^{m-1} C_k$, Alice must choose each x_{2k-1} to minimize C_k . The function $f(x) = x + \sqrt{2 - x^2}$ on $[0, \sqrt{2}]$ has a minimum value of $\sqrt{2}$, achieved only at $x = 0$ and $x = \sqrt{2}$. Any other choice would lead to a strictly larger sum S_{2m-2} for all $m > k+1$, making survival strictly harder. Thus, an optimal survival strategy for Alice must consist only of moves $x_{2k-1} \in \{0, \sqrt{2}\}$.

Let's compare these two choices at turn $2k-1$. Suppose Alice has survived so far, with sum S_{2k-2} . 1. If Alice chooses $x_{2k-1} = 0$: This move is valid if $S_{2k-2} \leq \lambda(2k-1)$. The resulting sum is $S_{2k} = S_{2k-2} + \sqrt{2}$. 2. If Alice chooses $x_{2k-1} = \sqrt{2}$: This move is valid if $S_{2k-2} + \sqrt{2} \leq \lambda(2k-1)$. The resulting sum is $S_{2k} = S_{2k-2} + \sqrt{2}$.

Both choices lead to the same future sums S_{2j} for $j \geq k$, meaning the survival conditions for all subsequent turns are identical regardless of which of the two is chosen. However, the condition to be allowed to make the choice at turn $2k-1$ is strictly easier for $x_{2k-1} = 0$. A strategy involving $x_{2k-1} = \sqrt{2}$ is only valid if the corresponding strategy with $x_{2k-1} = 0$ is also valid, but the converse is not true. Therefore, the strategy of always choosing $x_{2k-1} = 0$ is Alice's best hope for survival. If she cannot survive with this strategy, she cannot survive with any other.

We now analyze this specific line of play: Alice always plays $x_{2k-1} = 0$, and Bazza responds with $x_{2k} = \sqrt{2}$. 1. Let $h(k) = \frac{(k-1)\sqrt{2}}{2k-1}$. Since $h(k)$ is strictly increasing and approaches $\frac{\sqrt{2}}{2}$, and $\lambda < \frac{\sqrt{2}}{2}$, there exists a smallest integer $m \geq 2$ such that $\lambda < h(m)$. 2. For any $k < m$, we have $\lambda \geq h(k)$. Alice's move $x_{2k-1} = 0$ is valid, since $S_{2k-2} = (k-1)\sqrt{2}$ and the condition is $(k-1)\sqrt{2} \leq \lambda(2k-1)$, which is equivalent to $\lambda \geq h(k)$. 3. On turn $2m-1$, Alice has played according to her optimal survival strategy. The sum is $S_{2m-2} = (m-1)\sqrt{2}$. She must choose $x_{2m-1} \geq 0$ such that $(m-1)\sqrt{2} + x_{2m-1} \leq \lambda(2m-1)$. 4. By the choice of m , we have $\lambda < \frac{(m-1)\sqrt{2}}{2m-1}$, which is $\lambda(2m-1) < (m-1)\sqrt{2}$. 5. The condition for Alice's move becomes $(m-1)\sqrt{2} + x_{2m-1} \leq \lambda(2m-1) < (m-1)\sqrt{2}$. This implies $x_{2m-1} < 0$, which is impossible. Alice cannot make a move, so Bazza wins.

****Case 3: $\lambda = \frac{\sqrt{2}}{2}$ (Draw)****

In this case, neither player has a winning strategy. A player has a winning strategy if they can force a win in a finite number of moves against any of the opponent's strategies.

****Alice does not have a winning strategy.**** To prove this, we show that Bazza has a defensive strategy that prevents Alice from ever winning. Let Bazza adopt the strategy of

always choosing $x_{2k} = \sqrt{2k - Q_{2k-1}}$ (if possible). Alice wins if she can play x_{2m-1} on turn $2m - 1$ such that $Q_{2m-1} > 2m$. With Bazza's strategy, $Q_{2m-2} = 2(m - 1)$ (assuming Alice has not won before). Alice's winning condition becomes $2(m - 1) + x_{2m-1}^2 > 2m$, which simplifies to $x_{2m-1} > \sqrt{2}$. To play such a move, her budget must allow it: $S_{2m-2} + x_{2m-1} \leq \lambda(2m - 1) = \frac{\sqrt{2}}{2}(2m - 1)$. This requires $S_{2m-2} + \sqrt{2} < \frac{\sqrt{2}}{2}(2m - 1)$. Against Bazza's strategy, $S_{2m-2} = \sum_{k=1}^{m-1} (x_{2k-1} + \sqrt{2 - x_{2k-1}^2})$. The minimum value of each term is $\sqrt{2}$, so the minimum possible value of S_{2m-2} is $(m - 1)\sqrt{2}$. Thus, for Alice to win, it must be that $(m - 1)\sqrt{2} + \sqrt{2} < \frac{\sqrt{2}}{2}(2m - 1)$, which simplifies to $m\sqrt{2} < \frac{\sqrt{2}}{2}(2m - 1) \iff 2m < 2m - 1 \iff 0 < -1$. This is impossible. So, Alice can never satisfy her winning condition against this strategy from Bazza. Therefore, Alice does not have a winning strategy.

****Bazza does not have a winning strategy.**** To prove this, we show that Alice has a defensive strategy that prevents Bazza from ever winning. Let Alice adopt the strategy of always choosing $x_{2k-1} = 0$. Bazza wins if he can force $S_{2m-2} > \lambda(2m - 1) = \frac{\sqrt{2}}{2}(2m - 1)$ for some m . With Alice's strategy, $S_{2m-2} = \sum_{k=1}^{m-1} x_{2k}$. Bazza's moves x_{2k} are constrained by $Q_{2k} = \sum_{i=1}^k x_{2i}^2 \leq 2k$. To force a win, Bazza must maximize S_{2m-2} . As shown in the lemma in Case 1, the maximum possible value for S_{2m-2} is $(m - 1)\sqrt{2}$, achieved when Bazza plays $x_{2k} = \sqrt{2}$ for all $k < m$. So, the largest sum Bazza can force is $S_{2m-2} = (m - 1)\sqrt{2}$. Bazza wins if this sum exceeds Alice's budget limit, i.e., if $(m - 1)\sqrt{2} > \frac{\sqrt{2}}{2}(2m - 1)$. This simplifies to $2(m - 1) > 2m - 1 \iff 2m - 2 > 2m - 1 \iff -2 > -1$. This is impossible. So, Bazza can never force Alice to lose. Therefore, Bazza does not have a winning strategy.

Since neither player can force a win, neither has a winning strategy. The game can continue indefinitely, for example with Alice always playing $x_{2k-1} = 0$ and Bazza always playing $x_{2k} = \sqrt{2}$. This results in a draw. \square

A.6 Problem 6

Problem 6. Consider a 2025×2025 grid of unit squares. Matilda wishes to place on the grid some rectangular tiles, possibly of different sizes, such that each side of every tile lies on a grid line and every unit square is covered by at most one tile.

Determine the minimum number of tiles Matilda needs to place so that each row and each column of the grid has exactly one unit square that is not covered by any tile.

Flawed solution without hint. Let the grid be an $N \times N$ grid of unit squares, where $N = 2025$. Let the square in row i and column j be denoted by (i, j) for $1 \leq i, j \leq N$.

****1. The Structure of Uncovered Squares****

The condition is that each row and each column of the grid has exactly one uncovered square. Let U be the set of uncovered squares. For each row i , there is a unique column j such that $(i, j) \in U$. This allows us to define a function $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$, where the uncovered square in row i is $(i, \sigma(i))$. For each column j , there is a unique row i such that $(i, j) \in U$. This means that for any $j_0 \in \{1, \dots, N\}$, there is a unique i_0 such that $\sigma(i_0) = j_0$. This implies that the function σ is a bijection, i.e., a permutation. So, the set of uncovered squares is $U = \{(i, \sigma(i)) \mid i = 1, \dots, N\}$ for some permutation σ of $\{1, \dots, N\}$.

2. Partitioning the Tiled Region

Let C be the set of all squares covered by tiles. C is the complement of U in the grid. We can partition C into two disjoint sets: $C_L = \{(i, j) \mid j < \sigma(i)\}$, the set of squares in C that are to the left of the uncovered square in their respective rows. $C_R = \{(i, j) \mid j > \sigma(i)\}$, the set of squares in C that are to the right of the uncovered square in their respective rows.

It is clear that $C = C_L \cup C_R$ and $C_L \cap C_R = \emptyset$.

**3. Tiles are Confined to C_L or C_R **

Let a rectangular tile R cover the squares (i, j) for $i \in [r_1, r_2]$ and $j \in [c_1, c_2]$, where $[a, b]$ denotes the set of integers $\{a, a+1, \dots, b\}$. For R to be a valid tile, all squares it covers must be in C . This means for any $i \in [r_1, r_2]$, the uncovered square $(i, \sigma(i))$ is not in R . This implies that for any $i \in [r_1, r_2]$, $\sigma(i) \notin [c_1, c_2]$.

Let $S_\sigma = \{\sigma(i) \mid i \in [r_1, r_2]\}$. The condition for R to be a valid tile is $S_\sigma \cap [c_1, c_2] = \emptyset$. Let $m = \min(S_\sigma)$ and $M = \max(S_\sigma)$. The condition is equivalent to either $c_2 < m$ or $c_1 > M$.

* If $c_2 < m$, then for any square $(i, j) \in R$, we have $j \leq c_2 < m \leq \sigma(i)$ for all $i \in [r_1, r_2]$. Thus $j < \sigma(i)$, which implies $(i, j) \in C_L$. Therefore, the entire tile R is contained in C_L . * If $c_1 > M$, then for any square $(i, j) \in R$, we have $j \geq c_1 > M \geq \sigma(i)$ for all $i \in [r_1, r_2]$. Thus $j > \sigma(i)$, which implies $(i, j) \in C_R$. Therefore, the entire tile R is contained in C_R .

This proves that any tile must be entirely in C_L or entirely in C_R . Consequently, the regions C_L and C_R are tiled independently. The total minimum number of tiles, k , is the sum of the minimum number of tiles required to tile C_L (k_L) and C_R (k_R). So, $k = k_L + k_R$.

**4. Minimum Number of Tiles for C_L **

Let's determine k_L for an arbitrary permutation σ . Consider the set of squares $S_L = \{(i, \sigma(i) - 1) \mid i \in \{1, \dots, N\} \text{ such that } \sigma(i) > 1\}$. The size of this set is $|S_L| = N - 1$, since there is exactly one i_0 for which $\sigma(i_0) = 1$. Each square in S_L is in C_L .

Let $s_i = (i, \sigma(i) - 1)$ and $s_j = (j, \sigma(j) - 1)$ be any two distinct squares in S_L . Assume without loss of generality that $i < j$. Any tile that covers both s_i and s_j must contain the smallest rectangle R_{ij} that contains them. The rows of R_{ij} are $[i, j]$ and the columns are $[\min(\sigma(i) - 1, \sigma(j) - 1), \max(\sigma(i) - 1, \sigma(j) - 1)]$.

We investigate if R_{ij} contains any uncovered square from U . * **Case 1: $\sigma(i) < \sigma(j)$.** The column interval of R_{ij} is $[\sigma(i) - 1, \sigma(j) - 1]$. Consider the uncovered square $u_i = (i, \sigma(i))$. Its row i is in $[i, j]$. Its column $\sigma(i)$ is in $[\sigma(i) - 1, \sigma(j) - 1]$ because $\sigma(i) - 1 \leq \sigma(i)$ and $\sigma(i) < \sigma(j)$ implies $\sigma(i) \leq \sigma(j) - 1$ for integers. So, $u_i \in R_{ij}$. * **Case 2: $\sigma(i) > \sigma(j)$.** The column interval of R_{ij} is $[\sigma(j) - 1, \sigma(i) - 1]$. Consider the uncovered square $u_j = (j, \sigma(j))$. Its row j is in $[i, j]$. Its column $\sigma(j)$ is in $[\sigma(j) - 1, \sigma(i) - 1]$ because $\sigma(j) - 1 \leq \sigma(j)$ and $\sigma(j) < \sigma(i)$ implies $\sigma(j) \leq \sigma(i) - 1$. So, $u_j \in R_{ij}$.

In both cases, the rectangle R_{ij} contains an uncovered square. Therefore, s_i and s_j cannot be in the same tile. This holds for any pair of distinct squares in S_L . Thus, all $N - 1$ squares in S_L must belong to different tiles. This provides a lower bound on the number of tiles for C_L : $k_L \geq N - 1$.

To show this bound is achievable, we can tile C_L with exactly $N - 1$ tiles. For each i such that $\sigma(i) > 1$, the region to be tiled in row i is $\{(i, j) \mid 1 \leq j \leq \sigma(i) - 1\}$. This is a single rectangular region, which can be covered by one tile $T_i = [i, i] \times [1, \sigma(i) - 1]$. There

are $N - 1$ such values of i , so this construction uses $N - 1$ tiles to cover all of C_L .

Thus, for any permutation σ , the minimum number of tiles for C_L is exactly $k_L = N - 1$.

****5. Minimum Number of Tiles for C_R ****

We use a symmetry argument to determine k_R . Consider a 180-degree rotation of the grid. A square (i, j) is mapped to $(i', j') = (N + 1 - i, N + 1 - j)$. An uncovered square $(i, \sigma(i))$ is mapped to $(N + 1 - i, N + 1 - \sigma(i))$. Let's define a new permutation τ on $\{1, \dots, N\}$ by $\tau(i') = N + 1 - \sigma(N + 1 - i')$. The set of uncovered squares in the rotated grid is $\{(i', \tau(i')) \mid i' = 1, \dots, N\}$. The region $C_R(\sigma) = \{(i, j) \mid j > \sigma(i)\}$ in the original grid corresponds to the region of squares (i', j') where $N + 1 - j' > \sigma(N + 1 - i')$. This inequality is equivalent to $j' < N + 1 - \sigma(N + 1 - i')$, which is $j' < \tau(i')$. This is precisely the region $C_L(\tau)$ for the permutation τ . The minimum number of tiles required to tile a region is invariant under rotation. Thus, the minimum number of tiles for $C_R(\sigma)$ is equal to the minimum number of tiles for $C_L(\tau)$. Since we proved that $k_L(\pi) = N - 1$ for *any* permutation π , it holds for τ . So, $k_R(\sigma) = k_L(\tau) = N - 1$.

****6. Conclusion****

For any permutation σ , the minimum number of tiles is $k = k_L + k_R = (N - 1) + (N - 1) = 2N - 2$. Since the number of tiles is independent of the choice of permutation, the minimum number of tiles Matilda needs is $2N - 2$.

For $N = 2025$, the minimum number of tiles is $2 \times (2025 - 1) = 2 \times 2024 = 4048$. \square