"COMBINING HORIZONTAL AND VERTICAL DIFFERENTIATION: THE PRINCIPLE OF MAX-MIN DIFFERENTIATION"

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Abstract

A duopoly model encompassing both horizontal and vertical differentiation is analysed. We seek for sub-game perfect Nash equilibria in which firms choose, first, horizontal (variety) and vertical (quality) characteristics of their products and, then, compete in prices. It is shown that, whatever the products' characteristics, a price equilibrium exists. In the product selection stage, firms choose maximum differentiation along one characteristics and minimum differentiation along the other dimension. More specifically, when the quality range is broad enough relative to the variety range, firms choose the same variety but maximise differentiation in terms of quality. Otherwise, they both choose the maximum quality but maximise their difference in terms of variety.

1. Introduction

We combine horizontal and vertical product differentiation in a unified framework, in which we analyse price and product competition.

In the Lancasterian tradition, two prevalent models of product differentiation can be distinguished, namely horizontal and vertical differentiation. We say that two products are horizontally differentiated when there is no ranking among consumers based on their willingness-to-pay for the two products. By contrast, two products are said to be vertically differentiated when there exists such a ranking of consumers. Prototypes of the former class of models include Hotelling (1929) and his followers (see, e.g. d'Aspremont et al. (1979), Eaton and Wooders (1985), Neven (1985), Salop (1979)). The latter class was developed more recently by Gabszewicz and Thisse (1979) and Mussa and Rosen (1978) (see also Gabszewicz et al. (1986), Shaked and Sutton (1983)). These various contributions indicate that the nature of competition differs along several lines depending on the type of differentiated products.

Horizontal differentiation is associated with the existence of product varieties, while vertical differentiation occurs when products differ according to quality. Our purpose is to combine these two types of model. Indeed, casual observation suggests that most products cannot sorted out into either of these polar cases. Rather, they involve variety as well as quality differences. For example, consumer durables are typically sold under different designs, each of them offered with various degrees of reliability. The analysis of product and price competition when both dimensions matter is therefore warranted from an empirical prospective. As noticed above, such an inquiry is also justified from a theoretical standpoint, competition being affected by the prevailing type of product differentiation.

To the best of our knowledge, only two recent contributions have addressed this question. First, Ireland (1987) studies price competition in a model with two given products and two classes of consumers. According to which class they belong to, consumers have a different valuation of quality; inside each class, consumers are modelled along the lines of Gabszewicz and Thisse (1979), i.e. they are ranked according to their income. Second, Ginsburgh et al. (1987) consider a model in which given varieties can be produced under a continuum of qualities. Hence, they allow for price as well as quality competition. Our model differs in that firms compete in price, variety and quality. Specifically, we couple a vertically differentiated characteristics à la Mussa and Rosen with a horizontally differentiated characteristics à la Hotelling. Our model thus encompasses these two popular paradigms of product differentiation. However, it goes beyond a simple addtition of the two models : the interplay between quality and variety leads to results which are qualitatively different from those encountered in those two one-dimensional models[1].

Following a well established tradition, we formulate firms' choices as a two-stage game. In the first stage, both firms choose simultaneously the variety and quality characteristics of their product. In the second stage, they compete in prices, products being given. In other words, we have chosen to model the decisions on product characteristics as a simultaneous process because production will often require a joint specification of these characteristics.

1. Notice that our model is fundamentally different from the two-dimensional extension of Hotelling's model, proposed by Economides (1986). It is also different from Caplin and Nalebuff (1986) who consider a model with two horizontally differentiated characteristics.

Our results include the following. First, we show that a price equilibrium exists in duopoly for any product configuration. Second, two types of equilibrium obtain at the product selection stage : on the one hand, when the quality range is sufficiently large relative to the variety range, then both firms will select in equilibrium a pair of products which are maximally differentiated along the vertical characteristics and minimally differentiated along the horizontal characteristics. On the other hand, if the variety range is wide enough compared to the quality range, firms will choose in equilibrium products which are minimally differentiated along the vertical characteristics and maximally differentiated along the horizontal characteristics. Interestingly, there is an overlap in the ranges of parameters for which these equilibria occur.

The paper is organised as follows. We present the model in section 2. As usual, we proceed by backwards induction. Hence, section 3 deals with the price competition stage of the game. In section 4, we then analyse firms' product selection, contingent on the equilibrium prices. Some concluding remarks are provided in section 5.

2. The model

Assume there is an industry in which products can be defined along two dimensions. On the one hand, each product will be of some **"variety"**, y. There is a range of potential varieties which is represented by the [0,1] interval (the unit interval is selected, without loss of generality, by an adequate choice of the length unit). Consumers do not rank the product varieties in the same way and accordingly, this first dimension corresponds to some horizontal differentiation. On the other hand, each product will be of some **"quality"**, q. The range of potential qualities is represented by the interval $[\underline{q}, \bar{q}]$ and all consumers agree that a high quality is always preferable to a low quality. This second dimension then corresponds to some vertical differentiation. On the whole, each product i is thus characterised by its variety, y_i, and its quality, q_i, with $(y_i, q_i) \varepsilon [0,1]x[\underline{q},\bar{q}]$. In the present model, both horizontal and vertical differentiation are then present and orthogonal to each other.

Consumer preferences also vary along two dimensions. First, each consumer has a "most preferred" variety, say x, with $x \varepsilon [0,1]$. Second, each consumer is characterised by its valuation of quality, say θ, with $\theta \varepsilon [0,1]$ (the unit interval can be chosen without loss of generality by an adequate choice of the quality unit). We assume that a consumer with characteristics (x,θ) will derive the following utility (U) from buying one unit of product i :

$$U(y_i,q_i;x,\theta) = R + \theta q_i - (x - y_i)^2 - P_i \qquad (1)$$

in which P_i denotes the price of product i and R is a positive constant. We suppose that consumers will buy at most one unit of one differentiated product. They will select the product for which utility (1) is highest. Throughout, we shall also assume that R is large, so that all consumers will always find some product for which their utility is positive and hence, will never abstain from buying. Finally, consumers as represented by the parameters (x,θ) are supposed to be uniformly distributed over the unit square, with a total mass equal to one. These assumptions enable us to describe a product's aggregate demand as a subset of $[0.1]^2$.

We shall consider two single product firms (i = 1, 2), operating with constant (zero) marginal cost of production. Without loss of generality, we label the firm with the highest quality as firm 2, i.e. $q_2 \geq q_1$, and assume that $y_2 \geq y_1$. The opposite situation in which $y_1 \geq y_2$ can be dealt with in a symmetric way, by rotating the parameter space $[0,1]x[\underline{q},\bar{q}]$ around the axis $y = 0$ and making an appropriate change of variables.

Given (1), we can derive the set of consumers who are just indifferent between products 1 and 2. For any consumer type $x \in [0,1]$, the marginal consumer in terms of θ who is indifferent between the two products is written (by (1)) as :

$$\bar{\theta}(x) = \frac{(P_2 - P_1) + (y_2^2 - y_1^2) - 2\,(y_2 - y_1)\,x}{(q_2 - q_1)} \qquad (2)$$

For any $x \in [0,1]$, the consumers in the interval $[0,\bar{\theta}(x)]$ obtain a higher utility from, and hence purchase, product 1 while those for whom $\theta \in\ ]\bar{\theta}(x),1]$ prefer to buy product 2. We also observe that $\bar{\theta}(x)$ is a nonincreasing function of x. This is to say that consumers with a high valuation of quality will be attracted more easily by the (high quality) product 2 and the more so, the less attractive is product 1 (as compared to product 2) in terms of variety. An increase in P_1 (decrease in P_2) will have the effect of shifting up $\bar{\theta}(x)$, thereby reducing demand for product 1.

It is also worth noticing that the present model encompasses two popular paradigms of product differentiation : (i) By setting $y_1 = y_2$, $\bar{\theta}(x)$ becomes a horizontal segment and the model reduces to a duopoly à la Mussa and Rosen (as analysed by Gabszewicz and Thisse (1988)). (ii) If $q_1 = q_2$, $\bar{\theta}(x)$ is a vertical segment and our formulation boils down to Hotelling's model with quadratic transportation costs.

Let us now derive the aggregate demand for product 1, say D_1. We shall distinguish between three different segments in D_1, i.e. D_1^I, D_1^{II} and D_1^{III} respectively.

For any fixed $\bar{P}_2$, define P'_1 as the lowest price for which firm 1 meets no demand, i.e. such that $\bar{\theta}(x=0) = 0$. From (2) :

$$P'_1 = \bar{P}_2 + (y_2^2 - y_1^2) \tag{3}$$

For any price below P'_1, firm 1 will have a positive demand. Denote $\tilde{x}$ as the intersection point between $\bar{\theta}(x)$ and the lower side of the unit square, i.e. $\tilde{x}$ is the solution of $\bar{\theta}(x) = 0$. It follows immediately from (2) that

$$\tilde{x} = \frac{(\bar{P}_2 - P_1)}{2(y_2 - y_1)} + \frac{(y_2 - y_1)}{2}$$

For $P_1 < P'_1$ and as long as $\bar{\theta}(x) < 1$, i.e. as long as $\bar{\theta}(x)$ does not reach a corner of the unit square, the demand function can be written :

$$D_1 = D_1^I = \int_0^{\tilde{x}} \bar{\theta}(x)\, dx$$

$$= \frac{\left[(\bar{P}_2 - P_1) + (y_2^2 - {}^2y_1) \right]^2}{4(y_2 - y_1)(q_2 - q_1)} \tag{4}$$

Depending on the absolute value of its slope, $\bar{\theta}(x)$ will reach the upper left hand corner of the unit square before or after the bottom right hand corner, as P_1 is reduced. Hence, we have to distinguish between the following cases[2]: (i) If $\left| \frac{\partial \bar{\theta}}{\partial x} \right| < 1$ or equivalently $2(y_2 - y_1) < (q_2 - q_1)$, $\bar{\theta}(x)$ passes first through $x = 1$, $\theta = 0$, i.e. the bottom right hand corner. This situation is characterised by the fact that the quality difference **dominates** the difference in variety. It will be referred to as the case of **vertical dominance**.

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2. The terminology used in what follows is borrowed (i.e. stolen) from Ireland (1987). It encapsulates the same intuition but has a different formal content.

(ii) If $\left| \frac{\partial \bar{\theta}}{\partial x} \right| > 1$ or equivalently $(q_2 - q_1) < 2(y_2 - y_1)$, $\bar{\theta}(x)$ passes first through x=0, θ=1, i.e. the upper left hand corner of the unit square. This situation will be referred to as a case of **horizontal dominance.**

(iii) If $2(y_2 - y_1) = (q_2 - q_1)$, $\bar{\theta}(x)$ is parallel to the negative diagonal of the unit square and passes through both corners at the same time. In such case, neither horizontal nor vertical differentiation dominates.

We consider these cases in turn.

(i) **Vertical dominance** : denote P_1'' as the price for which $\bar{\theta}(x)$ passes through x = 1 θ = 0, i.e.,

$$P_1'' = \bar{P}_2 + (y_2^2 - y_1^2) - 2\,(y_2 - y_1) \tag{8}$$

Clearly, for $P_1 \in [P_1'', P_1']$, demand is defined by D_1^I. The corresponding region of parameters will be denoted R_I. For $P_1 < P_1''$, demand is given by :

$$D_1^{II} = \int_0^1 \bar{\theta}(x)\, dx$$

$$= \frac{(\bar{P}_2 - P_1) + (y_2^2 - y_1^2)}{(q_2 - q_1)} - \frac{(y_2 - y_1)}{(q_2 - q_1)} \tag{9}$$

This formulation of demand will be valid as long as $\bar{\theta}(x)$ does not go through x = 0, θ = 1. This will occur for some price P_1''' such that :

$$P_1''' = \bar{P}_2 + (y_2^2 - y_1^2) - (q_2 - q_1) \tag{10}$$

otice that, given (5), we have that $P_1''' < P_1''$. Hence, for $P_1 \in [P_1''', P_1'']$ D_1 is defined by (9) and the corresponding parameter region is denoted R_{II}'.

For $P_1 < P_1'''$, $\bar{\theta}(x)$ intersects both the upper and right hand sides of the unit square. Denote $\hat{x}$ as the intersection point of $\bar{\theta}(x)$ with the upper side of the unit square, i.e., $\hat{x}$ solves $\bar{\theta}(x) = 1$:

$$\hat{x} = \frac{(\bar{P}_2 - P_1) + (y_2^2 - y_1^2) - (q_2 - q_1)}{2\,(y_2 - y_1)}$$

The demand faced by firm 1 is then given by :

$$D_1^{III} = \hat{x} + \int_{\hat{x}}^{1} \bar{\theta}(x) \quad dx \tag{11}$$

$$= \left[\, 4(y_2 - y_1)\,(q_2 - q_1)\,\right]^{-1} \left[\, 2(q_2 - q_1)\left((\bar{P}_2 - P_1) + (y_2^2 - y_1^2) \right) - (q_2 - q_1)^2 - \left((\bar{P}_2 - P_1) + (y_2^2 - y_1^2) - 2(y_2 - y_1) \right)^2 \,\right]$$

This demand formulation will be valid as long as $\bar{\theta}(1) < 1$, i.e. before $\bar{\theta}(x)$ reaches the upper left hand corner. This will occur for some price P_1'''' such that :

$$P_1'''' = \bar{P}_2 + (y_2^2 - y_1^2) - (q_2 - q_1) - 2(y_2 - y_1) \tag{12}$$

Hence, for $P_1 \in [P_1'''', P_1''']$, $D_1 = D_1^{III}$ and the corresponding parameter region is R_{III}. Finally, for $P_1 < P_1''''$, firm 1's demand is equal to one.

So far, we have thus characterised demand for the case of vertical dominance, having identified three segments in the demand function. It is also easy to check that at each kink (P_1'' and P_1''') demand is continuous. A typical example of D_1 is depicted in figure 1.

<Insert figure 1>

We now turn to the case of horizontal dominance.

(ii) **Horizontal dominance** : we now assume that (6) holds. With respect to the first segment of the demand function, the expression given above (4) does also apply to the present case. However, the lowest price $\hat{P}_1''$ for which D_1^I is valid, is no longer given by (8). Indeed, when (6) holds, $\bar{\theta}(x)$ reaches the upper right hand corner of the unit square before the bottom left hand corner. Hence, the relevant expression for P_1'' is now given by the right hand side of (10). For $P_1 < \hat{P}_1''$, D_1^{II} is then written :

$$D_1^{II} = \bar{x} + \int_{\bar{x}}^{\bar{\bar{x}}} \bar{\theta}(x) \ dx$$

where $\bar{x}$ (resp. $\bar{\bar{x}}$) is the intersection point of $\bar{\theta}(x)$ with the upper (resp. lower) side of the unit square. Some simple calculations show that $\bar{x} = \hat{x}$ and $\bar{\bar{x}} = \tilde{x}$, so that :

$$D_1^{II} = \frac{(\bar{P}_2 - P_1) + (y_2^2 - y_1^2)}{2\,(y_2 - y_1)} - \frac{(q_2 - q_1)}{4(y_2 - y_1)} \qquad (13)$$

when horizontal dominance prevails ((6) rather than (5) holds). The corresponding parameter region is denoted R''_{II}.

With respect to the third segment of demand, the expression given above for D_1^{III} still applies. What is modified, however, is the interval of prices for which this expression holds. $\hat{P}_1'''$ is not defined by (10) anymore but by the right hand side of (8). Of course, the fact that in horizontal dominance, $\hat{P}_1'''$ (resp. $\hat{P}_1''$) is defined by the same equality as P_1'' (resp. P_1''') in vertical dominance, stems from the reversal of the inequality (5) which defines the two regimes.

(iii) Finally, when $2(y_2 - y_1) = (q_2 - q_1)$ holds, there is no intermediate region (II) in the demand function. The first and third segments directly connect.

To sum up, the demand function for firm 1 is in general composed of three segments. The expressions for demand in the first and third segments are independent of whether horizontal or vertical dominance prevails. The parameter regions for which these expressions apply will however be a function of the type of dominance. With respect to the intermediate segment (II), the expression of demand (and the parameter region) is affected by the type of dominance.

The aggregate demand for product 2, D_2, can be directly computed because, as assumed above, consumers never abstain from buying. Hence, $D_2 = 1 - D_1$. The various segments of D_2 can be derived along the lines of the analysis performed above for D_1, leading to a partition of the price segment $[P_2', P_2'''']$ akin to (3)-(8)-(10)-(12), in which P_1 and P_2 are permuted. The shape of D_2 is identical to that of D_1, mutatis mutandis.

We can now turn to the analysis of the second stage price equilibrium.

3. Price equilibrium

For each pair of products, the profit function of firm i (i = 1, 2) is defined as $\Pi_i(P_i, P_j) = P_i \; D_i(P_i, P_j)$ for $j \neq i$.

A noncooperative <u>price equilibrium</u> is a pair of prices (P_i^*, P_j^*) such that :

$$\Pi_i(P_i^*, P_j^*) \geq \Pi_i(P_i, P_j^*) \quad , \; \forall \; P_i \geq 0, \; i,j = 1, 2 \text{ and } i \neq j.$$

First, we shall prove that an equilibrium exists.

Proposition 1 : For any pair of products (y_1,q_1) and (y_2,q_2), there exists a price equilibrium in the corresponding subgame.

Proof : Consider the profit function Π_1 when vertical dominance prevails. On the interval $[P_1'',P_1''']$, the demand D_1 is concave and decreasing which implies that Π_1 is concave in P_1.

By contrast, D_1 is strictly convex on $[P_1',P_1'']$. However, it is straightforward to show that $\frac{\partial^3 \Pi_1}{\partial P_1^3} > 0$, so that $\frac{\partial \Pi_1}{\partial P_1}$ is strictly convex in P_1. Since the left hand side derivative of Π_1 is negative at P_1', $\frac{\partial \Pi_1}{\partial P_1} = 0$ has at most one solution in this region. Furthermore, at P_1'' (as given by (8)), it is easy to check that $\left.\frac{\partial \Pi_1}{\partial P_1}\right|_-$, using (4), and $\left.\frac{\partial \Pi_1}{\partial P_1}\right|_+$ are both equal to $\frac{3(y_2-y_1) + (y_2^2-y_1^2) - \bar{P}_2}{(q_2-q_1)}$. Accordingly, the sign of $\frac{\partial \Pi_1}{\partial P_1}$ is the same on both sides of P_1''.

Combining this result with the fact that Π_1 is concave on $[P_1'',P_1''']$ and zero outside of $[P_1',P_1''']$, we obtain that Π_1 has a unique maximum w.r.t. P_1 and therefore is quasi-concave in P_1.

The above argument can be repeated, mutatis mutandis, under horizontal dominance and for Π_2, under both types of dominance.

As the profit functions are continuous, it follows that an equilibrium exists. Q.E.D.

We now determine the equilibrium prices as a function of the product characteristics. This leads us to distinguish between six types of equilibria, namely three under vertical dominance and three under horizontal dominance. For each kind of dominance, we have the following types of equilibria : Case 1 The equilibrium occurs on the linear segments of both D_1 and D_2; Case 2 The equilibrium occurs on the strictly convex segment of D_1 and the strictly concave segment of D_2; Case 3 The equilibrium occurs on the strictly concave segment of D_1 and the strictly convex segment of D_2.

We shall analyse each case in turn and determine the region of parameters for which the equilibrium prices indeed fall on the corresponding segments. In each case, vertical and horizontal dominance are dealt with sequentially.

Case 1. (i) vertical dominance

We seek an equilibrium on the linear segment of D_1 and D_2. The appropriate expressions for demand are thus D_1^{II} as given by (9) and $D_2^{II} = 1 - D_1^{II}$. The first order conditions for maximisation of the corresponding profit functions have a single solution given by :

$$P_1^* = \frac{(q_2 - q_1) + (y_2^2 - y_1^2) - (y_2 - y_1)}{3} \qquad (14)$$

$$P_2^* = \frac{2(q_2 - q_1) - (y_2^2 - y_1^2) + (y_2 - y_1)}{3} \qquad (15)$$

As a price equilibrium exists and the FOC's are necessary, these prices are the equilibrium prices, provided that they belong to the intervals for which demand is indeed given by $D_1 = D_1^{II}$ and $D_2 = 1 - D_1^{II}$. This is :

$$P_1^* \in [P_1'''(P_2^*),\ P_1''(P_2^*)]$$

and

$$P_2^* \in [P_2'''(P_1^*),\ P_2''(P_1^*)]$$

First, $P_1^* \leq P_1''(P_2^*)$ (as given by (8)) is satisfied if and only if :

$$(y_2 - y_1)\left(4 - (y_1 + y_2)\right) \leq (q_2 - q_1) \qquad \text{(A)}$$

Second, $P_1^* \geq P_1'''(P_2^*)$ (as given by (10)) is satisfied if and only if :

$$(y_2 - y_1)\left(2 + (y_1 + y_2)\right) \leq 2\,(q_2 - q_1) \qquad \text{(B)}$$

Similarly, some simple computations show that $P_2^* \in [P_2'''(P_1^*), P_2''(P_1^*)]$ if and only if both conditions (A) and (B) are satisfied. As a result, (14)-(15) define the unique price equilibrium for the parameter region defined by conditions (A) and (B), i.e. region R'_{II}[3].

The heavy line in figure 2 illustrates the typical split of the market between the two firms, which will be encountered when the products' characteristics satisfy conditions (A) and (B).

<Insert figure 2>

(ii) **Horizontal dominance**

We now consider the linear segment of the demand functions, with the assumption that horizontal dominance prevails. Hence, D_1^{II} is now given by (13) and $D_2^{II} = 1 - D_1^{II}$. As before, solving the FOC's yields an unique pair of prices given by :

$$P_1^{**} = \frac{4(y_2 - y_1) + 2(y_2^2 - y_1^2) - (q_2 - q_1)}{6} \qquad (16)$$

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3. It is easy to check that there always exist admissible parameter values for which this region has a positive measure.

$$P_2^{**} = \frac{8(y_2 - y_1) - 2(y_2^2 - y_1^2) + (q_2 - q_1)}{6} \qquad (17)$$

Again, as an equilibrium exists and FOCs are necessary, these prices are equilibrium prices, provided they belong to the appropriate intervals, i.e.

$$P_1^{**} \in [\hat{P}_1'''(P_2^{**}), \hat{P}_1''(P_2^{**})]$$

and

$$P_2^{**} \in [\hat{P}_2'''(P_1^{**}), \hat{P}_2''(P_1^{**})]$$

The conditions $P_1^{**} \leq \hat{P}_1''(P_2^{**})$ and $P_1^{**} \geq \hat{P}_1'''(P_2^{**})$ are met respectively if and only if :

$$(y_2 - y_1)\Big(2 + (y_1 + y_2)\Big) \geq 2(q_2 - q_1) \qquad (C)$$

and

$$(y_2 - y_1)\Big(4 - (y_1 + y_2)\Big) \geq (q_2 - q_1) \qquad (D)$$

Similarly, the restrictions on P_2^{**} lead to the same conditions as (C) and (D). Hence, (16)-(17) provide the unique price equilibrium for the parameter region R''_{II}, defined by (C) and (D).

The heavy line in figure 3 represents a typical boundary between the firms' market when both conditions (C)-(D) are met.

<Insert figure 3 here>

Notice that condition (C) (resp. (D)) is the reverse of condition (B) (resp. (A)). Consequently, the intersection between R'_{II} and R''_{II} is a set of measure zero (see below).

(iii) Finally, when $2(y_2 - y_1) = (q_2 - q_1)$, so that neither horizontal, nor vertical dominance prevails, the four conditions (A)-(D) collapse into a single one. In this case, the equilibrium prices are such that

$$P_1^* = P_1^{**} = \frac{(y_2 - y_1) + (y_2^2 - y_1^2)}{3} \text{ and } P_2^* = P_2^{**} = \frac{5(y_2 - y_1) - (y_2^2 - y_1^2)}{3}.$$

Hence, the four conditions (A)-(D) are simultaneously satisfied if and only if $2(y_2 - y_1) = (q_2 - q_1)$ and the two price equilibria (in R'_{II} and R''_{II}) connect each other when this latter condition holds. This is the only situation for which the intersection between R'_{II} and R''_{II} is non empty but still has a zero measure.

Before turning to Case 2, we proceed with a few comments on the equilibrium prices just derived. First, notice that under vertical dominance, $P_2^* > P_1^*$. Hence, the high quality product is always sold at a premium. By contrast, under horizontal dominance it is not always true that $P_2^{**} > P_1^{**}$. Firm 1 might very well charge a higher price than firm 2, if it can secure a large market share by enjoying a better location than firm 2 along the horizontal characteristics. Last, the impact of changes in product characteristics are displayed in tables 1 and 2.

<Insert tables 1 and 2 here>

The following remarks are in order : first, we observe that equilibrium prices under vertical (resp. horizontal) dominance behave with respect to quality (resp. variety) parameters in the same way as they would in a model with vertical (horizontal) differentiation only (see Gabszewicz and Thisse (1988) (resp. Neven (1985))). Second, with respect to changes in the "dominated" characteristics, one finds non standard effects (which are underlined in the above tables) ; under vertical dominance, prices may increase as varieties get closer. Under horizontal dominance, the price of the low quality product always increases if the quality difference narrows down. These effects clearly show that our model is not simply the addition of two one-dimensional models ; some interactive effects come into play and generate unusual outcomes.

These unusual comparative statics results can be explained as follows. Consider, first, the case of vertical dominance (table 1). The more surprising result is $\frac{\partial P_1^*}{\partial y_1} > 0$, when $y_1 < \frac{1}{2}$. Inserting (14)-(15) in (2), we observe that in this case, the dividing line between firms' markets pivotes around the point $x^p = \frac{1 + y_1}{3}$ as y_1 increases. As $y_1 < \frac{1}{2}$, $x^p < \frac{1}{2}$ so that the demand D_1, evaluated at the equilibrium prices, increases (the loss on the LHS of x^p is more than compensated by the gain on the RHS). Hence, since $\frac{\partial D_1}{\partial P_1}$ is independent of y_1, in order to restore the FOC for firm 1, P_1^* must increase when $y_1 < \frac{1}{2}$ increases. On the other hand, when $y_1 > \frac{1}{2}$, the pivot x^p is larger than $\frac{1}{2}$ and an increase in y_1 leads to a deacrease in D_1 and hence, a fall in price for firm 1 to restore the FOC. When $y_1 < \frac{1}{2}$, the increase in demand, which leads to an increase in price as y_1 increases, can be said to arise because the price competition effect is dominated by the advantage of being centrally located. This is in contrast with pure horizontal models where the former effect always dominates. The price competition effect is here less stringent because firms are already differentiated along the vertical dimension. The advantage of being centrally located is then stronger. (The above argument applies, mutatis mutandis, to $\frac{\partial P_2^*}{\partial y_2}$).

Let us discuss the case of horizontal dominance (table 2). Here, the most striking result is $\frac{\partial P_1^{**}}{\partial q_1} > 0$. In a pure vertical model, competition would lead to a fall in P_1^{**} as q_1 increases, compensating and over the increase in price that one normally expects when quality increases. In the present case. the competitive effect is weakened because products are already differentiated along the horizontal characteristics. As a result, price increases with quality.

<u>Case 2.</u> **(i) vertical dominance**

In this case, the demand faced by firm 1 (on the strictly convex segment), D_1^I, is given by (4), while the demand for firm 2 is $D_2^I = 1 - D_2^I$. In terms of the parameter region, we have that (A) fails (i.e. $P_1^* > P_1''(P_2^*)$) and (B) holds.

Taking the FOC for firm 1 and solving yields :

$$P_1 = \frac{P_2 + (y_2^2 - y_1^2)}{3} \tag{18}$$

Inserting this expression into the FOC for firm 2 leads to the following quadratic expression (up to a positive factor) :

$$\frac{\partial \Pi_2}{\partial P_2} = - \frac{4}{9} P_2^2 - \frac{5}{9} (y_2^2 - y_1^2) P_2 + (y_2 - y_1) (q_2 - q_1) - \frac{1}{9} (y_2^2 - y_1^2) = 0 \tag{19}$$

The function $\frac{\partial \Pi_2}{\partial P_2}$ is a parabola which has one maximum. The relevant root of (19) is that one for which Π_2 is maximum, i.e. for which $\frac{\partial^2 \Pi_2}{\partial P_2^2} < 0$. This arises at the larger root :

$$\tilde{P}_2 = - \frac{5}{9} (y_2^2 - y_1^2) + \frac{3}{9} \left[(y_2^2 - y_1^2)^2 + 16 (y_2 - y_1) (q_2 - q_1) \right]^{1/2} \tag{20}$$

Using (18), one obtains :

$$\tilde{P}_1 = \frac{1}{9}(y_2^2 - y_1^2) + \frac{1}{8}\left[(y_2^2 - y_1^2)^2 + 16\,(y_2 - y_1)\,(q_2 - q_1)\right]^{1/2} \qquad (21)$$

As a price equilibrium exists, it follows that (20)-(21) is the unique pair of equilibrium prices, provided they fall in the appropriate region. That is :

$$\tilde{P}_1 \;\varepsilon\; [P_1''(\tilde{P}_2),\; P_1'(\tilde{P}_2)]$$

and

$$\tilde{P}_2 \;\varepsilon\; [P_2''''(\tilde{P}_1),\; P_2'''(\tilde{P}_1)]$$

It is straightforward to check that $\tilde{P}_1 < P_1'(\tilde{P}_2)$ and $\tilde{P}_2 > P_2''''(\tilde{P}_1)$ are always true. The other two inequalities are met if and only if condition (A) fails. Hence, (20)-(21) describe the equilibrium prices when (A) does not hold.

Notice that when condition (A) is met as an equality, $\tilde{P}_1 = P_1^*$ and $\tilde{P}_2 = P_2^*$, which means that equilibrium prices vary continuously when parameters change between R_I and R_{II}'.

(ii) **Horizontal dominance**

In this case, the functional form of the demands is not affected by the type of dominance, so that the demand expressions just used still apply. With respect to parameters, conditions (C) is not met (i.e. $P_1^{**} > \hat{P}_1(P_2^{**})$). The expressions of the equilibrium prices are still given by (20)-(21). However, the region of parameters for which these prices are valid is now different and corresponds to condition (C) (instead of (A)) being violated.

As before, $P_1^{**} = \tilde{P}_1$, $P_2^{**} = \tilde{P}_2$, when condition (C) is just met. Again, equilibrium prices change continuously when parameters move between R_I and R_{II}''.

Case 3 **Vertical (horizontal) dominance**

The demand faced by firm 1, D_1^{III} is now given by (11), with as usual, $D_2^{III} = 1 - D_1^{III}$. In terms of parameters, we have that condition (B) in case of vertical dominance, and condition (D) in case of horizontal dominance, are violated.

The FOC's yield a system of two quadratic equations (which is the same under both types of dominance) for which we have not determined a solution. However, as we shall see later, it will not preclude us to solve for the first stage of the game.

4. Product equilibrium

We now look at product choices, contingent on the equilibrium prices derived in the previous section. As mentioned in the introduction, we assume that each firm chooses the variety and quality of its product at the same time.

As a preliminary to the analysis of product equilibria, we describe how changes in each product characteristics affect the equilibrium prices. More specifically, we indicate what kind of price equilibria will be generated by changing y_i and/or q_i. This will enable us to construct the profit functions for the product selection game, by referring to the appropriate equilibrium prices.

At the outset, observe that any move from some $(y_i' \cdot q_i')$ to some (y_i'', q_i'') can be decomposed into two separate moves, namely $(y_i', q_i') \rightarrow (y_i', q_i'') \rightarrow (y_i'', q_i'')$ (or similarly, $(y_i', q_i') \rightarrow (y_i'', q_i') \rightarrow (y_i'', q_i'')$).

Let us first analyse the effect of changing q_i; starting from given values for y_1, y_2 and $q_2 = q_1$, we gradually increase q_2, up to $\bar{q}$. By so doing, we go through the following sequence :

1) When $q_2 = q_1$, horizontal dominance and both conditions (C) and (D) are met. In other words, this original parameter configuration lies in R''_{II} and the corresponding equilibrium prices are given by (16)-(17).

2) Increasing q_2, condition (C) is first violated (and, hence condition (B) is met), while horizontal dominance and condition (D) still hold. Accordingly, the parameter region is now R_I with horizontal dominance, so that the equilibrium prices are given by (20)-(21).

3) As q_2 is further increased, we reach the domain of vertical dominance. Condition (D) is still satisfied (so that (A) is violated). The parameter region is R_I with vertical dominance.

4) Ultimately, an increase in q_2 will lead condition (A) to become satisfied. As (B) is already met, the parameter region is now R'_{II} and the equilibrium prices are given by (14)-(15).

With respect to this sequence of equilibria, the following remarks are in order : first, as noticed above, the equilibrium prices change continuously as one moves across parameter regions. It follows that the profit functions evaluated at those prices are continuous in product characteristics. Second, the above sequence might be terminated at any step depending on the admissible increase in q_2 (given q_1, y_1, y_2 and $\bar{q}$). Third, the same path occurs if q_1 decreases from the initial configuration $q_1 = q_2$. This is so because all what matters in the above conditions with respect to quality, is the quality difference. Last, if $y_1 + y_2 = 2$ (so that both firms sell variety 1), we shift directly from R''_{II} to R'_{II}, without passing through R_I.

Let us now consider an increase in the variety y_2, up to 1, starting from given q_1, q_2 and $y_2 = y_1$:

1) For this original configuration, conditions (A)-(B) and vertical dominance hold. The region is R'_{II}.

2) Increasing y_2, condition (A) is first violated (and hence (D) is met), while vertical dominance and condition (B) are still valid. Accordingly, the parameter region is R_I with vertical dominance.

3) As y_2 is further increased, we enter the domain of horizontal dominance. Condition (B) is still satisfied, so that (C) is violated. Consequently, the parameter region is R_I with horizontal dominance.

4) Ultimately, an increase in y_2 will bring us in the region where (C) applies. The corresponding region is R''_{II}.

The four comments stated above, with respect to the sequence of equilibria following a quality change, remain valid, mutatis mutandis, if a variety change is considered.

We now turn to the product selection. Two types of product equilibria may emerge, described in propositions 2 and 3, respectively. For later reference, set $K_v \equiv \left(\frac{51}{32}\right)^2$.

Proposition 2 : **If $(\bar{q} - \underline{q}) \geq K_v$, there exists a product equilibrium given by $q_1^* = \underline{q}$, $q_2^* = \bar{q}$, $y_1^* = y_2^* = \frac{1}{2}$.**

__Proof__ : (i) First, assume that $q_2^* = \bar{q}$, $y_2^* = 1/2$ and show that $q_1^* = \underline{q}$, $y_1^* = 1/2$ is the best reply for firm 1, provided $(\bar{q} - \underline{q}) \geq K_v$.

- Consider region R'_{II} : firm 1's profit function evaluated at the corresponding price (P_1^*, P_2^*) is then given by :

$$\Pi_1^*(y_1, y_2, q_1, q_2) = \frac{\left[(q_2 - q_1) + (y_2^2 - y_1^2) - (y_2 - y_1) \right]^2}{9 (q_2 - q_1)} \tag{22}$$

The FOC implies that $y_1^* = 1/2$ (whatever y_2, q_1, q_2). Since $y_2^* = 1/2$, (22) reduces to $\frac{(q_2 - q_1)}{9}$, which is clearly maximised for $q_1 = \underline{q}$.

Thus, $q_1^* = \underline{q}$, $y_1^* = 1/2$ is a best reply in R'_{II} and the corresponding profit is : $\Pi_1^* = \frac{(\bar{q} - \underline{q})}{9}$ (23)

- By the lemma proved in the appendix, it turns out that the FOC of the profit function evaluated at $\tilde{P}_1$, $\tilde{P}_2$, with respect to q_1, can never be satisfied in the interior of R_I (under either horizontal or vertical dominance). Consequently, there is no best reply for firm 1 in this region (the possibility of a best reply on the boundaries is taken up by the analysis performed for R'_{II} and R''_{II} because the profit functions evaluated at the equilibrium prices are continuous in the characteristics variables).

- Consider R''_{II}. Firm 1's profit evaluated at (P_1^{**}, P_2^{**}) is given by :

$$\Pi_1^{**}(y_1, y_2, q_1, q_2) = \frac{\left[4(y_2 - y_1) + 2(y_2^2 - y_1^2) - (q_2 - q_1) \right]^2}{72 (y_2 - y_1)} \tag{24}$$

Given (24), it is clear that Π_1^{**} is maximised with respect to q_1, when $q_1 = q_2^* = \bar{q}$. Then, (24) reduces to :

$$\Pi_1^{**} = \frac{(y_2 - y_1)\left(y_1 + y_2 + 2\right)^2}{18} \tag{25}$$

in which we have used the fact that $(y_2 - y_1) > 0$, which holds in the domain of horizontal dominance. The first derivative of (25) w.r.t. y_1 leads the expression $(y_2 - 3y_1 - 2)$ which is always negative. Consequently, whatever y_2 (in particular, $y_2 = y_2^* = 1/2$), the best reply in terms of y_1 in R''_{II} is $y_1 = 0$.

Setting $y_1 = 0$, $y_2 = 1/2$ in (25) gives : $\Pi_1^{**} = \left(\frac{5}{12}\right)^2$ (26)

This is the maximum profit earned by firm 1 in R'_{II} (given $y_2^* = 1/2$, $q_2^* = \bar{q}$). Comparing (23) and (26), we see that $\Pi_1^* \geq \Pi_1^{**}$, if and only if

$(\bar{q} - \underline{q}) \geq \left(\frac{5}{4}\right)^2$, which is satisfied since $K_v > \left(\frac{5}{4}\right)^2$.

(ii) Suppose now that $q_1^* = \underline{q}$, $y_1^* = 1/2$ and show that $q_2^* = \bar{q}$, $y_2^* = 1/2$ is firm 2's best reply.

- In region R'_{II}, a reasoning similar to the one held above for firm 1 yields the maximum profit of firm 2 in this region :

$$\Pi_2^* = \frac{4}{9}\left(\bar{q} - \underline{q}\right)$$

- The lemma in the appendix is again used to show that there is no best reply in the interior of R_I for firm 2.

- In R''_{II}, the profit of firm 2, evaluated at (P_1^{**}, P_2^{**}) is written :

$$\Pi_2^{**}(y_1, y_2, q_1, q_2) = \frac{\left[8(y_2 - y_1) + 2(y_2^2 - y_1^2) + (q_2 - q_1)\right]^2}{72\,(y_2 - y_1)} \tag{27}$$

Taking the first order derivative of (27) w.r.t. y_2 yields an expression whose numerator (the denominator is always > 0) is equal to :

$$2(y_2 - y_1)\left[4(2 + y_2) - 4 - (y_1 + y_2)\right] - (q_2 - q_1) \qquad (28)$$

In order to stay in R'_{II} , condition (C) has to be satisfied. Accordingly, the upper bound on $(q_2 - q_1)$ is given by (C), met as an equality. Replacing $(q_2 - q_1)$ in (28) by this upper bound, and knowing that $(y_2 - y_1) > 0$ by the definition of horizontal domination, it is readily checked that (28) > 0, as long as (C) holds. Consequently, the best reply of firm 2 in terms of y_2 is $y_2 = 1$.

By inspection of (27), it is clear that the best reply of firm 2 in terms of q_2 is the maximum admissible value of q_2. Since condition (C) is the first one to be violated as q_2 increases, the maximum admissible value of q_2 is such that (C) is just met, i.e.

$$q_2 = \underline{q} + \frac{1}{2}(y_2 - y_1)\left(2 + (y_1 + y_2)\right)$$

Introducing $y_2 = 1$ and the above value of q_2 in (27) gives :

$$\Pi_2^{**} = \left(\frac{17}{16}\right)^2$$

Since $\Pi_2^* = \frac{4}{9}(\bar{q} - \underline{q})$, it follows that $\Pi_2^* \geq \Pi_2^{**}$, if and only if

$$\bar{q} - \underline{q} \geq \frac{9}{4}\left(\frac{17}{16}\right)^2$$

which holds by assumption. Consequently, $q_2^* = \bar{q}$ and $y_2^* = 1$ is the best reply of firm 2, given that $q_1^* = \underline{q}$ and $y_1 = 0$. Q.E.D.

Let us now define $K_h = 2 + \frac{\sqrt{63}}{8}$.

<u>Proposition 3</u> : If $\bar{q} - \underline{q} \geq K_h$, there exists a product equilibrium given by $q_1^{**} = q_2^{**} = \bar{q}$, $y_1^{**} = 0$, $y_2^{**} = 1$.

<u>Proof</u> : (i) First, assume that $q_2^{**} = \bar{q}$, $y_2^{**} = 1$ and show that $q_1^{**} = \bar{q}$, $y_1^{**} = 0$ is the best reply of firm 1, provided that $\bar{q} - \underline{q} \leq K_h$.

- Consider region R''_{II}. Firm 1's profit evaluated at the corresponding prices (P_1^{**}, P_2^{**}) is given by (24). It is immediate by inspection of this equation that firm 1's best reply in terms of q_1 is $q_1 = q_2^{**} = \bar{q}$. Setting q_1 to this value, (24) reduces to (25). Taking up the analysis performed above in terms of y_1 in R''_{II}, it follows that firm 1's best reply is $y_1 = 0$. Setting $y_1^{**} = 0$, $y_2^{**} = 1$ in (25) yields $\Pi_1^{**} = \frac{1}{2}$.

- The lemma proved in the appendix suffices to eliminate the possibility of best replies for firm 1 in the interior of R_I.

- Consider R'_{II}. Firm 1's profit is then given by (22). Assume first that $\bar{q} - \underline{q} \geq \frac{5}{4}$. Following the reasoning held above w.r.t. y_1, the best reply of firm 1 in this region is $y_1 = 1/2$. (Notice that when $y_1 = 1/2$, $y_2 = 1$, the restriction $\bar{q} - \underline{q} \geq \frac{5}{4}$ guarantees that condition (A) is satisfied.) Setting $y_1 = 1/2$, $y_2^{**} = 1$, $q_2^{**} = \bar{q}$ in (22), one obtains :

$$\Pi_1^* = \frac{\left[\frac{1}{4} + (\bar{q} - q_1)\right]}{9\,(\bar{q} - q_1)} \qquad (29)$$

The first derivative of (29) w.r.t. q_1 is :

$$\frac{\partial \Pi_1^*}{\partial q_1} = \frac{1}{9}\left[\frac{1}{16\,(\bar{q} - q_1)^2} - 1\right] \tag{30}$$

Given that $y_1 = 1/2$, $y_2^{**} = 1$, the restriction of vertical dominance implies that $\bar{q} - q_1 > 1$, for all admissible q_1. Hence, it follows that (30) is always negative, so that Π_1^* is decreasing in q_1. Accordingly, the best reply of firm 1, in terms of q_1 is $q_1 = \bar{q}$. Evaluating (29) at this value, one obtains :

$$\Pi_1^* = \frac{\left[\frac{1}{4} + (\bar{q} - \underline{q})\right]^2}{9(\bar{q} - \underline{q})} \tag{31}$$

Comparing this value with Π_1^{**}, it follows that $\Pi_1^* \leq \Pi_1^{**}$ if and only if

$\bar{q} - \underline{q} \leq K_h$.

Suppose now that $(\bar{q} - \underline{q}) < \frac{5}{4}$. The set of admissible characteristics values for (y_1, q_1) is then a proper subset of the set of admissible values considered above. As a result, the maximum profit that can be achieved is no greater than the profit level reached in (31).

(ii) Next, assume that $q_1^{**} = \bar{q}$, $y_1^{**} = 0$ and show that $q_2^{**} = \bar{q}$, $y_2^{**} = 1$ is firm 2's best reply.

- Consider R''_{II} : the profit function of firm 2 in this region can be expressed as firm 1's profit, i.e. (24), in which y_i is replaced by z_i, with $z_i = 1 - y_j$, $i = j$ (the decision variables of firm 2 are then z_1 and q_1) :

$$\Pi_2^{**}(z_1, q_1) = \frac{\left[8(z_2 - z_1) + 2(z_2^2 - z_1^2) - (q_2 - q_1)\right]^2}{72\,(z_2 - z_1)} \tag{32}$$

with $z_1 \leq z_2$, $q_1 \leq q_2$, $z_2 = 1$ and $q_2 = \bar{q}$. In view of (32), it is always profitable for firm 2 to set $q_1 = \bar{q}$. Setting $q_2 = \bar{q}$ in (32), simplifying by $(z_2 - z_1)$ and taking the first order derivative w.r.t. z_1 yield the expression $(z_2 - 3z_1 - 2)$, which is always negative. Hence, firm 2's best reply in terms of z_1 is $z_1 = 0$, or equivalently, $y_2 = 1$. As a best response to $q_1^{**} = \bar{q}$, $y_1^{**} = 0$, firm 2 will thus set $q_2^{**} = \bar{q}$, $y_2^{**} = 1$.

- The lemma in the appendix is again sufficient to rule out firm 2's best replies in the interior of R_I.

- Consider region R'_{II} : using the same change of variable as above in (22), one obtains that firm 2's profit is equal to :

$$\Pi_2^{**}(z_1,q_1) = \frac{\left[(q_2 - q_1) + (z_2^2 - z_1^2) - (z_2 - z_1) \right]^2}{9\,(q_1 - q_2)}$$

Setting $z_2 = 1$, $q_2 = \bar{q}$, the FOC shows that $z_1 = 1/2$ is the best reply of firm 2, whatever q_1. The corresponding profits are given by (29). The same profit comparison (between Π_2^{**} and Π_2^{*}) as for firm 1 can then be made. It shows that $\Pi_2^{*} \leq \Pi_2^{**}$ if and only if $\bar{q} - \underline{q} \leq K_h$. Q.E.D.

To sum up, we have identified two types of product equilibrium. If $\bar{q} - \underline{q} \geq K_v$, there exists an equilibrium such that both firms choose the same variety under respectively the lowest and the highest possible quality. Hence, they <u>**minimise**</u> differentiation along the horizontal dimension and <u>**maximise**</u> differentiation along the vertical dimension. The equilibrium outcome is characterised by vertical dominance. If $\bar{q} - \underline{q} \leq K_h$, there is an equilibrium such that both firms select the top quality but opposite varieties. Hence, they **maximise** differentiation along the horizontal dimension and **minimise** differentiation along the vertical dimension. The equilibrium now features horizontal dominance.

In addition, since $K_h > K_v$ there is a nondegenerate segment $[K_v, K_h]$ in which the two equilibria occur. This existence of multiple equilibria in some parameter region indicates that indeed the present model is not simply an addition of two models, one of the horizontal, and the other of the vertical type. A question then naturally arises : does one equilibrium Pareto dominate the other in terms of profits ? The answer is no. Indeed, firm 1 always prefers the equilibrium with horizontal dominance, while firm 2 always prefers the other. This reflects the asymmetry which arises at the equilibrium under vertical dominance (favorable to firm 2), whereas the equilibrium under horizontal dominance yields a symmetric outcome in terms of profits. As a final remark, notice that the sum of the equilibrium profits is always higher at the vertical dominance equilibrium. This might provide a very loose rationale for selecting this equilibrium in the segment $[K_v, K_h]$.

5. Conclusion

Our results with respect to product equilibria suggest that a "principle of max-min differentiation" might hold in markets involving horizontal and vertical characteristics, orthogonal to each other.[4] This principle has been derived admittedly in the context of a very bare model. The benefit of such exercise is to highlight the competitive forces

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4. This is reminiscent of a result by de Palma et al. (1985) : firms choose to agglomerate at the market centre when their products are perceived as sufficiently heterogeneous (along some other dimensions).

generated by the demand side. Presumably, sharply increasing production cost with respect to quality would reduce firms' incentive to differentiate themselves by increasing quality. It is doubtful, however, that (apart from this fairly intuitive modification) the nature of our equilibria would be drastically affected by the introduction of standard cost structures. With respect to our demand formulation, it seems reasonable to conjecture that existence of equilibrium will be guaranteed under more general specification of the utility function, at least regarding the vertical characteristics (see Champsaur and Rochet (1987)). As to the horizontal characteristics, it is well known that a significant departure from the quadratic case often poses serious problems concerning existence of equilibrium. Finally, some preliminary work by Champsaur and Rochet (1987) and Neven (1986) indicate that somewhat more general customer distributions can be considered without jeopardising equilibrium. In our context, more concentrated customer distribution will presumably lead to less differentiation along the horizontal dimension when horizontal dominance prevails.

All in all, despite some very stylised features, it is our contention that the present model gives a useful first insight into the interaction between price, variety and quality competition.

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Appendix

<u>Lemma</u> : **Given any (y_i, q_i), there exists no best reply in the interior of region R_I for firm $j \neq i$.**

<u>Proof</u> : (i) W.l.o.g., suppose $q_1 \leq q_2$ and $y_1 \leq y_2$. Start with firm 1 and assume that vertical dominance prevails. Using the equilibrium prices $\tilde{P}_1$ and $\tilde{P}_2$ given by (21) and (20), firm 1's profit function is written as :

$$\tilde{\Pi}_1(y_1,q_1,y_2,q_2) = \frac{(y_2^2 - y_1^2) + \left[\left(y_2^2 - y_1^2 \right)^2 + 16\ (y_2 - y_1)(q_2 - q_1) \right]^{1/2}}{12\ (y_2 - y_1)\ (q_2 - q_1)} \qquad (A.1)$$

Let $F \equiv \left[\left(y_2^2 - y_1^2 \right)^2 + 16\ (y_2 - y_1)(q_2 - q_1) \right]^{1/2}.$

Taking the first derivative of $\tilde{\Pi}_1$ w.r.t. q_1 yields (up to a positive factor):

$$(y_2 - y_1) \left[(y_2^2 - y_1^2) + F - \frac{24\ (y_2 - y_1)\ (q_2 - q_1)}{F} \right]$$

Since condition (A) is violated and vertical dominance holds, it follows that y_1 must be different from y_2. Hence, for the FOC $\frac{\partial \tilde{\Pi}_1}{\partial q_1} = 0$ to be satisfied, it must be that :

$$(y_2^2 - y_1^2)\ F - 8\ (y_2 - y_1)\ (q_2 - q_1) + (y_2^2 - y_1^2) = 0 \qquad (A.2)$$

Moving the last two terms of (A.2) to the RHS and, taking the square of both sides lead, after simplifications, to

$$(y_2^2 - y_1^2) = 32\,(y_2 - y_1)\,(q_2 - q_1) \tag{A.3}$$

As $(q_2 - q_1) > 2\,(y_2 - y_1)$, (A.3) implies that :

$$\left(y_2^2 - y_1^2\right)^2 > 64\,(y_2 - y_1)^2$$

or $(y_1 + y_2)^2 > 64$,

a contradiction. Accordingly, $\frac{\partial \Pi_1}{\partial q_1} = 0$ can never be satisfied in the interior of R_I under vertical dominance.

(ii) Assume now that horizontal dominance prevails. The expression of firm 1's profit is still given by (A.1). Since horizontal dominance holds, we must have y_1 different from y_2. Hence, the argument developed in (i) remains valid and leads to (A.3). As condition (C) is violated, it follows from (A.3) that :

$$(y_1 + y_2)^2 > 16\,(y_1 + y_2 + 2)$$

which is never satisfied. We thus reach a contradiction and $\frac{\partial \tilde{\Pi}_1}{\partial q_1} = 0$ cannot be verified in the interior of R_I.

(iii) Consider the case of firm 2. The argument presented below applies to both vertical and horizontal dominance. First, given (20), it is straightforward that $\frac{\partial \tilde{\Pi}_2}{\partial q_2} > 0$. Second, some simple manipulations show that

$\frac{\partial D_2(\tilde{P}_1, \tilde{P}_2)}{\partial q_2} > 0$ if and only if $\frac{(y_2^2 - y_1^2) + F}{(q_2 - q_1)}$ is decreasing in q_2.

Taking the first derivative of this expression w.r.t. q_2 gives (up to a positive factor) :

$$-\,2(y_2^2 - y_1^2) - 16\,(y_2 - y_1)\,(q_2 - q_1) - 2\,(y_2^2 - y_1^2)\,F$$

which is always negative. Consequently, as both $\tilde{P}_2$ and $D_2(\tilde{P}_1,\tilde{P}_2)$ increase with q_2, $\frac{\partial \tilde{\Pi}_2}{\partial q_2} = 0$ cannot hold in the interior of R_I. Q.E.D.

Table 1

Comparative statics under vertical dominance

	$y_1 < 1/2$, $y_2 < 1/2$	$y_1 < 1/2$, $y_2 > 1/2$	$y_1 > 1/2$, $y_2 > 1/2$
$\frac{\partial P_1^*}{\partial y_1}$	$\underline{> 0}$	$\underline{> 0}$	< 0
$\frac{\partial P_1^*}{\partial y_2}$	$\underline{< 0}$	> 0	> 0
$\frac{\partial P_2^*}{\partial y_1}$	< 0	< 0	$\underline{> 0}$
$\frac{\partial P_2^*}{\partial y_2}$	> 0	$\underline{< 0}$	$\underline{< 0}$

$$\frac{\partial P_1^*}{\partial q_1} < 0, \quad \frac{\partial P_1^*}{\partial q_2} > 0, \quad \frac{\partial P_2^*}{\partial q_1} < 0, \quad \frac{\partial P_2^*}{\partial q_2} > 0$$

Table 2

Comparative statics under horizontal dominance

$$\frac{\partial P_1^{**}}{\partial q_1} \ \underline{> 0}, \quad \frac{\partial P_1^{**}}{\partial q_2} \ \underline{< 0}, \quad \frac{\partial P_2^{**}}{\partial q_1} < 0, \quad \frac{\partial P_2^{**}}{\partial q_2} > 0$$

$$\frac{\partial P_1^{**}}{\partial y_1} < 0, \quad \frac{\partial P_1^{**}}{\partial y_2} > 0, \quad \frac{\partial P_2^{**}}{\partial y_1} < 0, \quad \frac{\partial P_1^{**}}{\partial y_1} < 0$$

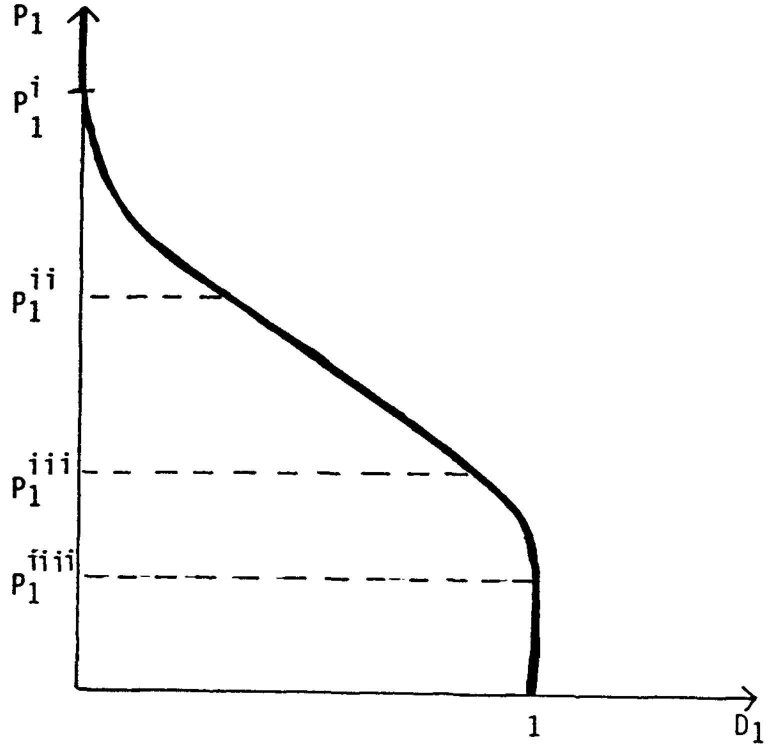

FIGURE 1

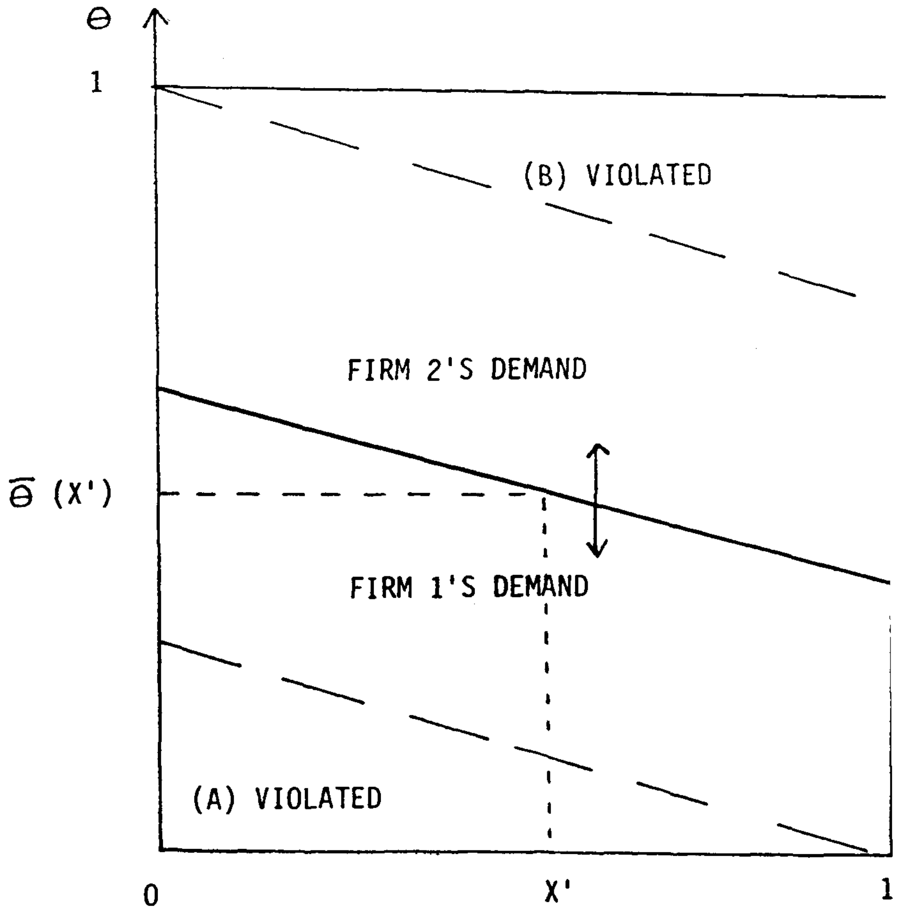

FIGURE 2

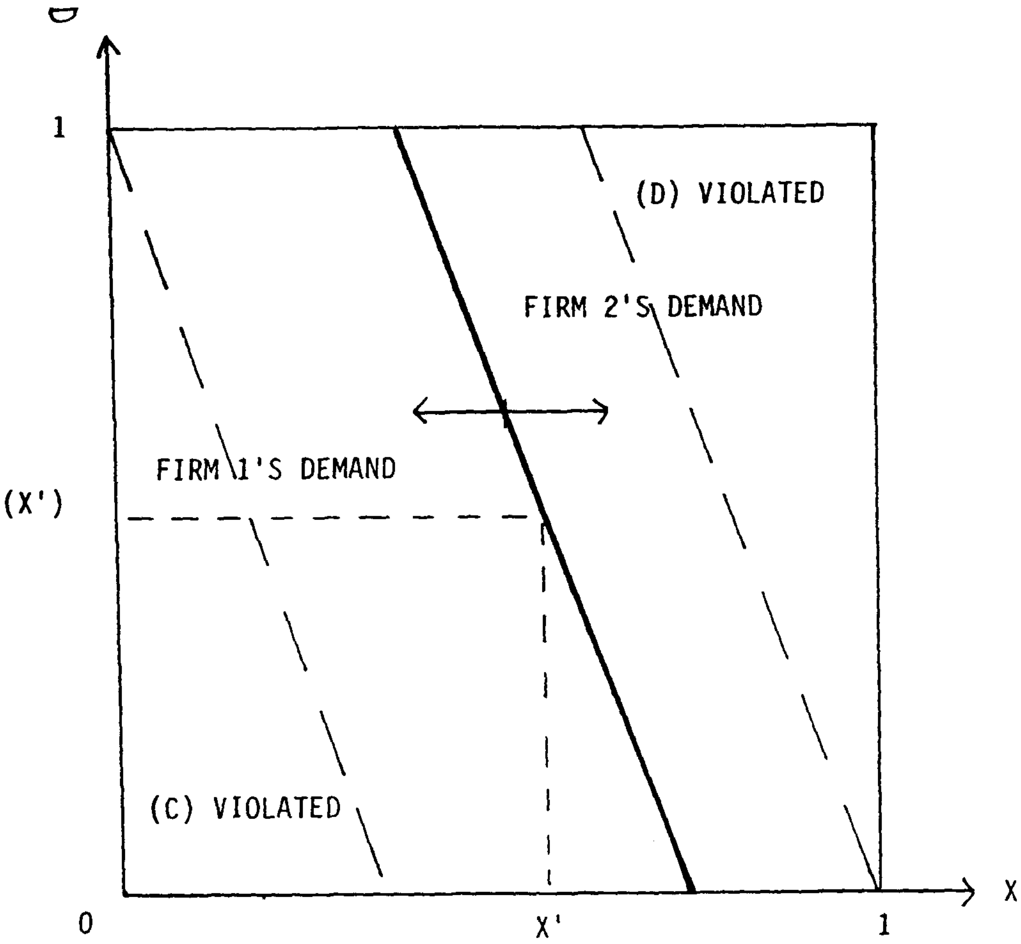

FIGURE 3

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