A characterization of the Taylor expansion of λ -terms CSL' 2013

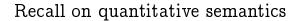




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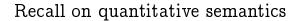
λ-calculus

$$X \longmapsto (F)X$$

$$\downarrow_{\beta},$$
 Y

Semantics

$$x \longmapsto f(x)$$





λ-calculus

$$\begin{array}{c} X \longmapsto (F)X \\ & \downarrow_{\beta *} \\ & Y \end{array}$$

Semantics

$$\begin{aligned} x &\longmapsto f(x) \\ & & \text{II} \\ & & \sum_{n=0}^{\infty} \frac{1}{n!} (\vartheta_x^n f \cdot x^n) 0 \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\partial_x^n f \cdot x^n) 0 \text{ is the } Taylor \ Expansion \text{ of } f$$

 λ -calculus $\xrightarrow{\text{Taylor Expansion}}$ Resource calculus

λ-calculus

Grammar: $\Lambda : T, U := x \mid \lambda x.T \mid (T)U$

 $(\lambda x.T)U \xrightarrow{\beta} T[U/x]$

 $\sigma \in S_n$

Resource calculus

Grammar: $\Delta: t, u := x \mid \lambda x.t \mid \langle t \rangle [u_1, \dots, u_n]$

$$\langle \lambda x.t \rangle [u_1, \dots, u_n] \xrightarrow{r} \sum \ t\{u_{\sigma(1)}/x_1, \dots, u_{\sigma(n)}/x_n\}$$

Substitutes each occurrence of x in t by only one u_i Reduces to 0 otherwise

 $M \xrightarrow{\text{Taylor Expansion}} \sum_{t \in \text{taylor}(M)} \alpha_t t \xrightarrow{\text{NF}} NF(\sum_{t \in \text{taylor}(M)} \alpha_t t)$

Goal: Characterize the image of this transformation



$$\exists M \; \lambda\text{-term s.t.} \; \sum_{t \in \Delta} \alpha_t \cdot t = NF(taylor(M)) \text{ iff }$$

- 0 ...
- 1 ...
- 2 ...
- 3 ...



$$\exists M \; \lambda\text{-term s.t.} \; \sum_{t \in \Lambda} \alpha_t \cdot t = NF(taylor(M)) \text{ iff }$$

O Theorem [Ehrhard - Regnier]:

$$\forall \alpha_t \in \text{NF}(taylor(M)), \text{ if } \alpha_t \neq 0 \text{ then } \alpha_t = \frac{1}{m(t)}$$

- $1 \cdots$
- 2 ...
- 3 ...

$$\forall \mathcal{T} \subseteq \Delta, \; \exists M \; \lambda\text{-term s.t.} \; \; \mathcal{T} = NF(\tau(M)) \; \text{iff} \; \;$$

 $\forall \alpha_t \in NF(taylor(M)), \text{ if } \alpha_t \neq 0 \text{ then } \alpha_t = \frac{1}{m(t)}$

$$\forall \ / \subseteq \Delta, \ \exists N i \land \text{-term s.t.} \ / = N F(\tau(N i))$$

$$\forall \mathcal{T} \subseteq \Delta, \exists M \lambda \text{-term s.t. } \mathcal{T} = NF(\tau(M)) \text{ iff}$$

$$\forall \alpha_t \in NF(taylor(M)), \text{ if } \alpha_t \neq 0 \text{ then } \alpha_t = \frac{1}{m(t)}$$

1 FV(
$$\mathcal{T}$$
) is finite

$$\mathcal{I}$$
 is r.e.

Conditions 1 and 2: based on Barendregt's theorem

Theorem [Barendregt]:

Let $\mathcal B$ be a Böhm-like tree. There is a λ -term M such that $\operatorname{BT}(M)=\mathcal B$ if, and only if, $\operatorname{FV}(\mathcal B)$ is finite and $\mathcal B$ is r.e..



$$\forall \mathcal{T} \subseteq \Delta, \; \exists M \; \lambda\text{-term s.t.} \; \mathcal{T} = NF(\tau(M)) \; \text{iff} \;$$

0 Theorem [Ehrhard - Regnier]: $\forall \alpha_t \in NF(taylor(M)), \text{ if } \alpha_t \neq 0 \text{ then } \alpha_t = \frac{1}{\mathfrak{m}(t)}$

- 1 FV(T) is finite
- 2 \mathcal{T} is r.e.
- 3 \mathcal{T} is an ideal

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Resource calculus and Taylor expansion

Ideal

Two corollaries and further works

Plan



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Two corollaries and further works

Resource calculus

 Δ : t, u ::= x | λx .t | $\langle t \rangle [u_1, \ldots, u_n]$ Grammar: Relation $\stackrel{r}{\rightarrow}$ (strongly normalizing, confluent):

$$\langle \lambda x.t \rangle [s_1, \dots, s_n] \stackrel{r}{\rightarrow} \begin{cases} \{t\{s_{\sigma(1)}/x_1, \dots, s_{\sigma(n)}/x_n\} \mid \sigma \in \mathcal{S}_n\} \\ \emptyset \text{ if } \deg_x(t) \neq n \end{cases}$$

Unique normal form: NF(t)

Taylor expansion:
$$\Lambda \longrightarrow \mathcal{P}(\Delta)$$

Taylor expansion:
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$$\tau(x) \stackrel{\triangle}{=} \{x\}$$

 $\tau(\lambda x.\mathsf{T}) \stackrel{\triangle}{=} \{\lambda x.\mathsf{t} \mid \mathsf{t} \in \tau(\mathsf{T})\}\$

$$exttt{NF}(\mathcal{T}) \stackrel{\triangle}{=} igcup exttt{NF}(exttt{t})$$

 $\tau((T)U) \stackrel{\triangle}{=} \{\langle t \rangle [u_1, \dots, u_k] \mid t \in \tau(T); k \in \mathbb{N}; u_1, \dots, u_k \in \tau(U) \}$

$$g_x(t) \neq n$$

A first example: S



$$S := \lambda xyz.((x)z)(y)z$$

Böhm tree of S:



Taylor expansion of S:

$$\begin{split} \tau(\mathbf{S}) &= \{\lambda x y z. \langle x \rangle 11, \lambda x y z. \langle x \rangle \underbrace{[z, \dots, z]}_{n} \underbrace{[\langle y \rangle 1, \dots, \langle y \rangle 1]}_{m}, \dots \} \\ &= \{\lambda x y z. \langle x \rangle [z^{n}] [\langle y \rangle [z^{n_{1}}], \dots, \langle y \rangle [z^{n_{k}}]] \; ; \; k, n, n_{1}, \dots, n_{k} \in \mathbb{N} \} \\ &= NF(\tau(\mathbf{S})) \end{split}$$



$$(\mathbf{S})\mathbf{II} = ((\lambda xyz.((x)z)(y)z)\lambda x.x)\lambda x.x \xrightarrow{\beta*} \lambda x.(x)x = \delta$$

$$\tau((\mathbf{S})\mathbf{II}) = \{\langle \lambda xyz.\langle x \rangle 11 \rangle 11, \\ \langle \lambda xyz.\langle x \rangle [z, \dots, z] [\langle y \rangle 1, \dots, \langle y \rangle 1] \rangle [I, \dots, I] [I, \dots, I], \dots \}$$

$$\mathsf{NF}(\tau((\mathbf{S})\mathbf{II})) = \{\langle \lambda x.\lambda yz.\langle x \rangle 11 \rangle 11, \dots \}$$



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$$= \{\lambda x.\langle x \rangle [x^n], n \in \mathbb{N}\} = \tau(\delta)$$



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$$= \{\lambda x.\langle x \rangle [x^n], n \in \mathbb{N}\} = \tau(\delta)$$

$$\mathbf{\Omega} = (\delta)\delta$$

$$\tau(\mathbf{\Omega}) = \{\langle \lambda x.\langle x \rangle [x^{n_0}] \rangle [\lambda x.\langle x \rangle [x^{n_1}], \dots, \lambda x.\langle x \rangle [x^{n_k}]]; k, n_0, \dots, n_k \in \mathbb{N}\}$$

 $NF(\tau(\Omega)) = \emptyset$

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Resource calculus and Taylor expansion

Ideal

Two corollaries and further works



Terms in normal form: $\Delta^{NF}: t := \lambda x_0 \dots x_{m-1}. \langle y \rangle \mu_0 \dots \mu_{n-1}$ μ_i : finite multisets of simple terms in normal form

Uniform approximation \leq

$$\lambda x_0 \dots x_{m-1} \cdot \langle y \rangle \mu_0 \dots \mu_{n-1} \leq t$$
 iff

(i)
$$t = \lambda x_0 \dots x_{m-1} \cdot \langle y \rangle v_0 \dots v_{n-1}$$

$$(ii) \ \forall i < n, |\mu_i| \neq \emptyset \implies \exists \nu \in |\nu_i|, \forall u \in |\mu_i|, u \preceq \nu$$

∹-ideal

- $\bullet \ \tau(\mathbf{S}) = \{\lambda xyz.\langle x\rangle[z^n][\langle y\rangle[z^{n_1}],\ldots,\langle y\rangle[z^{n_k}]] \ ; \ k,n,n_1,\ldots,n_k \in \mathbb{N}\}$
- \blacksquare { $\langle x \rangle [y, z]$ }
- $= \{ \chi[\chi] \},$



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- \blacksquare { $\langle x \rangle [y, z]$ }
- $\{x[x]\}, \{x_1, x[x]\}$? $x[x, x] \prec x[x]$



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- \blacksquare { $\langle x \rangle [y, z]$ }
- $\{x[x]\}, \{x_1, x[x]\}$? $x[x, x] \leq x[x] \Longrightarrow \{x[x^n] \mid n \in \mathbb{N}\}$



$$\forall \mathcal{T} \subseteq \Delta, \; \exists M \; \lambda\text{-term s.t.} \; \mathcal{T} = NF(\tau(M)) \; \text{iff} \;$$

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- 1 FV(T) is finite
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Two corollaries and further works

Corollary 1

Let $\mathcal{T} \in \mathcal{P}(\Delta^{NF})$.

There is a normalizable $\lambda\text{-term }M$ such that $NF(\tau(M))=\mathcal{T}$ iff

- (i) $height(\mathcal{T})$ is finite
- (ii) ${\mathcal T}$ is a maximal clique

 $\Delta^{\rm NF}: t ::= \lambda x_0 \dots x_{m-1}. \langle y \rangle \mu_0 \dots \mu_{n-1}, \ \mu_i \ {\rm finite \ multisets \ of \ simple \ terms \ in \ normal \ form.}$

Coherence \bigcirc on Δ^{NF} :

 $\lambda x_0 \dots x_{m-1}.\langle y \rangle \mu_0 \dots \mu_{n-1} \mathrel{{\raisebox{.3pt}{\sim}}} t$ iff

- (i) $t = \lambda x_0 \dots x_{m-1}. \langle y \rangle \nu_0 \dots \nu_{n-1}$
- (ii) $\forall i < n, \forall u, u' \in |\mu_i \cdot \nu_i|, u \circ u'$

Clique: subset of a \leq -ideal

 $\mathcal{T} \in \mathcal{P}(\Delta^{NF})$ clique: $\forall t, t' \in \mathcal{T}, t \circ t'$

Corollary 2

Let $\mathcal{T} \in \mathcal{P}(\Delta^{NF})$.

1. $FV(\mathcal{T})$ is finite

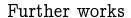
There is a total $\lambda\text{-term }M$ such that $NF(\tau(M))=\mathcal{T}$ iff

- 2. \mathcal{T} is r.e.
- 3. \mathcal{T} is a maximal clique

Total terms

(i)
$$M \xrightarrow{h*} \lambda x_0 \dots x_{m-1}.(y) M_0 \dots M_{n-1}$$

(ii) M_0, \dots, M_{n-1} are total





Bring the results to more expressive calculi:

- Λμ-calculus
 - □ Cannot use Barendregt's theorem
- Non-Deterministic settings
 - □ Cannot use Ehrhard Regnier's theorem
- . .