Termination and Logical Relations

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Goal

Use logical relations to prove termination of (call-by-name) Simply Typed Lambda Calculus, System **T** and System **F**.

The syntax of the Simply Typed Lambda Calculus is given by the following grammar:

$$
\begin{aligned}
\mathit{Typ}\ \tau\ &::= \mathit{nat} \mid \tau_1 \longrightarrow \tau_2 \\
\mathit{Exp}\ e\ &::= x \mid 0 \mid \mathbf{S}(e) \mid \lambda(x:\tau).e \mid e_1\ e_2
\end{aligned}
$$

Statics of the Simply Typed Lambda Calculus

The typing rules of the simply Typed Lambda Calculus are as follows:

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \quad \text{Typ-Var}$$

$$\frac{}{\Gamma \vdash 0 : \mathit{nat}} \quad \text{Typ-Zero}$$

$$\frac{\Gamma \vdash e : \mathit{nat}}{\Gamma \vdash \mathbf{S}(e) : \mathit{nat}} \quad \text{Typ-Succ}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \quad \text{Typ-Lam}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \longrightarrow \tau_2 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_2} \quad \text{Typ-App}$$

Dynamics of the Simply Typed Lambda Calculus

The closed values of **T** are defined by the following rules:

$$\frac{}{0 \ \textbf{Val}} \quad \text{Val-Zero}$$

$$\frac{e \ \textbf{Val}}{\textbf{S}(e) \ \textbf{Val}} \quad \text{Val-Succ}$$

$$\frac{}{\lambda(x : \tau).e \ \textbf{Val}} \quad \text{Val-Lam}$$

The transition rules for **T** are as follows:

$$\frac{e \longmapsto e'}{\textbf{S}(e) \longmapsto \textbf{S}(e')} \quad \text{Step-Succ}$$

$$\frac{e_1 \longmapsto e_1'}{e_1 \ e_2 \longmapsto e_1' \ e_2} \quad \text{Step-App}$$

$$\frac{}{(\lambda(x : \tau).e_1) \ e_2 \longmapsto e_1[x ::= e_2]} \quad \text{Step-Lam}$$

Some Key Lemmata

Proposition (Progress)

If $\vdash e : \tau$ *then either* e *is a value or there exists an* e' *such that* $e \longmapsto e'$.

Proposition (Preservation)

If $\vdash e : \tau$ *and* $e \longmapsto e'$ *then* $\vdash e' : \tau$.

Proposition (Canonical Forms)

If $\vdash v : \tau$ *and* v *is a value, then* $e = 0$ *or* $e = \mathbf{S}(v')$ *with* v' *a value if* $\tau = nat$, *and* $v = \lambda(x : \tau_1).e$ *if* $\tau = \tau_1 \longrightarrow \tau_2$.

First Attempt at Termination Proof

We try to prove the following using the obvious strategy:

Proposition

If $\vdash e : \tau$ then there exists a value v such that $e \mapsto^* v$.

Proof. The proof proceeds by induction over the derivation $\vdash e : \tau$

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$$\frac{}{x : \tau \vdash x : \tau}\ \text{Typ-Var}$$

  Does not apply since context is not empty.

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  0 is already a value.

- (Case Typ-Succ)

$$\frac{\vdash e : nat}{\vdash \mathbf{S}(e) : nat} \quad \text{Typ-Succ}$$

By the induction hypothesis $e \longmapsto^* v$ with v a value, so $\mathbf{S}(e) \longmapsto^* \mathbf{S}(v)$ which is a value.

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By the induction hypothesis $e \longmapsto^* v$ with v a value, so $\mathbf{S}(e) \longmapsto^* \mathbf{S}(v)$ which is a value.

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$$\frac{\vdash e_1 : \tau_1 \longrightarrow \tau_2 \qquad \vdash e_2 : \tau_2}{\vdash e_1\ e_2 : \tau_2} \quad \text{Typ-App}$$

By the induction hypothesis there exists a value f such that $e_1 \longmapsto^* f$. By canonical forms, $f = \lambda(x : \tau_1).e'$, hence by the Step-App rule,

$$e_1\ e_2 \longmapsto^* (\lambda(x : \tau_1).e')\ e_2 \longmapsto e'[x := e_2]$$

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$$e_1\ e_2 \longmapsto^* (\lambda(x : \tau_1).e')\ e_2 \longmapsto e'[x := e_2]$$

We are stuck because we don't know anything about e'!

Logical Relations As a Proof Technique

We (try to) define a unary relation $\mathcal{R}_\tau$ over expressions of a closed type τ such that if $e \in \mathcal{R}_\tau$ then there exists a value v such that $e \longmapsto^* v$. This relation should satisfy the property that if $e \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$ and $e' \in \mathcal{R}_{\tau_1}$, then $(e\ e') \in \mathcal{R}_{\tau_2}$. We then show that if $\vdash e : \tau$, then $e \in \mathcal{R}_\tau$, so there exists a value v such that $e \longmapsto^* v$.

Logical Relations As a Proof Technique

We (try to) define a unary relation $\mathcal{R}_\tau$ over expressions of a closed type τ such that if $e \in \mathcal{R}_\tau$ then there exists a value v such that $e \longmapsto^* v$. This relation should satisfy the property that if $e \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$ and $e' \in \mathcal{R}_{\tau_1}$, then $(e\ e') \in \mathcal{R}_{\tau_2}$. We then show that if $\vdash e : \tau$, then $e \in \mathcal{R}_\tau$, so there exists a value v such that $e \longmapsto^* v$.

Definition

We define a unary relation $\mathcal{R}_\tau$ on closed expressions of the type τ by the following rules: if e is a closed expression, then $e \in \mathcal{R}_\tau$ if and only if

1. if $\tau = nat$, then there exists a value v such that $e \longmapsto^* v$, and
2. if $\tau = \tau_1 \longrightarrow \tau_2$, then
   - there exists a v such that $e \longmapsto^* v$, and
   - if $e' \in \mathcal{R}_{\tau_1}$, then $(e\ e') \in \mathcal{R}_{\tau_2}$

Second Attempt at Termination Proof

Proposition

If $\vdash e : \tau$*, then* $e \in \mathcal{R}_\tau$*.*

Proof. Assume that $\vdash e : \tau$. The proof proceeds by induction over the derivation $\vdash e : \tau$.

Second Attempt at Termination Proof

Proposition

If $\vdash e : \tau$, then $e \in \mathcal{R}_\tau$.

Proof. Assume that $\vdash e : \tau$. The proof proceeds by induction over the derivation $\vdash e : \tau$.

- (Case Typ-Var)

$$\frac{}{x : \tau \vdash x : \tau}\ \text{Typ-Var}$$

Does not apply since context is not empty.

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If $\vdash e : \tau$, *then* $e \in \mathcal{R}_\tau$.

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  $$\frac{}{x : \tau \vdash x : \tau}\ \text{Typ-Var}$$

  Does not apply since context is not empty.

- (Case Typ-Zero)

  $$\frac{}{\vdash 0 : nat}\ \text{Typ-Zero}$$

  0 is already a value so $0 \in \mathcal{R}_{nat}$.

Second Attempt at Termination Proof

- (Case Typ-Succ)

$$\frac{\vdash e : nat}{\vdash \mathbf{S}(e) : nat} \quad \text{Typ-Succ}$$

By the induction hypothesis $e \in \mathcal{R}_{nat}$ so $e \longmapsto^* v$ with v a value, so $\mathbf{S}(e) \longmapsto^* \mathbf{S}(v)$ which is a value, so $\mathbf{S}(e) \in \mathcal{R}_{nat}$.

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By the induction hypothesis $e \in \mathcal{R}_{nat}$ so $e \longmapsto^* v$ with v a value, so $\mathbf{S}(e) \longmapsto^* \mathbf{S}(v)$ which is a value, so $\mathbf{S}(e) \in \mathcal{R}_{nat}$.

- (Case Typ-App)

$$\frac{\vdash e : \tau_1 \longrightarrow \tau_2 \qquad \vdash e' : \tau_1}{\vdash e\ e' : \tau_2} \quad \text{Typ-App}$$

By the induction hypothesis $e \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$ and $e' \in \mathcal{R}_{\tau_1}$, so $(e\ e') \in \mathcal{R}_{\tau_2}$.

Second Attempt at Termination Proof

- (Case Typ-Lam)

$$\frac{x : \tau_1 \vdash e : \tau_2}{\vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \quad \text{Typ-Lam}$$

$\lambda(x : \tau_1).e$ is already a value...

Second Attempt at Termination Proof

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$\lambda(x : \tau_1).e$ is already a value...
Induction hypothesis does not apply to e since e is not closed, so we are stuck. However $e[x := e']$ is closed ...

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$\lambda(x : \tau_1).e$ is already a value...
Induction hypothesis does not apply to e since e is not closed, so we are stuck. However $e[x := e']$ is closed ...
We need to define $\mathcal{R}_\tau$ for open terms by applying closing substitutions.

Extending the Relation

Notation

Let γ *be a function from a finite set* $\{x_1, \ldots, x_n\}$ *of expression variables to closed expressions. Let e be an arbitary expression. Then*

$$\hat{\gamma}(e) = e[(x_1, \ldots, x_n) := (\gamma(x_1), \ldots, \gamma(x_n))]$$

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Let γ be a function from a finite set $\{x_1, \ldots, x_n\}$ of expression variables to closed expressions. Let e be an arbitary expression. Then

$$\hat{\gamma}(e) = e[(x_1, \ldots, x_n) := (\gamma(x_1), \ldots, \gamma(x_n))]$$

Definition

Let e be an (arbitary) expression, τ a type and $\Gamma = \{x_1 : \tau_1, \ldots, x_n : \tau_n\}$ a context. Then $e \in \mathcal{S}(\Gamma, \tau)$ if and only if when γ is a function from $\{x_1, \ldots, x_n\}$ to closed expressions such that $\gamma(x_i) : \tau_i$ and $\gamma(x_i) \in \mathcal{R}_{\tau_i}$, then $\hat{\gamma}(e) \in \mathcal{R}_\tau$

Corollary

If $e \in \mathcal{S}(\varnothing, \tau)$, then e there exists a value v such that $e \longmapsto^* v$.

Proof of Termination

Proposition

If $\Gamma \vdash e : \tau$ *then* $e \in \mathcal{S}(\Gamma, \tau)$*.*

Proof. Assume that $\Gamma \vdash e : \tau$ with $\Gamma = \{x_1 : \tau_1, \ldots, x_n : \tau_n\}$ a context. Let γ be a function from $\{x_1, \ldots, x_n\}$ to closed expressions such that $\gamma(x_i) : \tau_i$ and $\gamma(x_i) \in \mathcal{R}_{\tau_i}$. The proof proceeds by induction on the derivation $\Gamma \vdash e : \tau$.

Proof of Termination

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If $\Gamma \vdash e : \tau$ then $e \in \mathcal{S}(\Gamma, \tau)$.

Proof. Assume that $\Gamma \vdash e : \tau$ with $\Gamma = \{x_1 : \tau_1, \ldots, x_n : \tau_n\}$ a context. Let γ be a function from $\{x_1, \ldots, x_n\}$ to closed expressions such that $\gamma(x_i) : \tau_i$ and $\gamma(x_i) \in \mathcal{R}_{\tau_i}$. The proof proceeds by induction on the derivation $\Gamma \vdash e : \tau$.

- (Case Typ-Var)

$$\frac{}{\Gamma', x : \tau \vdash x : \tau} \quad \text{Typ-Var}$$

  Then $x = x_i$ for some $1 \leq i \leq n$, so $\hat{\gamma}(x) = \gamma(x_i)$ which is in $\mathcal{R}_{\tau_i}$ by the definition of γ, so $x \in \mathcal{S}(\Gamma, \tau)$.

Proof of Termination

- (Case Typ-Zero)

$$\frac{}{\Gamma \vdash 0 : nat} \quad \text{Typ-Zero}$$

Then $\hat{\gamma}(0) = 0$ which is in $\mathcal{R}_{nat}$ so $0 \in \mathcal{S}(\Gamma, \tau)$.

Proof of Termination

- (Case Typ-Zero)

$$\frac{}{\Gamma \vdash 0 : nat} \quad \text{Typ-Zero}$$

Then $\hat{\gamma}(0) = 0$ which is in $\mathcal{R}_{nat}$ so $0 \in \mathcal{S}(\Gamma, \tau)$.

- (Case Typ-Succ)

$$\frac{\Gamma \vdash e : nat}{\Gamma \vdash \mathbf{S}(e) : nat} \quad \text{Typ-Succ}$$

Then $\hat{\gamma}(\mathbf{S}(e)) = \mathbf{S}(\hat{\gamma}(e))$. By the induction hypothesis, $e \in \mathcal{S}(\Gamma, nat)$, so $\hat{\gamma}(e) \in \mathcal{R}_{nat}$, so $\hat{\gamma}(e) \longmapsto^{*} v$ with v a value, so $\mathbf{S}(\hat{\gamma}(e)) \longmapsto^{*} \mathbf{S}(v)$ which is a value so $\mathbf{S}(\hat{\gamma}(e)) \in \mathcal{R}_{nat}$, so $\mathbf{S}(e) \in \mathcal{S}(\Gamma, nat)$.

Proof of Termination

- (Case Typ-App)

$$\frac{\Gamma \vdash e_1 : \tau_1 \longrightarrow \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}\ \text{Typ-App}$$

  Then $\hat{\gamma}(e_1\ e_2) = (\hat{\gamma}(e_1))\ (\hat{\gamma}(e_2))$. By the induction hypothesis, $e_1 \in \mathcal{S}(\Gamma, \tau_1 \longrightarrow \tau_2)$ and $e_2 \in \mathcal{S}(\Gamma, \tau_1)$, so $\hat{\gamma}(e_1) \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$ and $\hat{\gamma}(e_2) \in \mathcal{R}_{\tau_1}$, so $(\hat{\gamma}(e_1))\ (\hat{\gamma}(e_2)) \in \mathcal{R}_{\tau_2}$, so $(e_1\ e_2) \in \mathcal{S}(\Gamma, \tau)$.

Proof of Termination

- (Case Typ-Lam)

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \quad \text{Typ-Lam}$$

Assume without loss of generality that x is not in Γ since otherwise we can rename x to satisfy this condition. Then $\hat{\gamma}(\lambda(x : \tau_1).e) = \lambda(x : \tau_1).\hat{\gamma}(e)$ which is a value. Assume that $e' \in \mathcal{R}_{\tau_1}$. Let $\theta = \gamma \otimes [x \mapsto e']$. By the induction hypothesis, $e \in \mathcal{S}((\Gamma, x : \tau_1), \tau_2)$, so $\hat{\theta}(e) \in \mathcal{R}_{\tau_2}$. Now

$$(\lambda(x : \tau_1).\hat{\gamma}(e))\ e' \longmapsto \hat{\gamma}(e)[x := e'] = \hat{\theta}(e)$$

which is $\in \mathcal{R}_{\tau_2} \ldots$

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$$(\lambda(x : \tau_1).\hat{\gamma}(e))\ e' \longmapsto \hat{\gamma}(e)[x := e'] = \hat{\theta}(e)$$

which is $\in \mathcal{R}_{\tau_2} \ldots$

but $(\lambda(x : \tau_1).\hat{\gamma}(e))\ e'$ is not equal to $\hat{\theta}(e)$!

so we cannot conclude that $(\lambda(x : \tau_1).\hat{\gamma}(e))\ e' \in \mathcal{R}_{\tau_2}$. We need to prove that $\mathcal{R}_\tau$ is closed under converse evaluation for any τ.

A Final Piece

Proposition

Assume that $\vdash e : \tau$ *and* $\vdash e' : \tau$. *Assume that* $e \longmapsto e'$. *If* $e' \in \mathcal{R}_\tau$, *then* $e \in \mathcal{R}_\tau$.

Proof. The proof proceeds by induction over the type τ.

1. (Case $\tau = nat$) Since $e' \in \mathcal{R}_{nat}$, then there exists a value v such that $e' \longmapsto^* v$, thus $e \longmapsto^* v$ as $e \longmapsto e'$ by assumption, so $e \in \mathcal{R}_{nat}$.

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Assume that $\vdash e : \tau$ and $\vdash e' : \tau$. Assume that $e \longmapsto e'$. If $e' \in \mathcal{R}_\tau$, then $e \in \mathcal{R}_\tau$.

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2. (Case $\tau = \tau_1 \longrightarrow \tau_2$) Since $e' \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$, then there exists a value v such that $e' \longmapsto^* v$, thus $e \longmapsto^* v$ as $e \longmapsto e'$ by assumption. Now assume that $e_1 \in \mathcal{R}_{\tau_1}$. Then $(e\ e_1) \longmapsto (e'\ e_1)$. (This is why we use call-by-name!) Since $e' \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$, then $(e'\ e_1) \in \mathcal{R}_{\tau_2}$. By the induction hypothesis, $(e\ e_1) \in \mathcal{R}_{\tau_2}$, thus $e \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$. This completes the proof.

Completing the Termination Proof

- (Case Typ-Lam)

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \quad \text{Typ-Lam}$$

Assume without loss of generality that x is not in Γ since otherwise we can rename x to satisfy this condition. Then $\hat{\gamma}(\lambda(x : \tau_1).e) = \lambda(x : \tau_1).\hat{\gamma}(e)$ which is a value. Assume that $e' \in \mathcal{R}_{\tau_1}$. Let $\theta = \gamma \otimes [x \mapsto e']$. By the induction hypothesis, $e \in \mathcal{S}((\Gamma, x : \tau_1), \tau_2)$, so $\hat{\theta}(e) \in \mathcal{R}_{\tau_2}$. Now

$$(\lambda(x : \tau_1).\hat{\gamma}(e))\ e' \longmapsto \hat{\gamma}(e)[x := e'] = \hat{\theta}(e)$$

which is in $\in \mathcal{R}_{\tau_2} \ldots$

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$$(\lambda(x : \tau_1).\hat{\gamma}(e))\ e' \longmapsto \hat{\gamma}(e)[x := e'] = \hat{\theta}(e)$$

which is in $\in \mathcal{R}_{\tau_2} \ldots$ so $(\lambda(x : \tau_1).\hat{\gamma}(e))\ e' \in \mathcal{R}_{\tau_2}$ as $\mathcal{R}_{\tau_2}$ is closed under converse evaluation. Therefore $\hat{\gamma}(\lambda(x : \tau_1).e) \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$. This completes the proof.

System $\mathbf{T}$

System $\mathbf{T}$ is an extension of the Simply Typed Lambda Calculus.

$$\begin{aligned} Typ\ \tau\ &::= nat \mid \tau_1 \longrightarrow \tau_2 \\ Exp\ e\ &::= x \mid 0 \mid \mathbf{S}(e) \mid \lambda(x:\tau).e \mid e_1\ e_2 \mid \mathbf{Prec}(e_1, e_2)(e) \end{aligned}$$

We add the typing rule

$$\frac{\Gamma \vdash e : nat \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma \vdash e_1 : nat \longrightarrow (\tau \longrightarrow \tau)}{\Gamma \vdash \mathbf{Prec}(e_0, e_1)(e) : \tau}\ \mathrm{PR}$$

System $\mathbf{T}$

We also add the following transition rules:

$$\frac{e \longmapsto e'}{\mathbf{Prec}(e_0, e_1)(e) \longmapsto \mathbf{Prec}(e_0, e_1)(e')} \quad \text{Step-Prec-1}$$

$$\frac{}{\mathbf{Prec}(e_0, e_1)(0) \longmapsto e_0} \quad \text{Step-Prec-2}$$

$$\frac{e \ \mathbf{Val}}{\mathbf{Prec}(e_0, e_1)(\mathbf{S}(e)) \longmapsto (e_1 \ e) \ (\mathbf{Prec}(e_0, e_1)(e))} \quad \text{Step-Prec-3}$$

Termination Proof for System T

Suffices to prove the PR case. Uses an inner induction.

- (Case PR)

$$\frac{\Gamma \vdash e : nat \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma \vdash e_1 : nat \longrightarrow (\tau \longrightarrow \tau)}{\Gamma \vdash \mathbf{Prec}(e_0, e_1)(e) : \tau}\ \text{PR}$$

Then $\hat{\gamma}(\mathbf{Prec}(e_0, e_1)(e)) = \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e))$. By the induction hypothesis, $e \in \mathcal{S}(\Gamma, nat)$, $e_0 \in \mathcal{S}(\Gamma, \tau)$ and $e_1 \in \mathcal{S}(nat \longrightarrow (\tau \longrightarrow \tau))$, so $\hat{\gamma}(e) \in \mathcal{R}_{nat}$, $\hat{\gamma}(e_0) \in \mathcal{R}_{\tau}$ and $\hat{\gamma}(e_1) \in \mathcal{R}_{nat \longrightarrow (\tau \longrightarrow \tau)}$. Therefore, there exists a value v such that $\hat{\gamma}(e) \in \mathcal{R}_{nat} \longmapsto^* v$, so

$$\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e)) \longmapsto^* \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \ldots$$

Termination Proof for System T

Suffices to prove the PR case. Uses an inner induction.

- (Case PR)

$$\frac{\Gamma \vdash e : nat \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma \vdash e_1 : nat \longrightarrow (\tau \longrightarrow \tau)}{\Gamma \vdash \mathbf{Prec}(e_0, e_1)(e) : \tau} \text{ PR}$$

Then $\hat{\gamma}(\mathbf{Prec}(e_0, e_1)(e)) = \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e))$. By the induction hypothesis, $e \in \mathcal{S}(\Gamma, nat)$, $e_0 \in \mathcal{S}(\Gamma, \tau)$ and $e_1 \in \mathcal{S}(nat \longrightarrow (\tau \longrightarrow \tau))$, so $\hat{\gamma}(e) \in \mathcal{R}_{nat}$, $\hat{\gamma}(e_0) \in \mathcal{R}_\tau$ and $\hat{\gamma}(e_1) \in \mathcal{R}_{nat \longrightarrow (\tau \longrightarrow \tau)}$. Therefore, there exists a value v such that $\hat{\gamma}(e) \in \mathcal{R}_{nat} \longmapsto^* v$, so

$$\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e)) \longmapsto^* \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \ldots$$

We need to prove that $\mathcal{R}_\tau$ is closed under converse multistep evaluation!

Strengthening Converse Evaluation

Proposition

Assume that $\vdash e : \tau$ *and* $\vdash e' : \tau$. *Assume that* $e \longmapsto^* e'$. *If* $e' \in \mathcal{R}_\tau$, *then* $e \in \mathcal{R}_\tau$.

Proof. The proof proceeds by induction over the type τ.

1. (Case $\tau = nat$) Since $e' \in \mathcal{R}_{nat}$, then there exists a value v such that $e' \longmapsto^* v$, thus $e \longmapsto^* v$ as $e \longmapsto^* e'$ by assumption, so $e \in \mathcal{R}_{nat}$.
2. (Case $\tau = \tau_1 \longrightarrow \tau_2$) Since $e' \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$, then there exists a value v such that $e' \longmapsto^* v$, thus $e \longmapsto^* v$ as $e \longmapsto^* e'$ by assumption. Now assume that $e_1 \in \mathcal{R}_{\tau_1}$. Then $(e\ e_1) \longmapsto^* (e'\ e_1)$. Since $e' \in \mathcal{R}_{\tau_2}$, then $(e'\ e_1) \in \mathcal{R}_{\tau_2}$. By the induction hypothesis, $(e\ e_1) \in \mathcal{R}_{\tau_2}$, thus $e \in \mathcal{R}_{\tau_1 \longrightarrow \tau_2}$. This completes the proof.

Termination Proof for System T

- (Case PR)

$$\frac{\Gamma \vdash e : nat \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma \vdash e_1 : nat \longrightarrow (\tau \longrightarrow \tau)}{\Gamma \vdash \mathbf{Prec}(e_0, e_1)(e) : \tau} \text{ PR}$$

Then $\hat{\gamma}(\mathbf{Prec}(e_0, e_1)(e)) = \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e))$. By the induction hypothesis, $e \in \mathcal{S}(\Gamma, nat)$, $e_0 \in \mathcal{S}(\Gamma, \tau)$ and $e_1 \in \mathcal{S}(nat \longrightarrow (\tau \longrightarrow \tau))$, so $\hat{\gamma}(e) \in \mathcal{R}_{nat}$, $\hat{\gamma}(e_0) \in \mathcal{R}_{\tau}$ and $\hat{\gamma}(e_1) \in \mathcal{R}_{nat \longrightarrow (\tau \longrightarrow \tau)}$. Therefore, there exists a value v such that $\hat{\gamma}(e) \in \mathcal{R}_{nat} \longmapsto^* v$, so

$$\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e)) \longmapsto^* \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v)$$

Termination Proof for System T

- (Case PR)

$$\frac{\Gamma \vdash e : nat \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma \vdash e_1 : nat \longrightarrow (\tau \longrightarrow \tau)}{\Gamma \vdash \mathbf{Prec}(e_0, e_1)(e) : \tau} \text{ PR}$$

Then $\hat{\gamma}(\mathbf{Prec}(e_0, e_1)(e)) = \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e))$. By the induction hypothesis, $e \in \mathcal{S}(\Gamma, nat)$, $e_0 \in \mathcal{S}(\Gamma, \tau)$ and $e_1 \in \mathcal{S}(nat \longrightarrow (\tau \longrightarrow \tau))$, so $\hat{\gamma}(e) \in \mathcal{R}_{nat}$, $\hat{\gamma}(e_0) \in \mathcal{R}_{\tau}$ and $\hat{\gamma}(e_1) \in \mathcal{R}_{nat \longrightarrow (\tau \longrightarrow \tau)}$. Therefore, there exists a value v such that $\hat{\gamma}(e) \in \mathcal{R}_{nat} \longmapsto^* v$, so

$$\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e)) \longmapsto^* \mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v)$$

By canonical forms, $v = 0$ or $v = \mathbf{S}(e')$ with e' a value. So if we can show that $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \in \mathcal{R}_{\tau}$, then we can conclude that $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\hat{\gamma}(e))$ since $\mathcal{R}_{\tau}$ is closed under converse multistep evaluation.

Termination Proof for System T

We use natural number induction on v to show that $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \in \mathcal{R}_\tau$.

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- (Case $v = 0$) Then $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \longmapsto \hat{\gamma}(e_0)$ which is in $\mathcal{R}_\tau$, so $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \in \mathcal{R}_\tau$ as $\mathcal{R}_\tau$ is closed under converse evaluation

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- (Case $v = 0$) Then $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \longmapsto \hat{\gamma}(e_0)$ which is in $\mathcal{R}_\tau$, so $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(v) \in \mathcal{R}_\tau$ as $\mathcal{R}_\tau$ is closed under converse evaluation
- (Case $v = \mathbf{S}(e')$ with e' a value) Then

  $$\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\mathbf{S}(e')) \longmapsto ((\hat{\gamma}(e_1))\ e')\ (\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(e'))$$

  Since e' is a value, then by the (natural number) induction hypothesis, $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(e') \in \mathcal{R}_\tau$ (this is why we evaluate fully under **S**), and since $\hat{\gamma}(e_1) \in \mathcal{R}_{nat \longrightarrow (\tau \longrightarrow \tau)}$ and e' is a value (and hence is in $\mathcal{R}_{nat}$), it follows that

  $$((\hat{\gamma}(e_1))\ e')\ (\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(e')) \in \mathcal{R}_\tau,$$

  so $\mathbf{Prec}(\hat{\gamma}(e_0), \hat{\gamma}(e_1))(\mathbf{S}(e')) \in \mathcal{R}_\tau$ as $\mathcal{R}_\tau$ is closed under converse evaluation. This completes the proof.

System F

The syntax of **F** is given by the following grammar:

$$\begin{aligned} Typ\ \tau &::= t \mid \tau \longrightarrow \tau' \mid \forall t.\tau \\ Exp\ e &::= x \mid \lambda(x:\tau).e \mid e\ e' \mid \Lambda t.e \mid e\ [\tau] \end{aligned}$$

Let Δ be a set of type variables. The well-formedness of types is given by the following rules:

$$\frac{}{\Delta, t\ \mathbf{WF} \vdash t\ \mathbf{WF}} \quad \text{WF-Var}$$

$$\frac{\Delta \vdash \tau_1\ \mathbf{WF} \qquad \Delta \vdash \tau_2\ \mathbf{WF}}{\Delta \vdash \tau_1 \longrightarrow \tau_2\ \mathbf{WF}} \quad \text{WF-lam}$$

$$\frac{\Delta,\ t\ \mathbf{WF} \vdash \tau\ \mathbf{WF}}{\Delta \vdash \Lambda t.\tau\ \mathbf{WF}} \quad \text{WF-Lam}$$

System F

The typing rules of **F** are given by the following rules:

$$\frac{}{(\Delta, (\Gamma, x : \tau)) \vdash x : \tau} \quad \text{Typ-Var}$$

$$\frac{\Delta \vdash \tau_1 \ \textbf{WF} \qquad (\Delta, (\Gamma, x : \tau_1)) \vdash e : \tau_2}{(\Delta, \Gamma) \vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \quad \text{Typ-lam}$$

$$\frac{(\Delta, \Gamma) \vdash e_1 : \tau_2 \longrightarrow \tau \qquad (\Delta, \Gamma) \vdash e_2 : \tau_2}{(\Delta, \Gamma) \vdash (e_1 \ e_2) : \tau} \quad \text{Typ-app}$$

$$\frac{(\Delta \cup \{t\}, \Gamma) \vdash e : \tau}{(\Delta, \Gamma) \vdash \Lambda t.e : \forall t.\tau} \quad \text{Typ-Lam}$$

$$\frac{(\Delta, \Gamma) \vdash e : \forall t.\tau' \qquad \Delta \vdash \tau \ \textbf{WF}}{(\Delta, \Gamma) \vdash e\ [\tau] : \tau'[t := \tau]} \quad \text{Typ-App}$$

System F

The closed values of **F** are defined by the following rules:

$$\frac{}{\lambda(x:\tau).e\ \mathbf{Val}}\ \text{Val-lam}$$

$$\frac{}{\Lambda t.e\ \mathbf{Val}}\ \text{Val-Lam}$$

The dynamics of **F** are defined by the following rules:

$$\frac{e \longmapsto e'}{(e\ e_1) \longmapsto (e'\ e_1)}\ \text{Step-app}$$

$$\frac{}{(\lambda(x:\tau).e)\ e' \longmapsto e[x := e']}\ \text{Step-lam}$$

$$\frac{e \longmapsto e'}{(e\ [\tau]) \longmapsto (e'\ [\tau])}\ \text{Step-App}$$

$$\frac{}{(\Lambda t.e)\ [\tau] \longmapsto e[t := \tau]}\ \text{Step-Lam}$$

First Attempt at Defining a Relation for Closed Expressions of System F

Definition

Let e be a closed expression and τ a type. Define $\mathcal{R}_\tau$ by the following rules:

- if $\tau = \tau_1 \longrightarrow \tau_2$, then $e \in \mathcal{R}_\tau$ if and only if there exists a value v such that $e \mapsto^* v$, and if $e' \in \mathcal{R}_{\tau_1}$, then $(e\ e') \in \mathcal{R}_{\tau_2}$

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- if $\tau = \forall t.\tau'$, then $e \in \mathcal{R}_\tau$ if and only if for all closed types σ, $e\ [\sigma] \in \mathcal{R}_{\tau'[t:=\sigma]} \dots$

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- if $\tau = \forall t.\tau'$, then $e \in \mathcal{R}_\tau$ if and only if for all closed types σ, $e\ [\sigma] \in \mathcal{R}_{\tau'[t:=\sigma]} \dots$
  $\mathcal{R}_\tau$ is not well defined since $\tau'[t := \sigma]$ is not structurally smaller than τ!

First Attempt at Defining a Relation for Closed Expressions of System F

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Let e be a closed expression and τ a type. Define $\mathcal{R}_\tau$ by the following rules:

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- if $\tau = \forall t.\tau'$, then $e \in \mathcal{R}_\tau$ if and only if for all closed types σ, $e\ [\sigma] \in \mathcal{R}_{\tau'[t:=\sigma]} \dots$
  $\mathcal{R}_\tau$ is not well defined since $\tau'[t := \sigma]$ is not structurally smaller than τ!

We need to define some relation $\mathcal{S}_\tau$ such that if $\tau = \forall t.\tau'$, then there is a property P (which we can show holds!) such that for all closed types σ, P implies that $e\ [\sigma] \in \mathcal{S}_{\tau'[t:=\sigma]}$

Defining a Relation for Closed expressions of System F

Definition

Let e be a closed expression and τ an (arbitrary) type. Let δ be a function from a finite set of type variables to closed types. Let η be a function from a finite set of type variables to the set of unary relations $\mathcal{R}_\sigma$ over expressions of the closed type σ that are closed under converse evaluation. We define $\mathcal{CHT}(\delta, \eta, \tau)$ as follows:

1. if $\tau = t$, then $e \in \mathcal{CHT}(\delta, \eta, \tau)$ if and only if $e \in \eta(t)$,
2. if $\tau = \tau_1 \longrightarrow \tau_2$, then $e \in \mathcal{CHT}(\delta, \eta, \tau)$ if and only if
   - there exists a value v such that $e \longmapsto^* v$, and
   - if $e_1 \in \mathcal{CHT}(\delta, \eta, \tau_1)$, then $(e\ e_1) \in \mathcal{CHT}(\delta, \eta, \tau_2)$
3. if $\tau = \forall t.\tau'$, then $e \in \mathcal{CHT}(\delta, \eta, \tau)$ if and only if
   - there exists a value v such that $e \longmapsto^* v$, and
   - for any closed type σ and any relation $\mathcal{R}_\sigma$ over expressions of type σ that is closed under converse evaluation, we have

$$e\ [\sigma] \in \mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto \mathcal{R}_\sigma], \tau')$$

Two Important Lemmata

Let e, e' be closed expressions. Let τ be an (arbitary) type. Let σ be a closed type. Let $\mathcal{R}_\sigma$ be a unary relation over expressions of type σ that is closed under converse evaluation. Let δ be a function from a set finite set of type variables to closed types. Let η be a function from a finite set of type variables to the set of unary relations $\mathcal{R}_\sigma$ over expressions of the closed type σ that are closed under converse evaluation.

Proposition

Assume that $e \longmapsto e'$ *and that* $e' \in \mathcal{CHT}(\delta, \eta, \tau)$. *Then* $e \in \mathcal{CHT}(\delta, \eta, \tau)$.

Proposition

Assume that $e \in R_\sigma$ *if and only if* $e \in \mathcal{CHT}(\delta, \eta, \sigma)$. *Then* $e \in \mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto R_\sigma], \tau)$ *if and only if* $e \in \mathcal{CHT}(\delta, \eta, \tau[t := \sigma])$.

Some Notation

Notation

Let δ be a function from a finite set $\{t_1, \ldots, t_n\}$ of type variables to closed types. Let τ be an (arbitary) type. Let e be an (arbitary) expression. Then

$$\hat{\delta}(t) = t[(t_1, \ldots, t_n) := (\delta(t_1), \ldots, \delta(t_n))]$$

and

$$\hat{\delta}(e) = e[(t_1, \ldots, t_n) := (\delta(t_1), \ldots, \delta(t_n))]$$

Extending the Relation

Definition

Let Δ be a set of type variables, Γ a context and τ a type. We say that $e \in \mathcal{HT}(\Delta, \Gamma, \tau)$ if and only if the following condition holds:

- if δ is a function from a finite set of type variables to closed types, and
- η is a function from a finite set of type variables to the set of unary relations $\mathcal{R}_\sigma$ over expressions of the closed type σ that are closed under converse evaluation, and
- γ is a function from a finite set of expression variables to closed expressions such that if $(x : \tau') \in \Gamma$, then $\gamma(x) : \hat{\delta}(\tau')$ and $\gamma(x) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau'))$,

then $\hat{\gamma}(\hat{\delta}(e)) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau))$

Proving Termination

Proposition

Assume that $(\Delta, \Gamma) \vdash e : \tau$. *Then* $e \in \mathcal{HT}(\Delta, \Gamma, \tau)$

Proving Termination

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Assume that $(\Delta, \Gamma) \vdash e : \tau$*. Then* $e \in \mathcal{HT}(\Delta, \Gamma, \tau)$

Proof. Assume that $(\Delta, \Gamma) \vdash e$. Assume that

- δ is a function from a finite set of type variables to closed types,
- η is a function from a finite set of type variables to the set of unary relations $\mathcal{R}_\sigma$ over expressions of the closed type σ that are closed under converse evaluation, and
- γ is a function from a finite set of expression variables to closed types such that if $(x : \tau') \in \Gamma$, $\gamma(x) : \hat{\delta}(\tau')$ and $\gamma(x) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau'))$

We want to show that $\hat{\gamma}(\hat{\delta}(e)) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau))$. The proof proceeds by induction over the derivation $(\Delta, \Gamma) \vdash e$.

- (Case Typ-Var)

$$\frac{}{(\Delta, (\Gamma', x : \tau)) \vdash x : \tau} \quad \text{Typ-Var}$$

Then $\hat{\gamma}(\hat{\delta}(e)) = \gamma(x)$ with $\gamma(x) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau))$ by assumption, hence $x \in \mathcal{HT}(\Delta, \Gamma, \tau)$

Proving Termination

- (Case Typ-lam)

$$\frac{\Delta \vdash \tau_1 \ \textbf{WF} \qquad (\Delta, (\Gamma, x : \tau_1)) \vdash e : \tau_2}{(\Delta, \Gamma) \vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \ \text{Typ-lam}$$

  Assume without loss of generality that $x \notin dom(\gamma)$ since we may rename x to fulfil this condition. Then $\hat{\gamma}(\hat{\delta}(\lambda(x : \tau_1).e)) = \lambda(x : \tau_1.\hat{\gamma}(\hat{\delta}(e)))$ which is already a value.

Proving Termination

- (Case Typ-lam)

$$\frac{\Delta \vdash \tau_1 \ \mathbf{WF} \qquad (\Delta, (\Gamma, x : \tau_1)) \vdash e : \tau_2}{(\Delta, \Gamma) \vdash \lambda(x : \tau_1).e : \tau_1 \longrightarrow \tau_2} \ \text{Typ-lam}$$

Assume without loss of generality that $x \notin dom(\gamma)$ since we may rename x to fulfil this condition. Then $\hat{\gamma}(\hat{\delta}(\lambda(x : \tau_1).e)) = \lambda(x : \tau_1.\hat{\gamma}(\hat{\delta}(e)))$ which is already a value. Now assume that $e_1 \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_1))$. Let $\theta = \gamma \otimes [x \mapsto e_1]$. Observe that

$$\begin{aligned}(\lambda(x : \tau_1).\hat{\gamma}(\hat{\delta}(e)))\ e_1 &\longmapsto (\lambda(x : \tau_1).\hat{\gamma}(\hat{\delta}(e)))\ e_1 \\ &= \hat{\gamma}(\hat{\delta}(e))[x := e_1] = \hat{\theta}(\hat{\delta}(e))\end{aligned}$$

which is in $\mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_2))$ since $e \in \mathcal{HT}(\Delta, (\Gamma, x : \tau_1), \tau_2)$ by the induction hypothesis. Since $\mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_2))$ is closed under converse evaluation, it follows that $(\lambda(x : \tau_1).\hat{\gamma}(\hat{\delta}(e)))\ e_1 \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_2))$, thus $\lambda(x : \tau_1).e \in \mathcal{HT}(\Delta, \Gamma, \tau_1 \longrightarrow \tau_2)$.

Proving Termination

- (Case Typ-app)

$$\frac{(\Delta, \Gamma) \vdash e_1 : \tau_2 \longrightarrow \tau \qquad (\Delta, \Gamma) \vdash e_2 : \tau_2}{(\Delta, \Gamma) \vdash (e_1\ e_2) : \tau} \ \text{Typ-app}$$

  By the induction hypothesis, $e_1 \in \mathcal{HT}(\Delta, \Gamma, \tau_1 \longrightarrow \tau_2)$ and $e_2 \in \mathcal{HT}(\Delta, \Gamma, \tau_2)$, thus $\hat{\gamma}(\hat{\delta}(e_1)) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_1))$ and $\hat{\gamma}(\hat{\delta}(e_2)) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_2))$. Consequently, $\hat{\gamma}(\hat{\delta}((e_1\ e_2))) = (\hat{\gamma}(\hat{\delta}(e_1)))\ (\hat{\gamma}(\hat{\delta}(e_2))) \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau_2))$, thus $(e_1\ e_2) \in \mathcal{HT}(\Delta, \Gamma, \tau)$.

- (Case Type-Lam)

$$\frac{(\Delta \cup \{t\}, \Gamma) \vdash e : \tau}{(\Delta, \Gamma) \vdash \Lambda t.e : \forall t.\tau} \quad \text{Typ-Lam}$$

Assume without loss of generality that $t \notin dom(\delta)$ since otherwise we can rename t to fulfil this condition. Then $\hat{\gamma}(\hat{\delta}(\Lambda t.e)) = \Lambda t.\hat{\gamma}(\hat{\delta}(e))$. Now let σ be a closed type. Let $\mathcal{R}_\sigma$ be a unary relation over expressions of type σ that is closed under converse evaluation. We want to show that

$$(\Lambda t.\hat{\gamma}(\hat{\delta}(e)))\,[\sigma] \in \mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto \mathcal{R}_\sigma], \tau) \quad (1)$$

- (Case Typ-Lam)

$$\frac{(\Delta \cup \{t\}, \Gamma) \vdash e : \tau}{(\Delta, \Gamma) \vdash \Lambda t.e : \forall t.\tau} \quad \text{Typ-Lam}$$

Observe that $(\Lambda t.\hat{\gamma}(\hat{\delta}(e)))\ [\sigma] \longmapsto \hat{\gamma}(\hat{\delta}(e))[t := \sigma]$. Now let $\theta = \delta \otimes [t \mapsto \sigma]$. By the induction hypothesis, $e \in \mathcal{HT} \in (\Delta \cup \{t\}, \Gamma, \tau)$, thus

$$\hat{\gamma}(\hat{\theta}(e)) \in \mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto \mathcal{R}_\sigma], \tau)$$

but $\hat{\gamma}(\hat{\theta}(e)) = \hat{\gamma}(\hat{\delta}(e))[t := \sigma]$, thus

$$\hat{\gamma}(\hat{\delta}(e'))[t := \sigma] \in \mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto \mathcal{R}_\sigma], \tau)$$

Since $\mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto \mathcal{R}_\sigma], \tau)$ is closed under converse evaluation, it follows that Eq. (1) holds. Therefore, $\Lambda t.e \in \mathcal{HT}(\Delta, \Gamma, \forall t.\tau)$.

- (Case Type-App)

$$\frac{(\Delta, \Gamma) \vdash e : \forall t.\tau' \qquad \Delta \vdash \tau \ \textbf{WF}}{(\Delta, \Gamma) \vdash e\ [\tau] : \tau'[t := \tau]} \quad \text{Typ-App}$$

Assume without loss of generality that $t \notin dom(\delta)$ since otherwise we can rename t to fulfil this condition. Then

$$\hat{\gamma}(\hat{\delta}(e\ [\tau])) = (\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)]$$

and

$$\hat{\delta}(\tau'[t := \tau]) = (\hat{\delta}(\tau'))[t := \hat{\delta}(\tau)]$$

We want to show that

$$(\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)] \in \mathcal{CHT}(\delta, \eta, (\hat{\delta}(\tau'))[t := \hat{\delta}(\tau)]) \qquad (2)$$

- (Case Type-App)

$$\frac{(\Delta, \Gamma) \vdash e : \forall t.\tau' \qquad \Delta \vdash \tau \ \mathbf{WF}}{(\Delta, \Gamma) \vdash e\ [\tau] : \tau'[t := \tau]} \ \text{Typ-App}$$

Assume without loss of generality that $t \notin dom(\delta)$ since otherwise we can rename t to fulfil this condition. Then

$$\hat{\gamma}(\hat{\delta}(e\ [\tau])) = (\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)]$$

and

$$\hat{\delta}(\tau'[t := \tau]) = (\hat{\delta}(\tau'))[t := \hat{\delta}(\tau)]$$

We want to show that

$$(\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)] \in \mathcal{CHT}(\delta, \eta, (\hat{\delta}(\tau'))[t := \hat{\delta}(\tau)]) \qquad (2)$$

Define a unary relation $\mathcal{R}_{\hat{\delta}(\tau)}$ on expressions k of the closed type $\hat{\delta}(\tau)$ by $k \in \mathcal{R}_{\hat{\delta}(\tau)}$ if and only if $k \in \mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau))$. Since, $\mathcal{CHT}(\delta, \eta, \hat{\delta}(\tau))$ is closed under converse evaluation, it follows that $\mathcal{R}_{\hat{\delta}(\tau)}$ is closed under converse evaluation.

Recall the following lemma:

Lemma

Assume that σ is a closed type and that $\mathcal{R}_\sigma$ is a unary relation over closed expressions of type σ that is closed under converse evaluation. Assume that $e \in R_\sigma$ if and only if $e \in \mathcal{CHT}(\delta, \eta, \sigma)$. Then $e \in \mathcal{CHT}(\delta \otimes [t \mapsto \sigma], \eta \otimes [t \mapsto R_\sigma], \tau)$ if and only if $e \in \mathcal{CHT}(\delta, \eta, \tau[t := \sigma])$.

Since $\mathcal{R}_{\hat{\delta}(\tau)}$ is closed under converse evaluation, it follows that

$$(\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)] \in \mathcal{CHT}(\delta, \eta, (\hat{\delta}(\tau'))[t := \hat{\delta}(\tau)])$$

if and only if

$$(\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)] \in \mathcal{CHT}(\delta \otimes [t \mapsto \hat{\delta}(\tau)], \eta \otimes [t \mapsto \mathcal{R}_{\hat{\delta}(\tau)}], \hat{\delta}(\tau'))$$

- (Case Typ-App)

$$\frac{(\Delta, \Gamma) \vdash e : \forall t.\tau' \qquad \Delta \vdash \tau \ \textbf{WF}}{(\Delta, \Gamma) \vdash e\ [\tau] : \tau'[t := \tau]} \quad \text{Typ-App}$$

By the induction hypothesis, $e \in \mathcal{HT}(\Delta, \Gamma, \forall t.\tau')$, so $\hat{\gamma}(\hat{\delta}(e)) \in \mathcal{CHT}(\delta, \eta, \forall t.\tau')$. Consequently,

$$(\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)] \in \mathcal{CHT}(\delta \otimes [t \mapsto \hat{\delta}(\tau)], \eta \otimes [t \mapsto \mathcal{R}_{\hat{\delta}(\tau)}], \hat{\delta}(\tau')),$$

so

$$(\hat{\gamma}(\hat{\delta}(e)))\ [\hat{\delta}(\tau)] \in \mathcal{CHT}(\delta, \eta, (\hat{\delta}(\tau'))[t := \hat{\delta}(\tau)]),$$

which is what we wanted to show. Hence $e\ [\tau] \in \mathcal{HT}(\Delta, \Gamma, \tau'[t := \tau])$. This completes the proof.