Online Appendix:
Bias due to network misspecification under spatial dependence

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Appendix A Deriving the Bounds of the Bias in Non-Spatial Models

In the following we derive the bounds given in Betz, Cook, Hollenbach (n.d.). In keeping with the literature, we assume the weights matrix is a fixed, hollow matrix (no self ties) with exogenously determined non-negative elements. We consider both symmetric and asymmetric spatial weights matrices $\mathbf{W}$ that have been row standardized or scalar normalized using min-max normalization. To ease notation, we assume that $\mathbf{x}$ has mean zero. In detailing our derivation, it is more convenient to re-express the bias expression from the SLX model from the manuscript (equation 2) as[1]

$$(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{W}\mathbf{x} \leq 1. \tag{1}$$

For those familiar with spatial models, the parallel to Moran's I will be obvious, as it is the same ratio used in that measure. There are three virtues of this parallel for our purposes. First, for non-technical readers, it allows us to simplify our bounding condition as being satisfied whenever Moran's I produces values between -1 and 1, which is generally the case (Cliff and Ord, 1981).[2] Second, in our technical derivation, we are able to borrow from the literature on Moran's I in detailing the regularity conditions (i.e., assumptions) necessary for this inequality to obtain. Finally, when this condition is not satisfied, it typically also implies that the process is not stationary, meaning the straight-forward application of spatial econometric methods would be ill-advised without additional transformations of the data.

To identify the conditions for inequality (1) to hold, it is useful to distinguish between symmetric and non-symmetric $\mathbf{W}$ matrices. We first show that the condition holds for any arbitrary symmetric $\mathbf{W}$, and then detail sufficient conditions under which it holds for non-symmetric $\mathbf{W}$.

[1] We focus on the case where $\mathbf{x}'\mathbf{W}\mathbf{x}$ is positive, but the same conditions which ensure inequality (1) to hold also ensure that $(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{W}\mathbf{x} \geq -1$.

[2] Moran's I includes a scaling factor as well, which is the sample size N divided by the sum of all elements of $\mathbf{W}$. After row or min-max normalization, this ratio is always larger than one. The sum of all elements of $\mathbf{W}$ is equal to the sum of all row sums as well as to the sum of all column sums. With row-normalization, this sum is identical to N; with min-max normalization, it is at most N.

Symmetric W

Our derivation uses that, for any non-zero vector $\mathbf{x}$, the expression $(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{W}\mathbf{x}$ takes on values within the field of values (or the numerical range) of the matrix $\mathbf{W}$. For symmetric $\mathbf{W}$, the numerical range is on the real line, with endpoints determined by the largest and the smallest eigenvalues of $\mathbf{W}$. De Jong, Sprenger and Van Veen (1984), for example, use this feature when showing that for symmetric $\mathbf{W}$, $(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{W}\mathbf{x}$ lies within the smallest and largest eigenvalue of $\mathbf{W}$. Therefore, applying any familiar normalization strategy (row standardization, min-max, spectral) to a symmetric weights matrix (including common constructions based on contiguity, inverse distance, block group, common border length, fixed buffer) ensures that condition (1) holds, since the maximum eigenvalue of the normalized matrix is 1 (this is also the largest eigenvalue in absolute terms, that is, no eigenvalue is smaller than -1; and note that for real symmetric matrices, no eigenvalues are complex).

Using a different approach, we prove this independently by right multiplying and subtracting both sides by $\mathbf{x}'\mathbf{x}$. Re-arranging terms this can now be written as:

$$\mathbf{x}'(\mathbf{W} - \mathbf{I})\mathbf{x} \leq 0, \tag{2}$$

where $\mathbf{I}$ is an identity matrix of size N. This expression is now in the more-familiar quadratic form – i.e., $\mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is a vector and $\mathbf{A} = (\mathbf{W} - \mathbf{I})$ is a symmetric matrix – allowing us to exploit well-known results.

By definition, the expression in (2) is satisfied whenever $\mathbf{A}$ is negative semi-definite. As such, we need only demonstrate the conditions under which $\mathbf{A} = \mathbf{W} - \mathbf{I}$ is negative semi-definite to prove (1). One way to prove that $\mathbf{A}$ is negative semi-definite is to show that all eigenvalues of $\mathbf{A}$ are non-positive or, equivalently, that the largest eigenvalues of $\mathbf{W}$ is at most one. From Gershgorin's circle theorem, this condition holds for all $\mathbf{W}$ that have been normalized using min-max or row-normalization.[3] To see why, note that Gershgorin's circle theorem implies that all eigenvalues λ of

[3]Note that this is true trivially for spectral normalization, which normalizes $\mathbf{W}$ to ensure its largest eigenvalue is one.

$\mathbf{W}$ are located in discs with origin w_{ii}, such that:

$$|\lambda - w_{ii}| \leq \sum_{j \neq i} |w_{ij}|. \tag{3}$$

Since $w_{ii} = 0$ for all i and the off-diagonal elements are non-negative, this is equivalent to

$$|\lambda| \leq \sum_{i} w_{ij}, \tag{4}$$

which implies that the absolute value of the largest eigenvalue of $\mathbf{W}$ is bounded by the largest row-sum. Moreover, because eigenvalues are identical for the transpose of a matrix, Gershgorin's circle theorem implies that all eigenvalues must also be bounded by the largest column-sum. The minimum of the largest row-sum and the largest column-sum therefore provides a bound on the largest eigenvalue of $\mathbf{W}$. More simply, note that for symmetric matrices, $\sum_{j \neq i} w_{ij} = \sum_{i \neq j} w_{ij}$, such that row- and column-sums are identical. For min-max normalization it follows that the largest eigenvalue is bounded by one.[4] Similarly, for row-normalization, the largest eigenvalue is one, because all row-sums are equal to one (Ord, 1975). This proves that condition (1) holds for any symmetric $\mathbf{W}$. Note that inequality (4) also rules out eigenvalues smaller than -1, which ensures that the bound in condition (1) also holds when covariance between $\mathbf{Wx}$ and $\mathbf{x}$ is negative.

Before proceeding, we note that symmetric spatial weights matrices are frequently suggested from theoretical models, and commonly used in applied work. They include any matrix that is based on (undirected) attributes of pairs of observations, such as contiguity matrices, inverse distance matrices, matrices based on bilateral trade flows, and matrices based on distance thresholds.

Non-symmetric W

Extending this approach directly to non-symmetric matrices (as found in network-based ties) proves more challenging. Above we used the fact that negative semi-definite matrices always satisfy (1). For non-symmetric matrices $\mathbf{B}$, the quadratic form is instead given by $\mathbf{x}' \left[\frac{\mathbf{B}+\mathbf{B}'}{2}\right] \mathbf{x}$,

[4] Alternatively, note that because the largest eigenvalue is bounded by the minimum of the largest row-sum and largest column-sum, spectral normalization ensures a smaller normalization factor than min-max normalization.

such that our condition becomes

$$\mathbf{x}' \left[\frac{\mathbf{W} + \mathbf{W}'}{2} - \mathbf{I} \right] \mathbf{x} \leq 0. \tag{5}$$

As before, the goal is to identify which $\mathbf{W}$ satisfy this condition. Above, the corresponding condition (4) held anytime the row sums or column sums of the non-diagonal elements were one or less, which was guaranteed by min-max and row-normalization. Relying again on Gershgorin's circle theorem, the non-symmetric case instead requires that, for all eigenvalues λ of $\frac{\mathbf{W}+\mathbf{W}'}{2}$,

$$|\lambda| \leq \frac{\sum_{j \neq i} w_{ij} + \sum_{i \neq j} w_{ij}}{2} \leq 1. \tag{6}$$

That is, the non-symmetric case requires that the sum of the row and column sums needs to be less than 2 for each unit or that the largest eigenvalue of $\frac{\mathbf{W}+\mathbf{W}'}{2}$ is bounded by one. Note that, for non-symmetric matrices, it is not the case that the eigenvalues of the sum of matrices are identical to the sum of the eigenvalues. Condition (6) therefore does not hold in general after normalization of $\mathbf{W}$. However, it is satisfied in many cases, and in particular for common spatial weights matrices.

First, the above condition holds for all matrices that are doubly-stochastic, such that a unit's column sum equals its row sum, with elements adding up to one. These matrices need not be symmetric, but they ensure that the largest eigenvalue of $\frac{\mathbf{W}+\mathbf{W}'}{2}$ is bounded by one, which in turn guarantees that our condition holds. Doubly-stochastic matrices comprise a large number of weights matrices and are commonly used in theoretical work on the properties of spatial econometric estimators. They imply that each element of $\mathbf{Wx}$ is a weighted average of $\mathbf{x}$, where each $\mathbf{x}$ has the same total influence on the network (note that this influence can be distributed arbitrarily across units). Among others, and in addition to all symmetric variants of doubly-stochastic matrices (including inverse distance and contiguity matrices), this applies to many potentially asymmetric weights matrices based on nearest neighbors (LeSage and Pace, 2014). Indeed, the class of matrices that satisfy our bounds is more general than this and includes all line-sum symmetric matrices, such that the sum of elements in each row equals the sum of elements in each column (but row

sums need not be identical to each other).

Second, a set of possibly asymmetric matrices that satisfies the above conditions are spectral matrices. Min-max, row, and spectral normalization all ensure that the largest eigenvalue of $\mathbf{W}$ is at most one; this holds for arbitrary $\mathbf{W}$. For spectral matrices, the largest eigenvalue is identical to the numerical radius, and normalization thus ensures that $\mathbf{x}'\mathbf{W}\mathbf{x} \leq \mathbf{x}'\mathbf{x}$. Spectral matrices include all symmetric matrices (providing another approach to prove the above result for symmetric matrices), but they also include a large class of asymmetric matrices (for a characterization, see, e.g., Goldberg and Zwas 1975).

Third, note that for scalar normalizations, condition (6) always holds if we normalize by the maximum row or column sum – that is, the max-max. While not common to the literature, Kelejian and Prucha (2010) emphasize that *any* matrix norm $||\mathbf{W}||$ – e.g., the maximum eigenvalue, the maximum absolute row sum, the maximum absolute column sum, etc. – serves as a useful normalization factor since it bounds the spectral radius. The choice between different norms is theoretically arbitrary, since each is proportionally equivalent.

Fourth, a vast literature addresses the distribution of Moran's I. Cliff and Ord (1981) demonstrate that generally "the upper bound for $|I|$ will be less than unity, although it could exceed unity for an irregular pattern of weights if [observations] with extreme values of z_i are heavily weighted." Put differently, to obtain bounds of Moran's I larger than one in absolute value requires not only an unusual composition of $\mathbf{W}$, but that unusual $\mathbf{W}$ must also coincide in predictable ways with the structure of $\mathbf{x}$. In deriving the feasible range for Moran's I for tessellations, Boots and Tiefelsdorf (2000) have shown this rarely occurs. This is because the combinations of a matrix $\mathbf{W}$ and predictor $\mathbf{x}$ are so atypical that they are unlikely to hold in reasonable observational settings. Conversely, if $\mathbf{W}$ is sufficiently dense, these atypical cases cannot arise, because $\mathbf{W}$ effectively averages over x_i.

Importantly, if Moran's I is bounded by one – and we are outside the realm of 'irregular' cases

– our condition always holds. To see why, note that Moran's I is defined as

$$\mathbb{I} = \frac{N}{S}\frac{\text{cov}(\mathbf{Wx}, \mathbf{x})}{\text{var}(\mathbf{x})}, \tag{7}$$

where N is the sample size N as before and S is the sum of all elements of $\mathbf{W}$. For row-normalization and min-max normalization, $S \leq N$ because both normalizations ensure that either each row-sum or each column sum is at most one. Because the sum of all elements of $\mathbf{W}$ is at most the sum of all row-sums or the sum of all column sums, it follows that $S \leq N$. It follows that if Moran's I is bounded by 1 that $\frac{\text{cov}(\mathbf{Wx},\mathbf{x})}{\text{var}(\mathbf{x})} \leq 1$ must hold as well. In other words, the matrices that spatial econometric models typically envision ensure that our bounds hold.

Finally, and building on this notion of extreme and unusual cases, we can establish bounds for these outlier scenarios. Because these outlier scenarios depend on the specific realizations of $\mathbf{x}$, we can calculate a bound based on the sample. Observe that the worst case for our bounds is a scenario that creates the largest possible value for $\text{cov}(\mathbf{Wx}, \mathbf{x})$. With the mean of $\mathbf{x}$ being zero, this expression is identical to $\frac{1}{N}\sum_{i=1}^{N} x_i \sum_{j=1}^{N} w_{ij}x_j$ and is maximized if the most extreme values of x_i are paired with the largest values that can be produced by $\sum_{j=1}^{N} w_{ij}x_j$.

First consider a row-normalized $\mathbf{W}$ or $\mathbf{W}$ such that min-max normalization results in row-sums that are at most one. Note that in both cases, $\mathbf{W}$ preserves the range of $\mathbf{x}$. Then, $\text{cov}(\mathbf{Wx}, \mathbf{x})$ takes its maximum value if $\mathbf{W}$ is such that each observation is exclusively connected to one of the two most extreme cases realized in the sample – i.e., $\max\{x_i\}$ and $\min\{x_i\}$. This corresponds to a weights matrix that has almost all zero elements. We emphasize that these matrices present extreme forms of asymmetry (in particular, after normalization, the largest row-sum is one while the the largest column sum is $N-1$), violate standard assumptions about $\mathbf{W}$ (e.g., those presented in Anselin (1988)), and also represent a network with almost no interdependence: the two largest observations on $\mathbf{x}$ determine the values of $\mathbf{Wx}$ of all other observations in the sample, with no observation being exposed to more than one observation, and no path of any length that connects the two most extreme observations.

To obtain a bound for $\text{cov}(\mathbf{Wx}, \mathbf{x})$, sort $\mathbf{x}$ such that $x_1 \geq x_2 \geq x_3 \ldots \geq x_N$. Then, let k such that for $i \leq k$, $x_i \geq 0$ and for $i > k$, $x_i < 0$. Then we have that

$$\text{cov}(\mathbf{Wx}, \mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N} \left[x_i \sum_{j=1}^{N} w_{ij} x_j \right] \tag{8}$$

$$\leq \frac{1}{N} \sum_{i=1}^{k} x_i x_1 + \frac{1}{N} \sum_{i=k+1}^{N} x_i x_N, \tag{9}$$

which can be calculated in any given sample as an upper bound on $\text{cov}(\mathbf{Wx}, \mathbf{x})$. Note that $\sum_{i=1}^{k} x_i = -\sum_{i=k+1}^{N} x_i$ because $\sum_{i=1}^{N} x_i = 0$ (which implies that the largest positive covariance is identical to the largest negative covariance, such that we do not have to consider these cases separately).

This bound can be larger than one, but it need not be. For example, it is easy to verify that for any binary $\mathbf{x}$,[5] this expression simplifies to $\text{cov}(\mathbf{Wx}, \mathbf{x}) \leq 1$. That the bound is always smaller than one for binary $\mathbf{x}$, but not more generally, also reinforces the earlier point: for our main condition to fail, we would need to have a specific constellation of $\mathbf{W}$ *and* $\mathbf{x}$. Moreover, this bound allows calculating the largest possible value of $\text{cov}(\mathbf{Wx}, \mathbf{x})$ in any given sample and, from that, the minimum value of ρ that would be necessary to obtain a bias larger than β.

To obtain more intuition for the inequality for our worst-case bounds, we can also write

$$\frac{\text{cov}(\mathbf{Wx}, \mathbf{x})}{\text{var}(\mathbf{x})} \leq \left[\frac{(x_1 - x_N)^2}{4\text{var}(\mathbf{x})} \right] \left[\frac{\mathbb{E}[x_i | x_i \geq 0] + \mathbb{E}[x_i | x_i < 0]}{x_1 - x_N} \right],$$

The first term in this expression is a Popoviciu ratio: it shows how close $\text{var}(\mathbf{x})$ is to its possible maximum based on the largest and smallest value of $\mathbf{x}$. This ratio is at least one and attains the lower bound of 1 if $\mathbf{x}$ is dichotomous. The second term is an indicator of how spread out $\mathbf{x}$ is over its interior. This ratio is at most 1 and attains its upper bound again if $\mathbf{x}$ is dichotomous.

We next consider the case where $\mathbf{W}$ has been normalized with min-max normalization that resulted in column-sums of at most one but potentially larger row-sums (because the largest column-

[5] Recall that we assume that $\mathbf{x}$ has mean zero, which implies that a binary $\mathbf{x}$ takes on values $1 - p$ and $-p$, where p is the proportion of positive observations in the sample.

sum was smaller than the largest row-sum). The key here is to observe that in this case, each unit has a total influence of at most one on the entire network. Thus, the largest possible value for $\mathrm{cov}(\mathbf{Wx}, \mathbf{x})$ is obtained if $\mathbf{W}$ is a matrix that sums all positive values of $\mathbf{x}$ and associates them with x_1 and that sums all negative values of $\mathbf{x}$ and associates them with x_N. Put differently,

$$\begin{aligned}\mathrm{cov}(\mathbf{Wx}, \mathbf{x}) &= \frac{1}{N}\sum_{i=1}^{N}\left[x_i \sum_{j=1}^{N} w_{ij}x_j\right] \\ &\leq \frac{1}{N}\sum_{i=1}^{k} x_i x_1 + \frac{1}{N}\sum_{i=k+1}^{N} x_i x_N,\end{aligned}$$

which is identical to the expression in (9).

Recall that these bounds would imply that *all* units are *only* connected to the most extreme observations. If this were unreasonable, as it often is, tighter bounds would obtain. For example, if one is willing to assume that each unit is connected to a minimum number of observations, tighter bounds can be derived. Consequently, in almost all practical examples the bounds given by (1) will give a more typical approximation than those given by (9) – as we demonstrate using simulations for k-nearest neighbors and row standardization in the appendix. However, for completeness, we have presented both sets of bounds here.

SAR: powers of matrices and bounds

For any (symmetric or non-symmetric) matrix that satisfies $\mathrm{cov}(\mathbf{Wx}, \mathbf{x})/\mathrm{var}(\mathbf{x}) \leq 1$, it follows that $\mathrm{cov}(\mathbf{W}^k\mathbf{x}, \mathbf{x})/\mathrm{var}(\mathbf{x}) \leq 1$ for $k = 1, 2, 3, \ldots$. To prove this, note that the maximum of $\mathrm{cov}(\mathbf{W}^k\mathbf{x}, \mathbf{x})/\mathrm{var}(\mathbf{x})$ is equal to the numerical radius of $\mathbf{W}$. Denoting the numerical radius with $r(\mathbf{W})$, the Halmos inequality establishes that $r(\mathbf{W}^k) \leq r^k(\mathbf{W})$ – see, e.g., Goldberg and Zwas (1975). If $r(\mathbf{W}) \leq 1$, it follows that $r(\mathbf{W}^k) \leq r^k(\mathbf{W}) \leq 1$, which proves that $\mathrm{cov}(\mathbf{W}^k\mathbf{x}, \mathbf{x})/\mathrm{var}(\mathbf{x}) \leq 1$ whenever $\mathrm{cov}(\mathbf{Wx}, \mathbf{x})/\mathrm{var}(\mathbf{x}) \leq 1$.

Alternatively, an analogous approach to above can be used to derive bounds from the sample data on $\mathbf{x}$ and $\mathbf{y}$ to calculate a bound on the covariance $\mathrm{cov}(\mathbf{Wy}, \mathbf{x})$ for arbitrary $\mathbf{W}$. For $\mathbf{W}$ such that normalization results in a largest row-sum of at most one, the largest possible value for

$\mathrm{cov}(\mathbf{Wy}, \mathbf{x})$ obtains if $\mathbf{W}$ matches all positive x_i with the largest value of $\mathbf{y}$ and all negative values of x_i with $y_{\mathrm{T}} = \min\{0, \min\{y_i\}\}$. Note that if $y_i > 0$ for all i, this implies a matrix $\mathbf{W}$ such that observations with $x_i < 0$ are 'islands', with only zero elements in the corresponding rows (which is inconsistent with a large class of standard spatial weights matrices and hence results in a bound larger than what those matrices would permit). Then, defining k as before such that $x_i \geq 0$ for $i \leq k$ and $x_i < 0$ for $i > k$,

$$\mathrm{cov}(\mathbf{Wy}, \mathbf{x}) <= \frac{1}{N}\left[\sum_{i=1}^{k} y_{\max} x_i + \sum_{i=k+1}^{N} y_{\mathrm{T}} x_i\right],$$

which can be calculated from the data. A similar expression follows for $\mathbf{W}$ if normalization results in a column sum of at most one. The larger of the two expressions can then be used to derive an upper bound for the bias in the SAR case.

Appendix B Propagation of Measurement Error in the IV Estimation

To see that the measurement error in $\mathbf{W}$ is not solved by instrumental variable estimation of the SAR model, consider that pre-multiplying both sides of equation (4) by $\mathbf{W}$ and using repeated substitution for $\mathbf{Wy}$ yields

$$\mathbf{Wy} = \beta\mathbf{Wx} + \beta\rho\mathbf{W}^2\mathbf{x} + \beta\rho^2\mathbf{W}^3\mathbf{x} + \ldots + \mathbf{W}\epsilon + \rho\mathbf{W}^2\epsilon + \rho^2\mathbf{W}^3\epsilon\ldots, \tag{10}$$

which demonstrates how $\mathbf{Wx}$ and its powers have strength as instruments for $\mathbf{Wy}$. However, if we rely on $\widetilde{\mathbf{W}}$ from equation (8) we obtain:

$$\widetilde{\mathbf{W}}\mathbf{y} = \beta\widetilde{\mathbf{W}}\mathbf{x} + \beta\rho\widetilde{\mathbf{W}}^2\mathbf{x} + \beta\rho^2\widetilde{\mathbf{W}}^3\mathbf{x} + \ldots + \widetilde{\mathbf{W}}\epsilon + \rho\widetilde{\mathbf{W}}^2\epsilon + \rho^2\widetilde{\mathbf{W}}^3\epsilon\ldots \tag{11}$$

Due to the common transformation via $\widetilde{\mathbf{W}}$, the instrument $\widetilde{\mathbf{W}}\mathbf{x}$ is related to the measurement error in $\widetilde{\mathbf{W}}\mathbf{y}$. IV estimation will resolve the simultaneity bias – the usual concern with spatially-lagged outcomes – but not the bias due to measurement error.

Appendix C Additional Simulation Results

C.1 Additional SAR Simulation Results

In this section, we present results from the additional simulations of the SAR process.[6] First, Figure C.1 shows the results from the simulations where $\mathbf{W}$ is based on 10-nearest-neighbors but is row-normalized. Again, spatial dependence in $\mathbf{X}$ increases from left to right, while the spatial parameter of $\mathbf{Y}$ increases from zero for simulation results depicted in the top row to 0.6 in the bottom row. The results are quite similar to those presented in the paper. The biases in the standard linear models and the misspecified SAR models increase with higher spatial dependence in both $\mathbf{X}$ and $\mathbf{Y}$. Again, under all scenarios, the bias misspecified in SAR models is bounded from above by the bias in the non-spatial models.

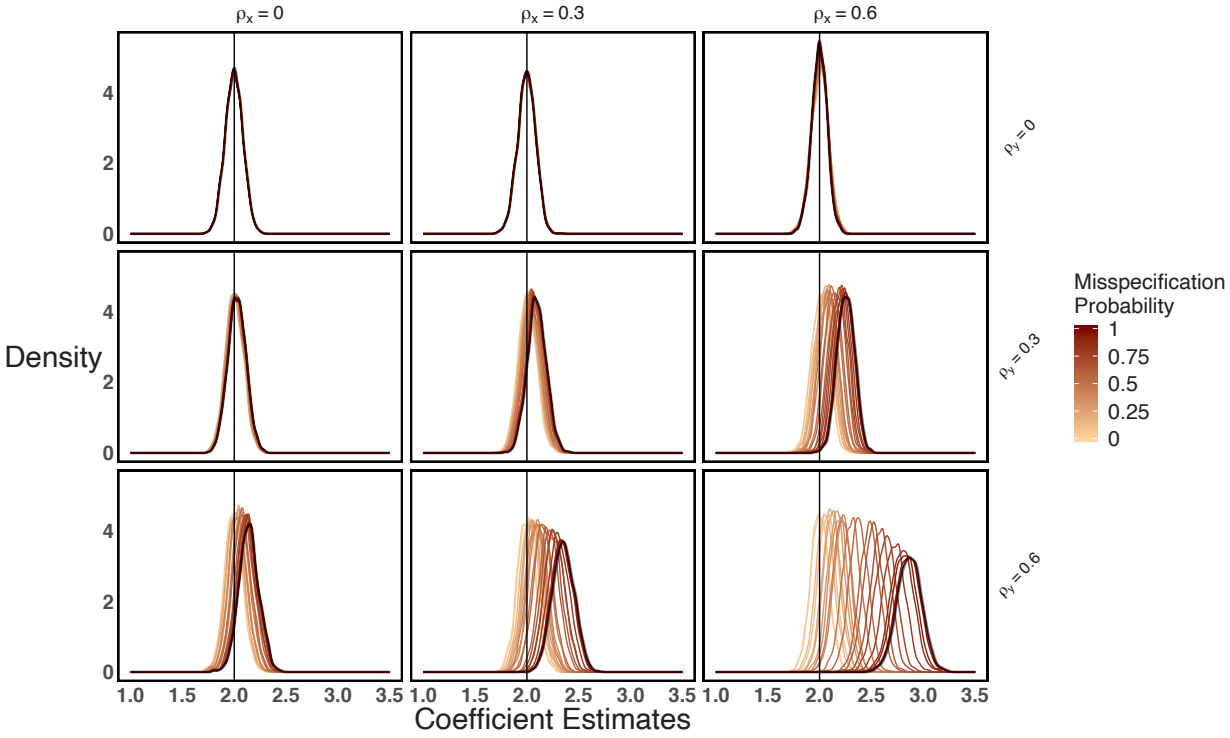

Figure C.1: Misspecification of W in SAR models – KNN & Row-Normalization

[6]The replication materials for the results presented both here and in the main text can be found at Hollenbach, Betz and Cook (2019).

The derivation of our analytical bounds allows us to calculate the expected bias for SAR models. In particular, equation 5 in the manuscript can be rearranged such that we can derive an expected $\hat{\beta}$ given the simulated scenarios: $\text{plim}_{n\to\infty}\,\hat{\beta}_{OLS} = \beta + \rho\frac{\text{cov}(\mathbf{Wy},\mathbf{x})}{\text{var}(\mathbf{x})}$.

In Table C.1, we compare the expected $\hat{\beta}$ given our analytical derivation to the average OLS estimate in our simulations given combinations of the spatial parameters in the simulations based on 10-nearest-neighbors and min-max normalization. The first two columns show the variation in the simulated spatial dependence (ρ_x and ρ_y), next we calculate the average $\text{cov}(\mathbf{Wy},\mathbf{x})$, and $\text{var}(\mathbf{x})$, and their ratio from the simulations. Based on the ratio $\frac{\text{cov}(\mathbf{Wy},\mathbf{x})}{\text{var}(\mathbf{x})}$ and true $\beta = 2$, we can then calculate the expected $\hat{\beta}$ given the analytical results. In contrast, the last column shows the average $\hat{\beta}$ estimated in the standard linear model at a given scenario. As one can see, the analytical results and the simulated quantities are effectively the same.

Table C.1: Analytical $\hat{\beta}$ & Mean $\hat{\beta}$ in Simulations – SAR

ρ_x	ρ_y	$\text{cov}(\mathbf{Wy},\mathbf{x})$	$\text{var}(\mathbf{x})$	$\frac{\text{cov}(\mathbf{Wy},\mathbf{x})}{\text{var}(\mathbf{x})}$	Expected $\hat{\beta}$	Mean $\hat{\beta}_{OLS}$
0.00	0.00	0.00	0.91	0.00	2.00	2.00
0.00	0.30	0.07	0.91	0.08	2.02	2.03
0.00	0.60	0.22	0.91	0.24	2.14	2.14
0.30	0.00	0.17	0.94	0.18	2.00	2.00
0.30	0.30	0.30	0.94	0.32	2.09	2.10
0.30	0.60	0.56	0.94	0.59	2.36	2.35
0.60	0.00	0.69	1.18	0.58	2.00	2.00
0.60	0.30	1.00	1.18	0.85	2.26	2.26
0.60	0.60	1.71	1.18	1.45	2.87	2.87

C.2 Additional SLX Simulation Results

To simulate the SLX models we begin with the following data generating process:

$$\mathbf{y} = \alpha + \beta\mathbf{x} + \theta\mathbf{Wx} + \epsilon, \tag{12a}$$

$$\mathbf{x} = (\mathbf{I} - \rho_x\mathbf{W})^{-1}u, \tag{12b}$$

where u and ϵ are $\mathcal{N}(0,1)$. The effect paramaters are β, θ_x, and ρ_x, with β reflecting the direct

(i.e., pre-spatial) effect of $\mathbf{x}$ on $\mathbf{y}$, θ_x the spillover effect, and ρ_x the spatial interdependence in $\mathbf{x}$. As in the main text, the binary $\mathbf{W}$ matrix is generated with ones assigned to each observation's ten nearest neighbors. We again hold $\mathbf{W}$, $\beta = 2$, $N = 150$, and u fixed across the simulations, focusing on variation in the spatial parameters ρ_x and θ. We vary ρ_x from 0 (no spatial interdependence) over 0.3 to 0.6 (high spatial interdependence). θ takes on the following values: 0, 1, and 2.[7] For each of these 9 experimental settings, we simulate $2,000$ data sets, which leads to $18,000$ in total. We again undertake the simulation excercise with $\mathbf{W}$ matrices normalized using two methods: row- and min-max normalization.

Using these simulated data for $\mathbf{y}$ and $\mathbf{x}$, we estimate a non-spatial linear model (via OLS) and the SLX models with the different $\widetilde{\mathbf{W}}$'s, i.e., the user-specificed weights matrix of varying accuracy (i.e., decreasing in p – the probability of misspecification). We record the estimated $\hat{\beta}$ based on the model's coefficient to assess potential bias.

Figure C.2 shows the results of the SLX simulation analysis for $\mathbf{W}$ based on 10 nearest neighbors and min-max normalization. Each cell in the plot shows the result for one experimental condition, ρ_x increases from 0 to 0.6 in cells going from left to right, while θ_x increases in cells moving from top to bottom. Each cell shows the densities of coefficient estimates for models estimated with OLS or SLX models at different levels of the misspecification probability p, where darker shading is indicative of higher levels of misspecification. The densities for the OLS models are plotted in black. As one can see, the bias in misspecified models increases in both θ and ρ_x, being largest in the bottom right cell. Again, the estimates of SLX models become increasingly worse with higher levels of misspecification, but is bounded from above by the standard linear model estimate.

Figure C.3 shows the simulation results when the 10-NN matrix is standardized using row-normalization. The results across these specifications are effectively the same as in the SAR process simulations.

Lastly, we again calculate the expected $\hat{\beta}$ based on our analytical derivation for the SLX model

[7] The parameter values from θ are larger than ρ_y in the main text because the implied effect on y from changes to $\mathbf{Wx}$ are much smaller than from changes to $\mathbf{Wy}$.

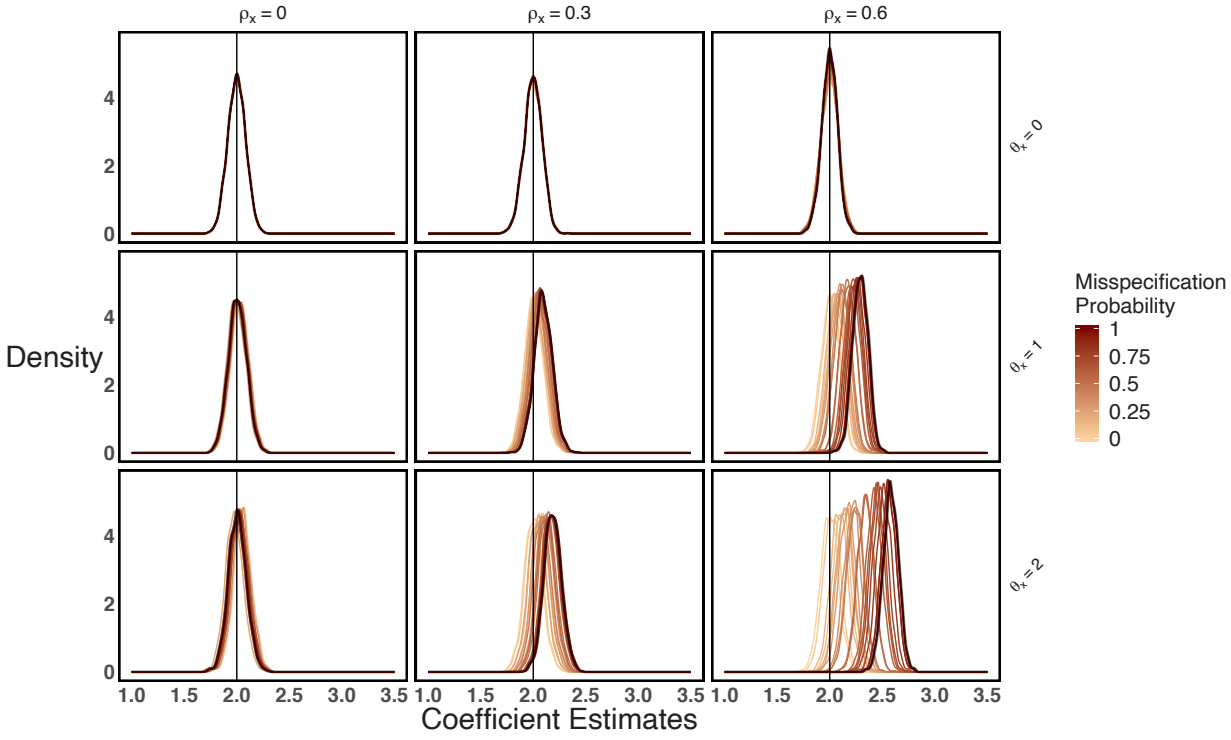

Figure C.2: Misspecification of W in SLX models – KNN & Min-Max Normalization

(equation 2 in the manuscript) for different scenarios in the simulation with 10 nearest neighbors and min-max normalization. Re-writing equation 2, we can express the expected estimate from the standard linear model as: $\text{plim}_{n\to\infty}\,\hat{\beta}_{OLS} = \beta + \theta\frac{\text{cov}(\mathbf{Wx},\mathbf{x})}{\text{var}(\mathbf{x})}$. As above, we calculate the expected $\hat{\beta}$ and compare it to the average OLS estimate for each combination of parameters in the simulation. As shown in Table C.2 the average simulation results for $\hat{\beta}$ are again quite similar to those derived analytically.

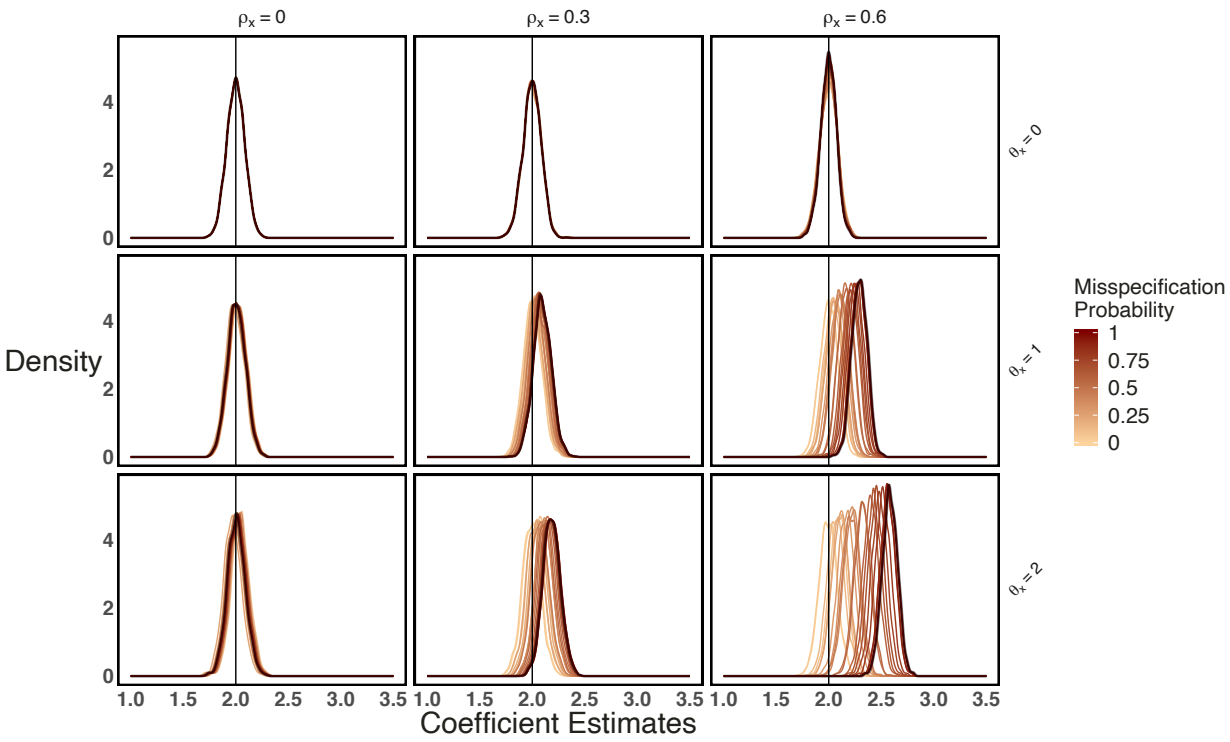

Figure C.3: Misspecification of W in SLX models – KNN & Row Normalization

Table C.2: Analytical $\hat{\beta}$ & Mean $\hat{\beta}$ in Simulations – SLX

ρ_x	θ	$\text{cov}(\mathbf{Wx}, \mathbf{x})$	$\text{var}(\mathbf{x})$	$\frac{\text{cov}(\mathbf{Wx},\mathbf{x})}{\text{var}(\mathbf{x})}$	Expected $\hat{\beta}$	Mean $\hat{\beta}_{OLS}$
0.00	0.00	0.00	0.91	0.00	2.00	2.00
0.00	1.00	0.00	0.91	0.00	2.00	2.01
0.00	2.00	0.00	0.91	0.00	2.00	2.00
0.30	0.00	0.09	0.94	0.09	2.00	2.00
0.30	1.00	0.09	0.94	0.09	2.09	2.10
0.30	2.00	0.09	0.94	0.09	2.18	2.18
0.60	0.00	0.34	1.18	0.29	2.00	2.00
0.60	1.00	0.34	1.18	0.29	2.29	2.29
0.60	2.00	0.34	1.18	0.29	2.59	2.58

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