Ec140 - Regression as Line Fitting and Conditional Expectation

(Part I)

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Regression Journey

- Regression as Matching on Groups. Ch2 of MM up to page 68 (not included).
- Regression as Line Fitting and Conditional Expectation. Ch2 of MM, Appendix + SoPo Econometrics. (Part I today)
- Multiple Regression and Omitted Variable Bias. Ch2 of MM pages 68-79.
- Regression Inference, Binary Variables and Logarithms. Ch2 of MM, Appendix + others.

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Regression as Line Fitting and Conditional Expectation

Regression as Line Fitting: Today's Goal

- The goals of today's class are two:

  1. Provide an explanation to what regression does when "it generate fitted values" (or "it fits a line"), and
  2. Provide some insight to a useful formula that represents the main coefficient of interest (β).

- Today's class will be a bit more technical than previous classes.

- For this reason it is important to always keep in mind what the goal is.

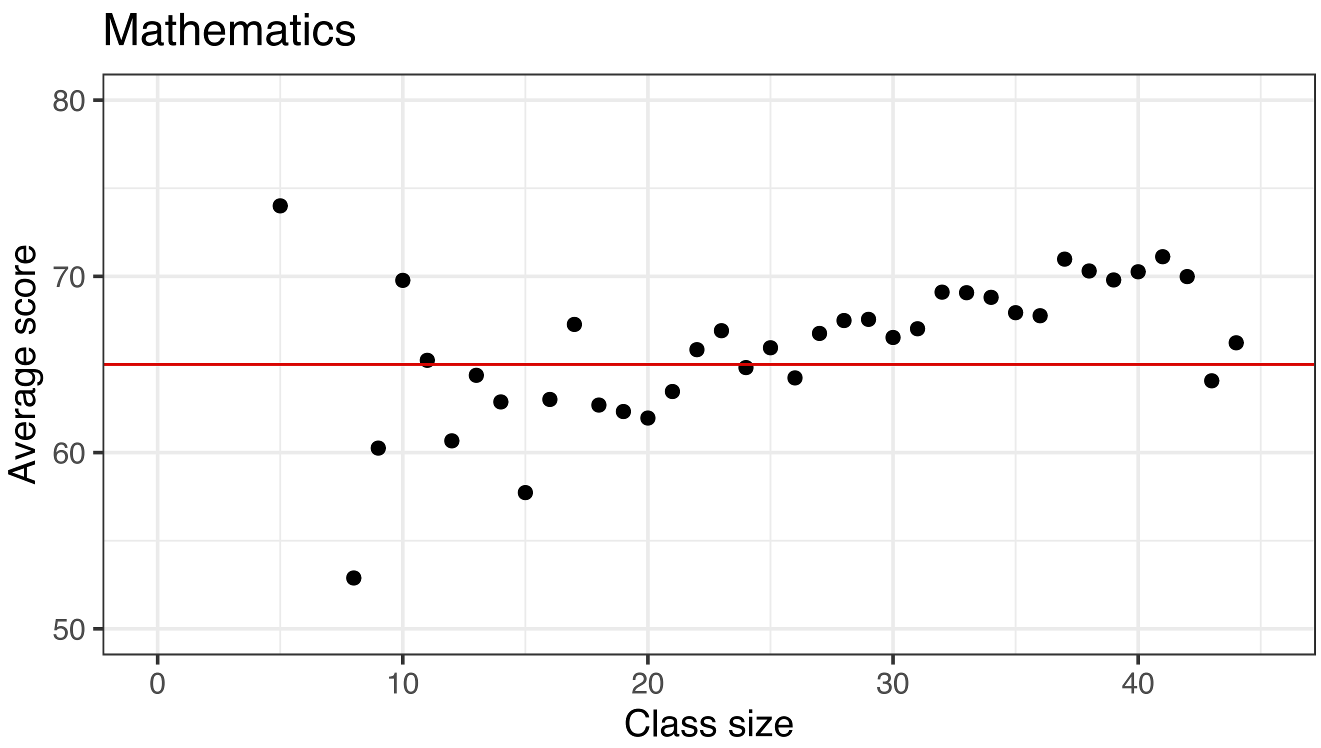
- Even if you end up completely lost about today's material, these explanations are not essential for you to do well in class.

Regression as Line Fitting

- Example: Class size and student performance (Slides adapted from SciencePo Econometrics course, and data from Raj Chetty and Greg Bruich's course)

Class size and student performance: Regression line

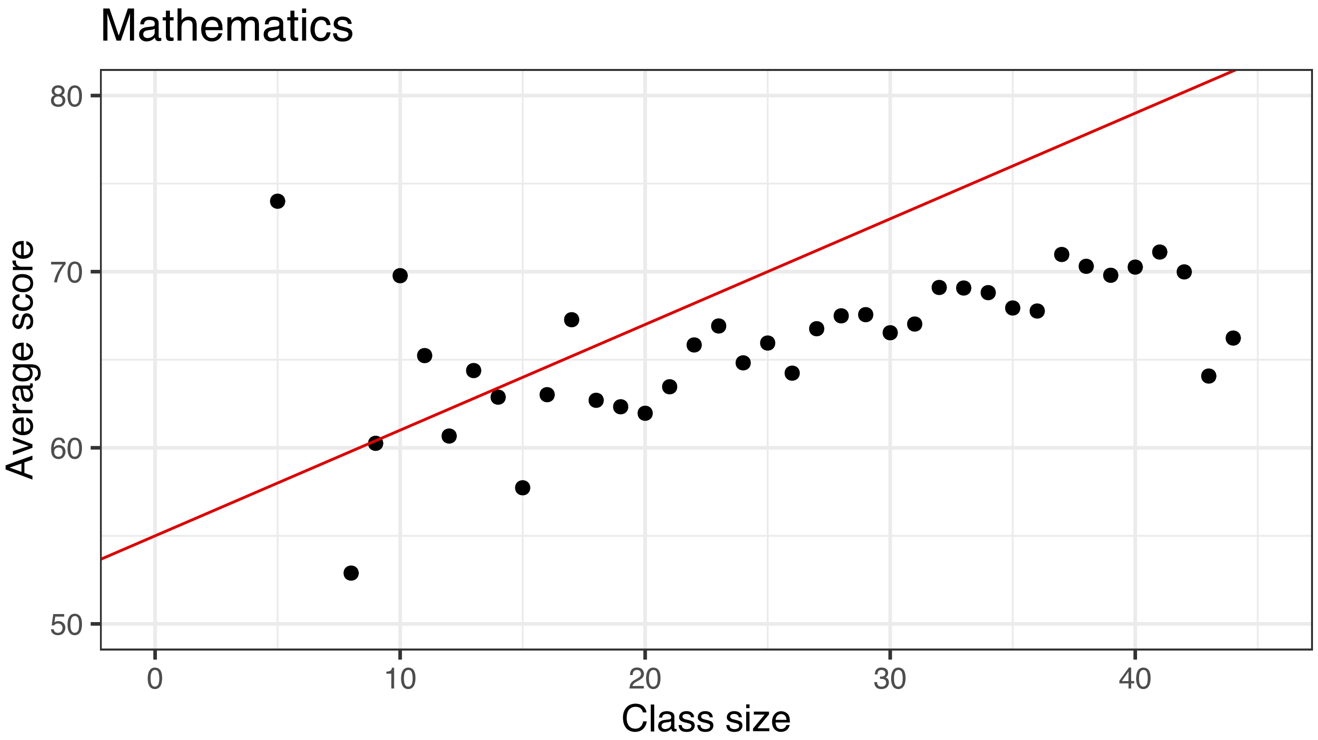
How to visually summarize the relationship: **a line through the scatter plot**



- A *line*! Great. But **which** line? This one?
- That's a *flat* line. But average mathematics score is somewhat *increasing* with class size.

Class size and student performance: Regression line

How to visually summarize the relationship: **a line through the scatter plot**



- **That** one?
- Slightly better! Has a **slope** and an **intercept**
- We need a rule to decide!

It's All About the Residuals

- In *Regression as Matching* we define the residuals, e_i, as the difference between the observed (Y_i) and fitted values ($\widehat{Y}_i$).

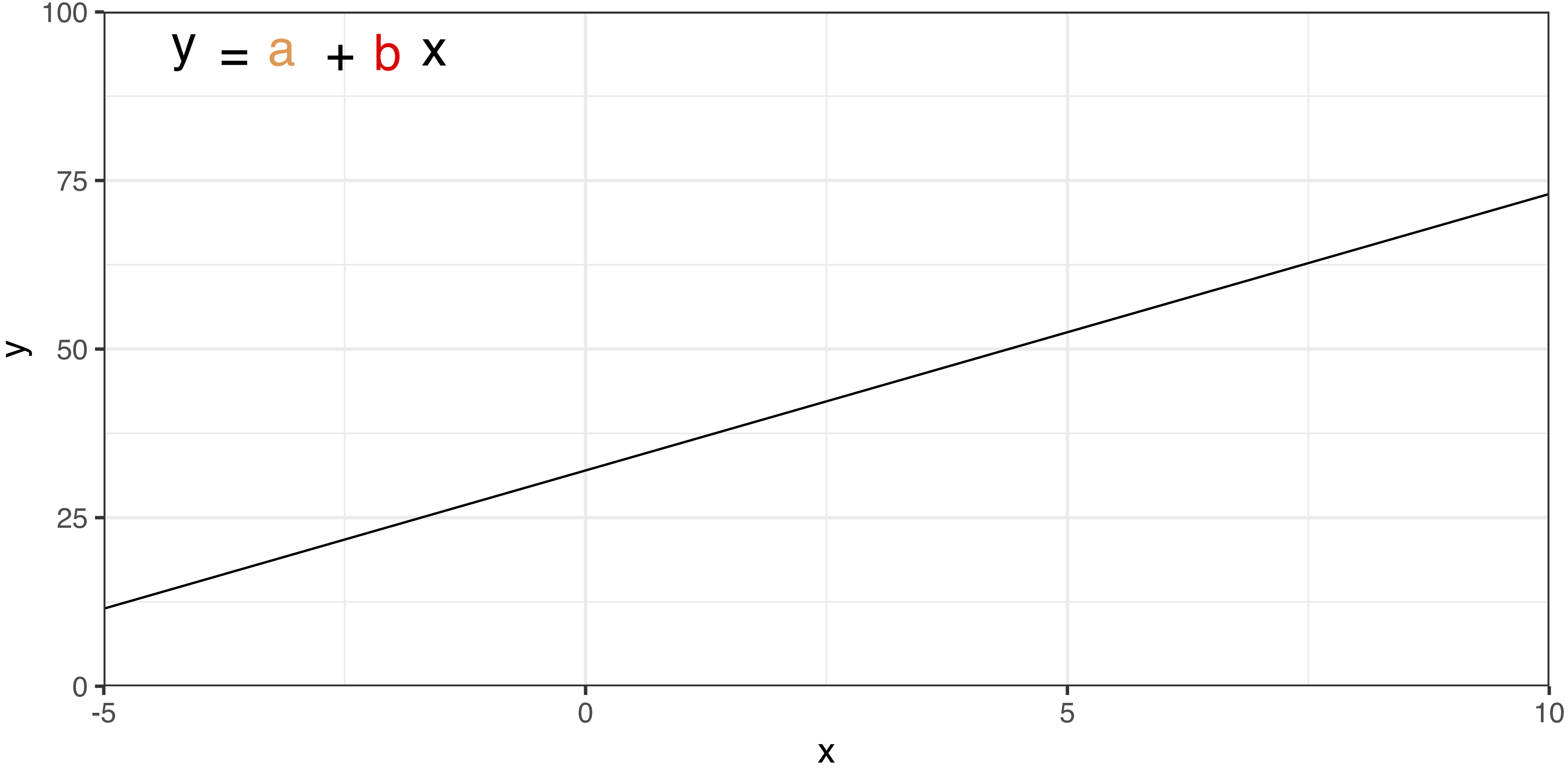
$$e_i \equiv Y_i - \widehat{Y}_i$$

- By fitted values, we mean a line (for now) that summarizes the relationship between X and Y.

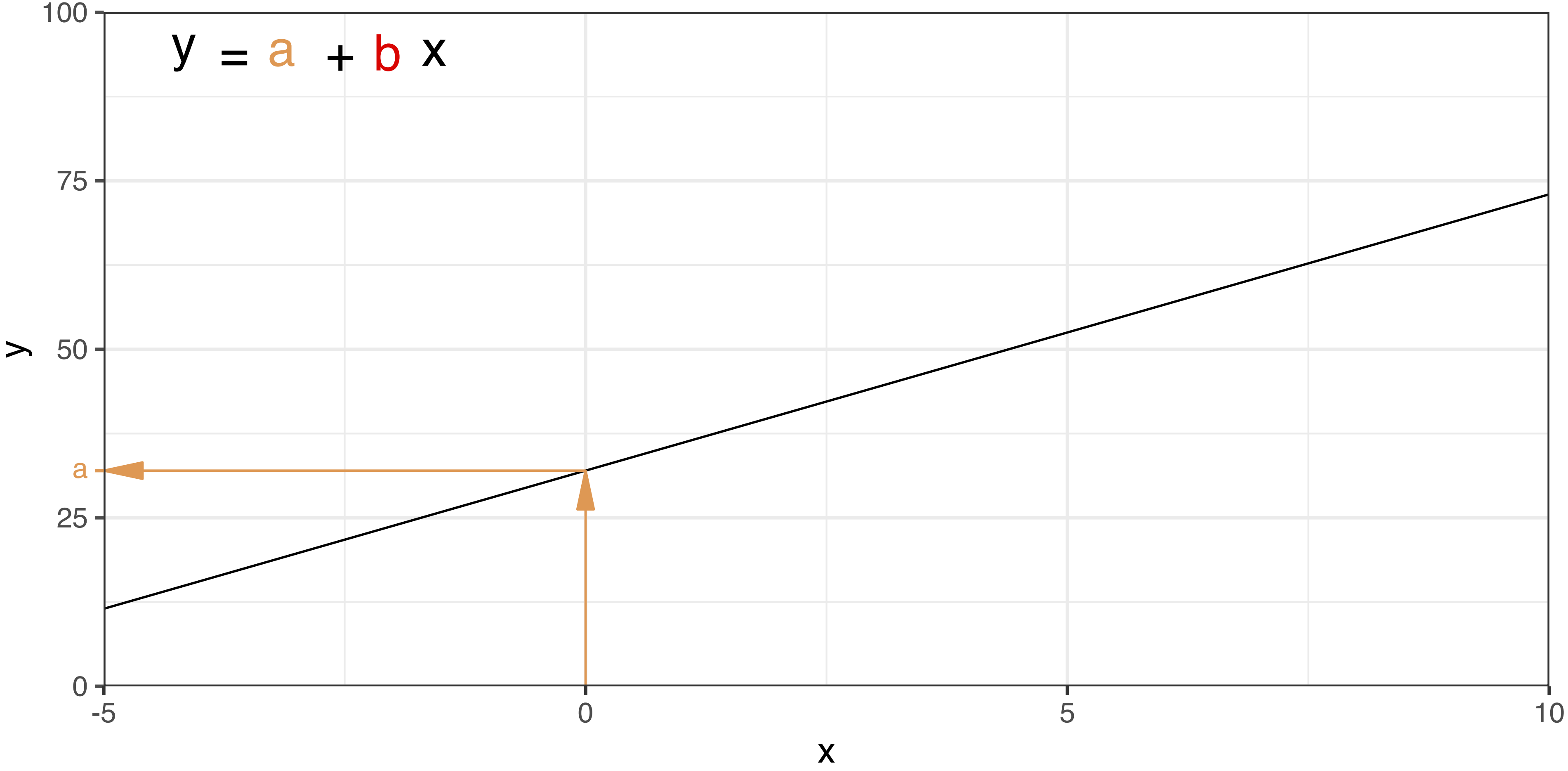
- The equation for such a line with an intercept a and a slope b is:

$$\widehat{Y}_i = a + bX_i$$

What's A Line: A Refresher



What's A Line: A Refresher



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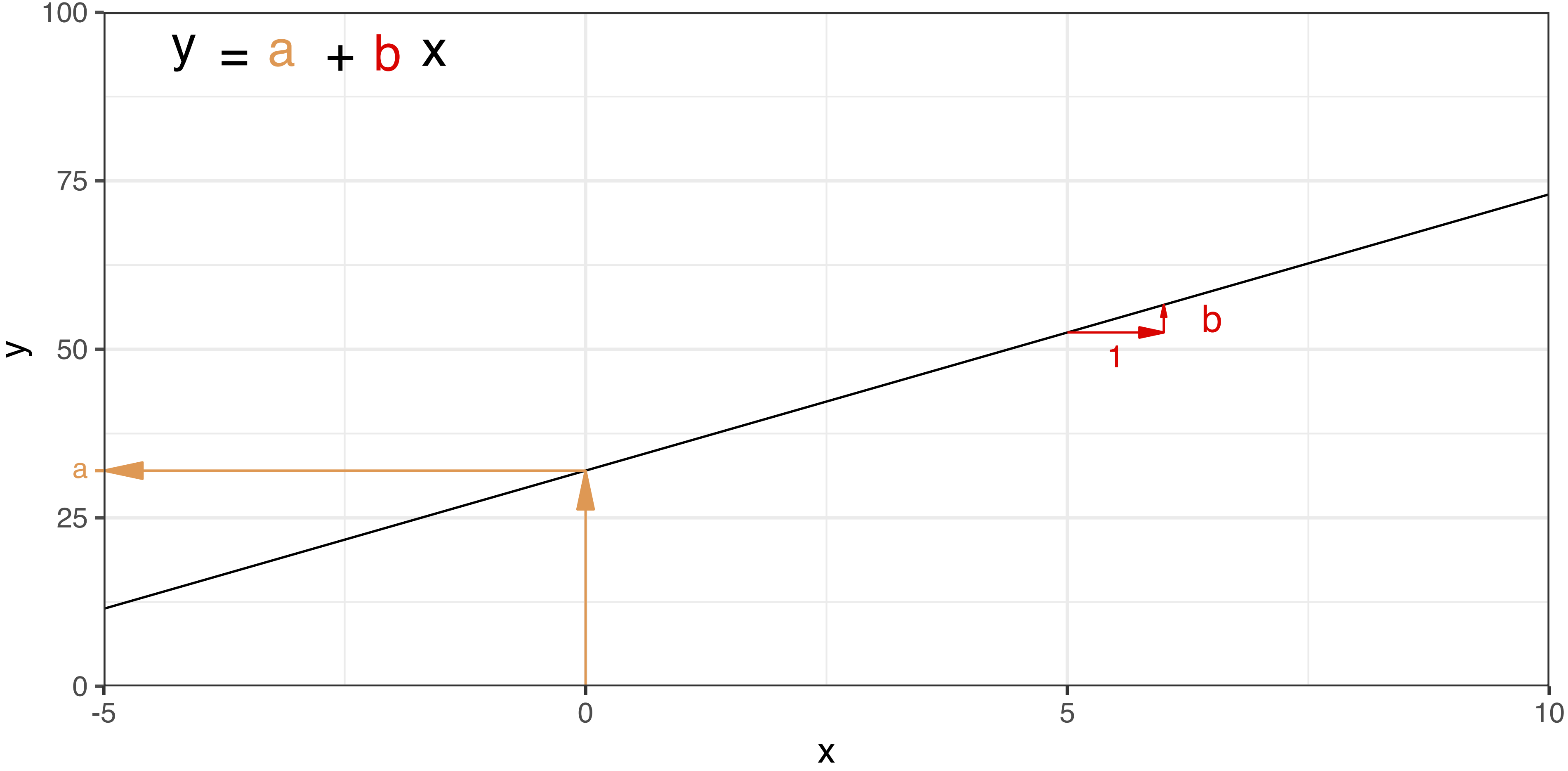
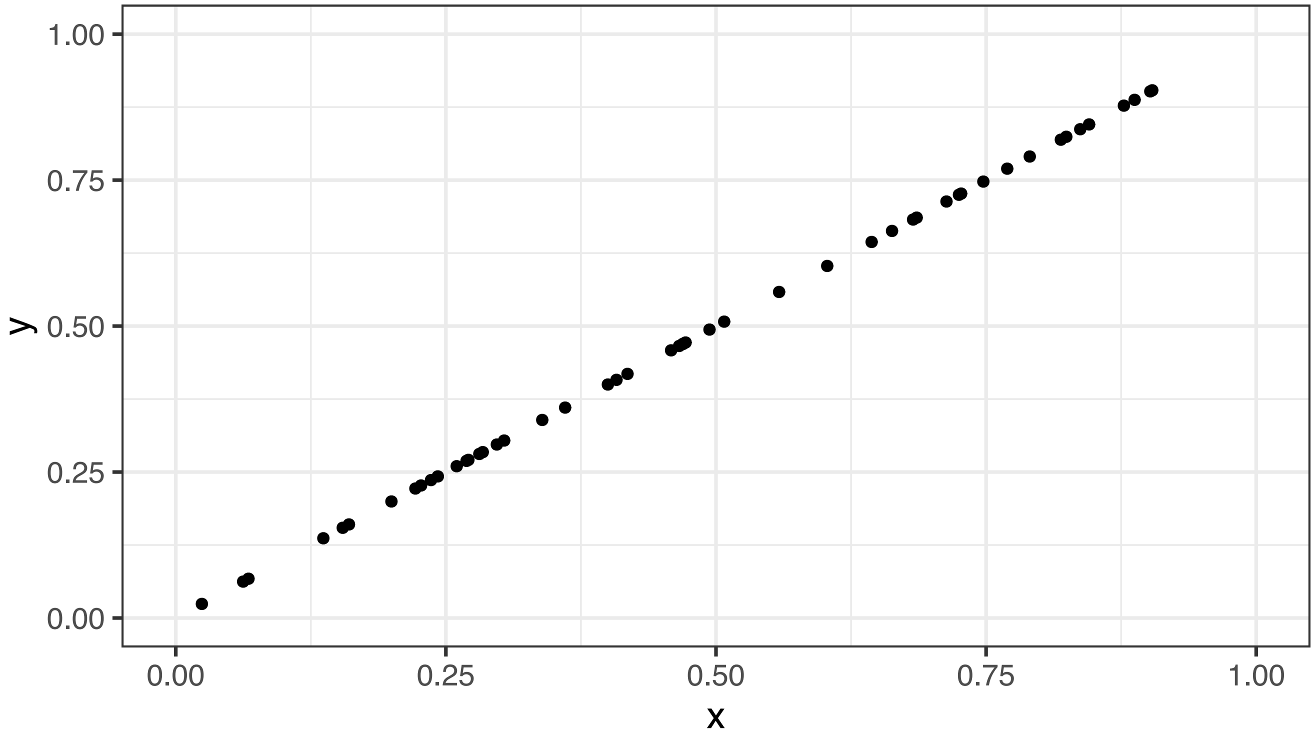


Simple Linear Regression: Residual

- If all the data points were **on** the line then $\widehat{Y}_i = Y_i$.



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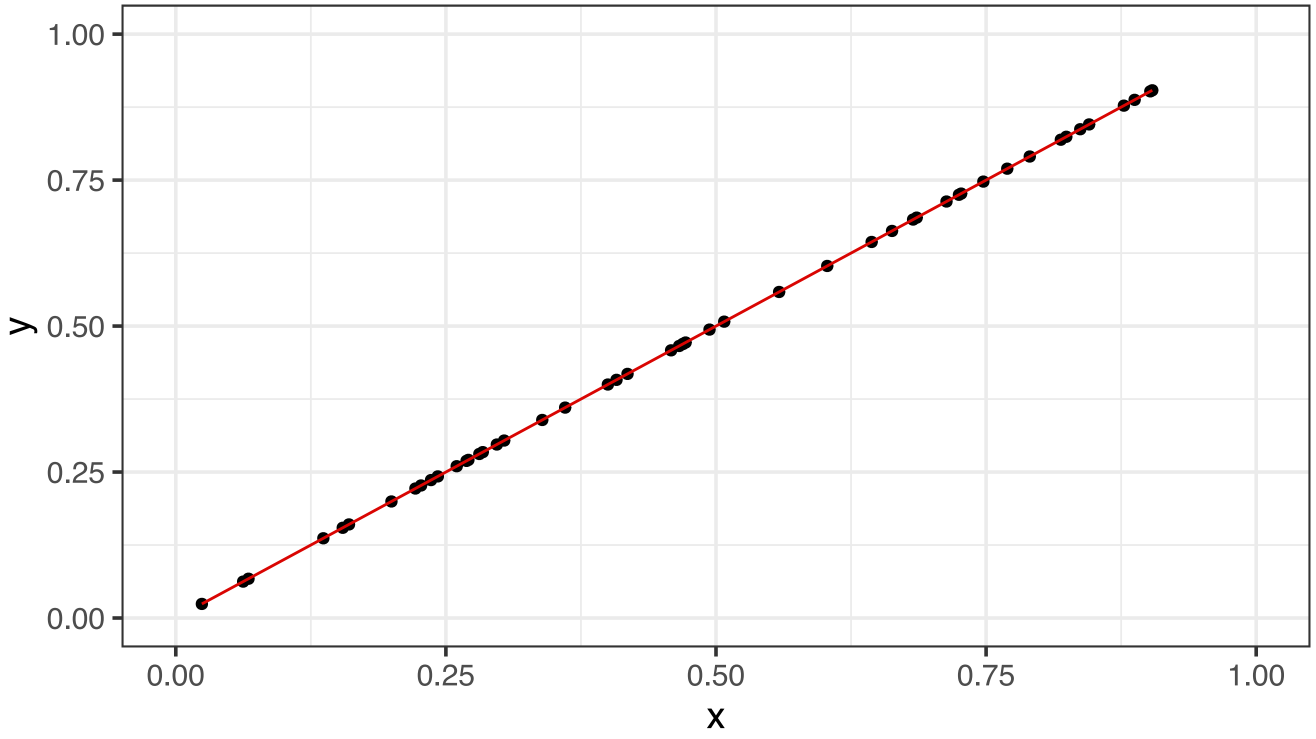
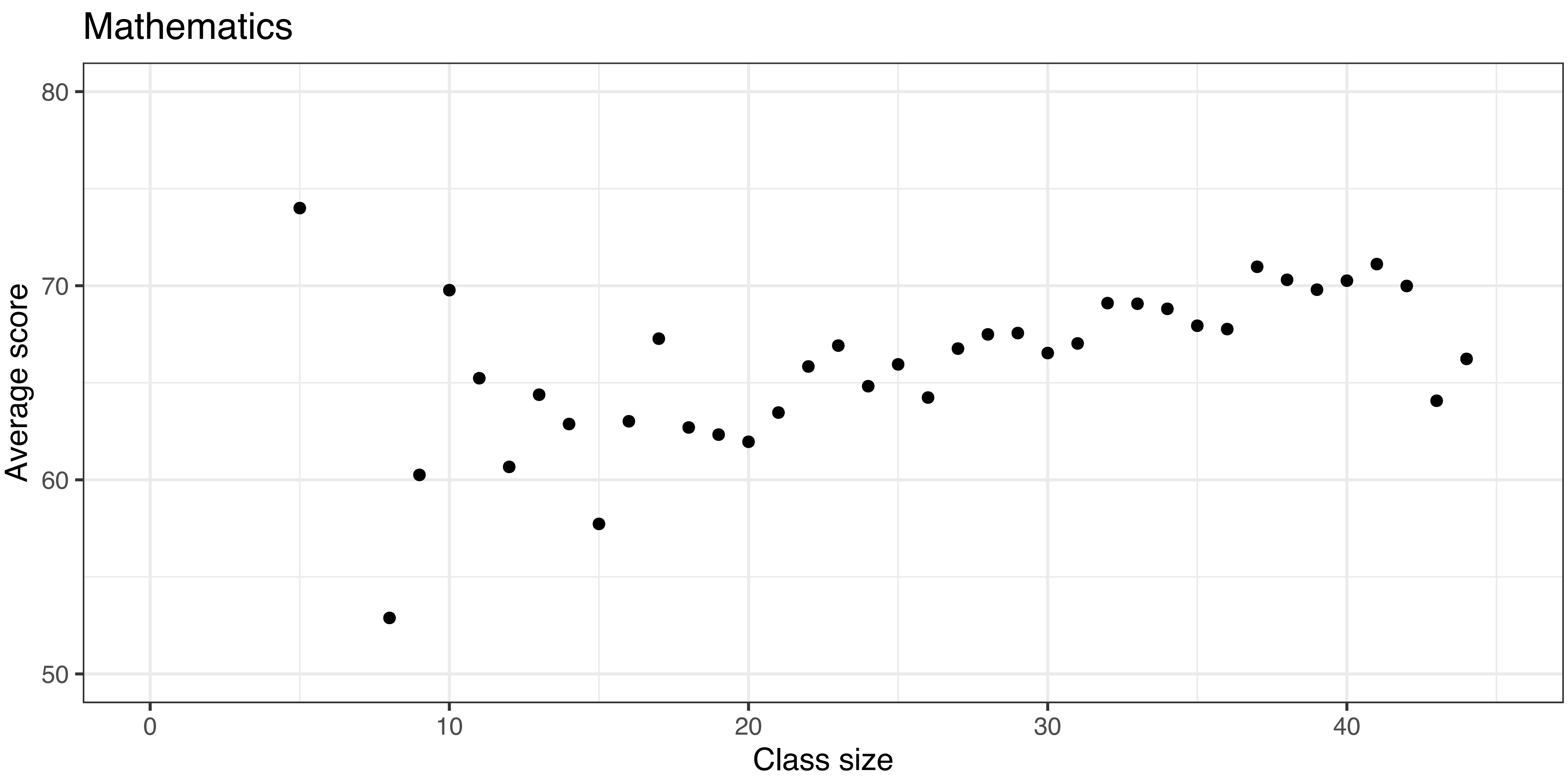


Simple Linear Regression: Graphically



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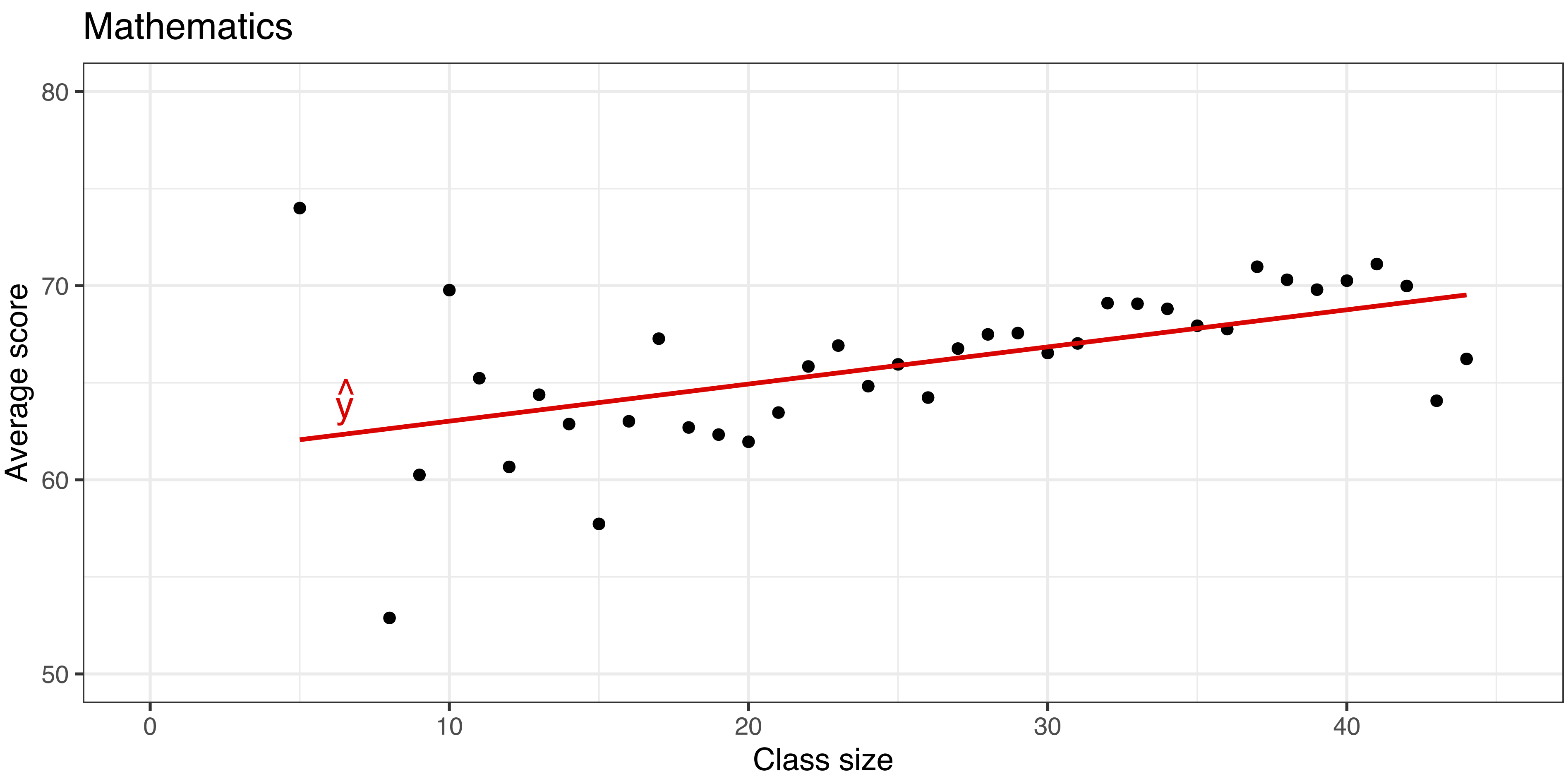

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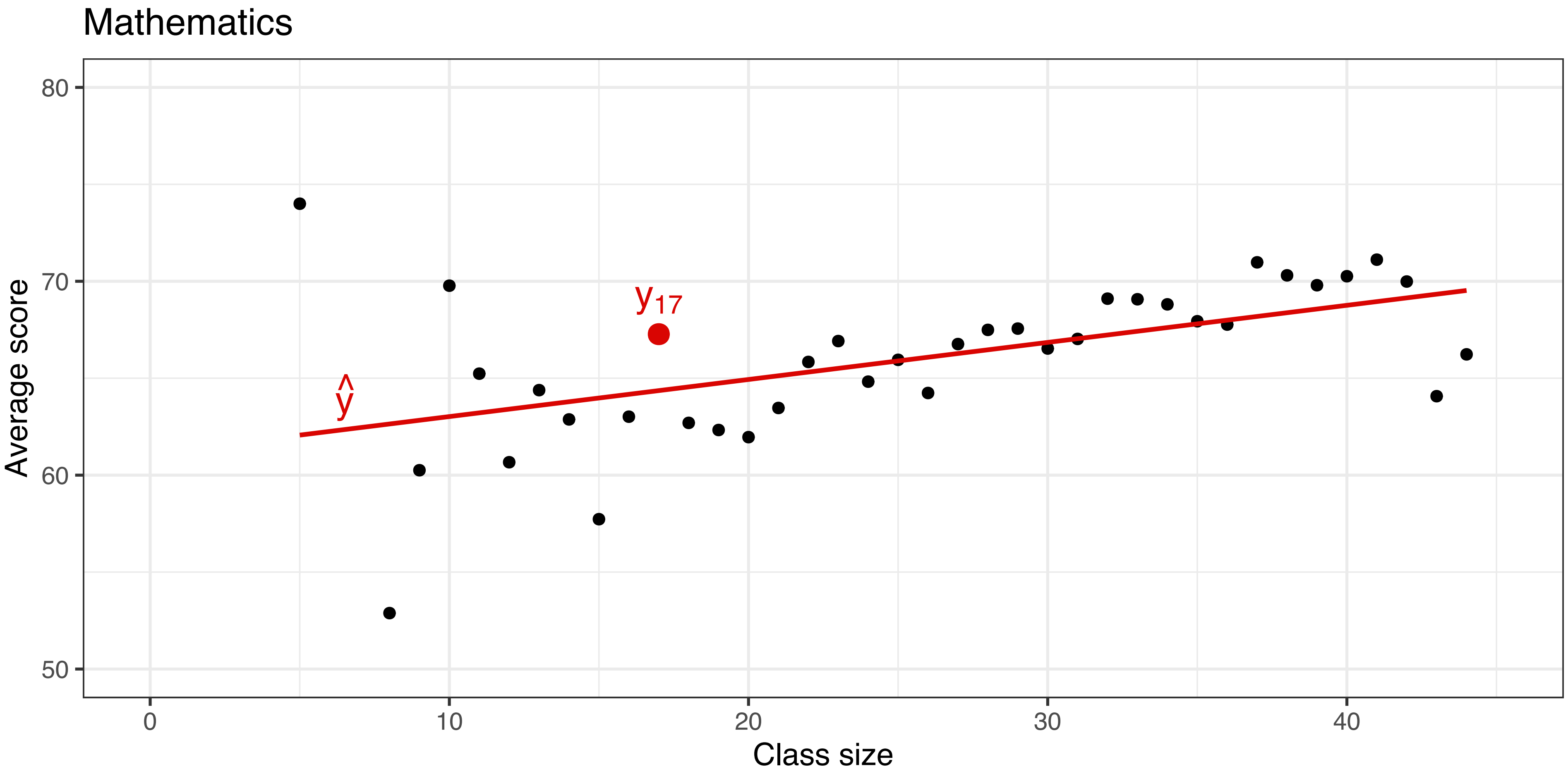

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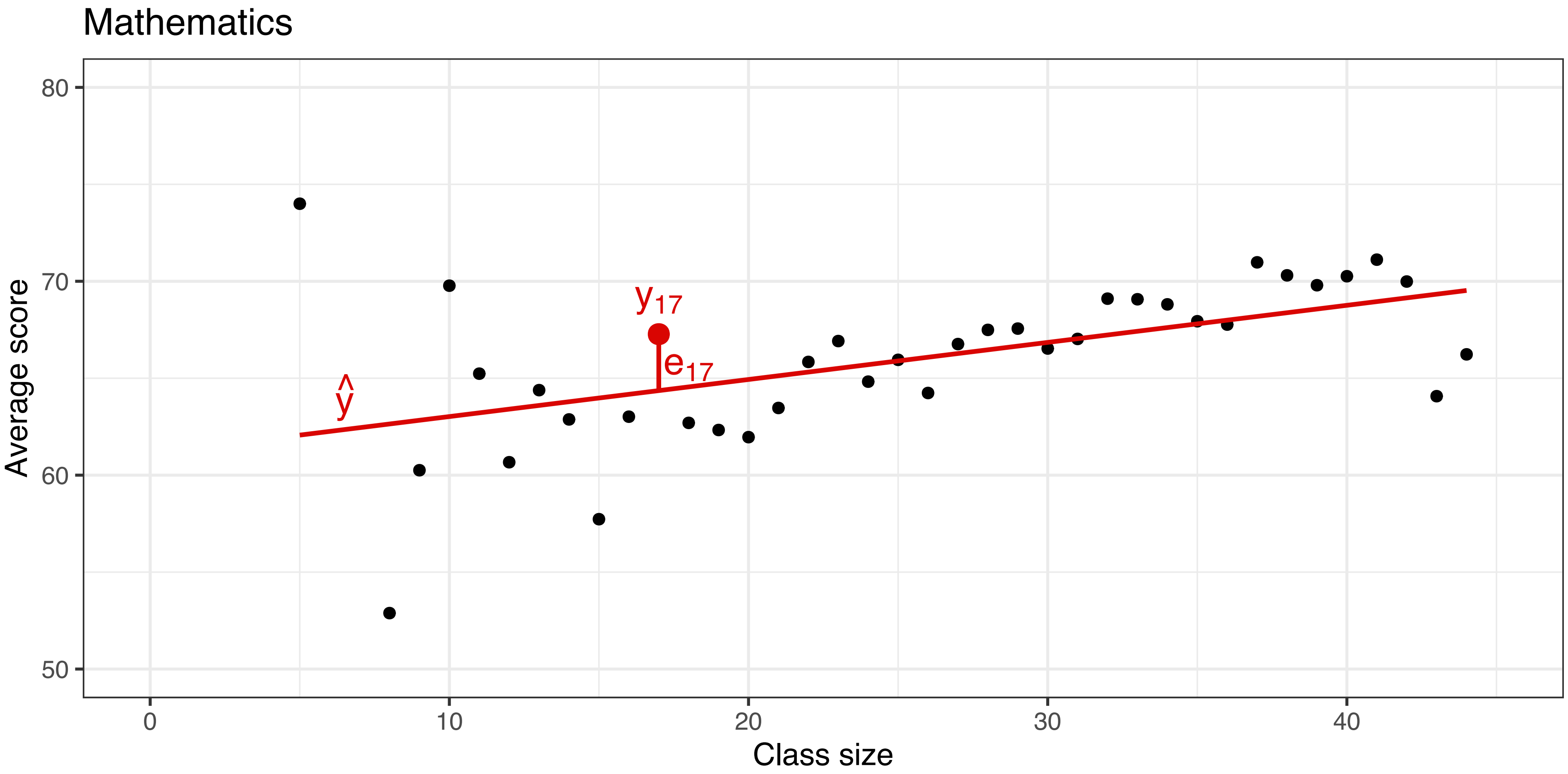

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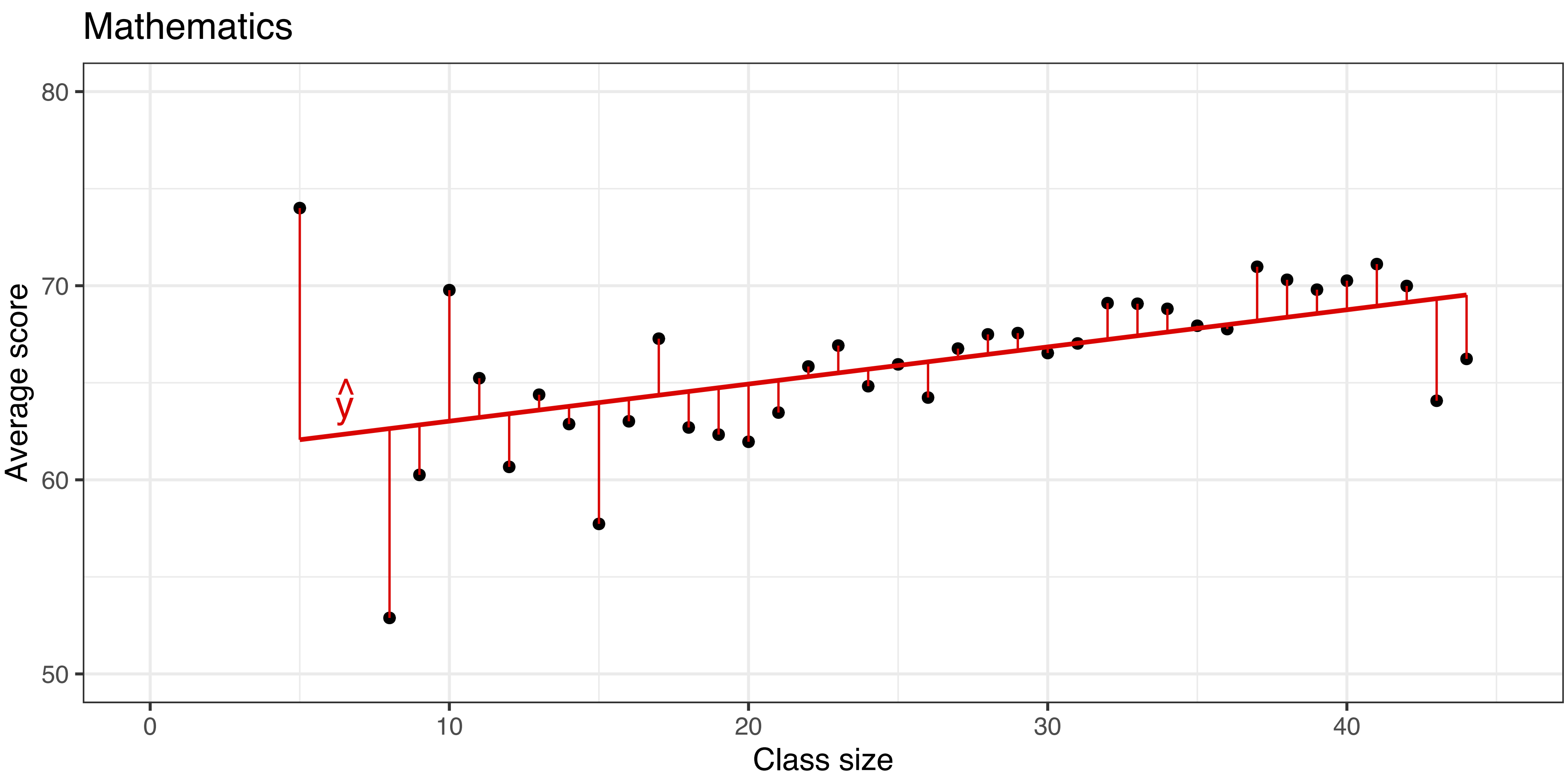

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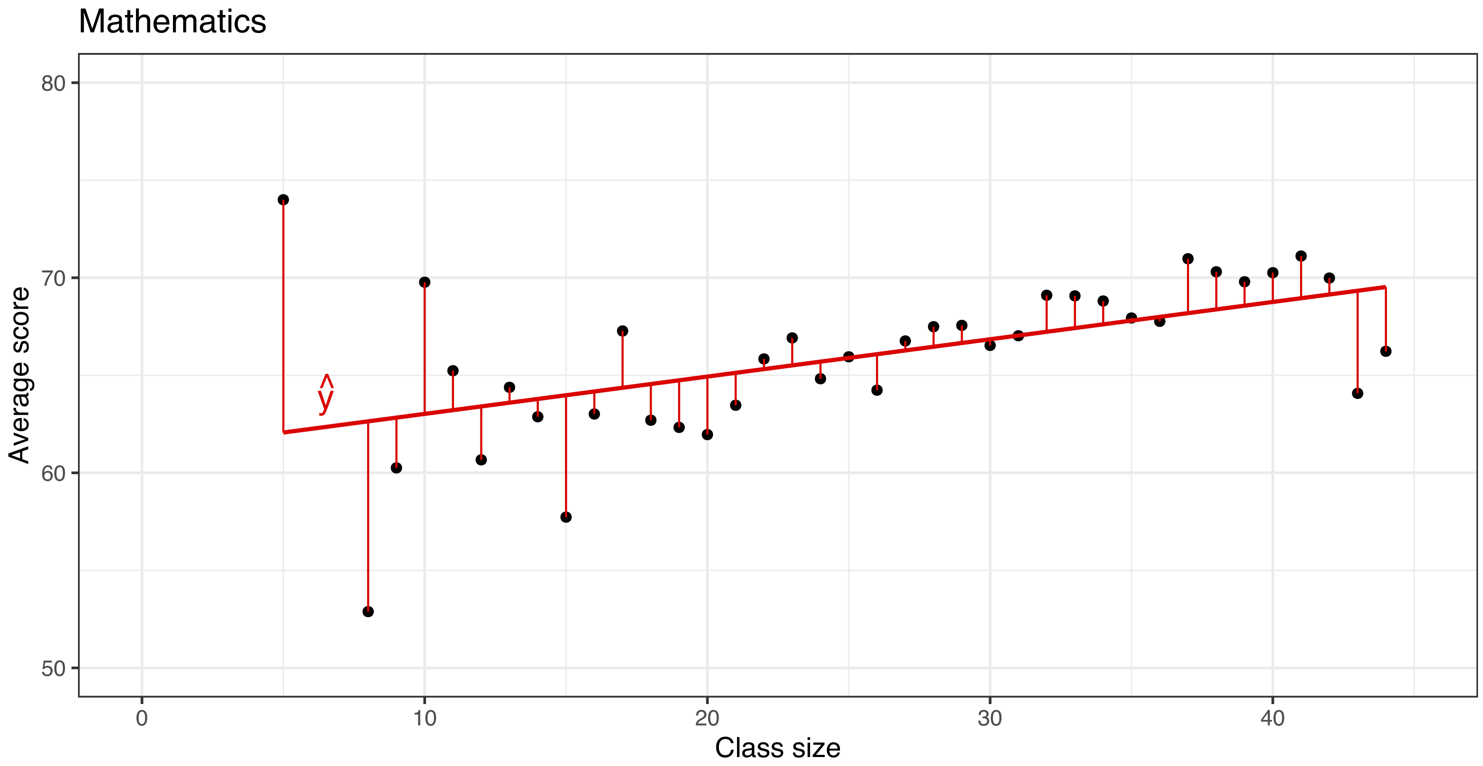


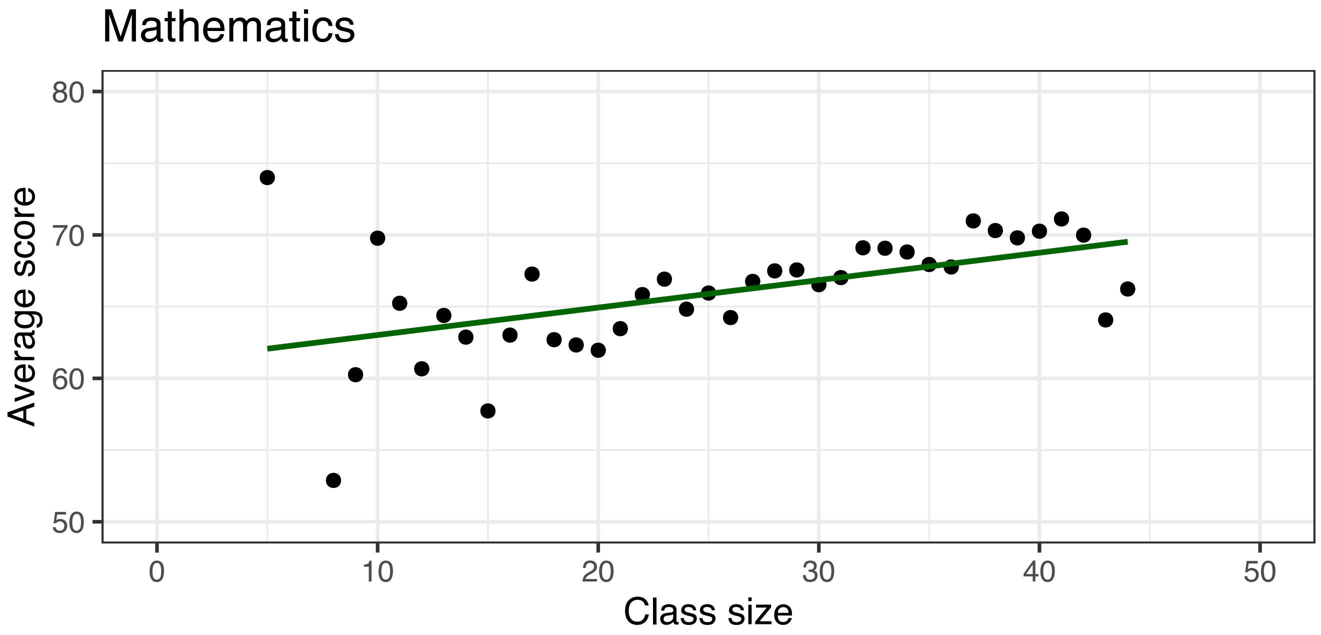
Which "minimisation" criterion should (can) be used?

Ordinary Least Squares (OLS) Estimation

- Errors of different sign $(+/-)$ cancel out, so we consider **squared residuals**

$$e_i^2 = (Y_i - \widehat{Y}_i)^2 = (Y_i - a - bX_i)^2$$

- Choose (a, b) such that $\sum_{i=1}^{N} e_1^2 + \cdots + e_N^2$ is **as small as possible**.



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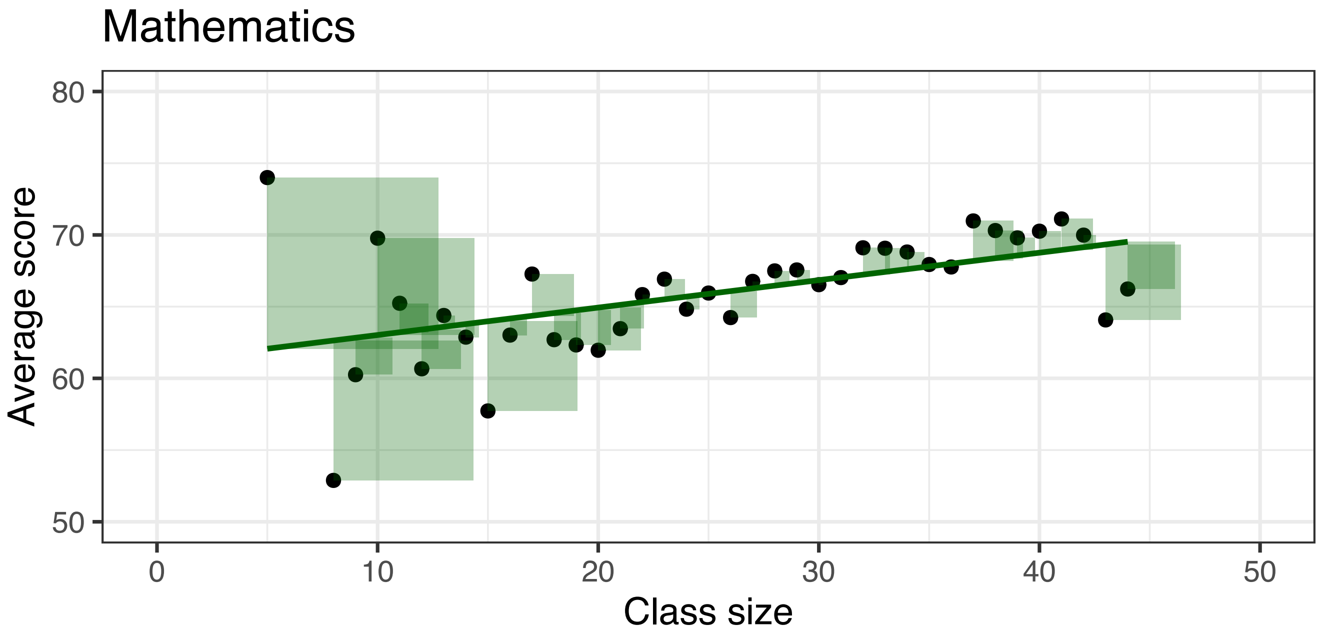

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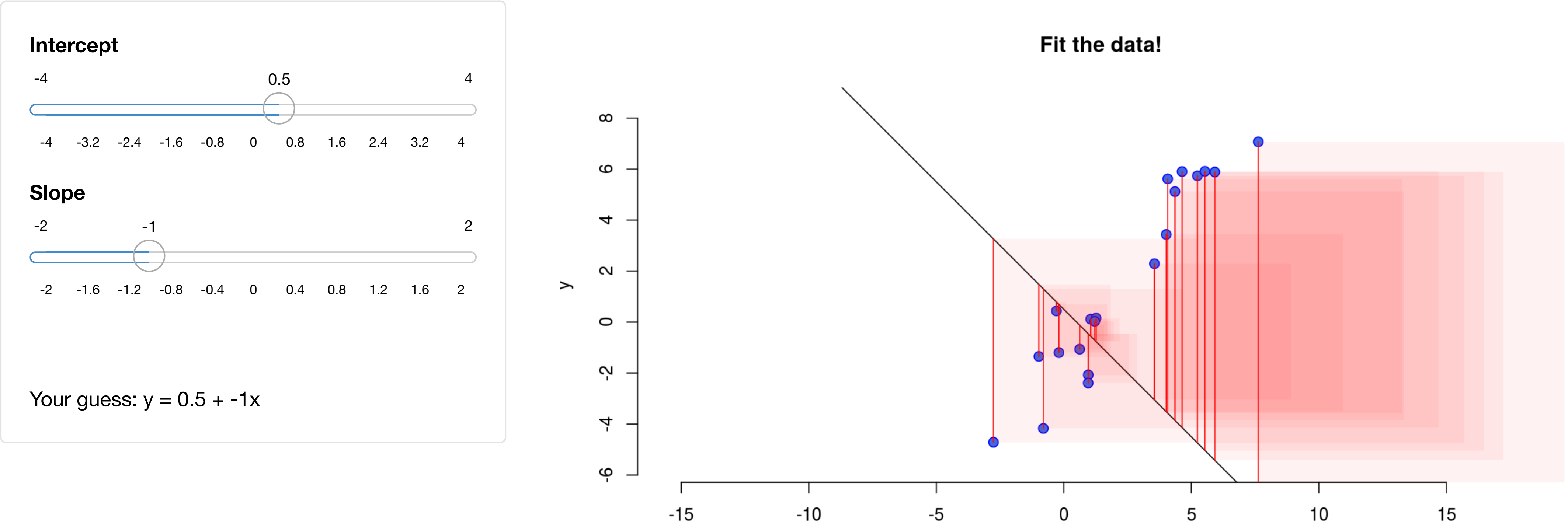

Link

Ordinary **L**east **S**quares (OLS) Estimation

Intercept

-4 0.5 4

-4 -3 -2 -1 0 1 2 3 4

Slope

-1 3

-1 -0.5 0 0.5 1 1.5 2 2.5 3

Link

Covariance: Brief Explainer 1/2

- The covariance is a measure of co-movement between two random variables (X_i, Y_i):

$$Cov(X_i, Y_i) = \sigma_{XY} = \mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])]$$

- With its sample counterpart (for the case of equally likely observations):

$$\widehat{\sigma}_{XY} = \frac{\sum(X_i - \overline{X_i})(Y_i - \overline{Y_i})}{n}$$

- If either formula looks weird, think of the variance, as the covariance between X_i and itself (X_i) and the above should look more familiar:
$\sigma_{XX} = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_i - \mathbb{E}[X_i])] = \mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right] = \sigma_X^2$

Covariance: Brief Explainer 2/2

In addition to $\sigma_{XX} = \sigma_X^2$, we might use two other properties of the covariance:

- If the expectation of either X_i or Y_i is 0, the covariance between them is the expectation of their product: $Cov(X_i, Y_i) = E(X_i Y_i)$
- The covariance linear functions of variables X_i and Y_i -- written as $W_i = c_1 + c_2 X_i$ and $Z_i = c_3 + c_4 Y_i$ for constants c_1, c_2, c_3, c_4 -- is given by:

$$Cov(W_i, Z_i) = c_2 c_4 Cov(X_i, Y_i)$$

- You are not asked to memorize any of these formulas. Just used them to understand many concepts in regression.

Ordinary Least Squares (OLS): Coefficient Formulas 1/4

- **OLS**: *estimation* method consisting in choosing a and b to minimize the sum of squared residuals.

- In the case of one regressor (and a constant), the result of this minimization generates the following formulas: (derivation in this video and these slides).

- So what are the formulas for a (intercept) and b (slope)?

- We can solve this problem for the population or for random sample.

- Warning: the next 3 slides are heavy on notation. If you lose track, the main takeaway is that we want an intuitive formula for the solution to this problem.

Ordinary **L**east **S**quares (OLS): Coefficient Formulas 2/4

Population

Problem to solve:

$$\arg\min_{a,b} \left\{\mathbb{E}[(Y_i - a - bX_i)^2]\right\}$$

Solution:

$$b = \beta = \frac{\mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])]}{\mathbb{E}[(X_i - \mathbb{E}[X_i])^2]}$$

$$a = \alpha = \mathbb{E}[Y_i] - b\,\mathbb{E}[X_i]$$

Sample

Problem to solve:

$$\arg\min_{a,b} \left\{\sum (Y_i - a - bX_i)^2\right\}$$

Solution:

$$b = \widehat{\beta} = \frac{\sum (Y_i - \overline{Y})(X_i - \overline{X})}{\sum (X_i - \overline{X})^2}$$

$$a = \widehat{\alpha} = \overline{Y} - b\overline{X}$$

- Let's bring the concept of Covariance to make this formulas more intuitive

Ordinary Least Squares (OLS): Coefficient Formulas 3/4

Population

$$b = \beta = \frac{Cov(X_i, Y_i)}{Var(X_i)} = \frac{\sigma_{XY}}{\sigma^2_X}$$

$$a = \alpha = \mathbb{E}[Y_i] - b\,\mathbb{E}[X_i]$$

Sample

$$b = \widehat{\beta} = \frac{\frac{\sum(Y_i - \overline{Y})(X_i - \overline{X})}{n}}{\frac{\sum(X_i - \overline{X})^2}{n}}$$

$$a = \widehat{\alpha} = \overline{Y} - b\overline{X}$$

Ordinary **L**east **S**quares (OLS): Coefficient Formulas 3/4

Population

$$b = \frac{Cov(X_i, Y_i)}{Var(X_i)} = \frac{\sigma_{XY}}{\sigma_X^2}$$

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Sample

$$b = \widehat{\beta} = \frac{Cov(X_i, Y_i)}{Var(X_i)} = \frac{\widehat{\sigma}_{XY}}{\widehat{\sigma}_X^2}$$

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Ordinary **L**east **S**quares (OLS): Coefficient Formulas 4/4

- The main takeaway:

$$b = \frac{Cov(X_i, Y_i)}{Var(X_i)}$$

Properties of Residuals 1/2

- As we saw at the beginning of this class, in a regression the observed outcome (Y_i) can be separated into a component "explained" by the regression equation (aka model) and a residual component:

$$Y_i = \underbrace{\widehat{Y}_i}_{\text{fitted values (explained)}} + \underbrace{e_i}_{residuals}$$

- Two important properties of the residuals:

  1. They have expectation 0. $E(e_i) = 0$
  2. They are uncorrelated with all the regressors that made them and with the corresponding fitted values. For each regressor X_{ki}:
     $E[X_{ki}e_i] = 0$ and $E[\widehat{Y}_i e_i] = 0$

Properties of Residuals 2/2

- We take this properties as given in this course (they come from the calculus of the minimization problem).

- One important point is that this properties are true always (regardless of biased coefficients).

- This does not imply however that we have solve the problem selection bias.

- In the traditional way of teaching econometrics this two concepts are mixed (hence required a distinction between residuals (e_i) and unobservables (u_i)).

(OLS with R)

- In `R`, OLS regressions are estimated using the `lm` function.

- This is how it works:

```
lm(formula = dependent variable ~  independent variable)
```

Let's estimate the following model by OLS: $\textbf{average math score}_i = a + b\textbf{class size}_i + e_i$

```r
# OLS regression of class size on average maths
lm(avgmath_cs ~ classize, grades_avg_cs)
```

```
#>
#> Call:
#> lm(formula = avgmath_cs ~ classize, data = grade
#>
#> Coefficients:
#> (Intercept)     classize
#>     61.1092       0.1913
```

Acknowledgments

- Ed Rubin's Undergraduate Econometrics II
- ScPoEconometrics
- MM