Decomposition of Permutations by Cost-Constrained Transpositions

Farzad Hassanzadeh

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Permutations

Permutations are ubiquitous combinatorial objects.

A permutation is an arrangement of a set of objects.

Example: [2431] : A permutation over the set $\{1, 2, 3, 4\}$

Large number of applications:

- Coding and information theory
- Computer science
- Biology and bioinformatics
- Recommendation systems
- Social sciences: competitions, voting
- Management and decision making

Rank Aggregation: Combining a set of rankings such that the resulting ranking is representative of the whole set.

Title	IMDB	FilmCrave	Aggregate
The Shawshank Redemption	1	1	?
The Godfather	2	3	?
Fight Club	10	2	?
The Godfather: Part II	3	11	?
Pulp Fiction	4	4	?
Schindler's List	5	8	?
The Dark Knight	7	5	?
One Flew Over the Cuckoo's Nest	6	13	?
LoR: The Fellowship of the Ring	13	6	?
LoR: The Return of the King	8	7	?
SWV: The Empire Strikes Back	9	10	?
Goodfellas	11	9	?
Star Wars	12	12	?

Rank Aggregation I

Rank Aggregation: Combining a set of rankings such that the resulting ranking is representative of the whole set.

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The Dark Knight	7	5	5/6/7?
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Star Wars	12	12	?

Rank Aggregation II

Requires a distance measure that:

1. Meaningfully handles difference between rankings
   - Top versus bottom

$$d(1234, 2134) > d(1234, 1243)$$

   - Similar items versus dissimilar items

$$d(ABA'B', B'BA'A) > d(ABA'B', A'BAB')$$

2. Can be efficiently calculated or approximated

Problem Statement

We are interested in a distance over permutations based on swaps.
Swap of two elements is called Transposition.
Transposition of a and b denoted by (ab).

Suppose that each transposition (ab) has cost $\varphi(a, b)$.

Distance between two permutations π and σ is the minimum cost of transforming π to σ using transpositions.

Problem Statement: For arbitrary non-negative cost function φ and permutations π and σ, find distance between π and σ.

Cost Functions and Distance

$M_\varphi(\pi, \sigma)$ = distance between π and σ based on φ.
Fact: For permutations π, σ, η, M_φ satisfies

1. $M_\varphi(\pi, \pi) = 0$
2. $M_\varphi(\pi, \sigma) = M_\varphi(\sigma, \pi) \geq 0$
3. $M_\varphi(\pi, \sigma) \leq M_\varphi(\pi, \eta) + M_\varphi(\eta, \sigma)$ (triangle inequality)
4. $M_\varphi(\pi, \sigma) = M_\varphi(\pi\eta, \sigma\eta)$ (right-invariance)

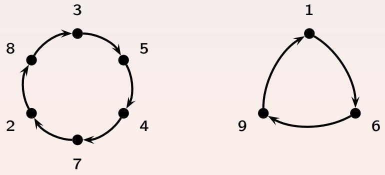
Decomposition is a sequence transposition that transforms identity permutation e to a permutation π.

Because of right invariance, it suffices to consider decompositions only: $M_\varphi(\pi, \sigma) = M_\varphi(e, \sigma\pi^{-1}) =: M_\varphi(\sigma\pi^{-1})$

Permutations and Cycles

1. Permutation as a bijection from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, n\}$. Example: $\pi = \begin{bmatrix} 123456789 \\ 685749231 \end{bmatrix}$. That is, $\pi(1) = 6, \pi(2) = 8$, etc. Composition of permutations is a permutation.
2. Functional digraph of permutation: Graph with vertex set $\{1, 2, \cdots, n\}$ and edges $(i, \pi(i))$.

The functional digraph of a permutation is a set of disjoint directed cycles: $\pi = (354728)(169)$.



An Example

MCD: Minimum Cost Decomposition
MLD: Minimum Length Decomposition: Cayley (1860) shows for a permutation with k cycles MLD has length $n-k$.

Consider $\sigma = (12345)$ with $\varphi(i,j) = \begin{cases} 3, & |i-j| = 1 \pmod 5 \\ 1, & \text{else} \end{cases}$

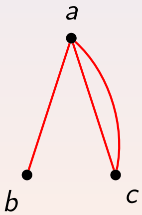
MLD has cost at least eight	MCD has cost six
12345	12345
21345	42315
23145	42135
23415	42153
23451	24153
	21453
	23451

Decompositions and Multigraphs I

A decompositions can be represented by a multigraph.

The vertices of the multigraph are the symbols $\{1, ..., n\}$, while the edges are specified by the decomposition in a natural manner.

For example, the decomposition $(ac)(bc)(ac)$ of $(ab)e$ induces the following multigraph on vertices $\{a, b, c\}$.

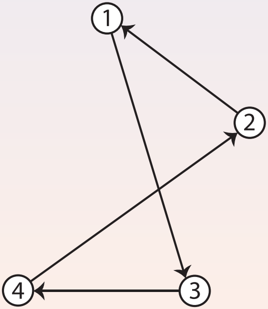


Decompositions and Multigraphs II

Consider transforming $\pi = 3142$ to $\sigma = 2341$ via $(23), (13)$.
Edges of functional digraph are black, edges of multigraph are red.

- Transposition (ab) switches the predecessors of a and b.
- Each arrow follows the path indicated by the edges of the multigraph.

3142



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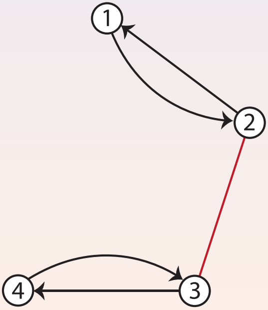

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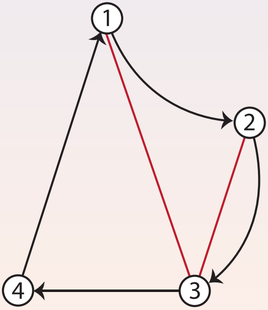

Outline of the Algorithm

- MCD of $\pi = (ab)e$.
  - Key idea: use decompositions of length three only.
  - Show recursive use of decompositions of length three is sufficient.
- Find min cost MLD of a cycle and connect it to MCD.
  - Key idea: connection between planar trees and MLDs of cycles. Search over planar trees instead of MLDs.
  - Approximate MCD using min cost MLD by lower bounding cost of MCD.
- Bound MCD of arbitrary permutations.
  - Direct bound using cycle decompositions
  - Merging cycles

MCD of Transpositions I

Characterizing decompositions of $\pi = (ab)e = bacd$: only odd lengths possible.

- Trivial: $abcd \xrightarrow{(ab)} bacd$
- Easy: triple-optimization:

$$abcd \xrightarrow{(ac)} cbad \xrightarrow{(bc)} bcad \xrightarrow{(ac)} bacd$$

$$(ab) = (ac)(bc)(ac)$$

Has smaller cost if $\varphi(a, b) > 2\varphi(a, c) + \varphi(b, c)$.

- Hard: Decompositions of lengths 5, 7, etc.

MCD of Transpositions II

Triple-optimization algorithm: Perform triple-optimization as many times as possible and update costs

$$(ab) = (ac)(bc)(ac)$$
$$\varphi(a, b) \leftarrow 2\varphi(a, c) + \varphi(b, c)$$

- Let φ^* be the resulting cost function. For all a, b, c we have

$$\varphi^*(a, b) \leq 2\varphi^*(a, c) + \varphi^*(b, c)$$

- φ^* is obtained in time $O(n^4)$
- Will show $\varphi^*(a, b) = M_\varphi(\pi)$ for $\pi = (ab)e$.

MCD of Transpositions III

Theorem

For $a, b \in [n]$ and the output φ^* of triple-optimization algorithm, $\varphi^*(a, b) = M_\varphi(\pi)$ where $\pi = (ab)e$.

Proof: Consider decomposition τ of $\pi = (ab)e$.

1. Multigraph of τ has at most one $a, b-$cut-edge.
2. Minimum cost graph with at most one $a, b-$cut-edge is of form

$$2 \times (\text{a path between } a \text{ and } b) - (\text{one edge}).$$

3. $\text{Cost}\Big(2 \times (\text{a path between } a \text{ and } b) - (\text{one edge})\Big) \geq \varphi^*(a, b)$

Cut-edge characterization: at most one cut-edge!

Lemma 1

Multigraph of decomposition of $(ab)e$ has $\leq$ one $a, b-$cut-edge.

Successive application of transpositions transforms e to $\pi = (ab)e$.

- $e(a) = a, e(b) = b$
- $\pi(a) = b, \pi(b) = a$

Suppose (uv) is an $a, b-$cut-edge

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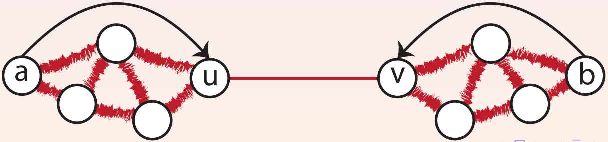
- Before applying (uv), must have $a \to u$ and $b \to v$.



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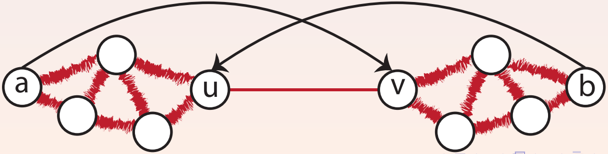

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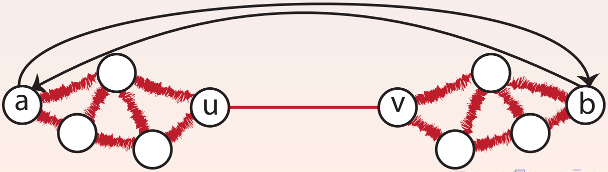


Cut-edge characterization: at most one cut-edge!

Lemma 1

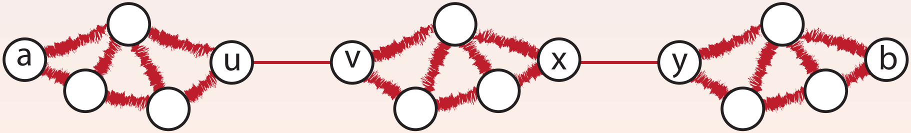
Multigraph of decomposition of $(ab)e$ has $\leq$ one $a, b-$cut-edge.

Suppose (uv) and (xy) are both $a, b-$cut-edges.

- Before applying (uv), must have $a \to u$ and $b \to v$.
- Before applying (xy), must have $a \to x$ and $b \to y$.

Cannot satisfy both conditions simultaneously.

- The multigraph of any decomposition of (ab) has at most one $a, b-$cut-edge!



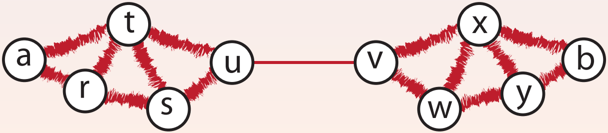
MCDs of Transpositions IV

Lemma 2

Minimum cost graph with at most one $a, b-$cut-edge is of form

$$2 \times (\text{a path between } a \text{ and } b) - (\text{one edge}).$$

- Can assume exactly one $a, b-$cut-edge. Suppose it is (uv).



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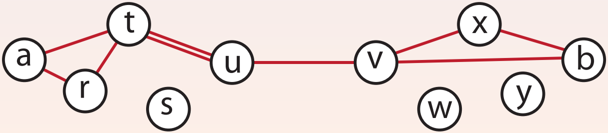

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- No $a, u-$cut-edge $\Rightarrow$ there exist two edge-disjoint a, u paths.
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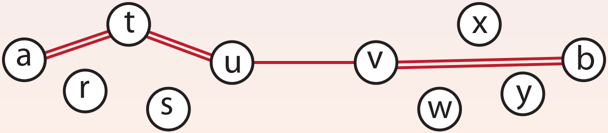
- From each pair pick path with smaller cost and duplicate it.

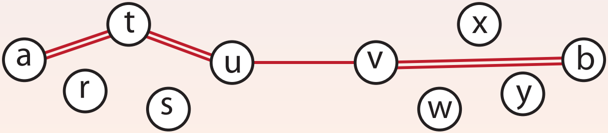


MCDs of Transpositions V

Lemma 3

Triple-optimization achieves a cost $\leq$ cost of any $2 \times$ (a path between a and b) $-$ (one edge).

$$2\varphi^*(a,t) + 2\varphi^*(t,u) + \varphi^*(u,v) + 2\varphi^*(v,b)$$



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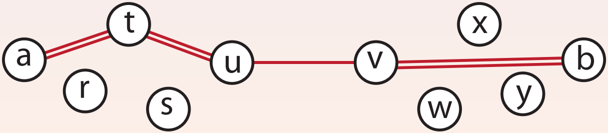

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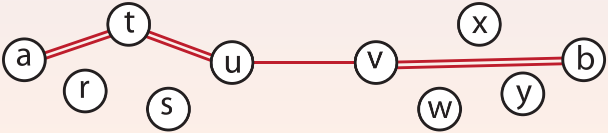

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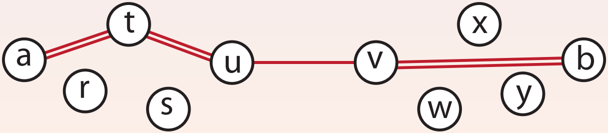

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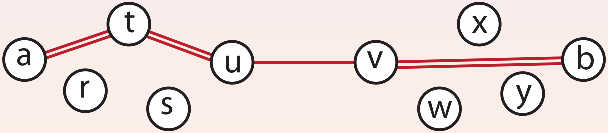

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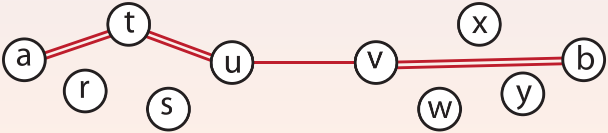

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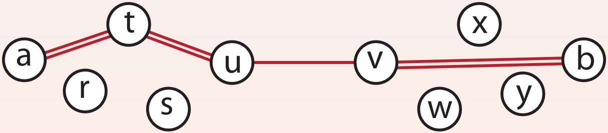

MCD of Transpositions VI

Lemma

Minimum cost G with
$G = 2 \times$ *(a path between a and b)* $-$ *(one edge)*
is the multigraph of some MCD of $\pi = (ab)e$.

```
a t u v b
```

MCD of Transpositions VI

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a	t	u	v	b
t	a	u	v	b
u	a	t	v	b
u	a	t	b	v

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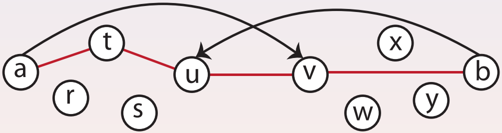
```
a t u v b
t a u v b
u a t v b
u a t b v
v a t b u
```



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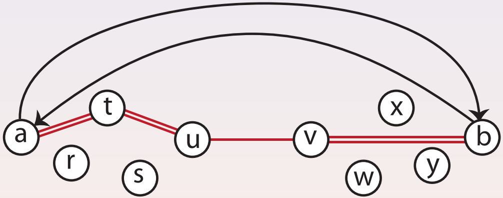
```
a t u v b
t a u v b
u a t v b
u a t b v
v a t b u
b a t v u
b a u v t
b t u v a
```



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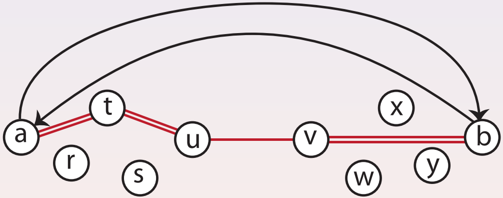
u	a	t	v	b
u	a	t	b	v
v	a	t	b	u
b	a	t	v	u
b	a	u	v	t
b	t	u	v	a



Corollary

$\varphi^*(a,b) \leq 2 \times ($ *cost of shortest path between a and b*$)$.

MCDs and MLDs of Cycles

Problem: Find min cost MLD of a given cycle with respect to optimized transposition costs φ^*.

- Can min cost MLD be found efficiently?
- Can min cost MLD be used to approximate MCD?

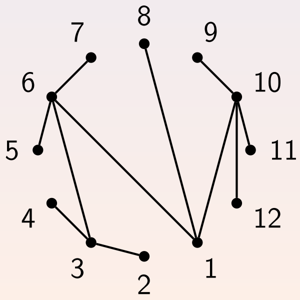
MLDs of Cycles

Is there a simple characterization of MLDs of cycles?
Given a cycle σ, arrange its vertices on a circle c.

Theorem

The multigraph of an MLD of σ is a planar (spanning) tree inside the circle c. Conversely, any such tree is the multigraph of at least one MLD.

We can find minimum cost planar tree in time $O(k^4)$, where k is the length of the cycle.



MCDs of Cycles: Approximation

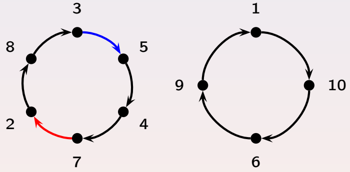
We need:

- a lower-bound for cost of MCD
- an upper-bound for min cost of MLD

MCD: a lower-bound I

Key idea: Divide a transposition to two components: each an h-transposition (half-transposition).

Example: h-transposition $2 \rightsquigarrow 5$: takes the predecessor of 2 to 5.



H-decomposition of a permutation is a sequence of h-transpositions equal to that permutation.

MCD: a lower-bound II

To an h-transposition assign half the cost of corresponding transposition. Thus

$$\text{cost of MCD} \geq \text{min cost of h-decomposition}$$

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What is cost of h-decomposition of cycle σ?

$$\frac{1}{2}\sum_{i} \mathrm{Cost}\Big(p^*(i, \sigma(i))\Big)$$

$p^*(a, b) =$ shortest path from a to b.
Proof Idea: Consider the cost needed to transform $e(i) = i$ to $\sigma(i) = j$ using h-transpositions.

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$p^*(a, b) =$ shortest path from a to b.
Proof Idea: Consider the cost needed to transform $e(i) = i$ to $\sigma(i) = j$ using h-transpositions.

$$\text{cost of MCD} \geq \frac{1}{2}\sum_i \text{Cost}\Big(p^*(i, \sigma(i))\Big)$$

Min cost MLD: an upper-bound

For $\sigma = (12 \cdots k)$, consider the MLD $(12)\,(23) \cdots (k-1\ k)$ with cost

$$\sum_{i=1}^{k-1} \varphi^* (i, \sigma(i))$$

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$$\text{min cost MLD} \leq 2 \sum_i \ \text{Cost}\Big(p^*(i, \sigma(i))\Big).$$

Constant-factor approximation

Theorem

The cost of *MCD* is at most 4 times the minimum cost of an MLD.

Proof:

$$\text{min cost MLD} \leq 2 \sum_i \text{Cost}\Big(p^*(i, \sigma(i))\Big),$$

$$\text{cost of MCD} \geq \frac{1}{2} \sum_i \text{Cost}\Big(p^*(i, \sigma(i))\Big).$$

Lemma

If the cost function is a metric, then the cost of *MCD* is at most 2 times the minimum cost of an MLD.

Proof Idea:

$$\text{min cost MLD} \leq \sum_i \text{Cost}\Big(p^*(i, \sigma(i))\Big).$$

MCDs of Permutations

Simple consequence of the cycle decomposition result:

Theorem

Results stated for individual cycles also hold for permutations.

An alternative approach– merging cycles and optimizing over one cycle.

- May give better answer than above theorem

Rank Aggregation I

Formally, given $\sigma_1, \cdots, \sigma_m$ and a distance function d, find

$$\arg\min_{\pi} \sum_{i=1}^{m} d(\pi, \sigma_i)$$

Several distance functions can be used [8]:

- Kendall's τ: $K(\pi, \sigma) = \#$ of transpositions of adjacent ranks. Equivalent to $\varphi_K(i, i+1) = 1$.
- Spearman's Footrule: $F(\pi, \sigma) = \sum_i |\pi(i) - \sigma(i)|$. Equivalent to the path cost function $\varphi_F(i, j) = |i - j|$.
- Cayley's distance: $T(\pi, \sigma) = \#$ of transpositions. Equivalent to $\varphi_T(i, j) = 1$.
- Spearman's rank correlation: $S^2(\pi, \sigma) = \sum_i (\pi(i) - \sigma(i))^2$

Rank Aggregation II

Cost-constrained transposition distance generalizes

- Kendall's τ to model the significance of top vs bottom of a ranking, e.g.,

$$\varphi(1,2) > \varphi(9,10)$$

- Cayley's distance to model similarities and dissimilarities of elements, e.g.,

$$\varphi(\text{God Father I}, \text{God Father II}) < \varphi(\text{God Father I}, \text{Goodfellas}) < \varphi(\text{God Father I}, \text{Star Wars})$$

Rank Aggregation III

To find

$$\pi^* = \arg\min_{\pi} \sum_{i=1}^{m} M_{\varphi}(\pi, \sigma_i)$$

we approximate M_{φ} by D where

$$D(\pi, \sigma) = \sum_{i=1}^{n} \mathrm{cost}\left(p^*(\pi(i), \sigma(i))\right).$$

We can find

$$\pi' = \arg\min_{\pi} \sum_{i=1}^{m} D(\pi, \sigma_i)$$

and can show that

$$\sum_{i=1}^{m} M_{\varphi}(\pi', \sigma_i) \leq 4 \sum_{i=1}^{m} M_{\varphi}(\pi^*, \sigma_i).$$

Future Work: Rank Aggregation

We intend to study the application of cost-constrained distance to rank aggregation and rank prediction.

- Application of cost-constrained distance to rank predictions (rank collaborative filters)
- Extension of costs-constrained distance when cost depends both on location and object: $\varphi(i, j, \pi^{-1}(i), \sigma^{-1}(j))$.
- Extension of costs-constrained distance to partial rankings.

Thank you!