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Fractals, fractal dimensions and landscapes — a review

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ABSTRACT

Mandelbrot's fractal geometry is a revolution in topological space theory and, for the first time, provides the possibility of simulating and describing landscapes precisely by using a mathematical model. Fractal analysis appears to capture some "new" information that traditional parameters do not contain. A landscape should be (or is at most) statistically self-similar or statistically self-affine if it possesses a fractal nature. Mandelbrot's fractional Brownian motion (fBm) is the most useful mathematical model for simulating landscape surfaces. The fractal dimensions for different landscapes and calculated by different methods are difficult to compare. The limited size of the regions surveyed and the spatial resolution of the digital elevation models (DEMs) limit the precision and stability of the computed fractal dimension. Interpolation artifacts of DEMs and anisotropy create additional difficulties in the computation of fractal dimensions. Fractal dimensions appear to be spatially variable over landscapes. The region-dependent spatial variation of the dimension has more practical significance than the scale-dependent spatial variation. However, it is very difficult to use the fractal dimension as a distributed geomorphic parameter with high "spatial resolution". The application of fractals to landscape analysis is a developing and immature field and much of the theoretical rigour of fractal geometry has not yet been exploited. The physical significance of landscape fractal characteristics remains to be explained. Research in geographical information theory and fractal theory needs to be strengthened in order to improve the application of fractal geometry to the geosciences.

Introduction

Over the last decade, fractal models and related analysis techniques have shifted from being matters for speculative coffee break discussions between sessions at geoscience meetings, to being primary topics for numerous technical sessions (Snow and Mayer, 1992). The term "fractal" was coined by Mandelbrot more than fifteen years ago. At that time it was not used primarily to describe landscapes. It has since become the most successful mathematical model for describing real landscapes because the fractal dimension appears to capture the essence of the surface topography of the earth in a way that other geomorphological attributes do not.

Fractal geometry makes possible a mathematical description of complex natural landscapes and provides a systematic way of characterizing landscapes in quantitative terms. Milne (1991) stated that, "fractal models of landscape structure provide the elements of a calculus for quantifying and predicting the multiscale dynamics of landscape processes." Meakin (1991) also stated that fractal geometry provides a much more complete and realistic description of most structures in geology and geophysics than does Euclidean geometry and will revolutionize the study of geomorphology. So, although fractal analysis is a relatively new field, it has been used in a large number of biological and geoscience studies, particularly for the characterization of landscapes. The concepts loosely associated with the term "fractal" have broad general appeal (Goodchild and Mark, 1987).

Fractal analyses for characterizing the spa-

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tial relationships of the earth's surface, as well as methods for calculating the fractal dimension, have been reported by Armstrong and Hopkins (1983), Shelberg et al. (1983), Mark and Aronson (1984), Brown and Scholz (1985), Clarke (1986), Kennedy and Lin (1986), Brown (1987), Goodchild and Mark (1987), Laverty (1987), Roy et al. (1987), Culling (1988, 1989), Milne (1988), Andrle and Abrahams (1989), Dubuc et al. (1989), Elliot (1989), Fox (1989), Gibert (1989), Hough (1989), Huang and Turcotte (1989), Jones et al. (1989), Longley and Batty (1989), Unwin (1989), Whalley and Orford (1989), Kumar and Bodvarsson (1990), Piech and Piech (1990), Klinkenberg (1992), Klinkenberg and Goodchild (1992), Ouchi and Matsushita (1992), Gallant et al. (1993). Hjelmfelt (1988), Tarboton et al. (1988), Gan et al. (1992), Hatano and Booltink (1992), and Karlinger and Troutman (1992) have used fractal analysis techniques to investigate the structure of river networks, and the river-length/catchment-area ratios. Their work has advanced our understanding of the geometry and composition of river networks (Tarboton et al., 1988).

Fractal analysis is now a growing field of research and an extensive literature concerned with the fractal nature of landscapes is developing. This paper summarizes fractal and related theories and reviews the last decade's research on the fractal analysis of landscapes.

Landscapes and landscape parameters

"We are now in an era of spatial modelling" (Moore et al., 1991). Most geo-processes occur on the land surface and they have close relationships with landscape properties. The relative magnitudes of many hydrological, geomorphological and biological processes are sensitive to topographic position (Moore et al., 1991). Many parameters have been developed for characterizing landscapes and related properties. Speight (1974) described over 20 topographic attributes that can be used to describe landforms. Klinkenberg (1992) assembled 24 traditional landscape morphometric attributes when he analysed the relationships between the fractal parameters and these attributes. They included the mean, standard deviation, skewness and kurtosis of elevation, slope gradient, aspect and plan and profile curvature and some special morphometric parameters such as elevation range and the coefficient of dissection. All of these attributes can be extracted from a digital elevation model.

In using landscape attributes in modelling, questions can be raised, such as: — are these parameters good enough to characterize landscape properties? — which of them is most efficient? — is it reasonable to use these parameters in the models encompassing them? — are there new landscape attributes that perform better? Evans (1984) carried out a principal component analysis of 23 landscape morphometric variables and his results indicated that the first seven components only incorporate 89% of the information of the original morphometric variables and the first component did not dominate the results. A landscape is therefore a complex surface and no single or simple linear combinations of these morphometric variables can describe it comprehensively. However, it has also been implied that the information content of landscape properties contained in each morphometric variable is low, with each one only able to describe a restricted aspect of landscape properties. New parameters that better characterize the variability of landscapes are continually being sought. The fractal dimension may be one of them.

There are explicit differences between the theoretical basis of fractals and those of traditional landscape morphometric parameters. The main characteristic is whether or not the parameter displays a homogeneous or continuous characteristic. Fractals treat the landscape region being investigated as a distinct entity and the fractal nature of the region is initially assumed to be homogeneous. Therefore, one can use mathematical functions (i.e., models) to describe certain characteristics of the landscape region (fractal characteristics) and one should obtain a uniform (consistent) fractal dimension. Traditional landscape parameters do not behave in the same way. For example, slope gradient depends on the characteristics of a small local area surrounding the point of measurement. Generally, the characteristic value at individual points can be extracted from the elevation values of a neighbouring point subset in the DEM matrix using a variety of algorithms. The computed slope gradient is scale-dependent and pixel-dependent and the entire slope "surface" of the investigated region is not continuous in a mathematical sense, but is perhaps continuous in a visual sense. Obviously other traditional landscape parameters also display similar properties to slope gradient.

Fractals and fractal dimensions of landscapes

"Fractal" is a word invented by Mandelbrot (1977) to bring together under one heading a large class of objects that have certain structural features in common, although they appear in diverse contexts in astronomy, geography, biology, fluid dynamics, probability theory, and pure mathematics (Dyson, 1978). According to Mandelbrot (1977, 1982), the term "fractal" comes from the Latin adjective "fractus", which has the same root as "fraction" and "fragment" and means "irregular and fragmented". Mandelbrot (1982) stated that he conceived and developed a new geometry of nature (fractals) and implemented its use in a number of diverse fields. It describes many of the irregular and fragmented patterns around us. Mandelbrot's (1977, 1982) mathematical definition of fractal is: "a set for which the Hausdorff–Besicovitch dimension strictly exceeds the topological dimension." The notion of a fractal not only is quite new for traditional applied mathematics and a revolution in topological space theory (this revolution did not originate from Mandelbrot), but also, for the first time, provides the possibility of describing and simulating landscapes precisely by using a mathematical model.

The key idea introduced into applied mathematics by Mandelbrot is that rugged and indeterminate systems can often be described by extending the classical concept of dimensional analysis to include a fractional number that describes the ruggedness of the system in the space spanned by the whole number dimensions encompassing its fractional magnitude (Kaye, 1989). It means that we can use the fractal dimension to describe the irregularity and roughness characteristics of natural landscapes.

The manifestation of a fractal is via the fractal dimension, usually expressed by the symbol D. The fractal dimension indicates the ability of a set to fill the Euclidean space in which it resides, and is the quantitative description for the fractal characteristics of the investigated objects. In other words, the fractal dimension is generally a measurement of the irregularity of the object in a mathematical sense. Mandelbrot called the Hausdorff–Besicovitch dimension the "fractal dimension" principally because the values of Hausdorff–Besicovitch dimensions are non-integer real numbers (could be integer) in most cases. Precise mathematical definitions of the Hausdorff–Besicovitch dimension can be found in the works of Mandelbrot (1977, 1982), Falconer (1991) and Tricot (1991).

There are less sophisticated, but nevertheless relevant, definitions of fractal dimensions. "In landscape research, fractal measurements provide information about the space-filling properties of a mosaic of patches at all scales" (Milne, 1991); "D provides a characteristic parameter whose variation can be usefully interpreted in terms of the processes that have influenced the entity's development" (Goodchild and Mark, 1987); "... fractal dimensions are real numbers in which the value of the whole number describes the nature of the data set under observation (0 being points, 1 being lines, 2 planes and so on) and the size of the decimal fraction represents the irregularity exhibited within the data" (Elliot, 1989). Intuitively, a small fractal dimension corresponds to a smooth surface, and a large value corresponds to a rough surface. Some people have concluded that it also captures other characteristics of the landscape (Klinkenberg, 1992; Klinkenberg and Goodchild, 1992).

Self-similarity, self-affinity and statistical fractals

The prerequisite for the measurement of the fractal dimension of an object is that it displays self-similarity or self-affinity. Therefore, self-similarity or self-affinity is regarded as a fundamental characteristic of fractal objects and is one of the central concepts of fractal geometry. Mandelbrot (1977, 1982) gave the following definition of self-similarity: "when each piece of a shape is geometrically similar to the whole, both the shape and the cascade (the generating mechanism for the details of the shape) that generate it are called self-similar." Self-similarity assumes that transformations in each direction of Euclidean coordinate space are the same, although objects may be rotated. If the transformations are different in each direction, then the object is self-affine rather than self-similar (Mandelbrot, 1982, 1986a; Shelberg et al., 1983; Goodchild and Mark, 1987; Roy et al., 1987; Milne, 1991; Ouchi and Matsushita, 1992).

Self-similarity is better termed scale-invariance (or scale-independence) in landscapes. This means that for any line or surface of a self-similar object, a portion of the whole is identical to the whole after suitable transformations, or "each small portion, when magnified, can reproduce exactly a larger portion" (Voss, 1985a). Therefore, the value of the fractal dimension should be stable when the range scale changes in a fractal landscape, and its fractal nature should be homogeneous. In fact there is currently great controversy over whether or not self-similar landscapes exist and if fractal dimensions are stable in real landscapes. In the fractal analysis of landscapes, self-affinity is generally more common and is applicable under a wider range of conditions than is self-similarity.

Landscapes are a composite of many competing and complicated geological processes, such as faulting, folding, flexure, erosion and sedimentation, etc. "A wide variety of natural phenomena exhibit complicated, unpredictable, and seemingly random behaviour" (Jensen, 1987). Milne (1991) indicated that, "exact fractal patterns are unlikely to occur in landscapes because contemporary patterns are the results of several processes that dominated in the past. So far, little is known about how the legacies of different processes (e.g., erosion versus deposition) accumulate to produce existing fractal landscape patterns." Shelberg et al. (1983) also commented that, "seldom in nature (crystals are one exception) does self-similarity occur and therefore a statistical form of self-similarity is often encountered." Under statistical self-similarity, each small portion of the object, when it is magnified, is not *exactly like* a larger portion or the whole, but *looks like* a larger portion or the whole. In other words, each portion of the object is statistically indistinguishable from the whole. It is clearly impossible that each portion of a landscape is geometrically (exactly) similar to the whole in most cases. Therefore, a landscape should be called statistically self-similar or statistically self-affine if it possesses a fractal nature. Stanley (1986) divided fractals into *exact fractals* and *statistical fractals.* According to his interpretation, exact fractals are regular fractals that display self-similarity, are highly artificial, and are not expected to appear in nature. Statistical fractals exhibit fractal characteristics when average properties are examined, are statistically self-similar, and can appear in nature. It suffices that the parts and the whole reduced by similarity should have identical (statistical) distributions (Mandelbrot, 1982).

As with statistically self-similar fractals, statistically self-affine objects have been proposed. The hypothesis of statistical self-affinity should be more appropriate than that of statistically self-similarity in determining the fractal characteristics of the surface of the earth. For land surfaces, the vertical size range is usually much less than the horizontal size range. Therefore, the vertical ratio under transformations is clearly less than the horizontal ratio.

Calculating the fractal dimension

Fractional Brownian model and landscapes

Estimating fractal dimensions and other fractal parameters of landscapes is the principal way of quantitatively describing the fractal properties of landscapes. Until now, Mandelbrot's fractional Brownian motion (fBm), or fractional Brown function, which is the mathematical generalization of fBm, probably provides the most useful mathematical model for the random fractals found in nature, particularly for application to landscape analysis.

The fundamental form of Mandelbrot's fractional Brownian model is the Brown line-to-line function $B(t)$, which is a random function and for t_1 and t_2 ($t_1 \neq t_2$) it satisfies:

$$P\{[B(t_2)-B(t_1)]/|t_2-t_1|^H<x\}=F(x) \tag{1}$$

where t and x are real numbers, $H=0.5$ and $F(x)$ is the cumulative contribution function for a random variable $X=[B(t_2)-B(t_1)]/|t_2-t_1|^H$ (or $X=B(t_2)-B(t_1)$, because for fixed (t_2-t_1), $|t_2-t_1|^H$ is a constant) that follows a Gaussian distribution with $E[X]=0$ (i.e., the expected value of X is equal to 0) and $E[X^2]=1$ ($E[X^2]=|t_2-t_1|^H$ for $X=B(t_2)-B(t_1)$). This means that:

$$E[(B(t_2)-B(t_1))^2]=|t_2-t_1|^{2H} \tag{2}$$

The $B(t)$ with $H=0.5$ is also called the ordinary Brown line-to-line function and its trail is a curve. When one changes the exponent H from $H=0.5$ to any real number satisfying $0<H<1$, then $B(t)$ is called the fractional Brown line-to-line function (fractional Brown function), and denoted by $B_H(t)$, for which:

$$E[(B_H(t_2)-B_H(t_1))^2]=|t_2-t_1|^{2H} \tag{3}$$

For a surface, the single variable t is replaced by point coordinates x and y on a plane to give $B_H(x,y)$ as the surface altitude at position x,y (i.e. Z). The surface that consists of these $B_H(x,y)$ points is usually called a *fractional Brownian surface.* A fractional Brownian surface $B_H(x,y)$ has the following properties (Mandelbrot, 1977, 1982, 1986a):

(a) The surface is continuous, but nondifferentiable, and everything concerning this surface is dependent on the single scaling parameter H.

(b) The plane (x,y) is *isotropic.*

(c) The fractal dimension of the surface is $D=3-H$.

(d) The section profile in any direction, i.e., the intersection of a vertical plane with the surface $B_H(x,y)$, is *self-affine* and $D=2-H$; i.e., the surface dimension minus one. Usually, the profile is a curve of the fractional Brown line-to-line function.

(e) The traverse at any altitude, i.e., the intersection of the horizontal plane with the surface $B_H(x,y)$, which produces a family of curves, is *self-similar* and $D=2-H$. This means that coastlines and contours are statistically self-similar when the $B_H(x,y)$ is used to simulate the surface of the landscape.

(f) $$\begin{aligned} E[(B_H(x_2,y_2)-B_H(x_1,y_1))^2] &= C[((x_2-x_1)^2 \\ &\quad +(y_2-y_1)^2)^{1/2}]^{2H} \\ &= C[(x_2-x_1)^2+(y_2-y_1)^2]^H \end{aligned} \tag{4}$$

where C is a constant that replaces the original constant 1 in the single variable fractional Brown function.

(g) The fractional Brownian surface is self-affine.

Methods of computing fractal dimensions

Mandelbrot (1977, 1982), Shelberg et al. (1983), Mark and Aronson (1984), Voss (1985a,b), Clarke (1986), Roy et al. (1987), Culling (1988, 1989), Milne (1988), Fox (1989), Yokoya et al. (1989), Ouchi and Matsushita (1992), and others have developed a range of methods for estimating the fractal dimension of a landscape. The principal methods are presented below.

Variogram method

The variogram method (Mandelbrot, 1977, 1982; Goodchild, 1980; Mark and Aronson, 1984; Pentland, 1984; Voss, 1985a,b; Roy et al., 1987; Gallant et al., 1993) is a well-known method to directly measure the surface fractal dimension of landscapes. It is based on the assumption that the landscape surface is a fractional Brownian surface and uses eq. (4) with C and H as constants for a given surface. Therefore, for a given surface, the mean square altitude difference $E[(B_H(x_2,y_2) - B_H(x_1,y_1))^2]$ depends on the horizontal distance h between points (x_1,y_1) and (x_2,y_2). The term $E[(B_H(x_2,y_2) - B_H(x_1,y_1))^2]$ is the variance of the random variable $X = B_H(x_2,y_2) - B_H(x_1,y_1)$ because the mean,

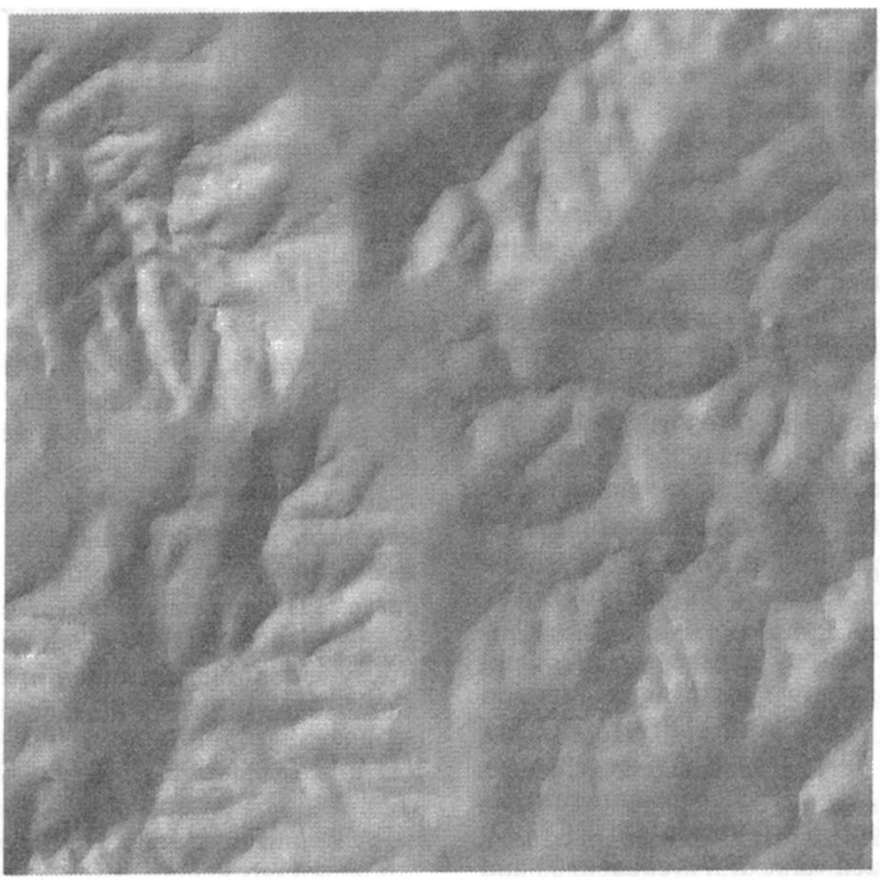

Fig. 1. Shaded relief diagram of the 512×512 pixel DEM from the northern section of the Brindabella Range, southeastern Australia (10 m×10 m grid spacing).

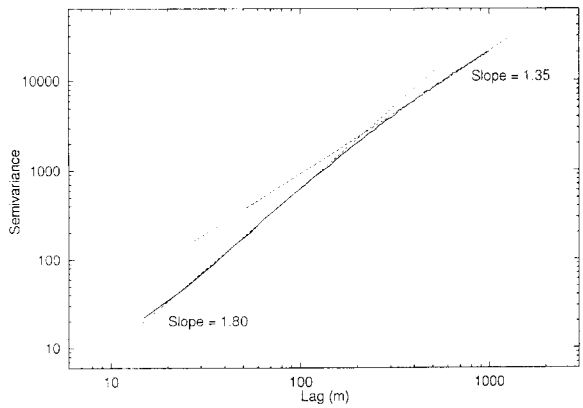

Fig. 2. Log–log semi-variogram plot of the DEM of Fig. 1. The slope at short lags (10 to 200 m) is 1.80 (D=2.10). For lags over 200 m, the slope is 1.35 (D=2.33).

$E[B_H(x_2,y_2)-B_H(x_1,y_1)]$, is equal to zero. If one can determine the parameter H from eq. (4), then, the fractal dimension D can be obtained from $D=3-H$.

Changing eq. (4) into logarithmic form, the following simpler relationship is obtained:

$$\log\{E[(B_H(x_2,y_2)-B_H(x_1,y_1))^2]\} = 2H\log(h)+\log(C) \tag{5}$$

The parameters H and C are usually estimated as the slope and intercept of a least-squares line fitted to a log–log plot of $E[(B_H(x_2,y_2)-B_H(x_1,y_1))^2]$ versus h. In order to make the estimate of $E[(B_H(x_2,y_2)-B_H(x_1,y_1))^2]$ meaningful, a sufficient number of random sampling point pairs are needed for each distance h. Oliver and Webster's (1986) semi-variogram is one of several methods of calculating eq. (5) (Gallant et al., 1993). A grid-based DEM is required for the variogram method.

Klinkenberg and Goodchild (1992) presented two auxiliary fractal characteristics, gamma and break-distance, in addition to the fractal dimension. Gamma is the ordinate intercept of the straight line (equation 5) in the variogram plot and represents the expected difference in elevation for point pairs a unit distance apart. The break-distance is the maximum distance to which a least-squares line could be fitted, with a correlation greater than 0.9, to the first linear segment in the variogram plot.

Figure 1 is a 512×512 pixel, 10 m×10 m grid DEM of a 26 km^2 area in the northern section of the Brindabella Range of southeastern Australia (35°22′S, 148°48′E). The total relief is 720 m (mean elevation=1090 m) and the slope gradient ranges from 0 to 200% with a mean slope gradient of 34%. The DEM was produced by ANUDEM (Hutchinson, 1989), which uses a spline interpolation of contour and spot height data while preserving sensible drainage properties input as digitized streamlines. Figure 2 is a log–log variogram plot of the DEM based on eq. (5) and illustrates the results that can be obtained with the variogram method using traditional data sources. The slope of this function is the coefficient term ($2H$) in eq. (5), and two distinct slopes are evident: $2H=1.80$ ($D=2.10$) at short lags (<200 m) and $2H=1.35$ ($D=2.33$) at larger lags. The reasons for the variable fractal dimension are discussed latter.

Divider method

Contour lines are regarded as statistically self-similar when fractional Brownian surfaces are used to simulate the landscape surface. The surface fractal dimension D_s can be obtained from the contour fractal dimension D_c using the relationship:

$$D_s = D_c + 1 \tag{6}$$

The divider method (Mandelbrot, 1977, 1982; Shelberg et al., 1983; Kennedy and Lin, 1986; Laverty, 1987; Hayward et al., 1989) can be used to measure D_c. The divider method, also called the Mandelbrot method or ruler method, is a well-known measurement technique for linear fractal characteristics. It determines the fractal dimension of linear phenomena such as contours by measuring their length using a "divider". The general mathematical form of divider method is:

$$L(r) = Cr^{1-D} \tag{7}$$

where D is the fractal dimension, $L(r)$ is the length of the linear object measured, r is the divider step and C is a constant. Equation (7) can also be written in log–log form as:

$$\log[L(r)] = (1-D)\log(r) + \log(C) \tag{8}$$

Similarly to the variogram method, D and C can be estimated from the slope and intercept of a linear regression equation fitted to a log–log plot of $L(r)$ versus r. This method yields meaningful results only when the linear object measured is self-similar. For a self-similar linear object, the smaller the step r, the longer the length $L(r)$, and the $L(r)\to\infty$ as $r\to 0$. The format of the primary data used with the divider method is usually a vectorial one with a string of point coordinate pairs on a plane.

Box counting method

The box counting method (Mandelbrot, 1977, 1982; Voss, 1985b; Milne, 1988; Jones, 1991; Meakin, 1991; Klinkenberg and Goodchild, 1992), also called the cell counting method, is another useful method for determining the fractal dimension of linear phenomena. It can be used to measure the fractal dimension of contour lines, Dc, and hence the surface fractal dimension using equation (6). The general mathematical form of the box counting method is:

$$N(r) = Cr^{-D} \tag{9}$$

where D is the fractal dimension, $N(r)$ is the number of boxes (cells) that cover the linear object measured, r is the side length of the square box and C is a constant. The log–log form of eq. (9) is:

$$\log[N(r)] = -D\log(r) + \log(C) \tag{10}$$

To obtain a believable value of D, one needs to count the $N(r)$ for different side lengths, r, then obtain the D from the data pairs $N(r)$ and r using least-squares regression, in the same way as the methods mentioned above. The box counting method is usually used with self-similar linear objects. The primary data form used with this method is a grid-based DEM.

Power spectral method

The power spectral method (Voss, 1985a,b; Fox and Hayes, 1985; Mandelbrot, 1986a; Fox, 1989; Huang and Turcotte, 1989) can be used to measure the fractal dimension of a self-affine linear object. The section profile of the landscape is statistically self-affine under the assumption of fractional Brownian motion and for which:

$$S_v(f) = Cf^{5-2D} \tag{11}$$

where D is the fractal dimension of the profile, $S_v(f)$ is the Fourier power spectral density of the profile data set measured, f is the frequency and C is a constant. The log–log form of eq. (11) is:

$$\log[S_v(f)] = (5-2D)\log(f) + \log(C) \tag{12}$$

D_{profile} is obtained from eq. (12) using least-squares regression, similar to the procedures described above. The surface fractal dimension of a landscape should be $D_{\text{profile}}+1$. This method includes the one-dimensional Fourier transform calculation.

Huang and Turcotte (1989) generalized eq. (11) into two-dimensions based on the work of Voss (1985a,b) and directly estimated the fractal dimension of a self-affine surface. Huang and Turcotte (1989) proposed that in two-dimensional form:

$$S_v(f) = Cf^{7-2D} \tag{13}$$

and:

$$\log[S_v(f)] = (7-2D)\log(f) + \log(C) \tag{14}$$

where D is the fractal dimension of the surface, $S_v(f)$ is the Fourier power spectral density of the surface data set measured, and C and f are the same as defined previously.

The maximum entropy method (MEM) of the spectral analysis technique was developed by Burg (1967, 1968) in response to the problem of truncation of the sample autocorrelation function. Burg's method essentially extrapolates the sample autocorrelation function to infinite length by choosing successive values that maximise the entropy of the power spectrum (Burg, 1967). This method was later improved (Burg, 1968) by avoiding the step of calculating the autocorrelation estimates and, instead, directly estimating the coefficients of the prediction error filter from the data. The spectrum is then easily calculated from the filter coefficients. The details of the computational method are described by Denham (1975). The number of filter coefficients is not defined by the method but must be chosen by experimentation, fewer coefficients giving smooth spectra with few spectral peaks and more coefficients giving noisier spectra with many spectral peaks. Gallant et al. (1993) found that twenty coefficients produced reasonable spectral estimates. The MEM spectral method gave more precise estimates of D for profiles than the traditional direct Fourier transform method (Gallant et al., 1993).

Questions and problems

Do fractals capture the essence of a landscape?

Fractal analysis, as a mathematical method for describing landscapes, seems unlikely to capture the essence of complex and diverse landscapes. However, it appears that the fractal dimension includes "new" information that traditional parameters do not contain because of its mathematical basis and methods of calculation are different from those of traditional morphometric parameters. "It differs from other measures of 'roughness' in that it provides a description theoretically independent of the sample — most other measures of roughness are sample dependent" (Brown and Scholz, 1985). The key issue is how to explain the significance of the information contained within the fractal dimension. In other words, how do we understand the physical significance of the fractal dimension? It is too sweeping a statement to say that it characterizes only "irregularity" or "roughness". The meanings of traditional morphometric parameters, such as altitude, slope, aspect, curvature, etc., which are called landform elements by Speight (1974), are intuitive and easy to understand. But fractal research is still too immature, and, "the links between the fractal dimensions and the physical processes which produce that characteristic of form captured by D have not been identified as of yet, although some studies have begun to explore those links" (Klinkenberg, 1992).

Are landscape surfaces self-similar?

There is insufficient definitive evidence to be able to conclude that landscapes are self-similar or self-affine over all ranges of scales. Klinkenberg and Goodchild (1992) concluded that, "it is not possible to provide blanket statements about the overall fit of the fractal model. Outright rejection of the self-similar fractal model does not appear to be warranted, but neither does a blind application." Their conclusions were from the very mixed results of a study of the fractal characteristics of 55 DEMs from seven different United States physiographic provinces using seven methods of calculating the fractal dimension. Research to date indicates that self-similarity is exhibited only in limited regions and over limited ranges of scale in real landscapes, although one can find statements in the literature like: "the applicability of fractal concepts to the Earth's topography should not be surprising since it only requires scale invariance, and it has long been recognized that topography is scale invariant in a variety of geological terrains" (Huang and Turcotte, 1989). Unfortunately, the unstable values of fractal dimensions and the non-linear log–log plots have demonstrated that categorical statements about the fractal properties of landscapes cannot be made. Moreover, "although fractional Brownian landscapes with dimension of 2.2–2.3 achieve realistic representations of the surface of the earth, little is known about the dimensionality of natural terrains" (Roy et al., 1987).

Generally, the prerequisite for the measurement of the fractal characteristics of landscapes is that the landscapes possess statistical self-similarity or self-affinity. Measurements of the fractal nature of landscapes are unbelievable if one fails to prove the self-similarity or self-affinity of the landscapes. However, to date there are no methods for proving this except by trying to calculate a constant fractal dimension of complex landscape surfaces.

Indeed, many people are infatuated with "magical" computer generated fractional Brownian surfaces. "The fractional Brownian process has been used as a convenient way of generating self-similar surfaces, and certainly such surfaces more closely resemble some types of real topography than do the results of any other available methods of simulation" (Goodchild and Mark, 1987). However, "the publicity that fractals and related mathematical concepts have received is largely due to their strong visual impact" (Goodchild and Mark, 1987). Mandelbrot (1977, 1982) held that visual appearance is the most important test of stochastic models of natural phenomena and that on this basis fractional Brownian model surfaces must be accepted as models of terrain. Pentland (1984) also stated that "... the natural appearance of fractals is strong evidence that they capture all of the perceptually relevant shape structure of natural surfaces." We are unable to deny the resemblance of fractal model surfaces to some natural landscape surfaces. Furthermore, the human eye (and brain) has the unsurpassed ability to observe and interpret complex phenomena. So it is easy to infer that landscapes possess self-similar properties.

There appears to be an inherent connection between fractional Brownian processes, landscapes, and other natural phenomena. However, it is inappropriate to define the self-similarity of landscapes only based on intuitive visual appearances. "The ability to generate a visually appealing display based on a fractional Brownian model does not necessarily mean that the landform being represented is actually fractal, or that it obeys quantitative predictions of the model" (Piech and Piech, 1990). It is also premature to reject the fractal model of landscapes based on some spatially variable fractal dimensions that have been obtained to date. It appears that self-similarity is exhibited only in limited regions and over limited ranges of scale in landscapes. In the next few years it will be the main task of research to test the self-similarity property of landscapes (and other natural phenomena).

There is a view that spatial variations of D over landscape surfaces, namely scale-dependent variations and region-dependent variations, are perhaps reasonable. Burrough (1981) indicated that, "although many natural phenomena do display certain degrees of statistical self-similarity over many spatial scales, there are others that seem to be structured and have their levels of variability clustered at particular scales. This behaviour does not exclude them from the fractal concept." Mandelbrot (1977, 1982) considered that it is quite acceptable to have a series of zones of distinct dimensions connected by transition zones. If this is reasonable, it means that the examination of the variability of D would be useful for trying to separate scales of variation that might be the result of particular natural processes. Roy et al. (1987) observed that, "in fact, one should anticipate that the dimensionality of most natural terrains should vary spatially. Variations in processes and/or structures may be responsible for these changes in dimensions. ... Systematic variations in the dimension within the surface bear important cartographic consequences. ... more attention should be given to the fractal signature of characteristic terrains." Most studies more or less confirm the spatial variation of D over landscape surfaces, particularly the scale-dependent variation. Theoretically, the spatial variation does contradict the hypothesis of self-similarity of landscapes. But, as yet there is no explicit criterion for defining how large the region with homogeneous fractal properties for which the landscape is (statistically) self-similar needs to be, and how to delimit the region. The challenge in the future is to relate the spatially variable fractal dimensions to spatially variable landscape formation processes, and climate and geographical characteristics.

We have only measured D in limited regions of landscapes, and over restricted ranges of scale. There is also a large number of methods of calculating D. Therefore, it is not yet appropriate to define the regularity of the spatial variation of the fractal nature. However, one should be aware that the spatial variation of D might imply a more significant usefulness of fractals. In particular, the spatial variation of D across a landscape surface with different topographic characteristics may be more significant than the spatial variation with the change of the measurement distance h of the variogram or divider step r of the divider method. In other words, the region-dependent spatial variation has more practical significance than the scale-dependent spatial variation.

The log–log plots used to define D often have multiple linear segments (Mark and Aronson, 1984; Roy et al., 1987; Klinkenberg and Goodchild, 1992; Ouchi and Matsushita, 1992), or are divergent at one end of the plot for the different methods of calculation. Are these reasonable or not? There is still insufficient definitive evidence to answer this. Mandelbrot (1986b, 1989) considered that the presence of multiple linear segments of a log–log plot may indicate that the land surface is in fact a multifractal surface. However, this conclusion may be dubious. Sometimes the object from which one extracted the log–log plot is non-fractal and the log–log plot is actually an arc and the so-called "linear segment" is just a segment of the arc. Gallant et al. (1993) found that synthetic Brownian profiles also produce log–log plots with multiple linear segments.

Several researchers have raised questions about the fractal dimensions of non-self-similar landscapes. Kennedy and Lin (1986) stated that being non-fractal in nature is no limitation to the fractal dimension technique and it could be simply seen as a method for extracting useful information from a log–log plot. As an analogy, normal distribution models have been applied in the geo-sciences for a long time, and many natural phenomena appear to follow the normal distribution. Normal distribution models are often applied to objects without proving the distributive characteristic of the objects. Furthermore, often one can obtain reasonable results from normal distribution models even when the objects do not strictly follow the normal distribution. Some models for discrete data sets can not even be proved mathematically. We use these models because they usually maintain stable and reasonable relationships between them and the objects being investigated.

How to explain the differences in fractal dimensions?

Landscapes with different surface characteristics are expected to have different fractal dimensions, and, many studies have demonstrated this. Table 1 is a brief summary of Klinkenberg and Goodchild's (1992) results. Differences in the characteristics of landscape surfaces are not, however, the only reason to lead to differences in the computed fractal dimensions. The following factors should be considered:

(a) The fractal dimension should distinguish between landscapes with different "roughness" characteristics and agree with the visual observations. "If the fractal dimension is to be a useful parameter, the measurement methods used to determine the dimensions must be robust, consistent, and have the capability of differentiating between visibly dis-

TABLE 1

Fractal dimensions for seven physiographic regions in the United States computed using seven methods (from Klinkenberg and Goodchild, 1992)

Physiographic region	D_{all}	D_s	D_{ang}	D_{cc}	$D_{s.d.}$	$D_{t.d.}$	D_e
Coastal Plain	2.65	2.66	2.62	2.67	2.63	1.20	1.29
Blue Ridge	2.32	2.42	2.31	2.19	2.23	1.18	1.26
Valley and Ridge	2.36	2.52	2.34	2.23	2.32	1.15	1.23
Appalachian Plateaus	2.30	2.38	2.31	2.16	2.23	1.17	1.24
Interior Low Plateaus	2.41	2.45	2.42	2.21	2.28	1.19	1.27
Colorado Plateau	2.34	2.32	2.31	2.16	2.21	1.18	1.27
Basin and Range	2.29	2.33	2.25	2.10	2.21	1.12	1.18

D_{all}: Entire-surface variogram dimension (first segment values only).
D_s: Mean of the dimensions of four (or less) sections.
D_{ang}: Mean of the dimensions of six (or less) angles.
D_{cc}: Mean of the dimensions of five cell counting values.
$D_{s.d.}$: Mean of the dimensions of five surficial dividers values.
$D_{t.d}$: Mean of the five (or less) traditional dividers dimensions obtained from the contours.
D_e : Mean of the five (or less) equipaced polygon dimensions obtained from the contours.

similar surfaces" (Klinkenberg and Goodchild, 1992).

(b) The fractal dimensions computed using different methods in the same region of a landscape are often different. Except for problems with the method of calculation itself, it also raises the question of whether landscapes are self-similar and have fractal properties. Theoretically, "in a fractal and self-similar terrain, the values of D should be in agreement regardless of method used" (Roy et al., 1987). In addition, some methods are strictly based on the hypothesis that the landscape possesses fractal properties. For example, the variogram method assumes that landscapes have statistical properties similar to those of fractional Brownian surfaces, namely that the altitude is a Brownian function of latitude and longitude. "Unless the fractional Brownian model is first validated for surfaces under consideration, the variogram analysis would not necessarily be expected to yield a correct fractal (surface) dimension" (Piech and Piech, 1990).

(c) There are many methods for determining the fractal dimensions of landscapes. Furthermore, every method can be divided into several subclasses. Having so many methods is unfortunate because it could conceal problems such as differences in their mathematical basis and mechanisms and understanding of the physical significance of fractals. Klinkenberg and Goodchild (1992) reported that the variability in the computed fractal dimensions is more a function of the methods used than it is a reflection of any theoretical inadequacy of the self-similar fractal model. We could, perhaps, see some differences between methods such as the oriented (angled) variogram methods in a new light. They must reflect the oriented distribution characteristics of valleys and ridges of landscapes, i.e., the anisotropy of the large terrain features. The distributional structure of valleys and ridges are mainly controlled by geological tectonism. The tectonic stress field controlling the tectonic processes has strong oriented characteristics.

(d) The limited size of the regions surveyed and the spatial resolution of the DEM data limit the precision and stability of the computed fractal dimension. Most of the investigations only use a limited DEM matrix of 256 by 256 pixels or less (e.g. Shelberg et al., 1983; Huang and Turcotte, 1989; Klinkenberg, 1992; Klinkenberg and Goodchild, 1992; Ouchi and Matsushita, 1992). Theoretically, scale invariance of fractal measures allows one to "extrapolate from properties observed at one scale to the properties of a scale which has not been observed" (Gibert, 1989). That is the case in an exactly self-similar object. However, landscapes are not exactly self-similar and, at most, are statistically self-similar. Thus, one needs sufficient pixels to obtain the effective statistical parameters for the fractal models.

The DEMs used can only represent finite details of landscape surfaces because of the limited size range and the resolution of the DEM data. For example, according to the "Data User Guide 5 of DEMs of the U.S. Geological Survey (USGS)" (1987), "within a standard DEM, most terrain features are generalized by being reduced to grid nodes spaced at regular intersections in the horizontal plane. This generalization reduces the ability to recover positions of specific features less than the interval spacing during testing and results in a de facto filtering or smoothing of the surface during gridding." It is obviously impossible that "big rills have little rills, and little rills have smaller rills, until spatial saturation" (Thornes, 1990) under the finite resolution of DEMs. There are not enough hierarchically recursive subdivisions (or not enough levels of cascade) to obtain relative steady "statistical averages" of the fractal dimension. The limited size range and resolution are two reasons why steady fractal dimensions have not been obtained.

(e) The variability of the estimated D is quite large, even for synthetic fractal data. Gallant et al. (1993) show that the 95% confidence interval of D for the variogram method is about ± 0.1. This places limits on the use of D as a discriminator of different landscape types.

(f) Some methods of calculating the fractal dimension of landscape surfaces, particularly the variogram methods, use grid DEM data. The accuracy of DEM data could be an influential factor of scale-dependent variation and directional bias of D (Polidori et al., 1991; Klinkenberg and Goodchild, 1992). The controlling factors of DEM accuracy are the scale and resolution of source data used to generate the DEM, and the interpolation method of generation. Usually, fine-scale terrain of interpolated DEMs is smoother than the real one, particularly when using linear interpolation methods and sparse source data. The interpolation process has a smooth filter function, as mentioned above, and lowers the value of small scale D while there is no smooth function for the large scale terrain. This results in a discrepancy between small scale and large scale D. Polidori et al. (1991) stated that the discrepancy may be interpreted as a consequence of the smoothing interpolation process. Moreover, based on knowledge that the directional interpolation methods (e.g., bilinear interpolation) and the uneven distribution of source data causes local artificial anisotropy of interpolated DEMs, Polidori et al. (1991) stated that the variations of D with direction for short distances may be interpreted as a consequence of the anisotropic interpolation process. That is, it may be possible to use fractal analysis to say something about the scale at which artifacts produced by the interpolation method end and the true behaviour of the landscape begins. The interpolation artifact of DEMs is not the only factor to cause the scale-dependent variation and directional bias of D, but is likely to be a considerable one. In fact, many traditional topographic attributes are closely related to the accuracy of the DEMs, particularly those based on second derivatives of the surface such as plan and profile curvature.

The 7.5-minute DEMs from the USGS are used to calculate the surface D of landscapes by many researchers (Mark and Aronson, 1984; Clarke, 1986; Roy et al., 1987; Huang and Turcotte, 1989; Klinkenberg and Goodchild, 1992). These DEMs also contain the artifacts described above. The USGS has used four processes to generate 7.5-minute DEMs: the Gestalt Photo Mapper II (GPM2), manual profiling from photogrammetric stereomodels, stereomodel digitizing of contours, and derivation from digital line graph hypsography and hydrography categories. Among them, the latter three processes generate the DEM from the source profile, contour or spot data by using interpolation (usually bilinear interpolation). For example, in order to obtain the final DEM with a 30 m grid spacing, the process of manual profiling from photogrammetric stereomodels produces two new profiles between every two source profiles with 90 m spacing, using the bilinear interpolation method. The process results in different degrees of the smoothness of new profile pixels in different directions and results in the directional bias of D (Klinkenberg and Goodchild, 1992). Obviously, for the DEMs generated by this process, the reliability of the fine scale D is reduced.

Polidori et al. (1991) concluded that fractal geometry can contribute to DEM quality assessment through a simple technique that does not require a reference DEM. However, their conclusion was based on the hypothesis that the landscape surface is a fractional Brownian surface. Although scale-dependent variations of D have been found by many people, because of the differences between the fractional Brownian model and real landscapes and the complex factors that cause scale-dependent variation of D, Polidori et al.'s conclusion is tenable only under very limited situations.

(g) We usually assume that a landscape is statistically self-similar or self-affine when calculating its fractal dimension. The methods of calculating the dimensions are based on the statistical characteristics of landscape properties. Sometimes the statistical characteristics of a region do not describe the local properties and the detailed structure of the region very well, thereby failing to distinguish the actual differences between the objects being investigated. For example, the "mean," a common statistical parameter, often cannot differentiate different data sets. The different permutations {1,2,3,4,5}, {5,4,3,2,1} and {3,1,2,5,4} have the same mean, 3, but the mean can not be used to distinguish these permutations. In fact, some statistical models (parameters) often mask the detailed characteristics of the objects being investigated. Kaye (1989), for example, reported that carbonblack profiles with different local structures have the same fractal dimensions and that the fractal dimension does not tell us anything about the overall gross shape of the profile.

(h) Using only the fractal model with a single parameter, the fractal dimension, to represent landscapes is not sufficient. "Despite the success of single parameter fractal models in creating visually realistic images representing mountains, coastlines, the distribution of landmasses, moonscapes, etc., it seems that a description of these systems in terms of a fractal model with a single fractal dimensionality is not completely realistic" (Meakin, 1991). Research on fractals has mainly concentrated on the fractal dimension and little attention has been paid to other fractal features, although Fox and Hayes (1985) and Klinkenberg and Goodchild (1992), appear to be aware of this problem. Klinkenberg and Goodchild (1992) stated that, "following Fox and Hayes (1985), it is anticipated that the combination of the fractal dimension and the Gamma value will capture the essential characteristics of the land surfaces. Thus, although two land surfaces may have similar forms (i.e. values of D) when considered from a scale-independent view, the magnitude of that roughness may well be very different, and that difference will be captured by the value of Gamma." A multi-parameter fractal model should have a stronger capacity to describe and differentiate the properties of landscapes than a single parameter fractal model. Fourier power spectral analysis has been successfully applied in the geo-sciences. There are three related parameters, the frequency, the power spectral energy intensity and their orientation, which characterize the Fourier spectral characteristics of objects in a two-dimensional Fourier spectral domain.

(i) There may be actual physiographic differences between landscapes. However, these differences may not be distinguishable by visual and conventional means. Fractals capture the differences and yield different fractal dimensions and so apparently similar landscapes have different fractal characteristics. These differences may reflect the new information of landscapes that are not reflected in the conventional morphometric parameters of landscapes. However, we cannot confirm this yet and we do not really know the reasons why fractal analysis of landscapes yields different fractal dimensions.

Fractal dimension as a spatially variable parameter

As mentioned earlier, the fractal nature of landscapes appears to vary spatially and this spatial variation has potentially practical significance. Furthermore, the spatial variation of fractal parameters could classify the land surface into fractally homogeneous regions (Klinkenberg 1992). Despite this, there is still some resistance to using D as a spatial parameter in geo-models. Indeed, it is still very difficult to use D as a distributed parameter with high "spatial resolution", even though fractals could be used to describe the profile of a fine particle. The fractally homogeneous regions proposed by Klinkenberg (1992) were quite large and the spatial resolution of them is very coarse and far from that of conventional landscape parameters like altitude, slope gradient, aspect, etc. In fact, only differentiating physiographic provinces is of more limited practical significance.

To obtain relative steady and believable values of the fractal dimension, one has to have a DEM data matrix with a very large number of cells. For example, in the variogram methods outlined previously, one needs a large number of random samples within a large area under each distance increment in order to calculate $E[(B_H(x_2,y_2)-B_H(x_1,y_1))^2]$. Therefore, at present D should be regarded as an areally lumped parameter.

Although researchers have estimated the region-dependent spatial variation of D, they have not yet found the border between regions with different fractal properties. Also, they cannot explain whether the spatial variation is continuous or not. Most of the results reported to date are from regular grid DEM matrices, and each matrix is usually treated as a unit with homogeneous fractal characteristics. In view of the definition of fractals and the present methods of calculating D, it is difficult to determine the spatial variation of D and the pattern of this variation within a DEM matrix. Perhaps we could assume that the fractal nature matches the geomorphological unit or hydrological unit? Perhaps we should calculate the fractal characteristics inside the irregular boundary of the hydrological or geomorphological unit, i.e., within a catchment?

We postulate that the fractal properties of landscapes should be more sensitive to spatial location. We therefore need a fractal model with multi-parameter (characteristics). Klinkenberg (1992) indicates that, "the fractal dimension and associated variogram parameters have potential as useful general geomorphometric parameters and, like the results of any other general geomorphometry study, they could be used to bring to light those spatial variations and structures in the land surface that geomorphology attempts to explain."

Developing trends in the fractal analysis of landscapes

The applications of fractals to landscape analysis is a developing and immature field and much of the theoretical rigour of fractal geometry has not yet been exploited. We are attempting to find better methods with which to estimate the fractal characteristics of landscapes. New methods or techniques need to be tested and improved before they can be proved to be effective and reliable. Remote sensing, a technology that arose in the early 1970s, has provided a good example of its developing course. We quote a paragraph from Molenaar's paper in 1991: "that [remote sensing] too was a promising field in the seventies, the expected potentials were great, it was almost treated as a kind of magic with its own priesthood. The deception came when it seemed that the potentials could not be realised as easily and quickly as expected. Now remote sensing people have to fight hard to get their research funds in competition with others. They are no wonder children anymore. This does not mean that remote sensing is not a powerful tool, but rather that it takes much more time and hard work to realize its potentials, to extract from remote sensing data the information required in the different application fields." The problem with remote sensing was that it "took much more time and hard work", and many people did not have the scientific basis and methods to study and apply remote sensing technologies.

The fractal analysis of landscapes, as a part of the developing field of fractals, is also in its preliminary and descriptive stage, and lacks theoretical explanation and rigour. The physical significance of landscape fractal characteristics remains to be explained. It seems that the development of fractal theory has lagged behind many of the potential applications. Mandelbrot (1986b) argued that this situation is reasonable and is the usual pattern of science. Obviously, measuring the fractal dimensions of landscapes is not our final goal. We hope that fractal analysis will provide effective tools for characterizing landscape properties, and that fractal parameters, which possess definite physical significance, can be integrated with diverse geo-models. In order to do this, we need to strengthen research in geographical information theory as well as fractal theory. In the age of mathematics and computers, the conventional geographical and geomorphological theories have performed poorly, and geo-models have developed independently of each other without any unified and systematic theoretical basis. Therefore, it is necessary to structure a systematic geographical information theory (geo-information theory) used to describe objects, phenomena or processes at the earth's surface. Improvement in geoinformation theory will lead to improvements in the application of fractal geometry to landscapes.

Until now studies of the fractal analysis of landscapes and the fractal analysis of river networks, the main two branches of the fractal analysis of landscapes, have developed almost in parallel. They need to be integrated with one another to describe the surface structure of landscapes effectively. River networks are essential features for structuring the land surface. We can not verify whether or not surface fractals contain definite information about river networks. Spatial structures of river networks exhibit some difference from the spatial structures of profiles or contours, and should have different fractal characteristics.

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