Article

Tensor Modeling and Analysis for Vehicle Traffic

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Abstract: A single paragraph of about 200 words maximum. For research articles, abstracts should give a pertinent overview of the work. We strongly encourage authors to use the following style of structured abstracts, but without headings: (1) Background: Place the question addressed in a broad context and highlight the purpose of the study; (2) Methods: Describe briefly the main methods or treatments applied; (3) Results: Summarize the article's main findings; and (4) Conclusion: Indicate the main conclusions or interpretations. The abstract should be an objective representation of the article, it must not contain results which are not presented and substantiated in the main text and should not exaggerate the main conclusions.

Keywords: keyword 1; keyword 2; keyword 3 (list three to ten pertinent keywords specific to the article, yet reasonably common within the subject discipline.)

1. Introduction

Content

1. Related work.
2. Contribution.
3. Content.

2. Tensor Algebra

Table 1. Tensor Algebra Notation Summary.

$\boldsymbol{\mathcal{X}}, \mathbf{X}, \mathbf{x}, x$	Tensor, matrix, vector scalar.
$\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$	A $I_1 \times \cdots \times I_N$ tensor.
$ord(\boldsymbol{\mathcal{X}})$	The order of a tensor.
$x_{i_1 \cdots i_N}$	The $(i_1 \cdots i_N)$ entry of an N^{th}-order tensor.
$\mathbf{X}^{(n)}$	The n^{th} matrix element from a sequence of matrices.
$\mathbf{X}_{(n)}$	The n-mode matricization of a tensor.
$\otimes$	Outer product of two vectors.
$\otimes_{kron}$	Kronecker product of two matrices.
$\odot$	Khatri Rao product of two matrices.
$\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \rangle$	Inner product of two tensors.
$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \times_n \mathbf{U}$	The n-mode product of a tensor $\boldsymbol{\mathcal{X}}$ times a matrix $\mathbf{U}$ along the n dimension.
$[\![\boldsymbol{\lambda}/\boldsymbol{\mathcal{G}}, \mathbf{U}^{(1)}, \cdots, \mathbf{U}^{(N)}]\!]$	Kruskal notation of tensor decomposition models.
$rank_D(\boldsymbol{\mathcal{X}}) = R$	Tensor decomposition/CP rank.
$rank_{tc}(\boldsymbol{\mathcal{X}}) = (R_1, \cdots, R_N)$	Tensor multilinear/Tucker rank, where $R_n = rank(\mathbf{X}_{(n)})$.
$rank_k(\boldsymbol{\mathcal{X}})$	Tensor Kruskal-rank
——	———————————
$\boldsymbol{\mathcal{X}} * \boldsymbol{\mathcal{Y}}$	t-product of two tensors.
$\boldsymbol{\mathcal{X}} *_{\Phi} \boldsymbol{\mathcal{Y}}$	Φ-product of two tensors.
——	———————————
$\mathcal{H}(\cdot)/\mathcal{H}^{-1}(\cdot)$	Hankelization direct/inverse transformation.
$\mathcal{L}(\cdot)/\mathcal{L}^{-1}(\cdot)$	Löwnerization direct/inverse transformation.
——	———————————
$\boldsymbol{\mathcal{V}}_{\tau}$	Video of duration τ, represented as a tensor.
$\boldsymbol{\mathcal{B}}$	Background tensor.
$\boldsymbol{\mathcal{F}}$	Foreground tensor.
$\boldsymbol{\mathcal{T}}^{(N)}$	Vehicle traffic feature tensor with N embedded models.

1. Notation.
2. Basic tensor concepts.
3. Operators on tensors.
4. Tensorization definition and methods.
5. Tensor decompositions (E.G.)
    (a) CANDECOM/PARAFAC Decomposition
    (b) Tucker Decomposition
    (c) Tensor Robust PCA

3. Problem Statement and Mathematical Definition

Current traffic surveillance systems employ different data models on each stage, so there is no such unified model which allows to capture relationships among all stages. Multidimensional models, have proven to be very powerful for explicitly representing and extracting multidimensional structures in several fields, including signal processing [], machine learning [], and telecommunications [], to name just a few. Unfortunately, despite its high potential in several fields, multidimensional models have not yet been exploited for the vehicle traffic modeling.

3.1. Problem Statement

Given a traffic surveillance video $\boldsymbol{\mathcal{V}}_{\tau}$ of duration τ, we seek to formulate a complete and flexible tensor modeling for the supervision of moving vehicles traffic, which allows us to link various data models involved during the analysis of the moving vehicle behavior and its intrinsic interactions among multiple tasks such as: vehicle detection, counting, tracking, occlusion management and classification, as well as to speed up or facilitate data transformation by using certain mathematical operations of tensor algebra.

3.2. Mathematical Definition

The problem raised above can be understood as a multidimensional modeling of moving vehicles which we called Vehicle Traffic Feature (VTF) tensor model, in such a way that each data model employed, can be represented as a mode or dimension, i.e., $\mathcal{T} \in \mathbb{R}^{Model\,1 \times Model\,2 \times \cdots \times Model\,N}$. Moreover, other representations could be also derived by either fixing certain dimensions or applying some multidimensional operators on it, in order to study the behavior of moving vehicles at specific modes.

4. Tensor-based Vehicle Traffic Modeling

Traditional vehicle traffic models treat data as a one-dimensional features vector [], to later be used to capture the internal correlation of historical data. However, vehicle traffic data is multi-mode by experiments, e.g., features mode, time mode, vehicle class mode, occlusion mode, among others, therefore, the current models turn out to be inadequate to capture these multidimensional interactions. The proposal of a multidimensional model will preserve the multi-mode nature of data, while the use of tensor methods such as decompositions, will help to better capture correlations among all modes.

4.1. Traffic Surveillance Video Modeling

Given a traffic surveillance video modeled as a four-order tensor $\mathcal{V}_\tau \in \mathbb{R}^{W \times H \times D \times Time}$ of width and height $W \times H$ resolution with a duration of τ seconds, and where each pixel is mapped in some color-space in $\mathbb{R}^D$, such as binary, grayscale, and RGB. Then, we will assume that there exist some tensor decomposition model such that Equation **??** holds (see Figure **??**):

$$\mathcal{V}_\tau = \mathcal{B} + \mathcal{F} \tag{1}$$

where $\mathcal{B} \in \mathbb{R}^{W \times H \times D \times Time}$ represents the background, which can be modeled as a low-rank tensor that capture the lowest frequency component of the video, while $\mathcal{F} \in \mathbb{R}^{W \times H \times D \times Time}$ represents the foreground modeled as a sparse tensor containing motion information of the video. Note that in Figure **??** exists another tensor denoted by $\mathcal{F}_m \in \mathbb{R}^{W \times H \times Time}$, which is the mask of $\mathcal{F}$ that is obtained by applying some binary operations on it.

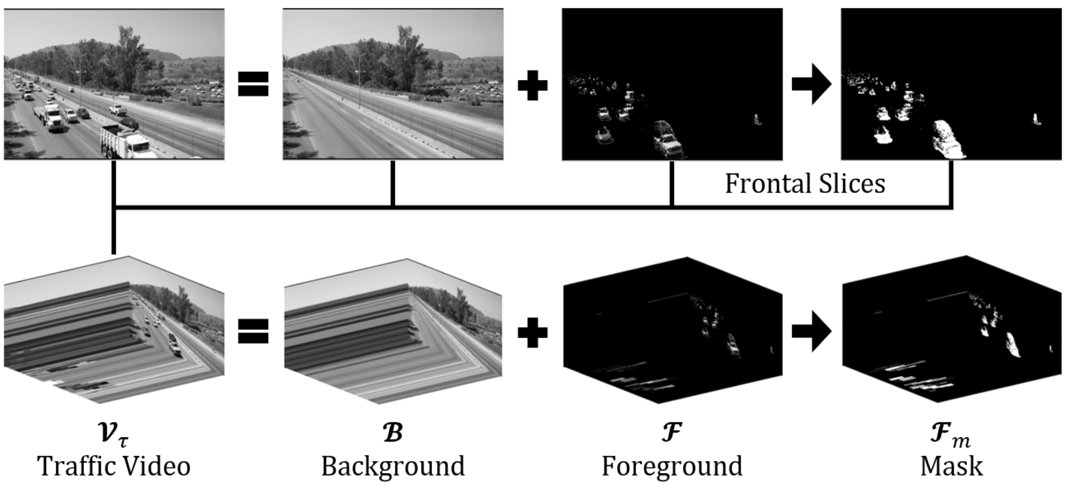

Figure 1. Illustration of the traffic surveillance video decomposition model.

For this model, there exist some methods and algorithms that successfully decompose a traffic surveillance video into the background and foreground components such as the Gaussian Mixture Model [X], Robust Principal Component Analysis [X], and the pixel-entropy [X] (see [X] for an extensive review on this decomposition). In this work, we will use the Tensor Robust Principal Component Analysis or t-RPCA in short, originally proposed by Lu, C., et. al., [X] to achieve such decomposition.

4.2. Moving Vehicle Traffic Tensor Modeling

From the foreground tensor $\mathcal{F}$, information about moving vehicles can be extracted such as their trajectory, geometry, kinematic or color information, which can be used later at a particular task of a VTS system. However, due to the high-volume of data and multimodality induced by multi-task VTS systems, a one-mode representation results to be not enough to exploit interactions among tasks.

To tackle the shortcomings of one-mode models, we proposed to arrange and group vehicle information as a high-order structure $\mathcal{T}^{(N)} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, here called Vehicle Traffic Feature (VTF) tensor, to model the vehicle behavior of a feature model over multiple tasks. Even though there is a vast set of features which can be extracted from $\mathcal{F}$, here we will focus on the geometric feature model.

In order to construct $\mathcal{T}^{(N)}$, for each detected vehicle on the road, we will record a set of geometric features modeled as vectors in $\mathbb{R}^n$ during a certain amount of time, while assuming that within the observation time, M vehicles will be detected and tracked. Based on these principles, we arrange the historical data of the ith vehicle into a matrix $\mathbf{T}_i \in \mathbb{R}^{F \times t}$, where F denotes the number of features, and t the time. Then, each $\mathbf{T}_i$ will be stacked along the third-dimension, such that the VTF tensor $\mathcal{T}^{(3)}$ of Equation **??** will be formed to model the temporal behavior of the vehicles geometric features.

$$\mathcal{T}^{(3)} \in \mathbb{R}^{Vehicle \times Geometric\ Features \times Time} \tag{2}$$

Following the above ideas, additional modes can also be added for a more generalized modeling of the vehicle geometric features. Specifically, in addition to the three modes of vehicles, geometric features and time, we also include the classification and occlusion modes using the VTF tensor model $\mathcal{T}^{(5)}$ as Equation **??** shows to extend our analysis to multiple stages. In this setting, the classification mode will categorize vehicles according to their sizes (e.g., small, midsize, and large), while the occlusion mode, the type of occlusion [] (e.g., non-occluded, lateral, and queue occlusions).

$$\mathcal{T}^{(5)} \in \mathbb{R}^{Vehicles \times Geometric\ Features \times Time \times Class \times Occlusion} \tag{3}$$

4.2.1. Multilinear Transformations over the Vehicle Traffic Feature Tensor

For any problem which involves ~~the use of~~ tensor structures, TD allow us to exploit the intrinsic ~~high-order~~ multidimensional structure of data, therefore, choosing an appropriate representation of data is a key ~~component~~ ingredient to improve any decomposition model. Multilinear ~~transformations, which aim to express a multidimensional vector space $V_1 \times \cdots \times V_n$ as their multilinear combination by applying some mapping function of several variables that is linear separately in each variable (see Equation ??) , allow to find other data representations~~ Transformations (MLT) [] allow us to represent high-order tensors in different ways by imposing some structured representation on data that could be more suitable for a particular TD.

$$f : V_1 \times \cdots \times V_n \to W$$

~~Similar to linear transformations, multilinear transformations also provide us different properties or structures that can be exploited according to the problem faced, hence the use of a particular transformation must be application dependent.~~ Here, we ~~will seek~~ are seeking for those functions which provide a higher-order representation of data while inducing well-known intrinsic structures~~. The first desired property will be needed for both preserving linear mixtures of the source space and avoids to introduce non-separable terms , while the second property will allow us to enjoy useful properties to be exploited in TD. For that purpose, we employed two deterministic multilinear transformations called~~ , where from all possible known MLT that hold the above, we focus on the deterministic Hankelization and Löwnerization (see Definition **??** and **??**), which ~~map the source space into a higher-order structure that have approximately an intrinsic low multilinear-rank x. On the other hand, these transformations enjoy some useful properties for data modeling, for example,~~

~~Hankel structures are known for representing exponential polynomials , while~~ transform the input space into an structured higher-order Hankel or Löwner ~~structures show a very close relationship with rational functions~~ tensor that intrinsically enforcing an approximately low multilinear-rank [x]~~,~~ ~~so that they can be used to model and approximate a wide variety of functions~~.

Definition 1 (Hankelization Transformation)**.** *The Kth-order Hankelization is a transformation that maps any vector* $\mathbf{x} \in \mathbb{R}^N$ *into a Kth-order* ~~*tensor*~~ *Hankel structured tensor* $\boldsymbol{\mathcal{H}}^{(K)} \in \mathbb{R}^{I_1 \times \cdots \times I_K}$ ~~*called Hankel tensor*~~ *with constant anti-diagonal hyperplanes as Equation* **??** *shows, where* $\mathcal{H}$ *is the Hankelization transformation, and* $N = \sum_{k=1}^{K} I_k - K + 1$.

$$\boldsymbol{\mathcal{H}}^{(K)} = \mathcal{H}(\mathbf{x}) : h^{(K)}_{i_1,\cdots,i_K} = x_{i_1+\cdots+i_K-K+1} \tag{4}$$

Definition 2 (Löwnerization Transformation)**.** *The Kth-order Löwnerization is a transformation that maps a vectorized function* $\mathbf{x}(\boldsymbol{\phi}) \in \mathbb{R}^N$ *evaluated at N points* $\boldsymbol{\phi} \in \mathbb{R}^N$ *into a Kth-order* ~~*tensor* $\boldsymbol{\mathcal{L}}^{(K)} \in \mathbb{R}^{I_1 \times \cdots \times I_K}$ *called*~~ *Löwner* ~~*tensor*~~ *structured tensor* $\boldsymbol{\mathcal{L}}^{(K)} \in \mathbb{R}^{I_1 \times \cdots \times I_K}$*, such that each entry is defined as Equation* **??** *shows, where* $\mathcal{L}$ *is the Kth-order Löwnerization transformation, while* $\mathbf{f}^{(p)}$ *is the pth disjoint subset of* $\boldsymbol{\phi}$*,* ~~*which holds that*~~ *i.e.,* $\mathbf{f}^{(p)} \bigcap \mathbf{f}^{(q)} = \emptyset \ \forall p \neq q$*, and* $\boldsymbol{\phi} = \bigcup_{k=1}^{K} \mathbf{f}^{(k)}$.

$$\boldsymbol{\mathcal{L}}^{(K)} = \mathcal{L}(\mathbf{x}) : \ell_{i_1,\cdots,i_K} = \sum_{k=1}^{K} \frac{x_{\phi_k^{(k)}}}{\prod_{p=1,p\neq k}^{K} (f_k^{(k)} - f_p^{(p)})} \tag{5}$$

~~An example of the second-order Hankelization~~

Both Hankel and Lö~~wnerization transformations are shown in Equations ?? and ?? respectively. In Equation ??, it is assumed that the set of evaluation points is~~ ~~$\mathbf{t} = \{1,\ 2,\ 3,\ 4\}$~~ ~~which is partitioned into the two disjoint subsets~~ ~~$\mathbf{f}_1 = \{1,\ 3\}$ and $\mathbf{f}_2 = \{2,\ 4\}$.~~

$$\boldsymbol{\mathcal{H}}^{(2)} = \mathcal{H}\left(\begin{bmatrix} 7 & 6 & 15 & 12 \end{bmatrix}\right) = \begin{bmatrix} 7 & 6 & 15 \\ 6 & 15 & 12 \end{bmatrix}$$

$$\boldsymbol{\mathcal{L}}^{(2)} = \mathcal{L}\left(\begin{bmatrix} 7 & 6 & 15 & 12 \end{bmatrix}\right) = \begin{bmatrix} \frac{7-6}{1-2} & \frac{7-12}{1-4} \\ \frac{15-6}{3-2} & \frac{15-12}{3-4} \end{bmatrix} = \begin{bmatrix} -1.0000 & 1.6667 \\ 9.0000 & -3.0000 \end{bmatrix}$$

~~While the use of~~ wner structures enjoy some useful properties for data modeling, for instance, Hankel structures are known for representing exponential polynomials with well-known ranks [], while Löwner structures show a very close relationship with rational functions that also show similar rank behaviors []. Therefore, they can be used to model and approximate a wide variety of functions. While these transformations appear to be beneficial ~~to exploit~~ for exploiting low-rank models in TD, ~~an increase in the volume of~~ the ~~target ambient space is inevitable due to the redundancy introduced by them~~number of elements in the transformed tensor increases exponentially with the number of dimensions. To alleviate the curse of dimensionality induced by these transformations, we must exploit the intrinsic structures of Hankel and Löwner ~~. For instance, a tensor–vector multiplication involving a Hankel tensor~~ ~~$\boldsymbol{\mathcal{H}}^K \in \mathbb{R}^{I_1 \times \cdots \times I_K}$~~ ~~can be performed in~~ ~~$\mathcal{O}(N \log(N))$~~ ~~flops using the FFT~~ in order to avoid the tensor construction at all, as proposed in [x]~~instead of the original~~ ~~$\mathcal{O}(\prod_{k=1}^{K} I_k)$~~ ~~flops required with the naive operation, where~~ ~~$N = \sum_{k=1}^{K} I_k$.~~ ~~Similar to the Hankel case, Löwner structures can avoid the curse of dimensionality in the case of equidistant points~~.

4.3. Factorization of the Vehicle Traffic Feature Tensor

Although we can extract a bunch set of multidimensional information to our VFT tensor in different ways, we will focus our attention on two fundamental problems: the pattern recognition for ~~ASDW~~ features anomaly detection and the use of multilinear transformations to enforce low-rank assumptions on tensor models. We start by analyzing the pattern recognition of anomaly features followed by low-rank models under multilinear transformations.

4.3.1. Pattern Recognition for Anomaly Features Detection

Through the VTF model we can exploit multi-mode relations by ~~Tensor Decompositions (TD)~~TD, which provide a powerful analytical tool for dealing with multidimensional statics. Common TD employed include the CP-decomposition, Tucker model, HoSVD, and an SVD-based on the t-product called t-SVD, where the choice of a particular decomposition must be application dependent. The CP model is generally used for latent factor extraction, while the Tucker model to uncover hidden pattern in data. On the other hand, the choice of the t-SVD is more suitable when dealing with oriented-tensors.

4.3.2. High-order Low-rank Model under Multilinear Transformations

A classical model employed in many real-world applications is the (multidimensional) low-rank model (LRM) [], which assumes that data have approximately low intrinsic dimensionality, e.g. lie either on some low-dimensional subspace or manifold [15,46], or is sparse in some basis [13]. Formally, LRM states that ~~a~~ any tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times \cdots I_N}$ can be decomposed as $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{N}}$, where $\boldsymbol{\mathcal{L}} \in \mathbb{R}^{I_1 \times \cdots I_N}$ is a low-rank tensor, and $\boldsymbol{\mathcal{N}} \in \mathbb{R}^{I_1 \times \cdots I_N}$ a residual tensor, a problem that can be posed as an optimization problem, and is commonly solved by minimizing the Frobenius norm of $\boldsymbol{\mathcal{N}}$ as Equation ~~**??**~~ **??** shows.

$$\cancel{\min_{\boldsymbol{\mathcal{L}}} \boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{L}}_F^2} \quad \begin{aligned} \min_{\boldsymbol{\mathcal{L}}} \quad & \|\boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{L}}\|_F^2 \\ \text{s.t.} \quad & rank_t(\boldsymbol{\mathcal{L}}) \leq \mathbf{r} \end{aligned} \tag{6}$$

where $rank_t(\cdot)$ is the high-order generalization of the matrix rank function which can take several forms, e.g., decomposition rank of the CPD model, the multi-linear rank from the Tucker decomposition, or the tensor tubal rank on the t-product operator. For a complete review on tensor ranks, see [x].

However, the brittleness of LRM with respect to grossly corrupted observations, often causes non optimal solutions []. For this reason, it is very common to replace ~~**??**~~ **??** by the robust model $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}$, called Robust LRM (RLRM), which is formulated as the optimization problem of Equation **??**, where $\|\cdot\|_\star$ denotes the tensor nuclear norm, $\|\cdot\|_1$ the L_1-norm, and $\boldsymbol{\mathcal{S}} \in \mathbb{R}^{I_1 \times \cdots I_N}$ is a sparse tensor which models grossly corruptions. It should be notice that this decomposition model can be consider as a generalization of Equation ~~**??**~~**??**, since in the case of free-corruption, Equation **??** reduces to ~~**??**~~**??**.

$$\begin{aligned} \min_{\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{S}}} \quad & \|\boldsymbol{\mathcal{L}}\|_\star + \|\boldsymbol{\mathcal{S}}\|_1 \\ \text{s.t.} \quad & \boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{S}} \end{aligned} \tag{7}$$

Although there exist many related works that successfully solve these models [15, 34], ~~the RLRM lacks~~ both LRM and RLRM lack optimality when the intrinsic structure of a tensor is of high-rank, a situation that often happen in practice, which leads to a global solution that is not exactly low-rank. However, recent studies based on matrix LRM [] show that we can obtain better performance by exploiting the intrinsic low-rank structure which ~~results~~ arises after transforming a matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$ into a new representation $\mathbf{Y} \in \mathbb{R}^{m \times n}$ either using linear or non-linear transformations. For second-order tensors (matrices), there exist some models that exploit the intrinsic low-rank structure of the transformed representation such as Structured Total Least Norm (STLN) [], and kernel PCA []. Furthermore, recently Li, C., et. al, [x], study a LRM-based Matrix Completion under Multiple linear Transformations (MCMT). Following these ideas, we propose ~~a~~ the Tensor RLRM under Multilinear Transformation (~~TRLRMMT~~TRLRM-MT).

We formulate the ~~TRLRMMT~~ TRLRM-MT problem as finding a low-rank approximation of a tensor $\boldsymbol{\mathcal{X}}$ under some multilinear transformation Φ, i.e., $\Phi(\boldsymbol{\mathcal{X}})$, while considering the low-rank structures of the transformed tensor. Formally, we can model the ~~TRLRMMT~~ TRLRM-MT problem as Equation **??** shows.

~~**********NOTA**********~~ **NOTA**. Para el problema MCMT, considera la estructura de bajo rango de la transformacion porque lo mete en el problema de optimizacion dentro de la norma, aqui tambien lo hacemos? A mi parecer si, puesto que la restriccion se termina introduciendo en la funcion de lagrange aumentada (sin restricciones), mas no en las normas, cosa contraria a MCMT.

$$\begin{aligned} \min_{\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{S}}} \quad & \|\boldsymbol{\mathcal{L}}\|_\star + \|\boldsymbol{\mathcal{S}}\|_1 \\ \text{s.t.} \quad & \Phi(\boldsymbol{\mathcal{X}}) = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{S}} \end{aligned} \tag{8}$$

~~*********~~ **NOTA**. ¿ESTO IRÍA EN RESULTADOS? ~~*********.~~

Here, we employ the ~~TRLRMMT~~ TRLRM-MT for modeling and approximation of features, and system complexity reduction []. For that purpose, we will assume that the behavior of any vehicle geometric feature over time can be ~~modeled~~ well-approximated by either polynomial or rational functions. Then, we ~~employ~~ combine the ideas of [5] ~~combined with the HoRLRM^{2}T~~ with our TRLRM-MT to model the VTF tensor $\boldsymbol{\mathcal{T}}^{(N)}$ as either polynomial or rational functions through the Hankelization or Löwnerization multilinear transformations respectively, in order to form a new space, i.e., $\boldsymbol{\mathcal{Z}} = \Phi(\boldsymbol{\mathcal{T}}^{(\mathcal{N})})$, while reducing the ~~transformed space complexity using~~ space complexity after approximating the transformed tensor as $\boldsymbol{\mathcal{Z}} \approx \hat{\boldsymbol{\mathcal{Z}}}$ via RLRM. Finally, we recover the original domain after applying the inverse transformation ~~previously employed~~ $\hat{\boldsymbol{\mathcal{T}}}^{(N)} \Phi^{-1}(\hat{\boldsymbol{\mathcal{Z}}})$.

~~**********NOTA**********~~ **NOTA**. EMPLEAMOS LOS MODELOS SEGUIDOS DE UN TRUNCAMIENTO DE LA T-SVD, DESCARTANDO AQUELLOS VALORES SINGULARES MULTIDIMENSIONALES QUE NO APORTEN SUFICIENTE INFORMACION DE ACUERDO A SU DISTRIBUCION. ADICIONALMENTE EL USO DE T-RPCA PARA GENERAR MODELOS MAS ROBUSTOS FRENTE A CORRUPCION EN LOS DATOS. PRESENTAR CONTRIBUCIONES EN EL MODELADO DE DATOS EN EL USO DE ESTAS DOS TRANSFORMACIONES, ASI COMO CARACTERISTICAS DE LOS TENSORES RESULTANTES COMO LO SON SUS RANGOS, TRABAJOS RELACIONADOS CON CP Y TUCKER.

5. Experiments

5.1. Test Evaluation

5.2. Modeling~~and~~, Approximation and Complexity Reduction of ~~Features~~the Geometric Feature Space

1. Modelado y approximación del espacio de caracteristicas geometricas.
2. Aplicacion a series temporales
3. Reduccion de la complejidad de sistemas.
4. Impacto de la reducción de la complejidad en tareas de clasificación (SVM, numero de vectores soporte, RF, NN, etc).
5. Evaluación de la perdida de información en base a entropía.

6. Discussion

Reconocimiento de patrones para ASDW, a partir de los cuales ...

Conexion de tensores hankel con la complejidad de sistenas multidimensionales multicanal, la complejidad de la tarea de clasificacion o del input space se ve reducida.

Hankel aproxima funciones exponenciales del tipo a^n, las caracteristicas geometricas tienen este comportamiento y pueden ser aproximadas por tensores hankel, explotamos el bajo rango que presenta esta transformacion.

Loewner aproxima funciones racionales, sin embargo existe una relacion entre hankel y loewner, de esta manera podemos emplearlo del mismo modo para modelar las caracteristicas geometricas empleando funciones racionales, en conjunto con t-RPCA, se logra una aproximacion mas robusta y exacta, pero no presenta bajo rango. Estudiar las propiedades de estos tensores.

7. Conclusions

CONCLUSIONES DEL LOW-RANK ~~Y~~ CON TRANSFORMACIONES. ~~CUANDO CONOCEMOS LA FORMA ALGEBRAICA DE LAS CARACTERISTICAS SE EMPLEA HANKEL+T-SVD, DE OTRO MODO LOEWNER + T-RPCA, EL USO DE ESTOS MODELOS DE BAJO RANGO NOS AYUDAN A ELIMINAR DISTURBIOS EN LOS DATOS, LO QUE SE TRADUCE EN UNA REDUCCION DE LA COMPLEJIDAD DE LOS DATOS, UNA MENOR COMPLEJIDAD EN LAS FUNCIONES DE CLASIFICACION (SIZE, OCCLUSION)~~La aplicación de una transformación multilineale apropiada (adecuada) a nuestros datos resulta ventajosa para explotar las propiedades que esta misma nos provee, por ejemplo, con Hankel y Loewner pudimos explotar las propiedades intrinsecas de bajo rango en modelos de descomposicón tensoriales. Los modelos de descomposicion robustos con Las estructuras Loewner resultan ser adecuadas cuando las propiedades en el espacio de funciones son desconocidas. Por otro lado, los modelos de descomposición LRM junto con estructuras Hankel resultan ser adecuadas cuando se conoce la geometria de las funciones a modelar. El uso de las aproximaciones de bajo rango bajo estos esquemas, permiten una reducción en la complejidad del espacio de caracteristicas, con lo cual, se pudo comprobar que las tareas de clasificacion muestran tambien una menor complejidad cuando el espacio de entrada es de baja complejidad.

TRABAJO FUTURO:

1. ~~Uso~~ Estudio de otras transformaciones multilineales como lo es ~~la~~ *Segmentation*, *Stochastic*, etc.
2. ~~Transformaciones~~ Estudio de transformaciones no lineales (*Kernel t-RPCA*).
3. Descomposiciones tensoriales con restricciones estructuradas (*Hankel*, *Loewner*, etc.): "*Structured Tensor Decompositions*" a fin de explotar las propiedades de las estructuras de una manera mas ~~eficientes~~ eficiente, asi mismo contemplandolas en el problema de minimización (Articulo: *Exploiting efficient representations in large-scale tensor decompositions*).
4. Empleo de nuestro modelo tensorial para la deteccion de anomalías en vehiculos con aplicacion para la detección de oclusiones (Nuestro Articulo).

Author Contributions: For research articles with several authors, a short paragraph specifying their individual contributions must be provided. The following statements should be used "conceptualization, X.X. and Y.Y.; methodology, X.X.; software, X.X.; validation, X.X., Y.Y. and Z.Z.; formal analysis, X.X.; investigation, X.X.; resources, X.X.; data curation, X.X.; writing–original draft preparation, X.X.; writing–review and editing, X.X.; visualization, X.X.; supervision, X.X.; project administration, X.X.; funding acquisition, Y.Y.", please turn to the CRediT taxonomy for the term explanation. Authorship must be limited to those who have contributed substantially to the work reported.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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Sample Availability: Samples of the compounds ...... are available from the authors.