An Introduction to Intuitionistic Propositional Logic

CIS 670 Lecture

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October 4, 2016

It's election season

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Me : So Mike, Josh and Harry, who are you voting for and why?

It's election season

Me : So Mike, Josh and Harry, who are you voting for and why?

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Josh : I think Trump would make a bad president. So yes, Hillary.

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Harry : Hey guys, this doesn't make sense to me.
How does one's being bad make the other good?

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Me : So who are you voting for then Harry?

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Mike : Trump. Because I think Hillary would make a bad president.

Josh : I think Trump would make a bad president. So yes, Hillary.

Harry : Hey guys, this doesn't make sense to me.
How does one's being bad make the other good?

Me : So who are you voting for then Harry?

Harry : I don't know.

I don't know

Source:www.tworoadsmarketing.com

Can the middle be excluded?

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Is $Q : P \vee \neg P$ true?

What is truth?

What is truth?

Truth is something that God knows.

What is truth?

Truth is something that I know.

What is truth?

Truth is something that I know.

Truth is something that I have seen.

What is truth?

Truth is something that I know.

Truth is something that I have seen.

Truth is something that I have evidence for.

Judgement

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- ϕ is a proposition.

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- ϕ_1 is true, ϕ_2 is true, ... , ϕ_n is true $\vdash \phi$ is a proposition.

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Meaning of hypothetical judgements

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- $$\frac{}{\phi_1 \text{ is true}, \phi_2 \text{ is true}, \ldots, \phi_i \text{ is true}, \ldots, \phi_n \text{ is true} \vdash \phi_i \text{ is true}}$$

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- $$\frac{\Gamma \vdash \phi_1 \text{ is true} \qquad \Gamma, \phi_1 \text{ is true} \vdash \phi_2 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

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- $$\frac{\Gamma \vdash \phi_2 \text{ is true} \qquad \Gamma \vdash \phi_1 \text{ is a proposition}}{\Gamma, \phi_1 \text{ is true} \vdash \phi_2 \text{ is true}}$$

Building complex propositions

- $\top$ is prop.

Building complex propositions

- $\top$ is prop.
- $\Gamma \vdash \top$ is true

Building complex propositions

- $$\frac{\Gamma \vdash \phi_1 \text{ is prop} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ is prop}}$$

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Building complex propositions

- $\perp$ is prop.

Building complex propositions

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- $$\frac{\Gamma \vdash \perp \text{ is true} \quad \Gamma, \perp \text{ is true} \vdash \phi \text{ is prop}}{\Gamma \vdash \phi \text{ is true}}$$

How do the connectives interact?

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Time for an exercise!

$\phi \lor (\psi_1 \land \psi_2)$ is true $\equiv (\phi \lor \psi_1) \land (\phi \lor \psi_2)$ is true

Sanity check

$\phi \wedge \neg\phi$ is true $\equiv \perp$ is true

Sanity check

$$\phi \land \neg\phi \text{ is true} \equiv \bot \text{ is true}$$

$$\frac{\phi \land \neg\phi \text{ is true}}{\bot \text{ is true}}$$

$$\frac{\bot \text{ is true}}{\phi \land \neg\phi \text{ is true}}$$

Sanity check

$$\phi \wedge \neg\phi \text{ is true} \equiv \bot \text{ is true}$$

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$$\frac{\bot \text{ is true}}{\phi \wedge \neg\phi \text{ is true}}$$

$$\frac{\dfrac{\phi \wedge \neg\phi \text{ is true}}{\neg\phi \text{ is true}} \qquad \dfrac{\phi \wedge \neg\phi \text{ is true}}{\phi \text{ is true}}}{\bot \text{ is true}}$$

More on interaction between connectives

$$\neg(\phi \vee \psi) \text{ is true} \equiv \neg\phi \wedge \neg\psi \text{ is true}$$

More on interaction between connectives

$$\neg(\phi \lor \psi) \text{ is true} \equiv \neg\phi \land \neg\psi \text{ is true}$$

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It's constructive!

$\neg(\phi \wedge \psi)$ is true $\equiv \neg\phi \vee \neg\psi$ is true

It's constructive!

$\neg(\phi \wedge \psi)$ is true $\equiv \neg\phi \vee \neg\psi$ is true

$\phi \vee \neg\phi$ is true

$\neg\neg(\phi \vee \neg\phi)$ is true

Time for another exercise

$\phi \implies \neg\neg\phi$ is true

Time for another exercise

$\phi \implies \neg\neg\phi$ is true

Problem

Absurdity of absurdity of absurdity is equivalent to absurdity. [BD81]

Classical vs. constructive

- Boolean algebra semantics = complemented distributive lattice

  $\forall\phi,\ \exists\ \neg\phi$ such that $\phi \wedge \neg\phi = 0$ and $\phi \vee \neg\phi = 1$

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  $\phi \implies \psi =$ weakest assumption which when adjoined with ϕ gives ψ

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  $\phi \implies \psi$ = weakest assumption which when adjoined with ϕ gives ψ

$$\begin{array}{c} 1 \\ \uparrow \\ \frac{1}{2} \\ \uparrow \\ 0 \end{array} \qquad (p \implies q) = \begin{cases} q, & \text{if } p > q \\ 1, & \text{otherwise} \end{cases}$$

Where is your evidence?

- ϕ is a proposition.
- ϕ is true.
- ϕ_1 is true, ϕ_2 is true, $\ldots$, ϕ_n is true $\vdash \phi$ is a proposition.
- ϕ_1 is true, ϕ_2 is true, $\ldots$, ϕ_n is true $\vdash \phi$ is true.

Where is your evidence?

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- $x : \phi$
- $x_1 : \phi_1, x_2 : \phi_2, \ldots, x_n : \phi_n \vdash ? : \phi$
- ϕ_1 is true, ϕ_2 is true, ... , ϕ_n is true $\vdash$ ϕ is true.

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Where is your evidence?

- $\top$ is prop.
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Where is your evidence?

- $$\frac{\Gamma \vdash ? : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash (?, ?') : \phi_1 \wedge \phi_2}$$

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- $$\frac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \pi_1(p) : \phi_1}$$

- $$\frac{\Gamma \vdash \phi_1 \wedge \phi_2 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

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- $$\frac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \pi_2(p) : \phi_2}$$

Where is your evidence?

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Where is your evidence?

- $$\frac{\Gamma \vdash ?' : \phi_1 \quad \Gamma, x : \phi_1 \vdash ? : \phi_2}{\Gamma \vdash \lambda x.? : \phi_1 \implies \phi_2}$$

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- $$\frac{\Gamma \vdash \phi_1 \implies \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

Where is your evidence?

- $$\frac{\Gamma \vdash ?' : \phi_1 \quad \Gamma, x : \phi_1 \vdash ? : \phi_2}{\Gamma \vdash \lambda x.? : \phi_1 \implies \phi_2}$$

- $$\frac{\Gamma, x : \phi_1 \vdash p_2 : \phi_2}{\Gamma \vdash \lambda x.p_2 : \phi_1 \implies \phi_2}$$

- $$\frac{\Gamma \vdash \phi_1 \implies \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is true}}{\Gamma \vdash \phi_2 \text{ is true}}$$

Where is your evidence?

- $$\frac{\Gamma \vdash ?' : \phi_1 \quad \Gamma, x : \phi_1 \vdash ? : \phi_2}{\Gamma \vdash \lambda x.? : \phi_1 \implies \phi_2}$$

- $$\frac{\Gamma, x : \phi_1 \vdash p_2 : \phi_2}{\Gamma \vdash \lambda x.p_2 : \phi_1 \implies \phi_2}$$

- $$\frac{\Gamma \vdash p : \phi_1 \implies \phi_2 \quad \Gamma \vdash p_1 : \phi_1}{\Gamma \vdash p(p_1) : \phi_2}$$

Where is your evidence?

- $$\frac{\Gamma \vdash \phi_1 \text{ is prop} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is prop}}$$

- $$\frac{\Gamma \vdash \phi_1 \text{ is true} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true} \quad \Gamma, \phi_1 \text{ is true} \vdash \phi \text{ is true} \quad \Gamma, \phi_2 \text{ is true} \vdash \phi \text{ is true}}{\Gamma \vdash \phi \text{ is true}}$$

- $$\frac{\Gamma \vdash ? : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash \text{ case } (\text{inl } ? \hookrightarrow ?|\ \text{inr } ?' \hookrightarrow ?') : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash \phi_1 \text{ is true} \quad \Gamma \vdash \phi_2 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true} \quad \Gamma, \phi_1 \text{ is true} \vdash \phi \text{ is true} \quad \Gamma, \phi_2 \text{ is true} \vdash \phi \text{ is true}}{\Gamma \vdash \phi \text{ is true}}$$

Where is your evidence?

- $$\frac{\Gamma \vdash ? : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash \ \text{case}\ (\text{inl}\ ? \hookrightarrow ?|\ \text{inr}\ ?' \hookrightarrow ?') : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash i_1(p_1) : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash \phi_2 \text{ is true} \quad \Gamma \vdash \phi_1 \text{ is prop}}{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true}}$$

- $$\frac{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true} \quad \Gamma, \phi_1 \text{ is true} \vdash \phi \text{ is true} \quad \Gamma, \phi_2 \text{ is true} \vdash \phi \text{ is true}}{\Gamma \vdash \phi \text{ is true}}$$

Where is your evidence?

- $$\frac{\Gamma \vdash ? : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash \ \text{case } (\text{inl } ? \hookrightarrow ?|\ \text{inr } ?' \hookrightarrow ?') : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash i_1(p_1) : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash p_2 : \phi_2 \quad \Gamma \vdash ? : \phi_1}{\Gamma \vdash i_2(p_2) : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash \phi_1 \vee \phi_2 \text{ is true} \quad \Gamma, \phi_1 \text{ is true} \vdash \phi \text{ is true} \quad \Gamma, \phi_2 \text{ is true} \vdash \phi \text{ is true}}{\Gamma \vdash \phi \text{ is true}}$$

Where is your evidence?

- $$\frac{\Gamma \vdash ? : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash \text{ case } (\text{inl } ? \hookrightarrow ?|\text{ inr } ?' \hookrightarrow ?') : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash ?' : \phi_2}{\Gamma \vdash i_1(p_1) : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash p_2 : \phi_2 \quad \Gamma \vdash ? : \phi_1}{\Gamma \vdash i_2(p_2) : \phi_1 \vee \phi_2}$$

- $$\frac{\Gamma \vdash p : \phi_1 \vee \phi_2 \quad \Gamma, x_1 : \phi_1 \vdash p_1 : \phi \quad \Gamma, x_2 : \phi_2 \vdash p_2 : \phi}{\Gamma \vdash \text{ case p } (\text{inl } x_1 \hookrightarrow p_1|\text{ inr } x_2 \hookrightarrow p_2) : \phi}$$

Where is your evidence?

- $\perp$ is prop.
- $$\frac{\Gamma \vdash \perp \text{ is true} \quad \Gamma, \perp \text{ is true} \vdash \phi \text{ is prop}}{\Gamma \vdash \phi \text{ is true}}$$

Where is your evidence?

- $? : \bot$
- $$\frac{\Gamma \vdash \bot \text{ is true} \quad \Gamma, \bot \text{ is true} \vdash \phi \text{ is prop}}{\Gamma \vdash \phi \text{ is true}}$$

Where is your evidence?

- $? : \perp$
- $$\frac{\Gamma \vdash p : \perp \quad \Gamma, p : \perp \vdash ? : \phi}{\Gamma \vdash \text{wild} \;\; : \phi}$$

Theorems Revisited

$$\phi \wedge (\psi_1 \vee \psi_2) \text{ is true} \equiv (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}$$

$$\frac{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}$$

$$\frac{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}$$

Theorems Revisited

$$\phi \wedge (\psi_1 \vee \psi_2) \text{ is true} \equiv (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}$$

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$$\dfrac{\dfrac{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}{\psi_1 \vee \psi_2 \text{ is true}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}{\phi \text{ is true}}}{\psi_1 \text{ is true} \vdash \phi \text{ is true}} \quad \dfrac{}{\psi_1 \text{ is true} \vdash \psi_1 \text{ is true}}}{\psi_1 \text{ is true} \vdash \phi \wedge \psi_1 \text{ is true}}}{\psi_1 \text{ is true} \vdash (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}{\phi \text{ is true}}}{\psi_2 \text{ is true} \vdash \phi \text{ is true}} \quad \dfrac{}{\psi_2 \text{ is true} \vdash \psi_2 \text{ is true}}}{\psi_2 \text{ is true} \vdash \phi \wedge \psi_2 \text{ is true}}}{\psi_2 \text{ is true} \vdash (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}}{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}$$

Theorems Revisited

$$\phi \wedge (\psi_1 \vee \psi_2) \text{ is true} \equiv (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}$$

$$\frac{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}$$

$$p : \phi \wedge (\psi_1 \vee \psi_2)$$

$$\text{case } (\pi_2 p)(\text{inl } x_1 \hookrightarrow i_1(\pi_1 p, x_1) \mid \text{inr } x_2 \hookrightarrow i_2(\pi_1 p, x_2)) : (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2)$$

$$\phi \wedge (\psi_1 \vee \psi_2) \text{ is true} \equiv (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}$$

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$$p : \phi \wedge (\psi_1 \vee \psi_2)$$

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$$\frac{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}$$

$$\dfrac{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ tr} \qquad \dfrac{\dfrac{\dfrac{}{\phi \wedge \psi_1 \text{ tr} \vdash \phi \wedge \psi_1 \text{ tr}}}{\phi \wedge \psi_1 \text{ tr} \vdash \phi \text{ tr}} \quad \dfrac{\dfrac{\dfrac{}{\phi \wedge \psi_1 \text{ tr} \vdash \phi \wedge \psi_1 \text{ tr}}}{\phi \wedge \psi_1 \text{ tr} \vdash \psi_1 \text{ tr}}}{\phi \wedge \psi_1 \text{ tr} \vdash \psi_1 \vee \psi_2 \text{ tr}}}{\phi \wedge \psi_1 \text{ tr} \vdash \phi \wedge (\psi_1 \vee \psi_2) \text{ tr}} \qquad \dfrac{\dfrac{\dfrac{}{\phi \wedge \psi_2 \text{ tr} \vdash \phi \wedge \psi_2 \text{ tr}}}{\phi \wedge \psi_2 \text{ tr} \vdash \phi \text{ tr}} \quad \dfrac{\dfrac{\dfrac{}{\phi \wedge \psi_2 \text{ tr} \vdash \phi \wedge \psi_2 \text{ tr}}}{\phi \wedge \psi_2 \text{ tr} \vdash \psi_2 \text{ tr}}}{\phi \wedge \psi_2 \text{ tr} \vdash \psi_1 \vee \psi_2 \text{ tr}}}{\phi \wedge \psi_2 \text{ tr} \vdash \phi \wedge (\psi_1 \vee \psi_2) \text{ tr}}}{\phi \wedge (\psi_1 \vee \psi_2) \text{ tr}}$$

Theorems Revisited

$$\phi \wedge (\psi_1 \vee \psi_2) \text{ is true} \equiv (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}$$

$$\frac{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}$$

$$p : \phi \wedge (\psi_1 \vee \psi_2)$$

$\text{case } (\pi_2 p)(\text{inl } x_1 \hookrightarrow i_1(\pi_1 p, x_1) \mid \text{inr } x_2 \hookrightarrow i_2(\pi_1 p, x_2)) : (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2)$

$$\frac{(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \text{ is true}}{\phi \wedge (\psi_1 \vee \psi_2) \text{ is true}}$$

$$p' : (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2)$$

$\text{case } p'(\text{inl } p_1 \hookrightarrow (\pi_1 p_1, i_1(\pi_2 p_1)) \mid \text{inr } p_2 \hookrightarrow (\pi_1 p_2, i_2(\pi_2 p_2))) :$
$\phi \wedge (\psi_1 \vee \psi_2)$

Exercise Revisited!

$$\phi \vee (\psi_1 \wedge \psi_2) \text{ is true} \equiv (\phi \vee \psi_1) \wedge (\phi \vee \psi_2) \text{ is true}$$

Progeny and ancestry of proofs

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"Nothing is lost ... Everything is transformed."

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- Principle of conservation of proof = introduction-elimination = β-rules

Progeny and ancestry of proofs

"Nothing is lost ... Everything is transformed."

- Principle of conservation of proof = introduction-elimination = β-rules
- Principle of reversibility of proof = elimination-introduction = η-rules

Progeny and ancestry of proofs $\wedge$

- β-rules

$$\dfrac{\dfrac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash p_2 : \phi_2}{\Gamma \vdash (p_1, p_2) : \phi_1 \wedge \phi_2}}{\Gamma \vdash \pi_1(p_1, p_2) : \phi_1}$$

- β-rules

$$\cfrac{\cfrac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash p_2 : \phi_2}{\Gamma \vdash (p_1, p_2) : \phi_1 \wedge \phi_2}}{\Gamma \vdash \pi_1(p_1, p_2) : \phi_1}$$

$$\Gamma \vdash p_1 \equiv \pi_1(p_1, p_2) : \phi_1 \qquad \Gamma \vdash p_2 \equiv \pi_2(p_1, p_2) : \phi_2$$

Progeny and ancestry of proofs $\wedge$

- β-rules

$$\dfrac{\dfrac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash p_2 : \phi_2}{\Gamma \vdash (p_1, p_2) : \phi_1 \wedge \phi_2}}{\Gamma \vdash \pi_1(p_1, p_2) : \phi_1}$$

$$\Gamma \vdash p_1 \equiv \pi_1(p_1, p_2) : \phi_1 \qquad \Gamma \vdash p_2 \equiv \pi_2(p_1, p_2) : \phi_2$$

- η-rule

$$\dfrac{\dfrac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \pi_1 p : \phi_1} \quad \dfrac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \pi_2 p : \phi_2}}{\Gamma \vdash (\pi_1 p, \pi_2 p) : \phi_1 \wedge \phi_2}$$

Progeny and ancestry of proofs $\wedge$

- β-rules

$$\dfrac{\dfrac{\Gamma \vdash p_1 : \phi_1 \quad \Gamma \vdash p_2 : \phi_2}{\Gamma \vdash (p_1, p_2) : \phi_1 \wedge \phi_2}}{\Gamma \vdash \pi_1(p_1, p_2) : \phi_1}$$

$$\Gamma \vdash p_1 \equiv \pi_1(p_1, p_2) : \phi_1 \qquad \Gamma \vdash p_2 \equiv \pi_2(p_1, p_2) : \phi_2$$

- η-rule

$$\dfrac{\dfrac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \pi_1 p : \phi_1} \quad \dfrac{\Gamma \vdash p : \phi_1 \wedge \phi_2}{\Gamma \vdash \pi_2 p : \phi_2}}{\Gamma \vdash (\pi_1 p, \pi_2 p) : \phi_1 \wedge \phi_2}$$

$$\Gamma \vdash p \equiv (\pi_1 p, \pi_2 p) : \phi_1 \wedge \phi_2$$

Progeny and ancestry of proofs $\implies$

- β-rule

$$\frac{\dfrac{\Gamma, x : \phi_1 \vdash p_2 : \phi_2}{\Gamma \vdash \lambda x.p_2 : \phi_1 \implies \phi_2} \quad \Gamma \vdash p_1 : \phi_1}{\Gamma \vdash (\lambda x.p_2)p_1 : \phi_2}$$

Progeny and ancestry of proofs $\implies$

- β-rule

$$\frac{\dfrac{\Gamma, x : \phi_1 \vdash p_2 : \phi_2}{\Gamma \vdash \lambda x.p_2 : \phi_1 \implies \phi_2} \quad \Gamma \vdash p_1 : \phi_1}{\Gamma \vdash (\lambda x.p_2)p_1 : \phi_2}$$

$$\Gamma \vdash [p_1/x]p_2 \equiv (\lambda x.p_2)p_1 : \phi_2$$

- β-rule

$$\dfrac{\dfrac{\Gamma, x : \phi_1 \vdash p_2 : \phi_2}{\Gamma \vdash \lambda x.p_2 : \phi_1 \implies \phi_2} \quad \Gamma \vdash p_1 : \phi_1}{\Gamma \vdash (\lambda x.p_2)p_1 : \phi_2}$$

$$\Gamma \vdash [p_1/x]p_2 \equiv (\lambda x.p_2)p_1 : \phi_2$$

- η-rule

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash p : \phi_1 \implies \phi_2}{\Gamma, x : \phi_1 \vdash p : \phi_1 \implies \phi_2} \quad \dfrac{}{\Gamma, x : \phi_1 \vdash x : \phi_1}}{\Gamma, x : \phi_1 \vdash p(x) : \phi_2}}{\Gamma \vdash \lambda x.p(x) : \phi_1 \implies \phi_2}$$

- β-rule

$$\dfrac{\dfrac{\Gamma, x : \phi_1 \vdash p_2 : \phi_2}{\Gamma \vdash \lambda x.p_2 : \phi_1 \implies \phi_2} \quad \Gamma \vdash p_1 : \phi_1}{\Gamma \vdash (\lambda x.p_2)p_1 : \phi_2}$$

$\Gamma \vdash [p_1/x]p_2 \equiv (\lambda x.p_2)p_1 : \phi_2$

- η-rule

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash p : \phi_1 \implies \phi_2}{\Gamma, x : \phi_1 \vdash p : \phi_1 \implies \phi_2} \quad \dfrac{}{\Gamma, x : \phi_1 \vdash x : \phi_1}}{\Gamma, x : \phi_1 \vdash p(x) : \phi_2}}{\Gamma \vdash \lambda x.p(x) : \phi_1 \implies \phi_2}$$

$\Gamma \vdash p \equiv \lambda x.p(x) : \phi_1 \implies \phi_2$

Progeny and ancestry of proofs $\vee$

- β-rules

$$\frac{\dfrac{\Gamma \vdash p : \phi_1}{\Gamma \vdash i_1 p : \phi_1 \vee \phi_2} \quad \Gamma, x_1 : \phi_1 \vdash p_1 : \psi \quad \Gamma, x_2 : \phi_2 \vdash p_2 : \psi}{\Gamma \vdash \ \text{case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi}$$

Progeny and ancestry of proofs $\vee$

- β-rules

$$\frac{\dfrac{\Gamma \vdash p : \phi_1}{\Gamma \vdash i_1 p : \phi_1 \vee \phi_2} \quad \Gamma, x_1 : \phi_1 \vdash p_1 : \psi \quad \Gamma, x_2 : \phi_2 \vdash p_2 : \psi}{\Gamma \vdash \text{ case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi}$$

$$\Gamma \vdash [p/x_1]p_1 \equiv \text{ case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi$$
$$\Gamma \vdash [p/x_2]p_2 \equiv \text{ case } (i_2 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi$$

- β-rules

$$\dfrac{\dfrac{\Gamma \vdash p : \phi_1}{\Gamma \vdash i_1 p : \phi_1 \vee \phi_2} \quad \Gamma, x_1 : \phi_1 \vdash p_1 : \psi \quad \Gamma, x_2 : \phi_2 \vdash p_2 : \psi}{\Gamma \vdash \text{ case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi}$$

$$\Gamma \vdash [p/x_1]p_1 \equiv \text{ case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi$$
$$\Gamma \vdash [p/x_2]p_2 \equiv \text{ case } (i_2 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi$$

- η-rule

$$\dfrac{\Gamma \vdash p : \phi_1 \vee \phi_2 \quad \dfrac{\Gamma, x : \phi_1 \vee \phi_2 \vdash q : \psi}{\Gamma, x_1 : \phi_1 \vdash [i_1 x_1/x]q : \psi} \quad \dfrac{\Gamma, x : \phi_1 \vee \phi_2 \vdash q : \psi}{\Gamma, x_2 : \phi_2 \vdash [i_2 x_2/x]q : \psi}}{\Gamma \vdash \text{ case } p(\text{inl } x_1 \hookrightarrow [i_1 x_1/x]q \mid \text{inr } x_2 \hookrightarrow [i_2 x_2/x]q) : \psi}$$

Progeny and ancestry of proofs V

- β-rules

$$\frac{\dfrac{\Gamma \vdash p : \phi_1}{\Gamma \vdash i_1 p : \phi_1 \vee \phi_2} \quad \Gamma, x_1 : \phi_1 \vdash p_1 : \psi \quad \Gamma, x_2 : \phi_2 \vdash p_2 : \psi}{\Gamma \vdash \text{ case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi}$$

$$\Gamma \vdash [p/x_1]p_1 \equiv \text{ case } (i_1 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi$$
$$\Gamma \vdash [p/x_2]p_2 \equiv \text{ case } (i_2 p)(\text{inl } x_1 \hookrightarrow p_1 \mid \text{inr } x_2 \hookrightarrow p_2) : \psi$$

- η-rule

$$\frac{\Gamma \vdash p : \phi_1 \vee \phi_2 \quad \dfrac{\Gamma, x : \phi_1 \vee \phi_2 \vdash q : \psi}{\Gamma, x_1 : \phi_1 \vdash [i_1 x_1/x]q : \psi} \quad \dfrac{\Gamma, x : \phi_1 \vee \phi_2 \vdash q : \psi}{\Gamma, x_2 : \phi_2 \vdash [i_2 x_2/x]q : \psi}}{\Gamma \vdash \text{ case } p(\text{inl } x_1 \hookrightarrow [i_1 x_1/x]q \mid \text{inr } x_2 \hookrightarrow [i_2 x_2/x]q) : \psi}$$

$$\Gamma \vdash [p/x]q \equiv \text{ case } p(\text{inl } x_1 \hookrightarrow [i_1 x_1/x]q \mid \text{inr } x_2 \hookrightarrow [i_2 x_2/x]q) : \psi$$

Problem

How to drink water?

The takeaways

- The technical stuff

The takeaways

- ~~The technical stuff~~

The takeaways

- The broader perspective of intuitionism

References

L. E. J. Brouwer and D. van Dalen, Brouwer's cambridge lectures on intuitionism / edited by d. van dalen, Cambridge University Press Cambridge [Eng.] ; New York, 1981.

A. Heyting, Intuitionism an introduction, North-Holland, 1956.

Per Martin-Löf, On the meanings of the logical constants and the justifications of the logical laws, Proceedings of the conference on mathematical logic, Vol. 2 (Siena, 1983/1984), Univ. Siena, Siena, 1985, pp. 203–281.

______, A path from logic to metaphysics, Nuovi problemi della logica e della filosofia della scienza, Vol. 2, CLUEB,Bologna,Italy, 1990, pp. 141–149.

Questions?

Thank you.