

From System T to Continuation-Passing Style

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1 Syntax and typing definitions

x, y	variables	
i	integer literals	
τ, σ	$::=$	types
	\mathbb{Z}	
	void	
	$\tau_1 \rightarrow \tau_2$	
	(τ)	S
e	$::=$	annotated terms
	u^τ	
u, v	$::=$	raw terms
	x	
	i	
	$\lambda x : \tau. e$	bind x in e
	$e_1 e_2$	
	$e_1 p e_2$	
	if0 (e_1, e_2, e_3)	
	let $x = v$ in u	bind x in u
	halt [τ] e	
	(u)	S
p	$::=$	primitives
	$+$	
	$-$	
Γ	$::=$	contexts
	$\Gamma, x : \tau$	

$\boxed{\Gamma \vdash_{\text{T}} e : \tau}$ annotated typing

$$\frac{\Gamma \vdash_{\text{T}} u : \tau}{\Gamma \vdash_{\text{T}} u^\tau : \tau} \quad \text{T_ANT_ANN}$$

$\boxed{\Gamma \vdash_{\text{T}} u : \tau}$ typing

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\text{T}} x : \tau} \quad \text{T_TERM_VAR}$$

$$\frac{}{\Gamma \vdash_{\text{T}} i : \mathbb{Z}} \quad \text{T_TERM_INT}$$

$$\frac{\Gamma, x_1 : \tau_1 \vdash_{\text{T}} e : \tau_2}{\Gamma \vdash_{\text{T}} \lambda x_1 : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \text{T_TERM_LAM}$$

$$\begin{array}{c}
\frac{\Gamma \vdash_{\text{T}} e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash_{\text{T}} e_2 : \tau_1}{\Gamma \vdash_{\text{T}} e_1 e_2 : \tau_2} \quad \text{T_TERM_APP} \\
\\
\frac{\Gamma \vdash_{\text{T}} e_1 : \mathbb{Z} \quad \Gamma \vdash_{\text{T}} e_2 : \mathbb{Z}}{\Gamma \vdash_{\text{T}} e_1 p e_2 : \mathbb{Z}} \quad \text{T_TERM_PRIM} \\
\\
\frac{\Gamma \vdash_{\text{T}} e_1 : \mathbb{Z} \quad \Gamma \vdash_{\text{T}} e_2 : \tau \quad \Gamma \vdash_{\text{T}} e_3 : \tau}{\Gamma \vdash_{\text{T}} \text{if0}(e_1, e_2, e_3) : \tau} \quad \text{T_TERM_IF0}
\end{array}$$

$\boxed{\Gamma \vdash_{\text{K}} e : \tau}$ annotated typing

$$\frac{\Gamma \vdash_{\text{K}} u : \tau}{\Gamma \vdash_{\text{K}} u^\tau : \tau} \quad \text{K_ANT_ANN}$$

$\boxed{\Gamma \vdash_{\text{K}} u : \tau}$ typing

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\text{K}} x : \tau} \quad \text{K_TERM_VAR}$$

$$\frac{}{\Gamma \vdash_{\text{K}} i : \mathbb{Z}} \quad \text{K_TERM_INT}$$

$$\frac{\Gamma, x : \tau_1 \vdash_{\text{K}} e : \tau_2}{\Gamma \vdash_{\text{K}} \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \text{K_TERM_LAM}$$

$$\frac{\Gamma \vdash_{\text{K}} v : \tau \quad \Gamma, x : \tau \vdash_{\text{K}} u : \text{void}}{\Gamma \vdash_{\text{K}} \text{let } x = v \text{ in } u : \text{void}} \quad \text{K_TERM_LET}$$

$$\frac{\Gamma \vdash_{\text{K}} e_1 : \mathbb{Z} \quad \Gamma \vdash_{\text{K}} e_2 : \mathbb{Z}}{\Gamma \vdash_{\text{K}} e_1 p e_2 : \mathbb{Z}} \quad \text{K_TERM_PRIM}$$

$$\frac{\Gamma \vdash_{\text{K}} e' : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash_{\text{K}} e : \tau}{\Gamma \vdash_{\text{K}} e' e : \tau_2} \quad \text{K_TERM_APP}$$

$$\frac{\Gamma \vdash_{\text{K}} e : \mathbb{Z} \quad \Gamma \vdash_{\text{K}} e_1 : \text{void} \quad \Gamma \vdash_{\text{K}} e_2 : \text{void}}{\Gamma \vdash_{\text{K}} \text{if0}(e, e_1, e_2) : \text{void}} \quad \text{K_TERM_IF0}$$

$$\frac{\Gamma \vdash_{\text{K}} e : \tau}{\Gamma \vdash_{\text{K}} \text{halt}[\tau]e : \text{void}} \quad \text{K_TERM_HALT}$$

2 Translation

2.1 Type translation

$$\begin{array}{ll}
\mathcal{K}[\mathbb{Z}] & \triangleq \mathbb{Z} \\
\mathcal{K}[\tau_1 \rightarrow \tau_2] & \triangleq \mathcal{K}[\tau_1] \rightarrow \mathcal{K}_{\text{cont}}[\tau_2] \rightarrow \text{void} \\
\mathcal{K}_{\text{cont}}[\tau] & \triangleq \mathcal{K}[\tau] \rightarrow \text{void}
\end{array}$$

2.2 Program translation

$$\begin{aligned}
\mathcal{K}_{\text{prog}}[u^\tau] &\triangleq \mathcal{K}_{\text{exp}}[u^\tau](\lambda x : \mathcal{K}[\tau]. \text{halt}[\mathcal{K}[\tau]]x^{\mathcal{K}[\tau]})^{\mathcal{K}_{\text{cont}}[\tau]} \\
\mathcal{K}_{\text{exp}}[y^\tau]k &\triangleq k(y^{\mathcal{K}[\tau]}) \\
\mathcal{K}_{\text{exp}}[i^\tau]k &\triangleq k(i^{\mathcal{K}[\tau]}) \\
\mathcal{K}_{\text{exp}}[(\lambda x_1 : \tau_1. u_2^{\tau_2})^\tau]k &\triangleq k((\lambda x : \mathcal{K}[\tau_1]. \lambda c : \mathcal{K}_{\text{cont}}[\tau_2]. \\
&\quad \mathcal{K}_{\text{exp}}[u_2^{\tau_2}]c^{\mathcal{K}_{\text{cont}}[\tau_2]})^{\mathcal{K}[\tau]}) \\
\mathcal{K}_{\text{exp}}[(u_1^{\tau_1} u_2^{\tau_2})^\tau]k &\triangleq \mathcal{K}_{\text{exp}}[u_1^{\tau_1}](\lambda x_1 : \mathcal{K}[\tau_1]. \\
&\quad \mathcal{K}_{\text{exp}}[u_2^{\tau_2}](\lambda x_2 : \mathcal{K}[\tau_2]. \\
&\quad x_1^{\mathcal{K}[\tau_1]} x_2^{\mathcal{K}[\tau_2]} k)^{\mathcal{K}_{\text{cont}}[\tau_2]})^{\mathcal{K}_{\text{cont}}[\tau_1]} \\
\mathcal{K}_{\text{exp}}[(e_1 \text{ p } e_2)^\tau]k &\triangleq \mathcal{K}_{\text{exp}}[e_1](\lambda x_1 : \mathbb{Z}. \\
&\quad \mathcal{K}_{\text{exp}}[e_2](\lambda x_2 : \mathbb{Z}. \\
&\quad \text{let } y = x_1 \text{ p } x_2 \text{ in } k(y^{\mathbb{Z}}))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}])^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} \\
\mathcal{K}_{\text{exp}}[\text{if0}(e_1, e_2, e_3)^\tau]k &\triangleq \mathcal{K}_{\text{exp}}[e_1](\lambda x : \mathbb{Z}. \\
&\quad \text{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\text{exp}}[e_2]k, \mathcal{K}_{\text{exp}}[e_3]k))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]}
\end{aligned}$$

3 Type correctness

3.1 Terms

Lemma 1. $\Gamma \vdash_{\text{T}} e : \tau \implies \mathcal{K} \circ \Gamma \vdash_{\text{K}} k : \mathcal{K}_{\text{cont}}[\tau] \implies \mathcal{K} \circ \Gamma \vdash_{\text{K}} \mathcal{K}_{\text{exp}}[e]k : \text{void}$

Proof. T_ANT_ANN is the only constructor of the hypothesis. Therefore,

$$e = u^\tau \tag{1}$$

$$\Gamma \vdash_{\text{T}} u^\tau : \tau \tag{2}$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_{\text{K}} k : \mathcal{K}_{\text{cont}}[\tau] \implies \mathcal{K} \circ \Gamma \vdash_{\text{K}} \mathcal{K}_{\text{exp}}[u^\tau]k : \text{void} \tag{3}$$

By induction on the typing judgement (2):

1. T_TERM_VAR

We know that

$$u = x \tag{4}$$

$$\Gamma(x) = \tau \tag{5}$$

$$\mathcal{K} \circ \Gamma \vdash_{\text{K}} k : \mathcal{K}[\tau] \rightarrow \text{void} \tag{6}$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_{\text{K}} \mathcal{K}_{\text{exp}}[x^\tau]k : \text{void} \tag{7}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\text{K}} k(x^{\mathcal{K}[\tau]}) : \text{void} \tag{8}$$

By K_TERM_APP and typing judgement (6), this follows

$$\mathcal{K} \circ \Gamma \vdash_{\text{K}} x^{\mathcal{K}[\tau]} : \mathcal{K}[\tau] \tag{9}$$

By K_ANT_ANN , this follows

$$\mathcal{K} \circ \Gamma \vdash_{\text{K}} x : \mathcal{K}[\tau] \tag{10}$$

By K_TERM_VAR , this follows

$$\mathcal{K} \circ \Gamma(x) = \mathcal{K}[\tau] \tag{11}$$

which is immediate from typing judgement (5).

2. T_TERM_INT

We know that

$$u = i \quad (12)$$

$$\tau = \mathbb{Z} \quad (13)$$

$$\mathcal{K} \circ \Gamma \vdash_K k : \mathbb{Z} \rightarrow \mathbf{void} \quad (14)$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[i^{\mathbb{Z}}]k : \mathbf{void} \quad (15)$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_K k(i^{\mathcal{K}[\mathbb{Z}]}) : \mathbf{void} \quad (16)$$

By K_TERM_APP and typing judgement (14), this follows

$$\mathcal{K} \circ \Gamma \vdash_K i^{\mathcal{K}[\mathbb{Z}]} : \mathbb{Z} \quad (17)$$

By K_ANT_ANN, this follows

$$\mathcal{K} \circ \Gamma \vdash_K i : \mathbb{Z} \quad (18)$$

which is immediate from K_TERM_INT.

3. T_TERM_LAM

We know that

$$u = \lambda x : \tau_1. e_2 \quad (19)$$

$$\tau = \tau_1 \rightarrow \tau_2 \quad (20)$$

$$\Gamma, x : \tau_1 \vdash_T e_2 : \tau_2 \quad (21)$$

$$\mathcal{K} \circ \Gamma \vdash_K k : \mathcal{K}[\tau_1 \rightarrow \tau_2] \rightarrow \mathbf{void} \quad (22)$$

T_ANT_ANN is only constructor of typing judgement (21). Therefore,

$$e_2 = u_2^{\tau_2} \quad (23)$$

$$\Gamma, x : \tau_1 \vdash_T u_2 : \tau_2 \quad (24)$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[\lambda x : \tau_1. u_2^{\tau_2}]k : \mathbf{void} \quad (25)$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_K k((\lambda x : \mathcal{K}[\tau_1]. \lambda c : \mathcal{K}_{\text{cont}}[\tau_2]. \mathcal{K}_{\text{exp}}[u_2^{\tau_2}]c^{\mathcal{K}_{\text{cont}}[\tau_2]})^{\mathcal{K}[\tau_1 \rightarrow \tau_2]}) : \mathbf{void} \quad (26)$$

By K_TERM_APP and typing judgement (22), this follows

$$\mathcal{K} \circ \Gamma \vdash_K (\lambda x : \mathcal{K}[\tau_1]. \lambda c : \mathcal{K}_{\text{cont}}[\tau_2]. \mathcal{K}_{\text{exp}}[u_2^{\tau_2}]c^{\mathcal{K}_{\text{cont}}[\tau_2]})^{\mathcal{K}[\tau_1 \rightarrow \tau_2]} : \mathcal{K}[\tau_1 \rightarrow \tau_2] \quad (27)$$

By K_ANT_ANN, this follows

$$\mathcal{K} \circ \Gamma \vdash_K \lambda x : \mathcal{K}[\tau_1]. \lambda c : \mathcal{K}_{\text{cont}}[\tau_2]. \mathcal{K}_{\text{exp}}[u_2^{\tau_2}]c^{\mathcal{K}_{\text{cont}}[\tau_2]} : \mathcal{K}[\tau_1] \rightarrow \mathcal{K}_{\text{cont}}[\tau_2] \rightarrow \mathbf{void} \quad (28)$$

By K_TERM_LAM, this follows

$$\mathcal{K} \circ \Gamma, x : \mathcal{K}[\tau_1] \vdash_K \lambda c : \mathcal{K}_{\text{cont}}[\tau_2]. \mathcal{K}_{\text{exp}}[u_2^{\tau_2}]c^{\mathcal{K}_{\text{cont}}[\tau_2]} : \mathcal{K}_{\text{cont}}[\tau_2] \rightarrow \mathbf{void} \quad (29)$$

which follows

$$\mathcal{K} \circ \Gamma, x : \mathcal{K}[\tau_1], c : \mathcal{K}_{\text{cont}}[\tau_2] \vdash_K \mathcal{K}_{\text{exp}}[u_2^{\tau_2}]c^{\mathcal{K}_{\text{cont}}[\tau_2]} : \mathbf{void} \quad (30)$$

which is immediate from K_ANT_ANN, K_TERM_VAR and the induction hypothesis.

4. T_TERM_APP

We know that

$$u = e_1 e_2 \quad (31)$$

$$\Gamma \vdash_T e_1 : \tau_1 \rightarrow \tau \quad (32)$$

$$\Gamma \vdash_T e_2 : \tau_1 \quad (33)$$

$$\mathcal{K} \circ \Gamma \vdash_K k : \mathcal{K}_{\text{cont}}[\tau] \quad (34)$$

T_ANT_ANN is only constructor of typing judgements (32)(33). Therefore,

$$e_1 = u_1^{\tau_1 \rightarrow \tau} \quad (35)$$

$$e_2 = u_2^{\tau_1} \quad (36)$$

$$\Gamma \vdash_T u_1 : \tau_1 \rightarrow \tau \quad (37)$$

$$\Gamma \vdash_T u_2 : \tau_1 \quad (38)$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[(u_1^{\tau_1 \rightarrow \tau} u_2^{\tau_1})^\tau] k : \text{void} \quad (39)$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[u_1^{\tau_1 \rightarrow \tau}] (\lambda x_1 : \mathcal{K}[\tau_1 \rightarrow \tau]. \mathcal{K}_{\text{exp}}[u_2^{\tau_1}] (\lambda x_2 : \mathcal{K}[\tau_1]. x_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k)^{\mathcal{K}_{\text{cont}}[\tau_1]})^{\mathcal{K}_{\text{cont}}[\tau_1 \rightarrow \tau]} : \text{void} \quad (40)$$

By the induction hypothesis, we need to show that

$$\mathcal{K} \circ \Gamma \vdash_K (\lambda x_1 : \mathcal{K}[\tau_1 \rightarrow \tau]. \mathcal{K}_{\text{exp}}[u_2^{\tau_1}] (\lambda x_2 : \mathcal{K}[\tau_1]. x_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k)^{\mathcal{K}_{\text{cont}}[\tau_1]})^{\mathcal{K}_{\text{cont}}[\tau_1 \rightarrow \tau]} : \mathcal{K}_{\text{cont}}[\tau_1 \rightarrow \tau] \quad (41)$$

By K_ANT_ANN, this follows

$$\mathcal{K} \circ \Gamma \vdash_K \lambda u_1 : \mathcal{K}[\tau_1 \rightarrow \tau]. \mathcal{K}_{\text{exp}}[u_2^{\tau_1}] (\lambda x_2 : \mathcal{K}[\tau_1]. u_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k)^{\mathcal{K}_{\text{cont}}[\tau_1]} : \mathcal{K}[\tau_1 \rightarrow \tau] \rightarrow \text{void} \quad (42)$$

By K_TERM_LAM, this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\tau_1 \rightarrow \tau] \vdash_K \mathcal{K}_{\text{exp}}[u_2^{\tau_1}] (\lambda x_2 : \mathcal{K}[\tau_1]. x_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k)^{\mathcal{K}_{\text{cont}}[\tau_1]} : \text{void} \quad (43)$$

By the induction hypothesis, we need to show that

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\tau_1 \rightarrow \tau] \vdash_K (\lambda x_2 : \mathcal{K}[\tau_1]. x_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k)^{\mathcal{K}_{\text{cont}}[\tau_1]} : \mathcal{K}_{\text{cont}}[\tau_1] \quad (44)$$

By K_ANT_ANN, this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\tau_1 \rightarrow \tau] \vdash_K \lambda x_2 : \mathcal{K}[\tau_1]. x_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k : \mathcal{K}[\tau_1] \rightarrow \text{void} \quad (45)$$

By K_TERM_LAM, this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\tau_1 \rightarrow \tau], x_2 : \mathcal{K}[\tau_1] \vdash_K x_1^{\mathcal{K}[\tau_1 \rightarrow \tau]} x_2^{\mathcal{K}[\tau_1]} k : \text{void} \quad (46)$$

which is immediate from K_TERM_APP, K_ANT_ANN, K_TERM_VAR and typing judgement (34).

5. T_TERM_PRIM

We know that

$$u = e_1 \ p \ e_2 \quad (47)$$

$$\tau = \mathbb{Z} \quad (48)$$

$$\Gamma \vdash_T e_1 : \mathbb{Z} \quad (49)$$

$$\Gamma \vdash_T e_2 : \mathbb{Z} \quad (50)$$

$$\mathcal{K} \circ \Gamma \vdash_K k : \mathbb{Z} \rightarrow \text{void} \quad (51)$$

T_ANT_ANN is only constructor of typing judgement (49)(50). Therefore,

$$e_1 = u_1^{\mathbb{Z}} \quad (52)$$

$$e_2 = u_2^{\mathbb{Z}} \quad (53)$$

$$\Gamma \vdash_T u_1 : \mathbb{Z} \quad (54)$$

$$\Gamma \vdash_T u_2 : \mathbb{Z} \quad (55)$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[(u_1^{\mathbb{Z}} p u_2^{\mathbb{Z}})^{\mathbb{Z}}]k : \text{void} \quad (56)$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[u_1^{\mathbb{Z}}](\lambda x_1 : \mathbb{Z}. \mathcal{K}_{\text{exp}}[u_2^{\mathbb{Z}}](\lambda x_2 : \mathbb{Z}. \text{let } y = x_1 p x_2 \text{ in } k(y^{\mathbb{Z}}))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]})^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \text{void} \quad (57)$$

By the induction hypothesis, we need to show that

$$\mathcal{K} \circ \Gamma \vdash_K (\lambda x_1 : \mathbb{Z}. \mathcal{K}_{\text{exp}}[u_2](\lambda x_2 : \mathbb{Z}. \text{let } y = x_1 p x_2 \text{ in } k(y^{\mathbb{Z}}))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]})^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \mathcal{K}_{\text{cont}}[\mathbb{Z}] \quad (58)$$

By K_ANT_ANN , this follows

$$\mathcal{K} \circ \Gamma \vdash_K \lambda x_1 : \mathbb{Z}. \mathcal{K}_{\text{exp}}[u_2](\lambda x_2 : \mathbb{Z}. \text{let } y = x_1 p x_2 \text{ in } k(y^{\mathbb{Z}}))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \mathbb{Z} \rightarrow \text{void} \quad (59)$$

By K_TERM_LAM , this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_K \mathcal{K}_{\text{exp}}[u_2](\lambda x_2 : \mathbb{Z}. \text{let } y = x_1 p x_2 \text{ in } k(y^{\mathbb{Z}}))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \text{void} \quad (60)$$

By the induction hypothesis, this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_K (\lambda u_2 : \mathbb{Z}. \text{let } y = x_1 p u_2 \text{ in } k(y^{\mathbb{Z}}))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \mathcal{K}_{\text{cont}}[\mathbb{Z}] \quad (61)$$

By K_ANT_ANN , this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_K \lambda x_2 : \mathbb{Z}. \text{let } y = x_1 p x_2 \text{ in } k(y^{\mathbb{Z}}) : \mathbb{Z} \rightarrow \text{void} \quad (62)$$

By K_TERM_LAM , this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z} \vdash_K \text{let } y = x_1 p x_2 \text{ in } k(y^{\mathbb{Z}}) : \text{void} \quad (63)$$

By K_ANT_LET , K_ANT_PRIM and K_ANT_VAR , we need to show that

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z}, y : \mathbb{Z} \vdash_K k(y^{\mathbb{Z}}) : \text{void} \quad (64)$$

By K_TERM_APP and typing judgement (51), this follows

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z}, y : \mathbb{Z} \vdash_K y^{\mathbb{Z}} : \mathbb{Z} \quad (65)$$

which is immediate from K_ANT_ANN and K_ANT_VAR .

6. T_TERM_IF0 We know that

$$u = \text{if0}(e_1, e_2, e_3) \quad (66)$$

$$\Gamma \vdash_T e_1 : \mathbb{Z} \quad (67)$$

$$\Gamma \vdash_T e_2 : \tau \quad (68)$$

$$\Gamma \vdash_T e_3 : \tau \quad (69)$$

$$\mathcal{K} \circ \Gamma \vdash_K k : \mathcal{K}_{\text{cont}}[\tau] \quad (70)$$

T_ANT_ANN is only constructor of typing judgements (67)(68)(69). Therefore,

$$e_1 = u_1^{\mathbb{Z}} \quad (71)$$

$$e_2 = u_2^{\tau} \quad (72)$$

$$e_3 = u_3^{\tau} \quad (73)$$

$$\Gamma \vdash_T u_1 : \mathbb{Z} \quad (74)$$

$$\Gamma \vdash_T u_2 : \tau \quad (75)$$

$$\Gamma \vdash_T u_3 : \tau \quad (76)$$

We need to show that

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[\text{if0}(u_1^{\mathbb{Z}}, u_2^{\tau}, u_3^{\tau})]k : \text{void} \quad (77)$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_K \mathcal{K}_{\text{exp}}[u_1^{\mathbb{Z}}](\lambda x : \mathbb{Z}. \text{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\text{exp}}[u_2^{\tau}]k, \mathcal{K}_{\text{exp}}[u_3^{\tau}]k))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \text{void} \quad (78)$$

By the induction hypothesis, we need to show that

$$\mathcal{K} \circ \Gamma \vdash_K (\lambda x : \mathbb{Z}. \text{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\text{exp}}[u_2^{\tau}]k, \mathcal{K}_{\text{exp}}[u_3^{\tau}]k))^{\mathcal{K}_{\text{cont}}[\mathbb{Z}]} : \mathcal{K}_{\text{cont}}[\mathbb{Z}] \quad (79)$$

By K_ANT_ANN , this follows

$$\mathcal{K} \circ \Gamma \vdash_K \lambda x : \mathbb{Z}. \text{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\text{exp}}[u_2^{\tau}]k, \mathcal{K}_{\text{exp}}[u_3^{\tau}]k) : \mathbb{Z} \rightarrow \text{void} \quad (80)$$

By K_TERM_LAM , this follows

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_K \text{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\text{exp}}[u_2^{\tau}]k, \mathcal{K}_{\text{exp}}[u_3^{\tau}]k) : \text{void} \quad (81)$$

By K_TERM_IF0 , this follows

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_K x^{\mathbb{Z}} : \mathbb{Z} \quad (82)$$

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_K \mathcal{K}_{\text{exp}}[u_2^{\tau}]k : \text{void} \quad (83)$$

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_K \mathcal{K}_{\text{exp}}[u_3^{\tau}]k : \text{void} \quad (84)$$

in which (82) is immediate from K_ANT_ANN and K_TERM_VAR . (83)(84) come from the induction hypotheses, K_ANT_ANN and typing judgements (70)(75)(76).

3.2 Programs

Theorem 1. $\vdash_T e : \tau \implies \vdash_K \mathcal{K}_{\text{prog}}[e] : \text{void}$

Proof. T_ANT_ANN is the only constructor of the hypothesis. Therefore,

$$e = u^{\tau} \quad (85)$$

$$\vdash_T u^{\tau} : \tau \quad (86)$$

We need to show that

$$\vdash_K \mathcal{K}_{\text{prog}}[u^{\tau}] : \text{void} \quad (87)$$

i.e.

$$\vdash_K \mathcal{K}_{\text{exp}}[u^{\tau}](\lambda x : \mathcal{K}[\tau]. \text{halt}[\mathcal{K}[\tau]]x^{\mathcal{K}[\tau]}))^{\mathcal{K}_{\text{cont}}[\tau]} : \text{void} \quad (88)$$

By Lemma 1, we need to show that

$$\vdash_K (\lambda x : \mathcal{K}[\tau]. \text{halt}[\mathcal{K}[\tau]]x^{\mathcal{K}[\tau]}))^{\mathcal{K}_{\text{cont}}[\tau]} : \mathcal{K}_{\text{cont}}[\tau] \quad (89)$$

By K_ANT_ANN , this follows

$$\vdash_K \lambda x : \mathcal{K}[\tau]. \text{halt}[\mathcal{K}[\tau]]x^{\mathcal{K}[\tau]} : \mathcal{K}[\tau] \rightarrow \text{void} \quad (90)$$

By K_TERM_LAM , this follows

$$x : \mathcal{K}[\tau] \vdash_K \text{halt}[\mathcal{K}[\tau]]x^{\mathcal{K}[\tau]} : \text{void} \quad (91)$$

By K_TERM_HALT , this follows

$$x : \mathcal{K}[\tau] \vdash_K x^{\mathcal{K}[\tau]} : \mathcal{K}[\tau] \quad (92)$$

which is immediate from K_ANT_ANN and K_TERM_VAR .