Article

Tensor Modeling and Analysis for Vehicle Traffic

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Abstract: A single paragraph of about 200 words maximum. For research articles, abstracts should give a pertinent overview of the work. We strongly encourage authors to use the following style of structured abstracts, but without headings: (1) Background: Place the question addressed in a broad context and highlight the purpose of the study; (2) Methods: Describe briefly the main methods or treatments applied; (3) Results: Summarize the article's main findings; and (4) Conclusion: Indicate the main conclusions or interpretations. The abstract should be an objective representation of the article, it must not contain results which are not presented and substantiated in the main text and should not exaggerate the main conclusions.

Keywords: keyword 1; keyword 2; keyword 3 (list three to ten pertinent keywords specific to the article, yet reasonably common within the subject discipline.)

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1. Introduction

Content

1. Related work.
2. Contribution.
3. Content.

2. Tensor Algebra

Table 1. Tensor Algebra Notation Summary.

$\boldsymbol{\mathcal{X}}, \mathbf{X}, \mathbf{x}, x$	Tensor, matrix, vector scalar.
$\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$	A $I_1 \times \cdots \times I_N$ tensor.
$ord(\boldsymbol{\mathcal{X}})$	The order of a tensor.
$x_{i_1 \cdots i_N}$	The $(i_1 \cdots i_N)$ entry of an N^{th}-order tensor.
$\mathbf{X}^{(n)}$	The n^{th} matrix element from a sequence of matrices.
$\mathbf{X}_{(n)}$	The n-mode matricization of a tensor.
$\otimes$	Outer product of two vectors.
$\otimes_{kron}$	Kronecker product of two matrices.
$\odot$	Khatri Rao product of two matrices.
$\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \rangle$	Inner product of two tensors.
$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \times_n \mathbf{U}$	The n-mode product of a tensor $\boldsymbol{\mathcal{X}}$ times a matrix $\mathbf{U}$ along the n dimension.
$[\![\lambda / \boldsymbol{\mathcal{G}}, \mathbf{U}^{(1)}, \cdots, \mathbf{U}^{(N)}]\!]$	Kruskal notation of tensor decomposition models.
$rank_D(\boldsymbol{\mathcal{X}}) = R$	Tensor decomposition/CP rank.
$rank_{tc}(\boldsymbol{\mathcal{X}}) = (R_1, \cdots, R_N)$	Tensor multilinear/Tucker rank, where $R_n = rank(\mathbf{X}_{(n)})$.
$rank_k(\boldsymbol{\mathcal{X}})$	Tensor Kruskal-rank
——	——————————
$\boldsymbol{\mathcal{X}} * \boldsymbol{\mathcal{Y}}$	t-product of two tensors.
$\boldsymbol{\mathcal{X}} *_\Phi \boldsymbol{\mathcal{Y}}$	Φ-product of two tensors.
——	——————————
$\mathcal{H}(\cdot) / \mathcal{H}^{-1}(\cdot)$	Hankelization direct/inverse transformation.
$\mathcal{L}(\cdot) / \mathcal{L}^{-1}(\cdot)$	Löwnerization direct/inverse transformation.
——	——————————
$\boldsymbol{\mathcal{V}}_\tau$	Video of duration τ, represented as a tensor.
$\boldsymbol{\mathcal{B}}$	Background tensor.
$\boldsymbol{\mathcal{F}}$	Foreground tensor.
$\cancel{\boldsymbol{\mathcal{F}}}\ \boldsymbol{\mathcal{T}}^{(N)}$	Vehicle traffic ~~features tensor~~ feature tensor with N embedded models.

1. Notation.
2. Basic tensor concepts.
3. Operators on tensors.
4. Tensorization definition and methods.
5. Tensor decompositions (E.G.)
   (a) CANDECOM/PARAFAC Decomposition
   (b) Tucker Decomposition
   (c) Tensor Robust PCA
   (d) Non-negative Tensor Decomposition

3. Problem Statement and Mathematical Definition

Current traffic surveillance systems employ different data models on each stage, so there is no such unified model which allows to capture relationships among all stages. Multidimensional models, have proven to be very powerful for explicitly representing and extracting multidimensional structures in several fields, including signal processing [], machine learning [], and telecommunications [], to name just a few. Unfortunately, despite its high potential in several fields, multidimensional models have not yet been exploited for the vehicle traffic modeling.

3.1. Problem Statement

Given a traffic surveillance video $\boldsymbol{\mathcal{V}}_\tau$ of duration τ, we seek to formulate a complete and flexible tensor modeling for the supervision of moving vehicles traffic, which allows us to link various data models involved during the analysis of the moving vehicle behavior and its intrinsic ~~relations in~~ interactions among multiple stages such as: detection, counting, tracking, occlusion management and classification; speed up or facilitate data transformation by using certain mathematical operations of tensor algebra.

3.2. Mathematical Definition

The problem raised above can be understood as a multidimensional modeling of moving vehicles which we called Vehicle Traffic Feature (VTF) tensor model, in such a way that each data model employed, can be represented as a mode or dimension, i.e., $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{Model\,1 \times Model\,2 \times \cdots \times Model\,N}$. From this model, other representations could be also derived by either fixing certain dimensions or applying some multidimensional operators on it, in order to study the behavior of moving vehicles at specific modes.

4. Tensor-based Vehicle Traffic Modeling

Traditional vehicle traffic models treat data as a one-dimensional features vector, to later be used to capture the internal correlation of historical data. However, vehicle traffic data is multi-mode by experiments, e.g., features mode, time mode, vehicle class mode, occlusion mode, among others, therefore, the current models turn out to be inadequate to capture these multidimensional interactions. The proposal of a multidimensional model will preserve the multi-mode nature of data, while the use of tensor methods such as decompositions, will help to better capture correlations among all modes.

4.1. Traffic Surveillance Video Modeling

Given a traffic surveillance video modeled as a four-order tensor $\boldsymbol{\mathcal{V}}_\tau \in \mathbb{R}^{W \times H \times D \times Time}$ of $W \times H$ resolution and a duration of τ seconds, and where each pixel is mapped in some color-space of dimension D, e.g., grayscale, RGB, then, we will assume that there exist some tensor decomposition model such that the following Equation holds (see Figure **??**):

$$\boldsymbol{\mathcal{V}}_\tau = \boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{F}} \tag{1}$$

where $\boldsymbol{\mathcal{B}} \in \mathbb{R}^{W \times H \times D \times Time}$ is a low-rank tensor which capture low-frequency components the video, i.e., the background, while $\boldsymbol{\mathcal{F}} \in \mathbb{R}^{W \times H \times D \times Time}$ is a sparse tensor that contains motion information on the video, in other words it represents the foreground. Note that in Figure **??** exists another tensor denoted by $\boldsymbol{\mathcal{F}}_m \in \mathbb{R}^{W \times H \times Time}$, which is the binary mask of $\boldsymbol{\mathcal{F}}$.

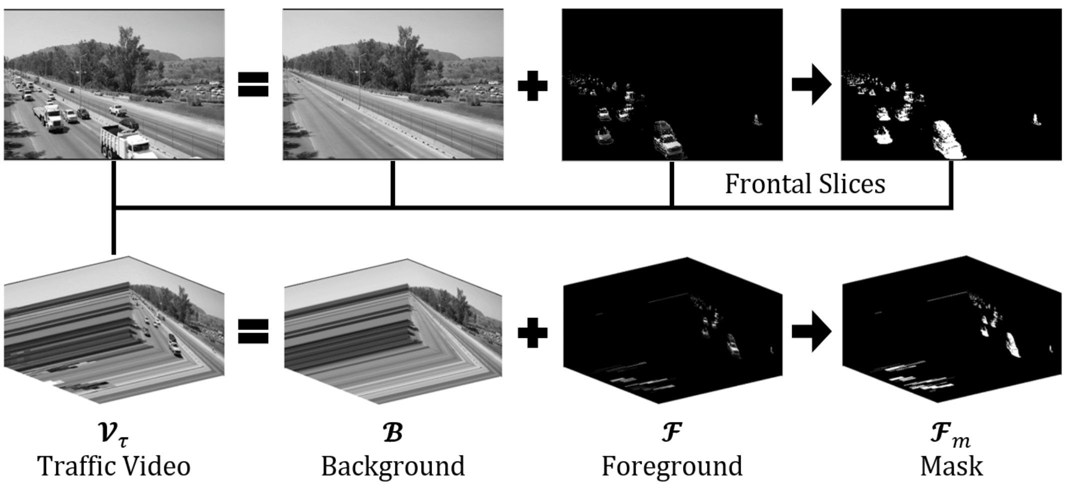

Figure 1. Illustration of the traffic surveillance video decomposition model.

For this model, there exist some methods and algorithms that successfully decompose a traffic surveillance video into the background and foreground components such as the Gaussian Mixture Model and Robust Principal Component Analysis (see [X] for an extensive review on this decomposition). In this work, we will use a modified version of the Tensor Robust Principal Component Analysis or ~~TRPCA~~ t-RPCA in short, originally proposed by Lu, C., et. al., [X] to achieve such decomposition.

4.2. Moving Vehicle Traffic Tensor Modeling

From the foreground tensor $\mathcal{F}$, information about moving vehicles can be extracted such as their trajectory, geometry, kinematic ~~and color informationby analyzing e.g., the connected components or the historical motion data in $\mathcal{F}_m$~~or color information, which can be used later at a particular stage ~~on a traffic surveillance~~ of a VTS system. However, due to the ~~high volume and the multi-mode nature of the data~~ high-volume of data and multimodality induced by multi-stage VTS systems, a one-mode representation results to be not enough to exploit ~~relations among all modes~~interactions among stages.

To tackle the shortcomings of one-mode models, we proposed to arrange and group ~~the moving vehicle data into~~ vehicle information as a high-order ~~tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, which will be called here~~ structure $\mathcal{T}^{(N)} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ called Vehicle Traffic Feature ~~tensor, where its order $ord(\mathcal{T}) = N$ will be equal to the number of data models to be used. Therefore, whenever we want to include a new data model, the order of the VTF tensor will increase by one. This makes it possible to have a flexible model in the face of new data models.~~ (VTF) tensor to model the vehicle behavior of a feature model over multiple stages. Even though there is a vast set of features which can be extracted from $\mathcal{F}$, here we will focus on the geometric feature model, for which we will assume that the behavior of any feature over time can be described by polynomial functions.

~~Unless otherwise stated, in the following sections we will refer to the VTF tensor as the five-order tensor with the following dimensions: $Vehicles \times Geometric\ Features \times Time \times Classification \times Occlusion$. QUIZA ESTO MEJOR EN LOS EXPERIMENTOS.~~

4.3. ~~*Factorization of the Vehicle Traffic Feature Tensor*~~

~~Although we can extract a bunch set of multidimensional information to our VFT tensor in different ways, we will focus our attention on two fundamental problems: the pattern recognition and the use of multilinear transformations to enforce low-rank assumptions on tensor models. We start by analyzing the pattern recognition followed by multilinear transformations.~~

4.2.1. ~~Pattern Recognition~~

~~Through the VTF tensor we can exploit the multi-mode correlations, for example, classification-mode, occlusion-mode, and temporal-mode, by Tensor Decompositions (TD), which provide a powerful analytical tool for dealing with multidimensional statics. Common TD employed include the CP-decomposition, Tucker model, HoSVD, and an SVD-based on the t-product called t-SVD, where the choice of a particular decomposition must be application dependent.The CP model is generally used for latent factor extraction, while the Tucker model to uncover hidden pattern in data.On the other hand, the choice of the t-SVD is more suitable when dealing with oriented-tensors~~In order to construct $\mathcal{T}^{(N)}$, for each detected vehicle on the road, we will record a set of geometric features modeled as vectors in $\mathbb{R}^n$ during a certain amount of time, while assuming that within the observation time, M vehicles will be detected and tracked. Based on these principles, we arrange the historical data of the ith vehicle into a matrix $\mathbf{T}_i \in \mathbb{R}^{F \times t}$, where F denotes the number of features, and t the time. Then, each $\mathbf{T}_i$ will be stacked along the third-dimension, such that the VTF tensor $\mathcal{T}^{(3)}$ of Equation **??** will be formed to model the temporal behavior of the vehicles geometric features.

$$\mathcal{T}^{(3)} \in \mathbb{R}^{Vehicle \times Geometric\ Features \times Time} \tag{2}$$

Following the above ideas, additional modes can also be added for a more generalized modeling of the vehicle geometric features. Specifically, in addition to the three modes of vehicles, geometric features and time, we also include the classification and occlusion modes using the VTF tensor model $\boldsymbol{\mathcal{T}}^{(5)}$ as Equation ?? shows to extend our analysis to multiple stages. In this setting, the classification mode will categorize vehicles according to their sizes (e.g., small, midsize, and large), while the occlusion mode, the type of occlusion [] (e.g., non-occluded, lateral, and queue occlusions).

$$\boldsymbol{\mathcal{T}}^{(5)} \in \mathbb{R}^{Vehicles \times Geometric\ Features \times Time \times Classification \times Occlusion} \quad (3)$$

4.2.1. ~~Low-rankness by~~ Multilinear Transformations over the Vehicle Traffic Feature Tensor

For ~~pattern learning in data, tensor decompositions~~ any problem which involves the use of tensor structures, TD allow us to exploit the ~~high-dimensional structure of the input space~~intrinsic high-order structure of data, therefore, choosing an appropriate representation of data is a key component to improve ~~pattern recognition~~any decomposition model. Multilinear transformations, which aim to express a multidimensional vector space $V_1 \times \cdots \times V_n$ as their multilinear combination by applying some mapping function ~~, i.e.,~~ of several variables $f : V_1 \times \cdots \times V_n \to W$ ~~, where V_i and W are vector spaces. It allows to find several representations of data that will~~ that is linear separately in each variable, allow to find new data representations that could be more suitable for a particular ~~tensor decomposition model~~TD.

~~As for~~ Similar to linear transformations, ~~a multilinear transformation provides different properties than~~ multilinear transformations also provide us different properties or structures [] that can be exploited according to the problem faced, hence the use of a ~~certain transformation will be also~~ particular transformation must be application dependent. Here, we will seek for those ~~multilinear~~ functions which provide a higher-order representation of data while ~~preserving intrinsic low-rank structures. These properties~~ inducing well-known intrinsic structures. The first desired property will be needed for both preserving linear mixtures of the ~~input~~ source space and avoiding to introduce non-separable terms ~~to~~ [], while the second property allows to enjoy useful properties to be exploited in ~~tensors decomposition~~TD. For that purpose, we ~~employ two~~ employed two deterministic multilinear transformations called Hankelization and Löwnerization [], which map the ~~input space to~~ source space into a higher-order ~~representations that approximately have~~ structure that have approximately a intrinsic low multilinear-rank~~, where~~. On the other hand, these transformations enjoy some useful properties for data modeling, for example, Hankel structures are known ~~to represent exponential polynomials~~ for representing exponential polynomials [], while Löwner structures show a very close relationship with rational functions [], so that they can be used to model and approximate a wide variety of functions~~such as sinusoidals~~.

The Kth-order Hankelization is a transformation that maps a vector ~~$\mathbf{x} \in \mathbb{K}^N$~~ $\mathbf{x} \in \mathbb{R}^N$ into a K-order tensor ~~$\boldsymbol{\mathcal{H}}^{(K)} \in \mathbb{K}^{I_1 \times \cdots \times I_K}$~~ $\boldsymbol{\mathcal{H}}^{(K)} \in \mathbb{R}^{I_1 \times \cdots \times I_K}$ called Hankel tensor ~~which contain~~ with constant anti-diagonal ~~hyperplanes with constant entries defined by Equation ??~~ hyperplanes as Equation ?? shows, where $\mathcal{H}$ represents the Hankelization transformation, and $N = I_1 + \cdots + I_K - K + 1$. On the other hand, the Kth-order Löwnerization is a transformation that maps a ~~vector $\mathbf{x} \in \mathbb{K}^N$~~ vectorized function $\mathbf{f} \in \mathbb{R}^N$ evaluated at N points $\boldsymbol{\phi} \in \mathbb{R}^N$ into a K-order tensor ~~$\boldsymbol{\mathcal{L}} \in \mathbb{K}^{I_1 \times \cdots \times I_K}$~~ $\boldsymbol{\mathcal{L}}^{(K)} \in \mathbb{R}^{I_1 \times \cdots \times I_K}$ called Löwner tensor such that each entry is defined as Equation ?? shows, where ~~$\phi^{(k)}_{i_k}$ represents ASASASAS, while $f^{(p)}_{i_p}$ BLABLABLA. The above ideas can also be extended to higher-order tensors in a straightforward way.~~ $\mathcal{L}$ represents the Kth-order Löwnerization transformation, while $\mathbf{f}^{(p)}$ is the pth subset of $\boldsymbol{\phi}$, which holds that $\mathbf{f}^{(p)} \bigcap \mathbf{f}^{(q)} = \emptyset$ and $\boldsymbol{\phi} = \bigcup_{k=1}^{K} \mathbf{f}^{(k)}$.

$$\boldsymbol{\mathcal{H}}^{(K)} = \mathcal{H}(\mathbf{x}) : \boldsymbol{\mathcal{H}}^{(K)}_{:,:,i_3,\cdots,i_K} = \mathcal{H}(\boldsymbol{\mathcal{H}}^{(K-1)}_{:,i_3,\cdots,i_K}),\ h^{(K)}_{i_1,\cdots,i_K} = x_{i_1+\cdots+i_K-K+1}$$

$$\boldsymbol{\mathcal{H}}^{(K)} = \mathcal{H}(\mathbf{x}) : h^{(K)}_{i_1,\cdots,i_K} = x_{i_1+\cdots+i_K-K+1} \quad (4)$$

$$\mathcal{L} = \mathcal{L}(\mathbf{x}) : \ell_{i_1,\cdots,i_K} = \sum_{k=1}^{K} \frac{x^{(k)}_{\phi_{i_k}}}{\prod_{m=1,m\neq k}^{K} (f^{(k)}_{i_k} - f^{(m)}_{i_m})}$$

$$\boldsymbol{\mathcal{L}}^{(K)} = \mathcal{L}(\mathbf{x}) : \ell_{i_1,\cdots,i_K} = \sum_{k=1}^{K} \frac{x_{\phi^{(k)}_{k}}}{\prod_{m=1,m\neq k}^{K} (f^{(k)}_{k} - f^{(m)}_{m})} \tag{5}$$

An example of the second-order Hankelization and Löwnerization transformations are shown in Equations ?? and ?? respectively. In Equation ??, it is assumed that the set of evaluation points is $\mathbf{t} = \{1, 2, 3, 4\}$ which is partitioned into the two disjoint subsets $\mathbf{f}_1 = \{1, 3\}$ and $\mathbf{f}_2 = \{2, 4\}$.

$$\boldsymbol{\mathcal{H}}^{(2)} = \mathcal{H}\left(\begin{bmatrix}7 & 6 & 15 & 12\end{bmatrix}\right) = \begin{bmatrix}7 & 6 & 15\\ 6 & 15 & 12\end{bmatrix} \tag{6}$$

$$\boldsymbol{\mathcal{L}}^{(2)} = \mathcal{L}\left(\begin{bmatrix}7 & 6 & 15 & 12\end{bmatrix}\right) = \begin{bmatrix}\frac{7-6}{1-2} & \frac{7-12}{1-4}\\ \\ \frac{15-6}{3-2} & \frac{15-12}{3-4}\end{bmatrix} = \begin{bmatrix}-1.0000 & 1.6667\\ 9.0000 & -3.0000\end{bmatrix} \tag{7}$$

While the use of these transformations appear to be beneficial to exploit low-rank models in TD, an increase in the volume of the ambient space is inevitable due to the introduced redundancy. To alleviate the curse of dimensionality induced by these transformations, we must exploit the intrinsic structures of Hankel and Löwner. For instance, a tensor–vector multiplication involving a Hankel tensor $\boldsymbol{\mathcal{H}}^{K} \in \mathbb{R}^{I_1\times\cdots\times I_K}$ can be performed in $\mathcal{O}(N\log(N))$ flops using the FFT [] instead of the original $\mathcal{O}(\prod_{k=1}^{K} I_k)$ flops required with the naive operation, where $N = \sum_{k=1}^{K} I_k$. Similar to the Hankel case, Löwner structures can avoid the curse of dimensionality in the case of equidistant points [].

4.3. *Factorization of the Vehicle Traffic Feature Tensor*

Although we can extract a bunch set of multidimensional information to our VFT tensor in different ways, we will focus our attention on two fundamental problems: the pattern recognition and the use of multilinear transformations to enforce low-rank assumptions on tensor models. We start by analyzing the pattern recognition followed by low-rank models under multilinear transformations.

4.3.1. Pattern Recognition

Through the VTF model we can exploit multi-mode correlations by Tensor Decompositions (TD), which provide a powerful analytical tool for dealing with multidimensional statics. Common TD employed include the CP-decomposition, Tucker model, HoSVD, and an SVD-based on the t-product called t-SVD, where the choice of a particular decomposition must be application dependent. The CP model is generally used for latent factor extraction, while the Tucker model to uncover hidden pattern in data. On the other hand, the choice of the t-SVD is more suitable when dealing with oriented-tensors.

4.3.2. Low-rank by Multilinear Transformations

A ~~common model used in many problems~~ classical model employed in many real-world applications is the (multidimensional) low-rank ~~signal model (see Equation ??)~~ model (LRM) [], which states that any ~~observable tensor $\boldsymbol{\mathcal{X}} \in \mathbb{K}^{I_1\times\cdots I_N}$ can be (approximately) decomposed~~ measurement tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1\times\cdots I_N}$ can be approximated as the sum of two components, a low-rank tensor ~~$\boldsymbol{\mathcal{L}} \in \mathbb{K}^{I_1\times\cdots I_N}$~~ $\boldsymbol{\mathcal{L}} \in \mathbb{R}^{I_1\times\cdots I_N}$, and a small dense noise ~~tensor $\boldsymbol{\mathcal{N}} \in \mathbb{K}^{I_1\times\cdots I_N}$.Although this decomposition can be solved by several tensor models, such as Tucker, CP, t-SVD or t-RPCA~~

~~decompositions (see Section ??), we only analyze the t-SVD and t-RPCA.~~ or approximation error tensor $\boldsymbol{\mathcal{N}} \in \mathbb{R}^{I_1 \times \cdots I_N}$, i.e., $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{L}} \simeq \boldsymbol{\mathcal{N}}$. This model is posed as an optimization problem, where in its simplest form, it can be formulated by minimizing the Frobenius norm as Equation **??** shows.

$$\min_{\boldsymbol{\mathcal{L}}} \|\boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{L}}\|_F^2 \tag{8}$$

However, the brittleness of LRM with respect to grossly corrupted observations, often causes non optimal solutions []. For this reason, it is very common to replace **??** by its robust version $\boldsymbol{\mathcal{X}} \simeq \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}$ called Robust LRM (RLRM), which is commonly formulated as the minimization problem of Equation **??**, where $\|\cdot\|_\star$ denotes the tensor nuclear norm, $\|\cdot\|_1$ the L_1-norm, and $\boldsymbol{\mathcal{S}} \in \mathbb{R}^{I_1 \times \cdots I_N}$ a sparse component which models grossly corruptions. This decomposition model can be consider as a generalization of Equation **??**, since in the case of free-corruption, Equation **??** is equivalent to **??**.

$$\cancel{\boldsymbol{\mathcal{X}} =} \min_{\boldsymbol{\mathcal{L}},\boldsymbol{\mathcal{S}}} \|\boldsymbol{\mathcal{L}}\|_\star + \cancel{\boldsymbol{\mathcal{N}}} \left\|\boldsymbol{\mathcal{S}}\right\|_1 \tag{9}$$

~~PREVIO A LA APLICACION DE LAS DESCOMPOSICIONES TENSORIALES, SE PRETENDE USAR LAS TRANSFORMACIONES DE HANKEL Y LOEWNER: HANKEL PROVE ESTRUCTURAS DE BAJO RANGO EN EL CASO DE SIN-RUIDO, SIN EMBARGO EN GENERAL ESTO NO SE TIENE EN LA PRACTICA, POR LO TANTO PARA MODELAR LAS CARACTERISTICAS GEOMETRICAS Y CINEMATICASDE NUESTRO TENSOR. POR OTRO LADO EL MODELADO DE LAS CARACTERISTICAS COMO FUNCIONES RACIONALES.~~

Although there exist many related works that successfully solve these models [15, 34], the RLRM lacks optimality when the intrinsic structure of a tensor is of high-rank, a situation that often happen in practice, which leads to a global solution that is not exactly low-rank. However, recently Li, C., et. al, [x], showed that a low-rank based matrix completion problem can obtain better performance by exploiting the low-rank structure resulted after transforming an observed matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$ into a new representation $\mathbf{Y} \in \mathbb{R}^{m \times n}$ by applying a set of K linear transformations $\{\mathcal{Q}_i(\cdot)\}_{i=1}^K$, ideas that established the foundations for a framework called Matrix Completion under Multiple linear Transformations (MCMT). MCMT is formulated as an optimization problem (see Equation **??**), where $\mathcal{P}_\Omega(\cdot)$ denotes a downsampling operation over the supporting set Ω, and $\delta \in \mathbb{R}^+$ a small constant.

$$\begin{aligned} \min_{\mathbf{X}} \quad & \sum_{i=1}^{K} \|\mathcal{Q}_i(\mathbf{X})\|_* \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\mathbf{X}) - \mathcal{P}_\Omega(\mathbf{Y})\|_F < \delta \end{aligned} \tag{10}$$

Following the ideas behind MCMT on matrices, we propose the high-order RLRM under Multilinear Transformations (HoRLRM^{2}T) with applications to modeling and approximation of functions, where we exploit the additional multi-mode low-rank structures of the transformed tensors. EMPLEAMOS LOS MODELOS SEGUIDOS DE UN TRUNCAMIENTO DE LA T-SVD, DESCARTANDO AQUELLOS VALORES SINGULARES MULTIDIMENSIONALES QUE NO APORTEN SUFICIENTE INFORMACION DE ACUERDO A SU DISTRIBUCION. ADICIONALMENTE EL USO DE T-RPCA PARA GENERAR MODELOS MAS ROBUSTOS FRENTE A CORRUPCION EN LOS DATOS.

PRESENTAR CONTRIBUCIONES EN EL MODELADO DE DATOS EN EL USO DE ESTAS DOS TRANSFORMACIONES, ASI COMO CARACTERISTICAS DE LOS TENSORES RESULTANTES COMO LO SON SUS RANGOS, TRABAJOS RELACIONADOS CON CP Y TUCKER.

5. Experiments

6. Discussion

Authors should discuss the results and how they can be interpreted in perspective of previous studies and of the working hypotheses. The findings and their implications should be discussed in the broadest context possible. Future research directions may also be highlighted.

7. Conclusions

CONCLUSIONES DEL LOW-RANK Y TRANSFORMACIONES, CUANDO CONOCEMOS LA FORMA ALGEBRAICA DE LAS CARACTERISTICAS SE EMPLEA HANKEL+T-SVD, DE OTRO MODO LOEWNER + T-RPCA, EL USO DE ESTOS MODELOS DE BAJO RANGO NOS AYUDAN A ELIMINAR DISTURBIOS EN LOS DATOS, CON LO CUAL SE PUEDE MEJORAR LOS MODELOS DE CLASIFICACION Y OCLUSION.

TRABAJO FUTURO: 1. USO DE TRANSFORMACIONES MULTI-LINEALES? ESTOCASTICAS, ASI COMO NO LINEALES. 2. USO DE LAS DESCOMPOSICIONES TENSORIALES CON RESTRICCIONES ESTRUCTURADAS (HANKEL, LOEWNER, ETC), ASI COMO OTRAS RESTRICCIONES (NO NEGATIVIDAD).

~~This section is not mandatory, but can be added to the manuscript if the discussion is unusually long or complex.~~

Author Contributions: For research articles with several authors, a short paragraph specifying their individual contributions must be provided. The following statements should be used "conceptualization, X.X. and Y.Y.; methodology, X.X.; software, X.X.; validation, X.X., Y.Y. and Z.Z.; formal analysis, X.X.; investigation, X.X.; resources, X.X.; data curation, X.X.; writing–original draft preparation, X.X.; writing–review and editing, X.X.; visualization, X.X.; supervision, X.X.; project administration, X.X.; funding acquisition, Y.Y.", please turn to the CRediT taxonomy for the term explanation. Authorship must be limited to those who have contributed substantially to the work reported.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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