On Computational Higher-Dimensional Type Theory

CIS 670 Project

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Motivation

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- The search for foundations of mathematics

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- Type theory

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- Austere notion of equality

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- Machine-checked proofs

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- Austere notion of equality
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- Connection between homotopy theory and type theory
- Univalent foundations
- Computational Higher-Dimensional Type theory

The Homotopy connection

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The Homotopy connection

‘Portrait of Lotte’ by Frans Hofmeester

The Homotopy connection

- Types as spaces
- Equality as paths

The Homotopy connection

- Types as spaces
- Equality as paths
- Paths between paths?

The Homotopy connection

- Types as spaces
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- Paths between paths?
- Paths between paths between paths?

Higher-Dimensional type theory

- Types have internal structure (Higher inductive types)

Higher-Dimensional type theory

- Types have internal structure (Higher inductive types)
- Type-theoretic operations respect the internal structures

From Martin-Löf type theory to Homotopy type theory

Judgements

- ϕ is a proposition.
- ϕ is true.
- ϕ_1 is true, ϕ_2 is true, ... , ϕ_n is true $\vdash \phi$ is a proposition.
- ϕ_1 is true, ϕ_2 is true, ... , ϕ_n is true $\vdash \phi$ is true.

Meaning of hypothetical judgements

- $$\frac{}{\phi_1 \text{ is true}, \phi_2 \text{ is true}, \ldots, \phi_i \text{ is true}, \ldots, \phi_n \text{ is true} \vdash \phi_i \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_1 \text{ is true} \qquad \Gamma, \phi_1 \text{ is true} \vdash \phi_2 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_2 \text{ is true} \qquad \Gamma \vdash \phi_1 \text{ is a proposition}}{\Gamma, \phi_1 \text{ is true} \vdash \phi_2 \text{ is true}}$$

From Martin-Löf type theory to Homotopy type theory

Building complex propositions - True

- $\top$ is prop.
- $\Gamma \vdash \top$ is true

Building complex propositions - And

- $$\frac{\Gamma \vdash \phi_1 \text{ is prop} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ is prop}}$$
- $$\frac{\Gamma \vdash \phi_1 \text{ is true} \quad \Gamma \vdash \phi_2 \text{ is true}}{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ is true}}$$
- $$\frac{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ is true}}{\Gamma \vdash \phi_1 \text{ is true}}$$
- $$\frac{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

Building complex propositions - Implication

- $$\frac{\Gamma \vdash \phi_1 \text{ is prop} \quad \Gamma, \phi_1 \text{ is true} \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \implies \phi_2 \text{ is prop}}$$

- $$\frac{\Gamma, \phi_1 \text{ is true} \vdash \phi_2 \text{ is true}}{\Gamma \vdash \phi_1 \implies \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_1 \implies \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

From Martin-Löf type theory to Homotopy type theory

Building complex propositions - Or

- $$\frac{\Gamma \vdash \phi_1 \text{ is prop} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is prop}}$$

- $$\frac{\Gamma \vdash \phi_1 \text{ is true} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_1 \vee \phi_2 \text{ true} \quad \Gamma, \phi_1 \text{ true} \vdash \phi \text{ true} \quad \Gamma, \phi_2 \text{ true} \vdash \phi \text{ true}}{\Gamma \vdash \phi \text{ true}}$$

Building complex propositions - False

- $\bot$ is prop.
- $$\frac{\Gamma \vdash \bot \text{ is true} \quad \Gamma, \bot \text{ is true} \vdash \phi \text{ is prop}}{\Gamma \vdash \phi \text{ is true}}$$

From Martin-Löf type theory to Homotopy type theory

What have we covered so for?

From Martin-Löf type theory to Homotopy type theory

What have we covered so for?

- $\top$ $\checkmark$

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓
- $\Rightarrow$ ✓

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓
- $\Rightarrow$ ✓
- $\vee$ ✓

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓
- $\Rightarrow$ ✓
- $\vee$ ✓
- $\bot$ ✓

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓
- $\Rightarrow$ ✓
- $\vee$ ✓
- $\perp$ ✓

What's missing?

From Martin-Löf type theory to Homotopy type theory

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓
- $\Rightarrow$ ✓
- $\vee$ ✓
- $\bot$ ✓

What's missing?

- $\forall$
- $\exists$

What have we covered so for?

- $\top$ ✓
- $\wedge$ ✓
- $\Rightarrow$ ✓
- $\vee$ ✓
- $\perp$ ✓

What's missing?

- $\forall$
- $\exists$
- $=$

From Martin-Löf type theory to Homotopy type theory

- $$\frac{\Gamma, x : A \vdash B(x) \text{ is prop}}{\Gamma \vdash \forall(x : A).B(x) \text{ is prop}}$$

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- $$\frac{\Gamma, x : A \vdash p : B(x)}{\Gamma \vdash \lambda(x : A).p : \forall(x : A).B(x)}$$

- $$\frac{\Gamma, x : A \vdash B(x) \text{ is prop}}{\Gamma \vdash \forall(x : A).B(x) \text{ is prop}}$$

- $$\frac{\Gamma, x : A \vdash p : B(x)}{\Gamma \vdash \lambda(x : A).p : \forall(x : A).B(x)}$$

- $$\frac{\Gamma \vdash p : \forall(x : A).B(x) \quad \Gamma \vdash a : A}{\Gamma \vdash p(a) : B(a)}$$

- $$\frac{\Gamma, x : A \vdash B(x) \text{ is prop}}{\Gamma \vdash \exists(x : A).B(x) \text{ is prop}}$$

From Martin-Löf type theory to Homotopy type theory

- $$\frac{\Gamma, x : A \vdash B(x) \text{ is prop}}{\Gamma \vdash \exists(x : A).B(x) \text{ is prop}}$$

- $$\frac{\begin{array}{l}\Gamma, x : A \vdash B(x) \text{ is prop} \\ \Gamma \vdash a : A \qquad\qquad \Gamma \vdash b : B(a)\end{array}}{\Gamma \vdash (a, b) : \exists(x : A).B(x)}$$

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- $$\frac{\begin{array}{l}\Gamma, z : \exists(x : A).B(x) \vdash C(z) \text{ is prop} \\ \Gamma, x : A, p : B(x) \vdash r : C((x, p))\end{array}}{\Gamma, z : \exists(x : A).B(x) \vdash r[x := \pi_1 z][p := \pi_2 z] : C(z)}$$

From Martin-Löf type theory to Homotopy type theory

- $$\frac{\Gamma \vdash x : A \quad \Gamma \vdash y : A}{\Gamma \vdash x =_A y \text{ is prop}}$$

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- $$\frac{\Gamma \vdash x : A \quad \Gamma \vdash y : A}{\Gamma \vdash x =_A y \text{ is prop}}$$

- $$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{refl}_a^A : a =_A a}$$

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- $$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}^A_a : a =_A a}$$

- $$\frac{\begin{array}{l}\Gamma, x : A, y : A, p : x =_A y \vdash C(x, y, p) \text{ is prop} \\ \Gamma, x : A \vdash d : C(x, x, \mathsf{refl}^A_x)\end{array}}{\Gamma, x : A, y : A, p : x =_A y \vdash d[x := x] : C(x, y, p)}$$

Non-Uniqueness of Identity proofs

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-

$$\frac{\begin{array}{c}\Gamma \vdash f, g : A \to B \\ \Gamma, x : A \vdash p : f(x) =_B g(x)\end{array}}{\Gamma \vdash \lambda(x : A).p : f =_{A \to B} g}$$

$$(f =_{A \to B} g) \triangleq \Pi_{(x:A)}(f(x) =_B g(x))$$

Non-Uniqueness of Identity proofs

- $$\frac{\Gamma \vdash f, g : A \to B \qquad \Gamma, x : A \vdash p : f(x) =_B g(x)}{\Gamma \vdash \lambda(x : A).p : f =_{A \to B} g}$$

  $$(f =_{A \to B} g) \triangleq \Pi_{(x:A)}(f(x) =_B g(x))$$

- $$\frac{\Gamma \vdash A, B : U \qquad \Gamma \vdash f : A \to B \qquad \Gamma \vdash g : B \to A \qquad \Gamma, x : A \vdash p_1 : g(f(x)) =_A x \qquad \Gamma, y : B \vdash p_2 : f(g(y)) =_B y}{\Gamma \vdash (f, g, (\lambda(x : A).p_1, \lambda(y : B).p_2)) : A =_U B}$$

  $$(A = B) \triangleq \Sigma_f \Sigma_g((\Pi_x(g(fx) = x)) \times (\Pi_y(f(gy) = y)))$$

Properties $=_{\_}$ must satisfy

- $$\frac{\Gamma \vdash a : A}{\Gamma \vdash r_a : a =_A a}$$

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- $$\frac{\Gamma \vdash a : A}{\Gamma \vdash r_a : a =_A a}$$

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Properties $=_{\_}$ must satisfy

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- $$\frac{\Gamma \vdash p : a_1 =_A a_2 \quad \Gamma \vdash q : a_2 =_A a_3}{\Gamma \vdash p \circ q : a_1 =_A a_3}$$

Properties $=_{\_}$ must satisfy

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- $$\frac{\Gamma \vdash P : \Pi_{(x:A)} B(x) \quad \Gamma \vdash q : a_1 =_A a_2}{\Gamma \vdash \mathrm{lift}_q^P : B(a_1) \to B(a_2)}$$

More properties

- $p \circ (q \circ r) =_{a_1 =_A a_4} (p \circ q) \circ r$

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Organizing the structure

$$x, x' : p =_{a_1 =_A a_2} p'$$

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We need a way to organize the structure of paths.

Organizing the structure

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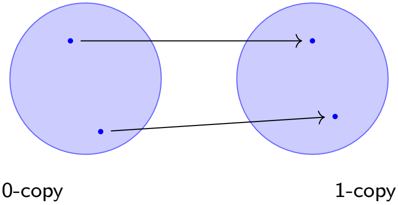

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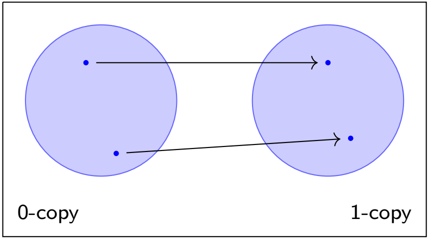

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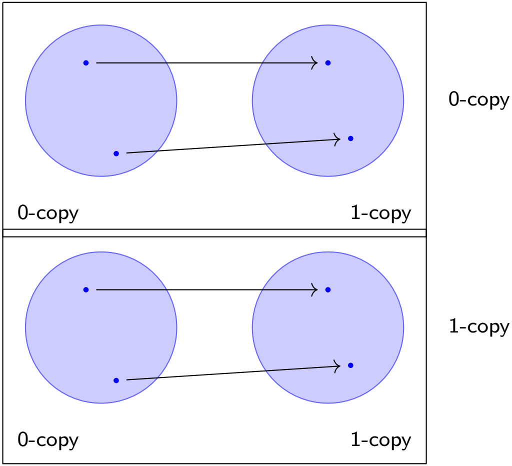

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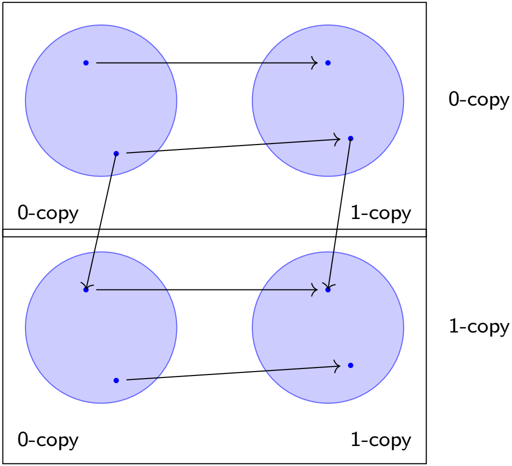

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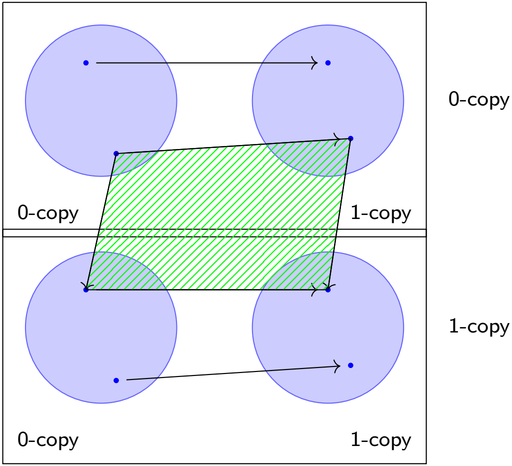

Cubical Type Theory

What is CTT?

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- Model of Homotopy Type Theory

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Essential concept: Dimensions

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- Based on cubical set
- i.e., types with “primitive” internal structure

Essential concept: Dimensions

- Abstract interval
- Name
- Endpoints 0 and 1
- Substitution

Cubical Types

What are cubical types?

Cubical Types

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Objects at dimensions:

Cubical Types

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Objects at dimensions:

- Dimension 0: usual points

Cubical Types

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Cubical Types

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Types at dimensions:

- Types of dim. 0: usual types
- Types of dim. $n+1$: type line between n-types
  - Elements: lines

Kan filling

What about open boxes?

Kan filling

What about open boxes?

- ($\rightarrow$ board)

Kan filling

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- Must be defined <u>for every type</u>

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  - ...depending on its shape

Kan filling

What about open boxes?

- ($\rightarrow$ board)
- Must be defined for every type
  - ...depending on its shape
- Explicit term hcom

Syntax

$$\begin{aligned} M := {} & (a{:}A) \to B \mid (a{:}A) \times B \mid \mathsf{Id}_{x.A}(M, N) \mid \mathsf{bool} \mid \mathsf{not}_r \mid \mathbb{S}^1 \\ & \mid \lambda a.M \mid \mathsf{app}(M, N) \mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \langle x \rangle M \mid M@r \\ & \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}_{a.A}(M; N_1, N_2) \mid \mathsf{notel}_r(M) \\ & \mid \mathsf{base} \mid \mathsf{loop}_r \mid \mathbb{S}^1\text{-}\mathsf{elim}_{a.A}(M; N_1, x.N_2) \\ & \mid \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \mathsf{hcom}_A^{\overrightarrow{r_i}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon}) \end{aligned}$$

Notations:

a	Term variable
x	Dim. variable
r	Dim. term (0, 1, *x*)

Typing judgments

- Shape: $\Psi; \Gamma \vdash a : A$ ($\Psi; \Gamma \vdash A : \mathcal{U}$)
- $$\frac{\Psi; \Gamma \vdash A : \mathcal{U} \qquad \Psi; \Gamma, a : A \vdash B : \mathcal{U}}{\Psi; \Gamma \vdash (a : A) \to B : \mathcal{U}}$$

Language: typing

Typing judgments

- Shape: $\Psi; \Gamma \vdash a : A$ $(\Psi; \Gamma \vdash A : \mathcal{U})$
- $$\frac{\Psi; \Gamma \vdash A : \mathcal{U} \qquad \Psi; \Gamma, a : A \vdash B : \mathcal{U}}{\Psi; \Gamma \vdash (a : A) \to B : \mathcal{U}}$$
- $$\frac{\Psi; \Gamma, a : A \vdash M : B}{\Psi; \Gamma \vdash \lambda a.M : A}$$

Typing judgments

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- $$\frac{\Psi, x; \Gamma \vdash A : \mathcal{U} \qquad \Psi; \Gamma \vdash M : A\langle r/x \rangle}{\Psi; \Gamma \vdash \mathrm{coe}_{x.A}^{r \rightsquigarrow r'}(M) : A\langle r'/x \rangle}$$

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- Shape: $\Psi; \Gamma \vdash a : A$ ($\Psi; \Gamma \vdash A : \mathcal{U}$)
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- $$\frac{\Psi, x; \Gamma \vdash A : \mathcal{U} \qquad \Psi; \Gamma \vdash M : A\langle r/x\rangle}{\Psi; \Gamma \vdash \mathrm{coe}^{r \rightsquigarrow r'}_{x.A}(M) : A\langle r'/x\rangle}$$

Language: reduction

Interesting reduction rules

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$$\frac{}{\mathsf{loop}_x \ \mathsf{val}}$$

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$$\overline{\mathsf{coe}^{r \rightsquigarrow r'}_{x.(a:A)\to B}(M) \longmapsto \lambda a.\mathsf{coe}^{r \rightsquigarrow r'}_{x.B[\mathsf{coe}^{r' \rightsquigarrow x}_{x.A}(a)/a]}(\mathsf{app}(M, \mathsf{coe}^{r' \rightsquigarrow r}_{x.A}(a)))}$$

Interesting reduction rules

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- Are your ready for *hcom*?

Language: reduction on the loose

$$\frac{F = \mathsf{hcom}_A^{\overrightarrow{r_i}}(r \rightsquigarrow z, \mathsf{fst}(M); \overrightarrow{y.\mathsf{fst}(N_i^\varepsilon)})}{\begin{array}{c}\mathsf{hcom}_{(a:A)\times B}^{\overrightarrow{r_i}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon}) \\ \longmapsto \\ \langle \mathsf{hcom}_A^{\overrightarrow{r_i}}(r \rightsquigarrow r', \mathsf{fst}(M); \overrightarrow{y.\mathsf{fst}(N_i^\varepsilon)}), \mathsf{com}_{z.B[F/a]}^{\overrightarrow{r_i}}(r \rightsquigarrow r', \mathsf{snd}(M); \overrightarrow{y.\mathsf{snd}(N_i^\varepsilon)})\rangle\end{array}}$$

$$\frac{}{\mathsf{hcom}_{(a:A)\to B}^{\overrightarrow{r_i}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon}) \longmapsto \lambda a.\mathsf{hcom}_B^{\overrightarrow{r_i}}(r \rightsquigarrow r', \mathsf{app}(M, a); \overrightarrow{y.\mathsf{app}(N_i^\varepsilon, a)})}$$

And now?

Why are these systems desirable?

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- Much easier to use than plain HoTT

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- Much easier to use than plain HoTT
- More integrated

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Why are these systems desirable?

- Much easier to use than plain HoTT
- More integrated
- For instance, no refl_t

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- With this, it might be easier to express formally and constructively more of mathematics and other languages.

References

Carlo Angiuli, Robert Harper, and Todd Wilson, Computational higher dimensional type theory, Proceedings of the 44th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (New York, NY, USA), POPL '17, ACM, 2017.

Martin Hofmann and Thomas Streicher, The groupoid interpretation of type theory, Twenty-five years of constructive type theory (Venice, 1995), Oxford Logic Guides, vol. 36, Oxford Univ. Press, New York, 1998, pp. 83–111.

Per Martin-Löf, An intuitionistic theory of types, Twenty-five years of constructive type theory, Oxford Logic Guides, vol. 36, Oxford University Press, 1998, pp. 127–172.

The Univalent Foundations Program, Homotopy type theory: Univalent foundations of mathematics, `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

Questions?

Thank you.