From System T to Continuation-Passing Style

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1 Introduction

1.1 System T

System T is an extension of simply typed lambda calculus with natural numbers. It provides *recursors* to define primitive recursive functions. [1, 2] This project focuses on a subset of system T that has `if0` functions. It can be extended by `fix` operators to define recursive functions. The syntax and typing rules are defined in Chapter 2.

1.2 Continuation-passing style

Continuation is a data structure that represents the computational process at a given point in the process' execution; the created data structure can be accessed by the programming language, instead of being hidden in the runtime environment. [3]

In direct style, and operation of boolean is defined as

```
andb = fun b1 b2 : bool => if b1 then b2 else false
     : bool -> bool -> bool
```

The expression (andb true (andb false true)) can step either call by value to:

```
if true then false else false
```

or call by name to:

```
if false then true else false
```

depending on the evaluation strategy of the language.

In continuation-passing style, the and operation can be defined as

```
andb1 b1 b2 k :=
  (fun x1 x2 : bool =>
     k (if x1 then x2 else false)) b1 b2
```

or

```
andb2 b1 b2 k :=
  (fun x1 : bool =>
     if x1
     then (fun x2 : bool => k x2) b2
     else k false) b1
```

The expression (andb1 false true k) steps to

```
(fun x2 : bool =>
   k (if false then x2 else false)) true
```

while (andb2 false true k) steps to

```
if false then (fun x2 : bool => k x2) true else k false
```

In this way, we can define the evaluation strategy within the function definition.

This project is to formalize a translation from system T to continuation-passing style and provide an informal proof of its type correctness. The translation is based on this paper [4].

2 Syntax and typing definitions

$x,\ y$	variables			
i	integer literals			
$\tau,\ \sigma$	$::=$			types
	$\mid$	$\mathbb{Z}$		
	$\mid$	`void`		
	$\mid$	$\tau_1 \to \tau_2$		
	$\mid$	(τ)	S	
e	$::=$			annotated terms
	$\mid$	u^τ		
$u,\ v$	$::=$			raw terms
	$\mid$	x		
	$\mid$	i		
	$\mid$	$\lambda x : \tau . e$	bind x in e	
	$\mid$	$e_1\ e_2$		
	$\mid$	$e_1\ p\ e_2$		
	$\mid$	$\mathsf{if0}(e_1, e_2, e_3)$		
	$\mid$	$\mathsf{let}\ x = v\ \mathsf{in}\ u$	bind x in u	
	$\mid$	$\mathsf{halt}\ [\tau] e$		
	$\mid$	(u)	S	
p	$::=$			primitives
	$\mid$	$+$		
	$\mid$	$-$		
Γ	$::=$			contexts
	$\mid$			
	$\mid$	$\Gamma, x : \tau$		

$\boxed{\Gamma \vdash_{\mathrm{T}} e : \tau}$ annotated typing

$$\frac{\Gamma \vdash_{\mathrm{T}} u : \tau}{\Gamma \vdash_{\mathrm{T}} u^\tau : \tau} \quad \textsc{T\_ant\_ann}$$

$\boxed{\Gamma \vdash_{\mathrm{T}} u : \tau}$ typing

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\mathrm{T}} x : \tau} \quad \textsc{T\_term\_var}$$

$$\frac{}{\Gamma \vdash_{\mathrm{T}} i : \mathbb{Z}} \quad \textsc{T\_term\_int}$$

$$\frac{\Gamma, x_1 : \tau_1 \vdash_{\mathrm{T}} e : \tau_2}{\Gamma \vdash_{\mathrm{T}} \lambda x_1 : \tau_1 . e : \tau_1 \to \tau_2} \quad \textsc{T\_term\_lam}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{T}} e_1 : \tau_1 \to \tau_2 \\ \Gamma \vdash_{\mathrm{T}} e_2 : \tau_1\end{array}}{\Gamma \vdash_{\mathrm{T}} e_1\ e_2 : \tau_2} \quad \textsc{T\_term\_app}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \\ \Gamma \vdash_{\mathrm{T}} e_2 : \mathbb{Z}\end{array}}{\Gamma \vdash_{\mathrm{T}} e_1\ p\ e_2 : \mathbb{Z}} \quad \textsc{T\_term\_prim}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \\ \Gamma \vdash_{\mathrm{T}} e_2 : \tau \\ \Gamma \vdash_{\mathrm{T}} e_3 : \tau\end{array}}{\Gamma \vdash_{\mathrm{T}} \mathsf{if0}(e_1, e_2, e_3) : \tau} \quad \textsc{T\_term\_if0}$$

$\boxed{\Gamma \vdash_{\mathrm{K}} e : \tau}$ annotated typing

$$\frac{\Gamma \vdash_{\mathrm{K}} u : \tau}{\Gamma \vdash_{\mathrm{K}} u^{\tau} : \tau} \quad \textsc{K\_ant\_ann}$$

$\boxed{\Gamma \vdash_{\mathrm{K}} u : \tau}$ typing

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\mathrm{K}} x : \tau} \quad \textsc{K\_term\_var}$$

$$\frac{}{\Gamma \vdash_{\mathrm{K}} i : \mathbb{Z}} \quad \textsc{K\_term\_int}$$

$$\frac{\Gamma, x : \tau_1 \vdash_{\mathrm{K}} e : \tau_2}{\Gamma \vdash_{\mathrm{K}} \lambda x : \tau_1 . e : \tau_1 \to \tau_2} \quad \textsc{K\_term\_Lam}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{K}} v : \tau \\ \Gamma, x : \tau \vdash_{\mathrm{K}} u : \mathtt{void}\end{array}}{\Gamma \vdash_{\mathrm{K}} \mathsf{let}\, x = v \,\mathsf{in}\, u : \mathtt{void}} \quad \textsc{K\_term\_let}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{K}} e_1 : \mathbb{Z} \\ \Gamma \vdash_{\mathrm{K}} e_2 : \mathbb{Z}\end{array}}{\Gamma \vdash_{\mathrm{K}} e_1 \, p \, e_2 : \mathbb{Z}} \quad \textsc{K\_term\_prim}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{K}} e' : \tau_1 \to \tau_2 \\ \Gamma \vdash_{\mathrm{K}} e : \tau\end{array}}{\Gamma \vdash_{\mathrm{K}} e' \, e : \tau_2} \quad \textsc{K\_term\_app}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathrm{K}} e : \mathbb{Z} \\ \Gamma \vdash_{\mathrm{K}} e_1 : \tau \\ \Gamma \vdash_{\mathrm{K}} e_2 : \tau\end{array}}{\Gamma \vdash_{\mathrm{K}} \mathsf{if0}(e, e_1, e_2) : \tau} \quad \textsc{K\_term\_if0}$$

$$\frac{\Gamma \vdash_{\mathrm{K}} e : \tau}{\Gamma \vdash_{\mathrm{K}} \mathsf{halt}\, [\tau] e : \mathtt{void}} \quad \textsc{K\_term\_halt}$$

3 Translation

3.1 Type translation

$$\begin{array}{lll}
\mathcal{K}[\![\mathbb{Z}]\!] & \triangleq & \mathbb{Z} \\
\mathcal{K}[\![\tau_1 \to \tau_2]\!] & \triangleq & \mathcal{K}[\![\tau_1]\!] \to \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \to \mathtt{void} \\
\mathcal{K}_{\mathtt{cont}}[\![\tau]\!] & \triangleq & \mathcal{K}[\![\tau]\!] \to \mathtt{void}
\end{array}$$

3.2 Program translation

$$
\begin{aligned}
&\mathcal{K}_{\mathsf{prog}}[\![u^\tau]\!] && \triangleq && \mathcal{K}_{\mathsf{exp}}[\![u^\tau]\!](\lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau]\!]}\\
&\mathcal{K}_{\mathsf{exp}}[\![y^\tau]\!]k && \triangleq && k(y^{\mathcal{K}[\![\tau]\!]})\\
&\mathcal{K}_{\mathsf{exp}}[\![i^\tau]\!]k && \triangleq && k(i^{\mathcal{K}[\![\tau]\!]})\\
&\mathcal{K}_{\mathsf{exp}}[\![(\lambda x_1 : \tau_1.u_2^{\tau_2})^\tau]\!]k && \triangleq && k((\lambda x : \mathcal{K}[\![\tau_1]\!].\lambda c : \mathcal{K}_{\mathsf{cont}}[\![\tau_2]\!].\\
& && && \quad \mathcal{K}_{\mathsf{exp}}[\![u_2^{t_2}]\!]c^{\mathcal{K}_{\mathsf{cont}}[\![\tau_2]\!]})^{\mathcal{K}[\![\tau]\!]})\\
&\mathcal{K}_{\mathsf{exp}}[\![(u_1^{\tau_1}u_2^{\tau_2})^\tau]\!]k && \triangleq && \mathcal{K}_{\mathsf{exp}}[\![u_1^{\tau_1}]\!](\lambda x_1 : \mathcal{K}[\![\tau_1]\!].\\
& && && \quad \mathcal{K}_{\mathsf{exp}}[\![u_2^{\tau_2}]\!](\lambda x_2 : \mathcal{K}[\![\tau_2]\!].\\
& && && \quad\quad x_1^{\mathcal{K}[\![\tau_1]\!]}x_2^{\mathcal{K}[\![\tau_2]\!]}k)^{\mathcal{K}_{\mathsf{cont}}[\![\tau_2]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau_1]\!]}\\
&\mathcal{K}_{\mathsf{exp}}[\![(e_1\ p\ e_2)^\tau]\!]k && \triangleq && \mathcal{K}_{\mathsf{exp}}[\![e_1]\!](\lambda x_1 : \mathbb{Z}.\\
& && && \quad \mathcal{K}_{\mathsf{exp}}[\![e_2]\!](\lambda x_2 : \mathbb{Z}.\\
& && && \quad\quad \mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]}\\
&\mathcal{K}_{\mathsf{exp}}[\![\mathsf{if0}(e_1, e_2, e_3)^\tau]\!]k && \triangleq && \mathcal{K}_{\mathsf{exp}}[\![e_1]\!](\lambda x : \mathbb{Z}.\\
& && && \quad \mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![e_2]\!]k, \mathcal{K}_{\mathsf{exp}}[\![e_3]\!]k))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]}
\end{aligned}
$$

4 Type correctness

An important property of the translation is that it translates well-formed λ^{T} expressions to well-formed λ^{K} expressions. This is a fundamental requirement of the semantic correctness of the translation.

4.1 Terms

Lemma 1. $\Gamma \vdash_{\mathrm{T}} e : \tau \implies \mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \implies \mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![e]\!]k : \mathtt{void}$

Proof. T_ANT_ANN is the only typing derivation of the hypothesis. Therefore,

$$e = u^\tau \tag{1}$$

$$\Gamma \vdash_{\mathrm{T}} u^\tau : \tau \tag{2}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \implies \mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u^\tau]\!]k : \mathtt{void} \tag{3}$$

By induction on a derivation of typing judgement (2):

1. If the last rule in the derivation is T_TERM_VAR, then

$$u = x \tag{4}$$

$$\Gamma(x) = \tau \tag{5}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}[\![\tau]\!] \to \mathtt{void} \tag{6}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![x^\tau]\!]k : \mathtt{void} \tag{7}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k(x^{\mathcal{K}[\![\tau]\!]}) : \mathtt{void} \tag{8}$$

By K_TERM_APP and typing judgement (6), this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} x^{\mathcal{K}[\![\tau]\!]} : \mathcal{K}[\![\tau]\!] \tag{9}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} x : \mathcal{K}[\![\tau]\!] \tag{10}$$

By K_TERM_VAR, this follows from

$$\mathcal{K} \circ \Gamma(x) = \mathcal{K}[\![\tau]\!] \tag{11}$$

which is immediate from typing judgement (5).

2. If the last rule in the derivation is T_TERM_INT

   We have

$$u = i \tag{12}$$

$$\tau = \mathbb{Z} \tag{13}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathbb{Z} \to \mathtt{void} \tag{14}$$

   We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathtt{exp}}[\![i^{\mathbb{Z}}]\!]k : \mathtt{void} \tag{15}$$

   i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k(i^{\mathcal{K}[\![\mathbb{Z}]\!]}) : \mathtt{void} \tag{16}$$

   By K_TERM_APP and typing judgement (14), this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} i^{\mathcal{K}[\![\mathbb{Z}]\!]} : \mathbb{Z} \tag{17}$$

   By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} i : \mathbb{Z} \tag{18}$$

   which is immediate from K_TERM_INT.

3. If the last rule in the derivation is T_TERM_LAM, then

$$u = \lambda x : \tau_1 . e_2 \tag{19}$$

$$\tau = \tau_1 \to \tau_2 \tag{20}$$

$$\Gamma, x : \tau_1 \vdash_{\mathrm{T}} e_2 : \tau_2 \tag{21}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}[\![\tau_1 \to \tau_2]\!] \to \mathtt{void} \tag{22}$$

   T_ANT_ANN is the only derivation of typing judgement (21). Therefore,

$$e_2 = u_2^{\tau_2} \tag{23}$$

$$\Gamma, x : \tau_1 \vdash_{\mathrm{T}} u_2 : \tau_2 \tag{24}$$

   We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathtt{exp}}[\![\lambda x : \tau_1 . u_2^{\tau_2}]\!]k : \mathtt{void} \tag{25}$$

   i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k((\lambda x : \mathcal{K}[\![\tau_1]\!] . \lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] . \mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!] c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]})^{\mathcal{K}[\![\tau_1 \to \tau_2]\!]}) : \mathtt{void} \tag{26}$$

   By K_TERM_APP and typing judgement (22), this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x : \mathcal{K}[\![\tau_1]\!] . \lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] . \mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!] c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]})^{\mathcal{K}[\![\tau_1 \to \tau_2]\!]} : \mathcal{K}[\![\tau_1 \to \tau_2]\!] \tag{27}$$

   By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda x : \mathcal{K}[\![\tau_1]\!] . \lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] . \mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!] c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]} : \mathcal{K}[\![\tau_1]\!] \to \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \to \mathtt{void} \tag{28}$$

   By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x : \mathcal{K}[\![\tau_1]\!] \vdash_{\mathrm{K}} \lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] . \mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!] c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]} : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \to \mathtt{void} \tag{29}$$

   which follows

$$\mathcal{K} \circ \Gamma, x : \mathcal{K}[\![\tau_1]\!], c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \vdash_{\mathrm{K}} \mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!] c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]} : \mathtt{void} \tag{30}$$

   which is immediate from K_ANT_ANN, K_TERM_VAR and the induction hypothesis.

4. If the last rule in the derivation is T_TERM_APP, then

$$u = e_1 e_2 \tag{31}$$

$$\Gamma \vdash_{\mathrm{T}} e_1 : \tau_1 \rightarrow \tau \tag{32}$$

$$\Gamma \vdash_{\mathrm{T}} e_2 : \tau_1 \tag{33}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\texttt{cont}}[\![\tau]\!] \tag{34}$$

T_ANT_ANN is the only derivation of typing judgements (32)(33). Therefore,

$$e_1 = u_1^{\tau_1 \rightarrow \tau} \tag{35}$$

$$e_2 = u_2^{\tau_1} \tag{36}$$

$$\Gamma \vdash_{\mathrm{T}} u_1 : \tau_1 \rightarrow \tau \tag{37}$$

$$\Gamma \vdash_{\mathrm{T}} u_2 : \tau_1 \tag{38}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\texttt{exp}}[\![(u_1^{\tau_1 \rightarrow \tau} u_2^{\tau_1})^{\tau}]\!] k : \texttt{void} \tag{39}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\texttt{exp}}[\![{u_1}^{\tau_1 \rightarrow \tau}]\!](\lambda x_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!]. \mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. \\ x_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]})^{\mathcal{K}_{\texttt{cont}}[\![\tau_1 \rightarrow \tau]\!]} : \texttt{void} \tag{40}$$

By the induction hypothesis, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!]. \mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. \\ x_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]})^{\mathcal{K}_{\texttt{cont}}[\![\tau_1 \rightarrow \tau]\!]} : \mathcal{K}_{\texttt{cont}}[\![\tau_1 \rightarrow \tau]\!] \tag{41}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda u_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!]. \mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. u_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]} : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!] \rightarrow \texttt{void} \tag{42}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!] \vdash_{\mathrm{K}} \mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. x_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]} : \texttt{void} \tag{43}$$

By the induction hypothesis, We must show

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!] \vdash_{\mathrm{K}} (\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. x_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]} : \mathcal{K}_{\texttt{cont}}[\![\tau_1]\!] \tag{44}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!] \vdash_{\mathrm{K}} \lambda x_2 : \mathcal{K}[\![\tau_1]\!]. x_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k : \mathcal{K}[\![\tau_1]\!] \rightarrow \texttt{void} \tag{45}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \rightarrow \tau]\!], x_2 : \mathcal{K}[\![\tau_1]\!] \vdash_{\mathrm{K}} x_1^{\mathcal{K}[\![\tau_1 \rightarrow \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k : \texttt{void} \tag{46}$$

which is immediate from K_TERM_APP, K_ANT_ANN, K_TERM_VAR and typing judgement (34).

5. If the last rule in the derivation is T_TERM_PRIM, then

$$u = e_1 \ p \ e_2 \tag{47}$$

$$\tau = \mathbb{Z} \tag{48}$$

$$\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \tag{49}$$

$$\Gamma \vdash_{\mathrm{T}} e_2 : \mathbb{Z} \tag{50}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathbb{Z} \rightarrow \texttt{void} \tag{51}$$

T_ANT_ANN is the only derivation of typing judgement (49)(50). Therefore,

$$e_1 = u_1^{\mathbb{Z}} \tag{52}$$

$$e_2 = u_2^{\mathbb{Z}} \tag{53}$$

$$\Gamma \vdash_{\mathrm{T}} u_1 : \mathbb{Z} \tag{54}$$

$$\Gamma \vdash_{\mathrm{T}} u_2 : \mathbb{Z} \tag{55}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![(u_1^{\mathbb{Z}}\ p\ u_2^{\mathbb{Z}})^{\mathbb{Z}}]\!]k : \mathtt{void} \tag{56}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_1^{\mathbb{Z}}]\!](\lambda x_1 : \mathbb{Z}.\mathcal{K}_{\mathsf{exp}}[\![u_2^{\mathbb{Z}}]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathtt{void} \tag{57}$$

By the induction hypothesis, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x_1 : \mathbb{Z}.\mathcal{K}_{\mathsf{exp}}[\![u_2]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!] \tag{58}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda x_1 : \mathbb{Z}.\mathcal{K}_{\mathsf{exp}}[\![u_2]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathbb{Z} \to \mathtt{void} \tag{59}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_2]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathtt{void} \tag{60}$$

By the induction hypothesis, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_{\mathrm{K}} (\lambda u_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ u_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!] \tag{61}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_{\mathrm{K}} \lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}) : \mathbb{Z} \to \mathtt{void} \tag{62}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z} \vdash_{\mathrm{K}} \mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}) : \mathtt{void} \tag{63}$$

By K_ANT_LET, K_ANT_PRIM and K_ANT_VAR, We must show

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z}, y : \mathbb{Z} \vdash_{\mathrm{K}} k(y^{\mathbb{Z}}) : \mathtt{void} \tag{64}$$

By K_TERM_APP and typing judgement (51), this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z}, y : \mathbb{Z} \vdash_{\mathrm{K}} y^{\mathbb{Z}} : \mathbb{Z} \tag{65}$$

which is immediate from K_ANT_ANN and K_ANT_VAR.

6. If the last rule in the derivation is T_TERM_IF0, then

$$u = \mathsf{if0}(e_1, e_2, e_3) \tag{66}$$

$$\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \tag{67}$$

$$\Gamma \vdash_{\mathrm{T}} e_2 : \tau \tag{68}$$

$$\Gamma \vdash_{\mathrm{T}} e_3 : \tau \tag{69}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \tag{70}$$

T_ANT_ANN is only constructor of typing judgements (67)(68)(69). Therefore,

$$e_1 = u_1^{\mathbb{Z}} \tag{71}$$

$$e_2 = u_2^{\tau} \tag{72}$$

$$e_3 = u_3^\tau \tag{73}$$

$$\Gamma \vdash_{\mathrm{T}} u_1 : \mathbb{Z} \tag{74}$$

$$\Gamma \vdash_{\mathrm{T}} u_2 : \tau \tag{75}$$

$$\Gamma \vdash_{\mathrm{T}} u_3 : \tau \tag{76}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![\mathsf{if0}(u_1^{\mathbb{Z}}, u_2^\tau, u_3^\tau)]\!]k : \mathtt{void} \tag{77}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_1^{\mathbb{Z}}]\!](\lambda x : \mathbb{Z}.\mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathtt{void} \tag{78}$$

By the induction hypothesis, We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x : \mathbb{Z}.\mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!] \tag{79}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda x : \mathbb{Z}.\mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k) : \mathbb{Z} \to \mathtt{void} \tag{80}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} \mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k) : \mathtt{void} \tag{81}$$

By K_TERM_IF0, this follows from

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} x^{\mathbb{Z}} : \mathbb{Z} \tag{82}$$

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k : \mathtt{void} \tag{83}$$

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k : \mathtt{void} \tag{84}$$

in which (82) is immediate from K_ANT_ANN and K_TERM_VAR. (83)(84) come from the induction hypotheses, K_ANT_ANN and typing judgements (70)(75)(76).

□

4.2 Programs

Theorem 1. $\vdash_{\mathrm{T}} e : \tau \implies \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{prog}}[\![e]\!] : \mathtt{void}$

Proof. T_ANT_ANN is the only typing derivation for the hypothesis. Therefore,

$$e = u^\tau \tag{85}$$

$$\vdash_{\mathrm{T}} u^\tau : \tau \tag{86}$$

We must show

$$\vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{prog}}[\![u^\tau]\!] : \mathtt{void} \tag{87}$$

i.e.

$$\vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u^\tau]\!](\lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau]\!]} : \mathtt{void} \tag{88}$$

By Lemma 1, this follows from

$$\vdash_{\mathrm{K}} (\lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \tag{89}$$

By K_ANT_ANN, this follows from

$$\vdash_{\mathrm{K}} \lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]} : \mathcal{K}[\![\tau]\!] \to \mathtt{void} \tag{90}$$

By K_TERM_LAM, this follows from

$$x : \mathcal{K}[\![\tau]\!] \vdash_{\mathrm{K}} \mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]} : \mathtt{void} \tag{91}$$

By K_TERM_HALT, this follows from

$$x : \mathcal{K}[\![\tau]\!] \vdash_{\mathrm{K}} x^{\mathcal{K}[\![\tau]\!]} : \mathcal{K}[\![\tau]\!] \tag{92}$$

which is immediate from K_ANT_ANN and K_TERM_VAR. □

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