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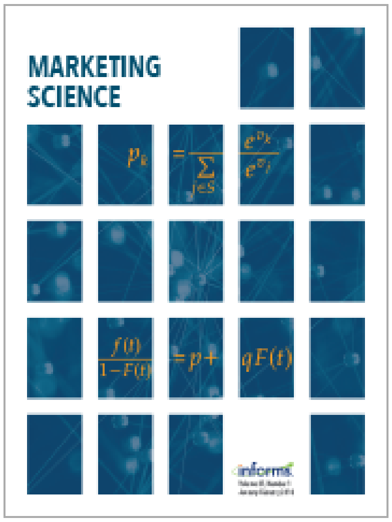

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The Impact of Discrete Bidding and Bidder Aggressiveness on Sellers' Strategies in Open English Auctions: Reserves and Covert Shilling

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Abstract

In practice, the rules in most open English auctions require participants to raise bids by a sizeable, discrete amount. Furthermore, some bidders are typically more aggressive in seeking to become the "current bidder" during competitive bidding. Most auction theory, however, has assumed bidders can place any tiny "continuous" bid increase, and recommend as optimal the tiniest possible increase.

This article examines how incorporating discrete bidding and bidder aggressiveness affect optimal strategies for an important decision for auction sellers, which is setting the lowest acceptable bid at which to sell the property. We investigate two alternative methods sellers often use to enforce this decision. These are setting an irrevocable *reserve* before the auction, and *covert shilling*, where the seller or confederates pose as bona fide bidders and raise bona fide bids, unsuspected by bidders. These optimal strategies interest auction participants, especially sellers who must recognize the bidding rules and bidder aggressiveness they will encounter in actual auctions. We also examine how these strategies change with the auction context, such as the number of bidders, and how they differ from corresponding strategies already identified for continuous bidding. Our model examines open English auctions where bidders have independent, private valuations. We find that discrete bidding does affect these strategies, as does the aggressiveness of the bidder with the highest valuation, *relative* to the average aggressiveness of all other remaining bidders.

We identify the seller's optimal discrete reserve, and show that if the highest valuator is relatively more (less) aggressive, this increases (decreases) from the optimal continuous reserve, and also increases (decreases) as the number of bidders increases. With continuous bidding, by contrast, this reserve is invariant to the number of bidders. As this bidder becomes relatively more aggressive, for a given number of bidders, the optimal discrete reserve increases, while as he or she becomes less aggressive, the seller's expected auction utility increases, which increases the set of auctions where discrete bidding generates higher seller welfare than continuous.

We propose a covert shilling model that requires shilling sellers, and any confederates and auctioneers, to outwardly act no differently than with reserves, to avoid detection. We identify cases where the seller optimally shills once the bona fide bidding has stopped, and identify the corresponding optimal point to stop shilling and accept the next bona fide bid, if offered. This stopping point does not depend on where bona fide bidding stops, or aggressiveness, or the number of bidders, or on whether shill bids alternate with bona fide bids or are consecutively entered. We also find that the optimal lowest acceptable bid with shilling can be higher (lower) than that with reserves if the highest valuator is sufficiently unaggressive (aggressive). By comparison, in continuous bidding shilling and reserves yield identical lowest acceptable bids.

Sometimes the seller using a shilling strategy optimally should not shill at all, and instead accept the bid where bona fide bidding stops. This can occur when that bid, or the number of bidders, is sufficiently high, or when the highest valuator is as, or less, aggressive than other bidders. Optimal shilling can be as practical to implement as reserves, because it does not require sellers to have any information beyond that needed in a reserve auction.

If sellers shill optimally, they can never be worse off compared to using a reserve, and can be better off. Shilling can make bidders worse off, but can also make them better off when the seller using a shilling strategy optimally accepts bids below the optimal reserve. In these latter cases, shilling Pareto dominates reserves, ex ante.

We provide numerical examples to illustrate these results. We discuss how our results might be affected if shilling is not covert, or bidders' valuations have a common value component rather than being independent, or by the rules used in many discrete bid Internet auctions.

(*Auctions*; *Internet Auctions*; *Discrete Bidding*; *Pricing*; *Bidder Behavior*)

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1. Introduction

One of an auction seller's most important decisions is setting the lowest acceptable price at which to sell the property.[1] Sellers often choose from two alternative methods to enforce this price. They can set an irrevocable *reserve* before the auction, and bid publicly "on behalf of the reserve" by raising lower bids, thereby rejecting them. This reserve can be public, or known only to the seller. Or, sellers can *shill covertly*, which is usually considered disreputable. Here the seller publicly claims to use a reserve, but actually rejects bona fide bids by raising them during the auction while covertly posing as a bona fide bidder (Cassady 1967, pp. 168–69, 174, 212–14; Graham et al. 1990, Smith 1989, pp. 148–55), and bidders do not anticipate or detect this. We do not model, but do discuss later, bidders who suspect or can detect that sellers are shilling, and modify their bidding.

Optimal strategies for sellers and bidders depend on an auction's rules. One of the most popular auctions is the open English auction (OE), where bidders raise each others' bids until only one bidder is left. Most normative models of OE auctions have assumed rules of "continuous" bidding that permit any bid increase, no matter how tiny; in such models the bidders' optimal bid raise is usually just greater than zero (e.g., McAfee and McMillan 1987; p. 707), a result that affects sellers' optimal reserve and shilling strategies.

In actual practice, however, most OE auctions use rules that require bids to increase by at least a specified amount (see Smith 1989, pp. 94–98). These rules have existed for centuries (Brough 1963, p. 26, provides an example from 1744), and courts have upheld their use (*Taylor v. Harnett*, 55 N.Y. Suppl. 988). Many auctioneers specify these increments in advance, and they can be considerable. Christie's and Sotheby's catalogs specify increments of approximately 5% to 10% of the current bid. Most Internet auctions also use discrete bid rules (e.g., www.eBay.com). Although auctioneers typically have discretion to recognize smaller increases, few would accept anything near the "epsilon" increases optimal in auction models that assume continuous bid rules.

Existing models of discrete bid auctions have also assumed that all remaining OE bidders seek to become the current bidder with equal aggressiveness. However, auction observers suggest that bidders often have different levels of aggressiveness. This can also affect sellers' optimal strategies. Continuous bid models have not studied aggressiveness because it does not affect their outcomes.

The present research identifies optimal reserve and covert shilling strategies for sellers in OE auctions with discrete bid rules and allows aggressiveness to vary among bidders. We study this for two reasons. First, whereas research on discrete bidding has examined such questions as optimal increments and allocation efficiency, assuming equal aggressiveness (Rothkopf and Harstad 1994b, Yamey 1972), optimal reserve and shilling strategies in these auctions have not been examined, nor has the impact of varying aggressiveness. Investigating these issues can increase the realism of auction theory and bring it closer to actual practice and bidder behavior, an approach urged for auction research (Rothkopf and Harstad 1994a).

Second, we also provide useful qualitative insights for auction practitioners, especially sellers. These show that discrete bidding and aggressiveness often do change optimal reserve and shilling strategies compared to those with continuous bidding. We identify the nature of these changes and the auction contexts in which they occur. We also examine how directional changes in auction characteristics (such as the number of bidders, the increment, and aggressiveness) affect optimal strategies. This can help sellers adapt strategies to a particular auction.

We study these issues in the context of the independent private value (IPV) model of bidder valuation (Harris and Raviv 1981, Myerson 1981, Riley and Samuelson 1981, Vickrey 1961), where bidders draw valuations from a common, continuous distribution. We find that optimal reserve and shilling strategies in these discrete bid auctions differ from those for continuous bidding as follows:

1. When the bidder with the highest valuation is relatively more (less) aggressive compared with all

[1] Sellers also should try to attract as many auction bidders as possible (Bulow and Klemperer 1996, Engelbrecht-Wiggans 1987). Our model does not cover situations where the number of bidders depends on the reserve.

other remaining bidders as a whole, the seller's optimal reserve increases (decreases) past the optimal continuous reserve and increases (decreases) with the number of bidders. When this aggressiveness equals the other bidders', the optimal continuous and discrete reserves are equal.

2. The seller's expected utility with a reserve increases as this bidder becomes less aggressive, for any given number of bidders. This increases the set of auctions where discrete bidding produces higher expected seller utility than continuous bidding.

3. If the seller uses a covert shilling strategy, optimal reserve and shilling can differ:

a. If it is optimal to shill even once, the optimal lowest acceptable bid can differ from the optimal reserve, if this bidder's relative aggressiveness differs from others', and does not depend on where bona fide bidding stops or the number of bidders.

b. Sometimes the seller's optimal shilling strategy is to accept the highest bona fide bid and not submit even one shill bid. Whether to do so depends on the number of bidders, provided the highest valuator is relatively less, or equally aggressive, compared with others.

c. Auctions exist where using shilling Pareto-dominates reserves. In other cases, shilling's disrepute for benefitting sellers at bidders' expense is justified.[2] With continuous bidding, shilling and reserves yield the same expected seller utility. The differences in 3a–3c do not occur in continuous bid auctions, where optimal shilling and reserve strategies yield the same lowest acceptable bid and the same expected utilities for seller and bidders (Graham et al. 1990).

Shilling is illegal in many jurisdictions, but its existence has long been acknowledged and continues today. Our purpose is not to advocate it, but rather to study how optimal shilling depends on the discrete bid rules used in essentially all progressive auctions and on bidder aggressiveness.

Discrete bidding is such an entrenched auction practice that all participants usually must accept these rules and cannot employ them only when they benefit. Many Internet auctions also require discrete bidding, providing further evidence that these rules are widely accepted. Thus, this article addresses the scenario in which sellers must accept discrete bidding and determine their optimal reserve and shilling strategies accordingly. We do not seek to model a single motive for discrete bidding, nor claim that sellers use this practice only when it increases the seller's welfare. Several such motives have been proposed, and more than one may operate in any given auction.

Yamey (1972) shows that discrete bidding decreases the seller's OE auction utility in two auction outcomes, but increases it in another, compared with continuous bidding. The net impact on the seller's expected utility depends on these outcomes' relative probabilities. Bidders can prefer discrete bidding in the first two outcomes, which also decrease the winning bidder's expected payment, but this also lowers allocation efficiency. Discrete bidding also speeds up auctions, which saves bidders time and lowers their participation costs (Rothkopf and Harstad 1994b). The "epsilon" bid increases assumed by continuous bid models create very long auctions, of infinite duration if taken to the limit.

The auction models we present describe OE auctions with all bidders either present or bidding by telephone. They do not subsume all possible discrete bid rules or every issue that might arise with reserves or shilling. So many variants of OE rules exist in practice that no single model and set of assumptions can include them all. We discuss important limitations of our assumptions and suggest how departures might change our results, for example, when assuming bidders do not anticipate or detect shilling or in the case of Internet auctions. For reasons of space, we do not consider sellers who start shilling once their prior reserve price is met, such as might occur if a bidder's outright comments suggest a willingness to pay a higher price.

2. The Optimal Reserve with Discrete Bidding

The Seller. A risk-neutral or risk-averse seller desires to sell at an OE auction a single item that he or she values at v_0, where $v_0 \geq 0$ and is private to the seller.

[2]If bidders draw valuations from different distributions and the seller knows these distributions and how many bidders draw valuations from each, the optimal lowest bid to accept differs with shilling versus reserves, sellers can increase their utility by shilling, and the optimal point to stop shilling depends on where bona fide bidding stopped (Graham et al. 1990, Figure 1).

We assume either the seller holds the auction or pays an auctioneer a fixed fee. We do not examine auctioneers that receive a commission based on the sales price. Such commissions' implications are discussed later.

The seller's auction profit, π_S, is his auction revenue less v_0. His utility function for this profit is $U(\pi_S)$, where $U(\pi_S) = 0$ for revenue v_0. Only the seller knows $U(\cdot)$. We assume $U(\pi_S)$ is linear or concave, strictly increasing in π_S, and the seller seeks to maximize expected utility $E[U(\pi_S)]$. The seller rejects any bid below v_0.

The Reserve. We define the reserve, v_D, as the lowest bid at which the seller will sell the property. We assume the seller sets v_D irrevocably before the auction, announces to all bidders that a reserve is in effect, and can either keep its amount private or public. Reserves are typically considered ethical provided bidders are told a reserve exists. We assume if the property fails to sell the seller will not sell it in the near future, because this makes the reserve less credible.

The Bidders. We assume there are n bona fide bidders, with $2 \leq n < \infty$. The bidders need not know n, and we will examine whether the seller must know n to determine optimal reserve and shilling. Bidders draw valuations independently from the same continuous distribution, with c.d.f. $F(v)$, p.d.f. $f(v)$, and range $[\underline{v}, \bar{v}]$. Only the ith bidder knows his or her v_i. Thus, bidders have independent, private valuations (Myerson 1981, Riley and Samuelson 1981, Vickrey 1961).[3] We assume the seller knows $F(v)$, but our results apply regardless of whether bidders do.[4] The v_i are ordered $v_{(1)} < v_{(2)} < \ldots < v_{(n)}$, associated respectively with bidders $B_1, \ldots, B_n$. Bidders can have any risk orientation, which can vary across bidders. We assume these bidders use their optimal bidding strategy, which in a continuous, OE auction with IPV valuations is to bid against competition and reserves up to their valuations, regardless of risk orientation, $F(v)$, and whether they know $F(v)$ (Vickrey 1961), or n or v_D (Riley and Samuelson 1981).[5] This applies, also, to discrete auctions, where B_i optimally bids up to and including the highest level not exceeding $v_{(i)}$ (Rothkopf and Harstad 1994b, Yamey 1972).

Many auction models assume IPV bidders, including those addressing discrete bid auctions (Rothkopf and Harstad 1994b) and shilling with continuous bidding (Graham et al. 1990). However, no single model of bidder valuation always holds (McAfee and McMillan 1987, p. 705). The IPV assumption is appropriate when bidders form valuations independently of other bidders' information, and cannot increase their expected auction utility by changing v_i as they observe bids. IPV bidders are not concerned with a "common value" component of property's value, determined by subsequent use or resale (Capen et al. 1971; see Milgrom and Weber 1982 on "affiliated value" auctions with both common and private value components). IPV valuations might occur for property bought only for personal use.

The Bidding Increment. We assume the auction rules require bidding to proceed using an exogenously determined, public, and constant increment I, where $I \gg 0$ to reflect the not inconsequential increments usually used in auction practice. Auctioneers usually "open" the auction with a bid s_a, which is well below v_D, and then will only recognize bids of $s_a + kI$, where k is a positive integer (see Definition 1 in the appendix). For expositional clarity we assume I is constant, but our results hold when I varies with the bid level in a manner made public in advance. This occurs in some

[3]The IPV model has been extended to include bidders drawing valuations from different distributions (e.g., Graham et al. 1990, Myerson 1981). Our model does not cover such situations.

[4]We assume $f(v)/(1 - F(v))$ is strictly increasing (Appendix Assumption 1). Assumption 1 holds in most auction models—otherwise they can yield the anti-intuitive results that the winning bidder's expected payment decreases as his valuation increases (Myerson 1981), which can imply that the seller's optimal strategy is to randomly choose a winning bidder (McAfee and McMillan 1987, p. 708). This assumption occurs in many other economics and industrial organization models (Tirole 1988, p. 156). Distributions meeting it include the uniform, normal, exponential, logistic, and any with increasing density. Certain log-normal, Weibull, and generalized gamma functions meet this assumption, whereas others do not.

[5]When bidders are risk-averse, first-price, continuous auctions produce higher expected revenue for the seller than OE auctions (Matthews 1980, 1987; Riley and Samuelson 1981). Because OE auctions are so popular, however, we examine OE results for risk-averse bidders, while acknowledging that in such cases sellers might want to consider a first price auction.

actual auctions. Since varying I as bids increase can sometimes maximize $E[U(\pi_S)]$ (Rothkopf and Harstad 1994b), sellers might wish to set I endogenously. This may be difficult to implement, however, because in practice I is traditionally a round number such as $100 or $1,000.

We assume $I < (\bar{v} - v_0)/2$. A very large I forces the seller to accept the first bid exceeding v_0, since no v_i is likely to exceed the next, much greater bid level. We also assume the current bidder will not raise his own bid, and that bidders do not raise bids by more than one increment. Although this latter "jump bidding" is not optimal for discrete bidders where $f(v)$ is nonincreasing, we do not examine its optimality here (see Rothkopf and Harstad 1994b, who show that jump bidding can be optimal when $f(v)$ is increasing, in which case this decision can also depend on knowing n). Thus, bidding ends unless another bidder raises this bid or the seller bids on behalf of the reserve.

Bidding on Behalf of Reserve. Let v_D denote the seller's discrete reserve and v_D^* the optimal reserve that maximizes $E[U(\pi_S)]$. If bona fide bidding stops below v_D, we assume the auctioneer "bids for the reserve" by raising one increment. This is standard procedure in most reserve auctions.[6] Although bids below v_D are typically recognized, they cannot buy the property.

We assume that once the current bid is $v_D - 2I$ the auctioneer immediately bids $v_D - I$. This practice, known as keeping the bidding "in step" (Cassady 1967, p. 153) or on the "right foot" (i.e., correct foot), is standard in many OE auctions and required to invoke the reserve.[7]

Bidding Aggressiveness. A person's bidding aggressiveness describes the probability that he secures the bid at the next level when bidding is below his optimal highest bid, and j other bidders also remain ($1 \leq j \leq n - 1$). We refer to this probability as that person's "bid probability," denoted $\wp(j)$. Aggressiveness does not affect any outcomes or utilities in continuous OE auctions, where bidders can raise the current bid by an infinitesimal, essentially costless amount, and so cannot gain or lose by holding, or not holding, the current bid. Aggressiveness does affect outcomes and utilities with discrete bidding, since raising a discrete bid increases the required payment by I, and any B_i cannot raise a bid below $v_{(i)}$ if the next discrete increment exceeds $v_{(i)}$. Prior work on discrete bidding (Rothkopf and Harstad 1994b, Yamey 1972) has assumed that all bidders have equal aggressiveness. This is certainly a reasonable starting assumption.

However, many auction writers observe that aggressiveness often varies across bidders, based on behavior such as the following: "whether to be bold or cautious in initial bidding" (Cassady 1967, p. 147); "When to jump in is up to [the bidder]. Neophytes often get in early, then fade. . . . It's probably better to wait . . . then pounce" (Dobrzynski 1995); "There are two schools of thought . . . aggressive and jumping in early [versus] . . . the 'hang back and see what develops method'" (Roberts 1986, pp. 41–42). Auctioneers have also observed aggressiveness differences—"'There'll be ten bids the first few seconds . . . then it's usually down to two or three. Then some guy who sits in the background jumps in . . .'" (Rubin 1972, pp. 28–29).

Bidding aggressiveness may vary across IPV bidders for several reasons. Bidders may find the competition and public attention from aggressive bidding either enjoyable or stressful. Some bidders may not mind bidding unsuccessfully, whereas others may feel defeat. Some may flamboyantly attract the auctioneer's attention by calling bids loudly or waving their paddles, whereas others are timid. In each case the former bidder is more likely to obtain the current bid, all else equal. Because our model uses IPV bidders, we do not examine aggressiveness variations caused by common

[6]E.g., rules in Christie's New York American Art sale No. 8886 on May 21, 1998, under "Bidding," state, "The auctioneer may also execute bids on behalf of the consignor to protect the reserve . . . up to the amount of the reserve. . . . Under no circumstances will the auctioneer place any bid on behalf of the consignor at or above the reserve . . ." (p. 5).

[7]This ensures that bidders can purchase at v_D. If a bidder raises to $v_D - I$ but no other bidder subsequently raises, the auctioneer must bid v_D, which blocks the bidder from winning at v_D. This violates the reserve. This is why the bidding rules in footnote 6 prohibit auctioneers from bidding *at* the reserve. If the reserve is alternatively defined as one increment *below* the lowest acceptable bid, and thus the last point where the auctioneer bids on its behalf, v_D^* must be decreased one increment.

value factors, such as avoiding early bids to prevent revealing your interest to others.

Low bidder aggressiveness creates "sniping" in timed Internet auctions, where a new bidder waits until the last second to bid. Irritated comments on sniping often appear in online bulletin boards. Although the time limits of OE Internet auctions are not included in our model, the rise of sniping in these relatively new Internet auctions suggests that aggressiveness varies even in some new auction types.

We do not claim that aggressiveness always differs across bidders, but it often does, and this affects optimal reserve and shilling. If aggressiveness matters, sellers will want to obtain information that helps estimate aggressiveness differences. Although we could separately model each behavioral factor affecting aggressiveness, we feel that to obtain useful qualitative insights from a tractable model, it is more appropriate to discuss the several reasons aggressiveness may vary, as just done, and incorporate their influence in a single aggressiveness variable. We show that the seller's optimal strategies depend on the aggressiveness of B_n, the bidder with the highest valuation, *relative to* that of the other remaining bidders collectively.

If all bidders strategically adjust aggressiveness to the level that most benefits them and if this is the same for all bidders, an equilibrium can arise where all bidders will be equally aggressive. Although some bidders may regard aggressiveness this way, we feel it is reasonable to assume that often one or more bidders do not and have certain bidding habits or personality characteristics that make their aggressiveness differ from that of other bidders. Consequently, we specify aggressiveness as an exogenous behavioral factor in the model rather than as a behavior that bidders adjust strategically. Our model examines all three situations where B_n is relatively more aggressive, less aggressive, or equally aggressive compared with the other remaining bidders collectively.

Modeling B_n's Relative Aggressiveness. We assume neither the seller nor any bidder knows who is B_n. Let $\wp(j)$ represent the "bid probability" that B_n obtains the current bid at $s_a + kI$ when bidding against j *other* remaining bidders who also have $v_i \geq s_a + kI$, where $1 \leq j \leq n - 1$. Note that $\wp(j)$ includes both the probability that B_n does not obtain the bid at $s_a + kI$, and that B_n holds the bid at $s_a + (k - 1)I$ and so does not bid. We define that $\wp(\cdot)$ depends on j, and assume that it does not depend on n (note that bidders need not know n). If B_n has "neutral" relative aggressiveness, defined as equalling the average aggressiveness of these j remaining bidders, then $\wp(j) = [1/(j + 1)]$.

We define that $\wp(j)$ strictly increases with B_n's relative aggressiveness for any given j. We also assume that $\wp(j)$ cannot be constant for all j (this eliminates $\wp(j) = 0$ or 1 anywhere in j). A constant $\wp(j)$ implies that B_n's chances of obtaining the current bid do not depend on how many bidders compete against him at $s_a + kI$, which is counterintuitive. Even the most aggressive possible B_n has at most $\wp(j)$ just less than 0.5, since he must be the underbidder at at least every other level (except immediately after an auctioneer's "right foot" bid, in which case $\wp(j)$ could be just below 1).

We represent B_n's relative aggressiveness as $\wp(j)$ given her actual aggressiveness *minus* $\wp(j)$ given neutral aggressiveness, defined as $\chi(j) = \wp(j) - (1/(j + 1))$. To allow a succinct yet general exposition, the proofs focus on the variable ξ, which is the minimum of $\chi(j)$ for aggressiveness (thus, $\chi(j) > 0$) or the maximum of $\chi(j)$ for unaggressiveness, (thus, $\chi(j) < 0$) for $j = 1, 2, \ldots, n - 1$. $\xi = 0$ represents a B_n with "neutral" relative aggressiveness (as assumed in Rothkopf and Harstad 1994b, Yamey 1972), $\xi > 0$ a relatively aggressive B_n, and $\xi < 0$ a relatively unaggressive B_n. Note that $\wp(\cdot)$ is a function of j, while ξ can be either dependent or independent of n. To accommodate a wide range of aggressiveness behavior, we do not impose any particular functional form on the relationship between $\wp(\cdot)$ and j or between ξ and n, and allow ξ to be either independent or dependent on n. We sometimes refer to B_n's relative aggressiveness as "aggressiveness."

We do assume that as j changes, a given bidder must always stay relatively aggressive, or unaggressive, or neutral, and cannot "switch" among these three categories. Our model is not intended to cover switching bidders. They do not have any consistent aggressiveness, and a more complex form of $\wp(j)$ is required to accommodate their switches. Formally, then, for each bidder, $\wp(j) - (1/(j + 1))$ is *always* either equal to zero, or less than zero, or greater than zero, for all j and n.

We translate this behavioral condition into the sufficient model condition on $\wp(j)$ that $(j + 1)\wp(j)$ must be nondecreasing (nonincreasing) in j for aggressive (unaggressive) bidders. By definition $(j + 1) \cdot \wp(j) = 1$ for neutrally aggressive bidders. As described in the technical appendix under "Definition and Properties of $\wp(j)$," a B_n who violates this condition has the inconsistent aggressiveness just described.

We do not assume that sellers always know ξ. Rather, we seek to show that aggressiveness affects their optimal strategies and $E[U(\pi_S)]$, making it worthwhile to estimate ξ. Sellers may be able to do so based on prior experience with bidders for particular kinds of property. Even if they cannot determine it precisely, we show that their optimal strategies differ based just on knowing whether ξ is positive or negative, since ξ's sign drives our basic model results.

The Optimal Discrete Reserve, v_D^*

Before the auction, the seller first examines whether to set v_D at the first discrete level at or above v_0, denoted $v - I$, or one increment higher, at v, where $v < \bar{v}$. Raising v_D has four possible, mutually exclusive, and exhaustive consequences: The property (a) sells at the same price $\geq v$ with both reserves, (b) does not sell with either, (c) sells for $v - I$ when $v_D = v - I$, and sells for v when $v_D = v$, and (d) sells for $v - I$ when $v_D = v - I$ but does not sell when $v_D = v$.

In (a) and (b), raising v_D does not change $E[U(\pi_S)]$. In (c), the seller gains by raising v_D since revenue increases from $v - I$ to v. Case (c) can happen via two events: If $v_{(n)} \geq v$ and $v_{(n-1)} < v - I$, or, if $v_{(n)} \geq v$ and at least one other bidder has $v - I \leq v_{(i)} < v$, but B_n would obtain the bid when $v_D = v - I$. Only the latter event is affected by aggressiveness. Thus, the seller's expected gain from raising v_D, denoted $q(v, v_0, n)$, is (see proof of Proposition 1 in the technical appendix for derivation):

$$q(\nu, \nu_0, n) = [U(\nu - \nu_0) - U(\nu - I - \nu_0)] \left\{\frac{1 - F(\nu)}{F(\nu) - F(\nu - I)} [F^n(\nu) - F^n(\nu - I)] + n\xi[1 - F(\nu)][F^{n-1}(\nu) - F^{n-1}(\nu - I)]\right\}, \quad (1)$$

which increases with B_n's relative aggressiveness. In (d) the seller loses by raising v_D, and his revenue decreases from $v - I$ to v_0. Thus, the seller's absolute expected loss from raising v_D, denoted $r(v, v_0, n)$, is:

$$r(\nu, \nu_0, n) = U(\nu - I - \nu_0)\,[F^n(\nu) - F^n(\nu - I)], \quad (2)$$

and does not depend on aggressiveness. If this analysis supports raising v_D, the seller repeats it with v now one interval higher. Given any permissible $v - I$, the seller should raise v_D to v if $q(v, v_0, n) > r(v, v_0, n)$, should not raise if $q(v, v_0, n) < r(v, v_0, n)$, and is indifferent if $q(v, v_0, n) = r(v, v_0, n)$. The optimal reserve, v_D^*, is the highest discrete bid v with $q(v, v_0, n) > r(v, v_0, n)$, characterized as follows (see the technical appendix for proofs of propositions):

PROPOSITION 1. *There exists an optimal discrete bid reserve for the seller, v_D^*, such that:*

a) If B_n is relatively more aggressive than other bidders, so $\xi > 0$, then v_D^* has a minimum value equal to the optimal continuous bid reserve, v_C^*, increases with n, and can assume any higher discrete bid level $< \bar{v}$ if n is large enough.

b) If B_n is relatively unaggressive, so $\xi < 0$, then v_D^* has a maximum value of v_C^*, decreases as n increases, and can assume any lower discrete bid level $> v_0$ if n is large enough.

c) If B_n has neutral aggressiveness, then $v_D^* = v_C^*$ and does not depend on n.

d) For any ξ allowed, $v_D^* > v_0$ and increases with v_0, and v_D^* converges to v_C^* as $I \to 0$. For any given n, v_D^* increases as ξ increases. Further, when $\xi \leq 0$, v_D^* decreases as the seller's absolute risk aversion increases.

We next show that B_n's aggressiveness affects $E[U(\pi_S)]$:

PROPOSITION 2. *When the seller sets the discrete bid reserve at v_D^*, for any given n:*

a) $E[U(\pi_S)]$ increases as B_n's relative aggressiveness decreases (as ξ decreases).

b) As ξ decreases (increases), this increases (decreases) the set of possible auctions, defined as combinations of $F(v)$, v_0, n, and I, where $E[U(\pi_S)]$ is higher with discrete than continuous bidding.

These results should prove useful to auction sellers. Proposition 1 identifies v_D^*, the optimal discrete reserve, and shows it depends on B_n's aggressiveness (ξ) and on n (unless $\xi = 0$). Aggressiveness affects v_D^* due to outcomes where v_D equals the highest bid level not exceeding $v_{(n-1)}$, and $v_{(n)}$ at least equals the next higher level. Here, for a given n, increasing ξ increases the probability B_n obtains the current bid at v_D in competition against the $j \geq 1$ remaining bidders. If B_n does not achieve this, she must bid one increment higher, increasing $q(v, v_0, n)$. However, the associated loss, $r(v, v_0, n)$, does not depend on ξ. Thus, v_D^* increases. Sellers require more information to set discrete than continuous reserves, since ξ and n affect v_D^* but not v_C^*.

$E[U(\pi_S)]$ increases as ξ decreases, for a given n, since it becomes more likely that B_n must pay $v_D^* + I$, as in Proposition 2a, and will not obtain the bid at v_D^*. Proposition 2b shows that the set of auctions where sellers are better off with discrete than with continuous bidding increases (decreases) as ξ decreases (increases) for a given n. This suggests that one explanation for the popularity of sizeable bid increments among sellers, beyond their more basic, operational ability to eliminate small, time-consuming raises, could be that many sellers may perceive (whether correctly or not) that B_n tends to be relatively unaggressive.

Contrast with Continuous Bidding. By contrast, v_C^* does not depend on n in OE auctions with continuous bidding, when IPV bidders draw v_i from the same distribution (Matthews 1980, Riley and Samuelson 1981). This difference can be explained as follows. In both continuous and discrete auctions, the seller should continue increasing v_D as long as:

$$\frac{\textit{absolute loss}}{\textit{gain}} < \frac{P(\textit{seller gains by raising reserve})}{P(\textit{seller loses by raising reserve})}. \quad (3)$$

The ratio in the left-hand side of (3) does not change with n in either the continuous or discrete auction. In the continuous, the ratio in the right-hand side also does not change with n, so v_C^* does not change with n (see (A14) and subsequent discussion, in proof of Proposition 1d in the technical appendix, for this ratio's behavior as $I \to 0$). However, with discrete bidding, when $\xi > (<)0$, the right-hand side ratio increases (decreases) unboundedly with n, even if I is very small (see proofs of Propositions 1a–1c in the technical appendix). This makes v_D^* increase (decrease) with n. It also implies that sellers require more information to set v_D^* than v_C^*, since they must know n and ξ. The numerical example in § 5 shows that although v_D^* can remain constant for some successive values of n, when $\xi > (<)0$, it eventually increases (decreases) as n increases further, and can be several increments higher (lower) than v_C^*. Thus, the impact of discrete bidding is not just to, at most, round v_C^* to the nearest higher (lower) discrete bid level—the effect can be greater. In the limit, as $I \to 0$, v_D^* converges to v_C^*, for all ξ, as in Proposition 1d. Thus, for any aggressiveness, when bid increments are small a seller could use v_C^* to approximate closely v_D^*.

3. Optimal Covert Shilling in Discrete Bid Auctions

Shilling occurs when the seller, or a confederate, covertly poses as a bona fide bidder and raises bona fide bids. A shilling seller accepts a bid when he stops raising and lets the item sell. Shilling is traditionally regarded as an unethical attempt by sellers to raise prices at bidders' expense. That shilling is a long-established practice, and continues today, is supported by comments from auction observers (Cassady 1967, pp. 212–13; Cicero 106–43 B.C., Book III, XV, on "a bogus bidder"; Smith 1989, pp. 151–55) and by discussions in legal references of laws prohibiting shilling (*Corpus Juris Secundum* 1980, §7–15), where shilling is probably the subject of more laws than any other auction practice. In a much-publicized 1993 case, the court refused to dismiss charges that a bank shilled in a real estate auction (*Stormy Weathers Inc. v. FDIC,* 834 F. Supp. 519; Hewett 1994). Shilling is also a problem in Internet auctions (Schwartz 1998). In January 1999, eBay, a popular Internet auction firm, announced new rules to combat shilling (Anders 1999). Subsequent investigations in the press, prompted by the suspicious sale of a high-priced painting, revealed the existence of Internet "shill rings" that were extremely difficult for most bidders to detect and that flourished despite the new rules (Dobrzynski 2000a). These revelations soon motivated the FBI to start a formal investigation of online shilling (Dobrzynski 2000b), and shortly afterwards, allegations of further shilling activities, this time in coin auctions, appeared in the press (Simpson 2000). Prosecutions of shilling are rare, however, suggesting much shilling remains undetected. As one observer comments, shilling "is not as common as many auction cynics would assert but more common than most auction houses would admit" (Smith 1989, p. 150).

Only a few auction models have examined shilling. Graham et al. (1990) analytically model the practice in continuous bid, OE auctions where sellers can determine when only one bidder remains. They contrast results when IPV bidders draw v_i from the same versus different distributions. Rothkopf and Harstad (1995) examine bid-taker cheating where sellers submit false bids after the bona fide sealed bidding ends in Vickrey auctions. However, optimal shilling strategy for sellers has not been examined in the context of discrete bid, OE auctions.

Assumptions of the Shilling Model

We modify the reserve model as follows:

The Seller. A seller using a covert shilling strategy is *ready* to reject bona fide bids by submitting covert shill bids, even if he eventually makes no shill bids. We assume the seller announces falsely that a private reserve is in effect and does not expect that bidders can detect shilling. We assume sellers who "buy back" their property while shilling will not reoffer it in the near future, since this can signal bidders that shilling occurred (also assumed by Graham et al. 1990). While we refer to the *seller's* shilling decisions, we assume the seller can use a covert agent(s) to shill, and that any auctioneer cooperates with the seller's shilling strategy.

Bona Fide Bidders' Optimal Bidding Strategy. We examine covert shilling, where bidders believe a reserve is in effect, do not anticipate or detect shilling, and perceive shill bids as bona fide. Thus, bidders use the same bidding strategy as with reserves—to bid up to and including the highest level not exceeding their v_i. Because this strategy does not depend on n in an OE auction, it does not change with the addition of an apparent extra bidder(s) if shilling occurs.

We feel it is important to examine covert, rather than overt, shilling because we believe the former is more prevalent in actual auctions. Shilling is extremely difficult for bidders to detect. Although many auction bidders may be aware that shilling sometimes occurs, this does not necessarily imply that they believe shilling is occurring at the present auction, or can detect it, or attempt to adjust their bidding strategies accordingly. Furthermore, discussions of shilling in legal references and evidence from legal cases make clear that laws exist to protect bidders from shills precisely because it is a practice carried out *in secret*, and that often bidders do not detect shilling while bidding in auctions where later they found out it had occurred.[8] From a consumer protection perspective, bidders who do not detect or anticipate shilling are consumers legally entitled to trust the (in fact) false claims of shilling sellers or auctioneers that all bidders are bona fide. Thus, examining covert shilling allows us to assess shilling's full potential to injure consumer welfare, whereas examining overt or anticipated shilling cannot.

We could assume bidders always believe shilling may occur and modify their bidding accordingly. This creates an information symmetry with some appeal—the seller may shill, the bidder knows this and acts accordingly, the seller knows that the bidder knows, etc. We are faced with a choice between assuming that in actual auctions most bidders expect and can detect shilling, or that in such auctions most bidders do not anticipate or detect shilling, as supported by legal evidence. We feel the latter occurs more frequently, making it a useful starting point for examining shilling. At the same time, we acknowledge that the former "savvy" bidders may exist, and even unsavvy bidders may detect clumsy shilling. Our model is not intended to cover these situations, some of which are examined in the discussion section.

To justify the assumption that bidders cannot detect shilling, our model requires that (1) shills must bid in a manner consistent with optimal bidding behavior for bona fide bidders in a reserve auction, and (2) the manner in which bona fide bidders' bids are recognized as the current bid, or rejected, must be outwardly identical to a reserve auction.

[8]Legal references supporting that lawmakers and courts assume shilling can be kept secret include "[the shill] is secured from risk [of paying] by a secret understanding with the seller"; "the secret employment, by the seller, of a puffer or by-bidder" (C.J.S. 1980, Sect. 7-15); "such secret arrangements are offensive to fair dealing" (*Burdon v. Seitz,* 267 S.W. 219). Legal references supporting that lawmakers and courts assume bona fide bidders do not anticipate or detect shilling include "Mr. Sowa [the bidder] says he was not aware, at the time [of the auction], that an agent of the owner was bidding against him" (*Stormy Weathers Inc. v. FDIC*, 834 F. Supp. 519, p. 521); "At that time [soon after the auction] he [the buyer] knew nothing of the fraud which had been perpetrated upon him" (*Edmunds v. Gwynn,* 161 S.E. 892).

The Seller's Decisions With Shilling

We assume the seller or auctioneer allows bidding to proceed among bona fide bidders, without shilling, until no further raises occur. Instead of selling the property or bidding for the reserve, however, the auctioneer in a shilling auction covertly signals to any shill(s) that now they *may* start shilling. This should not arouse bidder suspicion, since auctioneers in reserve auctions often exchange confidential signals with secretive bidders. Moreover, in almost all OE auctions the auctioneer takes a few seconds to determine whether bidding is finished by searching for last-second bids. Thus, our shilling model does not require auctioneers to take any public actions or make any judgments that they would not make in a reserve auction.

Covert shilling also cannot allow shills to "steal" bona fide bids by quickly calling out the same bid just made by a bona fide bidder and obtaining the current bid. This is possible because most auction rules give auctioneers authority to determine the current bidder. However, deliberate attempts to ignore bona fide bidders are likely eventually to trigger shilling suspicions. For the same reason, we do not consider the well-known "backup" practice where, if a shill bid is not followed by a bona fide raise, the auctioneer announces a "mistake" in recognizing an apparent bidder and declares that the previous, bona fide bidder still has the bid (Smith 1989, p. 151).

Let the random variable S_0 denote the bid where bona fide bidding stops, with realized value s_0. We assume $s_0 \geq v_0$, since the seller rejects bids below v_0. Once bidding stops the seller must decide whether to sell the property for s_0, or reject s_0 by entering the first shill bid, of $s_0 + I$ (Graham et al. 1990 also assume shilling begins at this point). The first shill "fails" if no bona fide bidder raises it to $s_0 + 2I$, so the property does not sell and $U(\pi_S) = 0$. The first shill "succeeds" if a bona fide bidder raises to $s_0 + 2I$. Here, the seller must decide whether to stop shilling and accept that bid or shill again by bidding $s_0 + 3I$. The process repeats until either a shill fails or the seller stops shilling. We define the "terminal shill" as the last shill bid, made at a level denoted s_t ($s_t \geq s_0 + I$), so the seller accepts bona fide bid $s_t + I$, if offered. To maximize $E[U(\pi_S) \mid s_0]$, which is conditional on $S_0 = s_0$, the seller using a shilling strategy must decide (i) whether to shill even once when bona fide bidding stops, and (ii) if so, finding the optimal terminal shill, termed s_t^*.

The previous discussion assumes that shills raise bona fide bids by only one increment. For completeness, the technical appendix examines the more general case where shills can raise by one or more increments consecutively.[9] However, since we show (proofs of Propositions 3a–3c in technical appendix) that optimal shilling does not depend on whether intermediate shill bids occur singly or successively, for brevity the text examines shills that raise bids by one increment, which reflects bidding in most auctions.

The Seller's Expected Utility from Ending Shilling at s_t, Given s_0

The probability that ending shilling at s_t "succeeds" is the product of the probabilities of three events that must all occur: (i) $v_{(n)}$ is the only v_i (here using v_i as a generic valuation) in the bidding increment it occupies; (ii) the first shill succeeds, conditional on (i); (iii) any subsequent shills needed to reach s_t all succeed, conditional on (ii). The seller can calculate these probabilities from additional information on each bidder's v_i that he obtains only from observing s_0 (and cannot use in a reserve auction) as follows:

i) $\mathbf{v}_{(n)}$ is the only $\mathbf{v}_i$ in its increment. This probability is denoted $P(v_{(n),only} \mid s_0)$. Bidding can stop at s_0 due to any one of three exhaustive, mutually exclusive outcomes (enumerated in the technical appendix Equations (A1a)–(A1c)). First, $v_{(n)} \geq s_0 + I$ and at least one other bidder has $s_0 \leq v_i < s_0 + I$, but B_n obtains the bid at s_0. Second, only $v_{(n)} \geq s_0$, but at least one other bidder has $s_0 - I \leq v_i < s_0$ and one such bidder obtains the bid at $s_0 - I$, forcing B_n to bid s_0. Third, $v_{(n)}$ and at least $v_{(n-1)}$ share the same increment. The shill can possibly succeed in the first two cases, which happen if and only if $v_{(n)}$ is the only v_i in its increment, but certainly fails in the third, because $v_{(n)} < s_0 + I$.

[9] To covertly place two or more successive shill bids requires using two shills, who raise each other's bids in quick succession as often as necessary (they must prearrange who bids first). This allows each shill to raise by only one interval, which bona fide bidders will perceive as other bona fide bidders bidding quickly, which can occur in actual auctions.

Once s_0 is known, the seller can compute each outcome's probability, conditional on $S_0 = s_0$, and compute $P(v_{(n),only} \mid s_0)$ as the ratio of the sum of the first two probabilities to that of all three. Only the first two probabilities depend on B_n's relative aggressiveness (technical appendix Equations (A2a)–(A2c)). By observing s_0, sellers know that at least one bidder has $v_i \geq s_0$, at least one other bidder (the previous current bidder) has $s_0 - I \leq v_i < s_0 + I$, and the $n - 2$ other bidders have $\underline{v} \leq v_i < s_0 + I$.

In continuous bid auctions, $P(v_{(n),only} \mid s_0) = 1$ always; B_n can always raise any bona fide bid since $v_{(n)} > v_{(n-1)}$ strictly. However, with discrete bidding $P(v_{(n),only} \mid s_0) < 1$ always (see technical appendix Lemma 6b), because the probability that $v_{(n)}$ and at least $v_{(n-1)}$ are in the same increment, which can prevent B_n from placing the last bid, is strictly positive always. If the first shill succeeds, however, on all subsequent shills it is certain only B_n remains bidding.

ii) The probability the first shill succeeds if $\mathbf{v}_{(n),only}$ occurs. Equations (A17)–(A19) in the technical appendix derive this probability, denoted $P(v_{(n)} \geq s_0 + 2I \mid v_{(n),only})$. This depends on $v_{(n-1)}$, since the distribution of $v_{(n)}$ has lower support at $v_{(n-1)}$. However, the seller does not know $v_{(n-1)}$ when deciding whether to begin shilling, but knows only that $s_0 - I \leq v_{(n-1)} < s_0 + I$. If the first shill succeeds, the seller knows for certain that $v_{(n)} \geq s_0 + 2I$, so $v_{(n-1)}$'s value is no longer relevant to $v_{(n)}$'s value.

iii) The probability the second through last shills succeed if the first succeeds. If s_j is the most recent *successful* shill bid, where $j \geq 1$, the seller knows $v_{(n)} \geq s_j + I$. Thus, after each successful shill the seller revises the c.d.f. of $v_{(n)}$ as $F(v)$ truncated on the left at $s_j + I$ and normalized to a probability of 1. Thus, the probability a subsequent shill bid of $s_k = s_j + 2I$ succeeds, conditional on the previous shill succeeding, is $P(v_{(n)} \geq s_k + I \mid v_{(n)} \geq s_j + I)$, which is $(1 - F(s_k + I))/(1 - F(s_j + I))$.

Optimal Shilling Strategy. $E[U(\pi_S) \mid s_0]$, the expected utility from ending shilling at s_t, given s_0, can be expressed using the three probabilities in (i) through (iii), as in the appendix (under sketch of proof of Proposition 3). The seller maximizes $E[U(\pi_S) \mid s_0]$, subject to the constraint that it must exceed $U(\pi_S)$ from accepting s_0 (see Equations (A21)–(A22) in the technical appendix). This leads to two propositions that together define optimal shilling. Proposition 3 describes optimal shilling if it is optimal to shill at least once, and Proposition 4 identifies cases where sellers using a shilling strategy should not shill even once, but instead accept s_0.

PROPOSITION 3. *When the bona fide discrete bidding stops at $s_0 \geq v_0$, if it is optimal for the seller to shill at least once, then:*

a) There exists a unique value of s_t^* that maximizes the seller's expected utility from shilling, as defined in the technical appendix Equation (A24). If $s_0 < s_t^*$ the seller should shill until reaching s_t^* as the last shill bid, and then accept a bid of $s_t^* + I$, if offered. This optimal shilling strategy can use any combination of shill bids as long as the last is s_t^* and does not depend on intermediate shill bids.

b) s_t^* does not depend on s_0, or B_n's relative aggressiveness, or n. It increases with v_0, and decreases as the seller's absolute risk aversion increases.

c) $s_t^* + I$ can be higher or lower than the optimal discrete reserve, v_D^*, depending on ξ. Their relationship depends on B_n's aggressiveness and on n, and they are always equal only when $\xi = 0$. When $\xi > 0$ ($\xi < 0$), v_D^* can be strictly greater (less) than $s_t^* + I$ provided n is large enough. Moreover, $s_t^* + I = v_C^*$, the optimal continuous reserve, provided v_C^* is a permitted discrete bid.

Proposition 3a shows that if optimally the seller shills at least once, optimal shilling can use any sequence of shill bids provided the last is s_t^*, so the lowest acceptable bid is $s_t^* + I$, regardless of the order of intermediate shill bids.[10] Proposition 3b implies sellers can determine s_t^* *before* the auction, and even without knowing n, since s_0, n, and B_n's aggressiveness do not affect s_t^*. An interesting question given shilling's reputation as a method to increase prices is whether the minimum acceptable bid with shilling always exceeds that with reserves. Proposition 3c shows that, *provided* the seller shills at least once, the minimum acceptable

[10]If $s_0 < s_t^*$ and $s_t^* - s_0 = kI$ where k is odd, the seller can obtain the bid at s_t^* by submitting a single shill bid at $s_0 + I$ and alternating single bids with B_n. If k is even, the seller is on the "wrong" shilling foot because this gives B_n the bid at s_t^*. To covertly get on the "right" shilling foot requires that two confederates shill in quick succession, as in footnote 9.

bid is not consistently higher or lower than v_D^*. This occurs because v_D^* increases with ξ, and can either increase or decrease with n (as per Propositions 1a and 1b). Proposition 3c also shows that $s_t^* + I = v_C^*$ as long as the latter is a permissible bid, so that the lowest optimal discrete bid the seller accepts, provided he shills, is the same as the optimal continuous reserve. Next, we examine when sellers should not shill even once.

PROPOSITION 4. *When the seller uses a shilling strategy, and B_n is relatively less, or as aggressive, than other bidders, discrete bid auctions exist where the seller optimally should not shill even once but instead accept s_0. These scenarios depend on s_0, n, and ξ.*

Proposition 4 shows that in some auctions $U(\pi_S)$ from accepting s_0 exceeds $E[U(\pi_S) \mid s_0]$ when shilling to s_t^*, so the seller should not shill even once. These auctions can occur when B_n is relatively less aggressive, or as aggressive, than the other bidders. They occur because when $\xi \leq 0$ and n increases, $P(v_{(n),only} \mid s_0)$ monotonically decreases to a lower bound greater than zero, which lowers the probability that the first shill succeeds. This lowers $E[U(\pi_S)]$ from shilling to s_t^*. At some sufficiently large n, the seller's certain utility from accepting s_0 exceeds $E[U(\pi_S) \mid s_0]$ from shilling to s_t^*. Note that when $\xi > 0$, $P(v_{(n),only} \mid s_0)$ is not monotonic w.r.t. n, so Proposition 4 does not address this case. Propositions 3 and 4 together imply that the optimal lowest acceptable bid with shilling, however, is either s_0 or $s_t^* + I$, and never a bid between them.

Contrast with Continuous Bid Auctions. By contrast, in continuous bid auctions (i.e., the case of homogeneous distribution bidders in Graham et al. 1990), the seller should *always* shill to the same optimal stopping point provided it exceeds s_0, and this point does not depend on n or s_0. The discrete and continuous results differ because once s_0 is known, the seller in the continuous auction knows $v_{(n-1)}$, the lower support of $v_{(\mathbf{n})}$, and that $P(v_{(n),only} \mid s_0) = 1$, which also implies that B_n has the current bid (Graham et al. 1990), regardless of s_0 or n. However, at this point the seller in the discrete auction still does not know $v_{(n-1)}$ with certainty, but does know that $P(v_{(n),only} \mid s_0) < 1$. Both of these terms' values depend on s_0, n, and ξ, which in some cases make the seller worse off if he shills even once. If the first shill succeeds, however, the discrete bid seller knows the lower support of $v_{(n)}$ with certainty and that only B_n remains. Thus, in discrete auctions the decision whether to shill even once depends upon s_0, n, and aggressiveness, but s_t^* does not.

Comparing Information and Implementation Demands of Shilling and Reserves. Propositions 3 and 4 imply that optimal shilling does not require discrete bid sellers to have information beyond that needed to set optimal reserves, except observing s_0, which they must also do in a reserve auction to declare bidding ended. Sellers in both auctions must know n, $F(v)$, v_0, and ξ. This means shilling is not more demanding than reserves from an information perspective.

Optimal shilling is also reasonably practical to implement for the following reasons. Before the auction the seller can calculate s_t^* and determine, for each possible s_0, whether it is optimal to shill to s_t^* or accept s_0. The seller can summarize these decisions into a fairly simple "shilling schedule," and give it to any shilling confederates before the auction. These need only state s_t^* for the entire auction and, for each possible s_0, whether to shill to s_t^* or accept s_0. Neither the seller nor confederates need make any complicated decisions or calculations under time pressure during the auction, which would reduce shilling's practicality. Thus, our results do not preclude that shilling represents a reasonably practical strategy for sellers to implement at many actual auctions. As mentioned earlier, our shilling model does not reflect all possible factors that may arise in auctions, which may create additional information or implementation demands.

4. Impact of Reserves and Shilling on Seller's and Bidders' Welfare

Given shilling's disrepute as benefitting sellers at bidders' expense, it is useful to examine how it affects both parties' expected auction utility compared with using reserves:

PROPOSITION 5. *In a discrete bid, open English auction:*

a) Optimal shilling always gives the seller an expected utility either (i) higher or (ii) the same as with the optimal *reserve. The latter occurs only if $s_t^* + I = v_D^*$, and if optimally the seller shills for every possible $s_0 < s_t^*$.*

b) Auctions exist where bidders earn higher expected utility when the seller shills optimally rather than using the optimal reserve. In these, shilling Pareto dominates reserves, ex ante. Such auctions occur when, for some n, (i) $s_t^* + I < v_D^*$, which can occur in auctions where B_n is relatively aggressive ($\xi > 0$), or (ii) $s_t^* + I = v_D^*$ and the seller does not shill for at least one $s_0 < s_t^*$, which Proposition 4 posits can occur for an unaggressive or neutral B_n ($\xi \le 0$). When shilling is not Pareto dominant, bidders are as well off, or worse off, than with a reserve.

Proposition 5a holds because shilling sellers can observe s_0 before determining the lowest acceptable bid. This information is not available before the auction, when the reserve is set, but can increase $E[U(\pi_S)]$. IPV bidders in OE auctions do not change their highest optimal bid when observing bidding, however (Harris and Raviv 1981, Myerson 1981, Riley and Samuelson 1981, Vickrey 1961). Because shilling does not constrain which bids sellers can accept or reject, compared with reserves, and sellers can change actions upon observing s_0 but bidders will not, this information cannot make sellers worse off, by Blackwell's Theorem (Blackwell 1951). Furthermore, shilling does not change $E[U(\pi_S)]$ *if* $s_t^* + I = v_D^*$, and optimal shilling, for all $s_0 < s_t^*$, is shill to s_t^*.

Whereas shilling can make bidders worse off compared to reserves, Proposition 5b also shows that auctions exist where it makes bidders better off, in which case shilling Pareto dominates reserves. These occur when the shilling seller accepts some bids below v_D^* (but exceeding v_0), as in Proposition 5b, i, and ii. If the seller shills, these bids are at least $s_t^* + I$ but still less than v_D^*. This can be optimal when $B_n > 0$, since increased aggressiveness causes v_D^* to increase and to increase with n (but s_t^* remains unchanged). If the seller decides not to shill, which we show can occur when $\xi \le 0$ (while not ruling this out when $\xi > 0$), these bids are $s_0 < v_D^*$ (Proposition 5bii). In both cases, the winning bidder's expected payment is lowered if the property sells, compared with reserves. Here, shilling also lowers the probability the property does not sell, which gives all bidders zero surplus. These factors together make bidders better off. Meanwhile, because in all such auctions the shilling seller accepts bids lower than v_D^*, the seller is better off (by Proposition 5a).

Contrast with Continuous Bid Auctions. With continuous bidding, by contrast, shilling and reserves always yield the same welfare for bidders and sellers, since both yield the same optimal lowest acceptable bid (i.e., a homogeneous bidder distribution in Graham et al. 1990). That shilling can increase $E[U(\pi_S)]$ will tend to increase its attraction to sellers in discrete bid auctions.[11]

5. Numerical Examples to Illustrate Propositions

These examples are intended to describe discrete bid auctions within the range of parameter values often encountered in actual practice, but are not comprehensive. They examine a risk-neutral seller with $2 \le n \le 40$, and $v_0 = \$2{,}000$ or \$4,000. The $f(v)$ are beta distributions scaled to have range [\$1,000, \$10,000]. Two unimodal $f(v)$ are used for reserve examples and two for shilling. In each pair, one $f(v)$ is close to symmetric, and the other has a longer right tail. We choose $I = \$500$, an increment often specified at art and antiques auctions when bids are between \$5,000 and \$10,000, and $I = \$100$ to represent smaller increments used in many corresponding Internet auctions.

B_n's relative aggressiveness is represented by an exponent, γ, on the neutral bid probability—$\wp(j)$ takes the form: $[1/(j + 1)]^\gamma$. Here, ξ depends on n. We use $\gamma = 0.4$ for a relatively aggressive B_n (thus, $\xi > 0$), $\gamma = 3$ for an unaggressive B_n ($\xi < 0$), and $\gamma = 1$ for neutral aggressiveness ($\xi = 0$).

These $f(v)$ are designed to place v_C^* on a discrete bid level. This creates a very conservative test for the seller's utility advantages of using v_D^* instead of v_C^*, since v_C^* is now a permissible discrete bid level. This

[11]While shilling can increase sellers' expected utility with continuous bidding when IPV bidders draw valuations from different distributions, this occurs only if the seller knows each distribution and the number of bidders drawing valuations from each (Graham et al. 1990). Such sellers must have considerably more information than in the discrete bid auctions where shilling increases their welfare, where bidders draw valuations from the same distribution.

constraint prevents $f(v)$ from exact symmetry and requires changing its parameters when v_0 changes.

Reserve Strategy. Table 1 illustrates how v_D^* depends on ξ and n (Proposition 1). For example, with the "symmetric" $f(v)$, when B_n is unaggressive and $I =$ \$500, v_D^* decreases with n, so that while $v_C^* =$ \$6,000, $v_D^* =$ \$6,000 when $n \le 5$, $v_D^* =$ \$5,500 when $6 \le n \le 13$, and $v_D^* =$ \$5,000 when $14 \le n \le 40$. When B_n is aggressive, v_D^* increases with n, so that $v_D^* =$ \$6,000 when $n \le 9$, $v_D^* =$ \$6,500 when $10 \le n \le 36$, and $v_D^* =$ \$7,000 when $37 \le n \le 40$. These changes in v_D^* also occur when $I =$ \$100 and for the right-skewed $f(v)$.

Shilling Strategy. Table 2 illustrates results from Propositions 3–5 for particular s_0. These alternate v_0 between \$2,000 and \$4,000 to illustrate how shilling becomes less attractive to sellers as v_0 decreases, which increases the cost of failed shills. Note s_t^* does not depend on n, or aggressiveness, or s_0 (Proposition 3). For both $f(v)$ here, instances exist where the seller who is prepared to shill to s_t^* should instead accept s_0 (Proposition 4).

Consider the right-tailed $f(v)$, when $v_0 =$ \$2,000, B_n is unaggressive, and $I =$ \$500. Here, $s_t^* =$ \$3,500, and the seller does not shill when $s_0 =$ \$3,000 if $n \ge 9$, or when $s_0 =$ \$2,500 if $n \ge 13$. If B_n has neutral aggressiveness, the seller always shills when $I =$ \$500 and $s_0 =$ \$3,000, and never shills when $I =$ \$100 and $s_0 =$ \$3,800 (in the latter case $s_t^* =$ \$3,900). For some examples the seller always shills, or never shills, regardless of n, whereas for others this depends on n. We include an aggressive B_n for completeness, although Proposition 4 does not address such bidders.

These examples contain both instances in Proposition 5bi–ii where shilling Pareto dominates reserves, evident from comparing v_D^* from Table 1 with s_t^* from Table 2. One is the result in 5bi, where $s_t^* + I < v_D^*$ which occurs for the "symmetric" $f(v)$ when B_n is aggressive and $I =$ \$100 or \$500, if n is sufficiently large. The other is that in 5bii, where $s_t^* + I = v_D^*$ and shilling is not optimal for at least one $s_0 < s_t^*$ which occurs for the "symmetric" $f(v)$ and $I =$ \$100 when $\gamma = 3$ and $2 \le n \le 5$, when $\gamma = 1$ for all n, and when $\gamma = 0.4$ and $2 \le n \le 8$. However, the examples also include instances where $s_t^* + I \ge v_D^*$ and the seller shills, which makes sellers better off but bidders worse off, e.g., $I =$ \$500 and $\gamma = 3$, for "symmetric" $f(v)$. Here, shilling does not Pareto dominate a reserve strategy.

Impact on Seller's Utility. Table 1 compares $E[U(\pi_S)]$ at v_D^* (termed Strategy D for "discrete"), for illustrative values of n, with two other reserve strategies: Strategy N (for "null") sets $v_D^* = v_0$, which is a useful comparison reference since it is intuitively obvious and simple, requiring no normative auction theory and Strategy C (for "continuous") sets $v_D^* = v_C^*$, and uses existing continuous bid theory. Hence, $(E[U(\pi_S)]_C - E[U(\pi_S)]_N)$ indicates how much strategies from existing continuous bid models increase the seller's welfare compared to the "null" strategy, and $(E[U(\pi_S)]_D - E[U(\pi_S)]_N)$ compares our discrete bid models with the "null." The ratio of these differences, $(E[U(\pi_S)]_D - E[U(\pi_S)]_N)/(E[U(\pi_S)]_C - E[U(\pi_S)]_N)$, creates a useful "improvement ratio" for the optimal discrete strategy.

For the "symmetric" $f(v)$, when B_n is relatively unaggressive, $I =$ \$500, and $n = 8$ (where $v_D^* < v_C^*$), this ratio is 1.98. When B_n is relatively aggressive and $n =$ 12 or 15 (both cases where $v_D^* > v_C^*$), this ratio is 1.21 and 1.66, respectively.

The bottom of Table 1 illustrates how $E[U(\pi_S)]$ increases as ξ decreases (Proposition 2), for $n = 5$, 10, and 15. For example, with the right-tailed $f(v)$, $I =$ \$500, and $n = 10$, $E[U(\pi_S)] =$ \$1,927.20 in a discrete auction when B_n is relatively aggressive, increases to \$2,001 when neutrally aggressive, and increases further to \$2,102.50 when unaggressive. These values of $E[U(\pi_S)]$ are also compared with $E[U(\pi_S)]$ from continuous bid auctions (as modeled succinctly in Matthews 1980). Whereas for a given n continuous bidding generates higher $E[U(\pi_S)]$ than discrete when B_n is aggressive, and slightly higher when B_n has neutral aggressiveness, discrete bidding generates higher $E[U(\pi_S)]$ when B_n is unaggressive. Aggressiveness' impact is less when $I =$ \$100.

For shilling, Table 2 compares $E[U(\pi_S) \mid s_0]$ when the seller always shills to s_t^*, with the certain $U(\pi_S)$ from accepting s_0. (Note $E[U(\pi_S) \mid s_0]$ here is conditional on s_0, and so not comparable to $E[U(\pi_S)]$ in Table 1.) For example, with the right-tailed $f(v)$, when $s_0 =$ \$3,000, B_n is relatively unaggressive, and $I =$ \$500, the seller earns \$1,000 for certain by accepting s_0. It is, however,

Table 1 Numerical Examples for Discrete Bid Reserve

	Distribution 1: Right-tailed						Distribution 2: Close to Symmetric					
B_n's Aggressiveness	Unaggressive ($\gamma = 3$)		Neutral ($\gamma = 1$)		Aggressive ($\gamma = 0.4$)		Unaggressive ($\gamma = 3$)		Neutral ($\gamma = 1$)		Aggressive ($\gamma = 0.4$)	
I	500	100	500	100	500	100	500	100	500	100	500	100
ν_0	4000	4000	4000	4000	4000	4000	4000	4000	4000	4000	4000	4000
ν_C^*	5500	5500	5500	5500	5500	5500	6000	6000	6000	6000	6000	6000
n for which $\nu_D^* =$												
$\nu_C^* - 2I$	20 to 40	21 to 40			—	—	14 to 40	15 to 40			—	—
$\nu_C^* - I$	9 to 19	8 to 20			—	—	6 to 13	6 to 14			—	—
ν_C^*	2 to 8	2 to 7	all n	all n	2 to 13	2 to 11	2 to 5	2 to 5	all n	all n	2 to 9	2 to 8
$\nu_C^* + I$	—	—			14 to 40	12 to 34	—	—			10 to 36	9 to 23
$\nu_C^* + 2I$	—	—			Note 4	35 to 40	—	—			37 to 40	24 to 40
$E[U(\pi_S)]$, with reserve at	$\nu_D^* = \nu_C^* - I$				$\nu_D^* = \nu_C^* + I$		$\nu_D^* = \nu_C^* - I$				$\nu_D^* = \nu_C^* + I$	
n	9	8			14	12	6	6			10	9
ν_0	1960.1	1726.9			2229.5	2164.7	2478.7	2385.6			2948.2	2921
ν_C^*	1979.3	1800.9			2263	2194.5	2520.1	2465.7			2978.5	2946.4
ν_D^*	1981.9	1800.9			2263.6	2194.5	2521	2465.7			2979.4	2946.5
Improvement Ratio	1.14	1.00			1.02	1.00	1.02	1.00			1.03	1.00
n	11	10			16	14	8	8			12	11
ν_0	2202.5	2007.4			2373.6	2339.4	2905.3	2819.1			3171.7	3178.8
ν_C^*	2205	2043.3			2394.6	2356	2910.4	2846			3186.2	3189.3
ν_D^*	2210.6	2043.7			2397.8	2356.2	2915.4	2846.3			3189.3	3189.4
Improvement Ratio	3.24	1.01			1.15	1.01	1.98	1.01			1.21	1.01
n	14	13			19	17	11	11			15	14
ν_0	2474	2316.7			2550.1	2547.5	3315.4	3239.2			3420.7	3458.7
ν_C^*	2470.2	2328.4			2560.6	2554.6	3312.2	3244.3			3425.7	3461.7
ν_D^*	2475.7	2328.9			2565.5	2554.8	3316.3	3244.6			3429	3461.8
Improvement Ratio	Note 5	1.04			1.47	1.03	Note 5	1.06			1.66	1.03
$E[U(\pi_S)] \mid \nu_D^*$												
$n = 5$ (*Discrete*)	1348.7	1321.7	1302.9	1310.4	1270	1302.6	2269.6	2223.4	2192.8	2204.9	2137.1	2192
(*Continuous bidding*)	1310.7	←	←	←	←	←	2205.4	←	←	←	←	←
$n = 10$ (*Discrete*)	2102.5	2043.7	2001	2018.7	1927.2	2001.4	3201.3	3129.1	3073.6	3097.5	2979.4	3075.7
(*Continuous bidding*)	2019.4	←	←	←	←	←	3098.5	←	←	←	←	←
$n = 15$ (*Discrete*)	2548.8	2480.8	2425.3	2449.3	2333.3	2427.5	3658.6	3593.8	3529.2	3559.5	3429	3535.3
(*Continuous bidding*)	2450.3	←	←	←	←	←	3560.7	←	←	←	←	←

Note 1: $\underline{\nu} = \$1{,}000$; $\bar{\nu} = \$10{,}000$

Note 2: Distribution 1 is based on Beta(2, 3.65); Distribution 2 is based on Beta(2, 2.46).

Note 3: Computations were performed for n from 2 to 40.

Note 4: Not found for n from 2 to 40.

Note 5: In these instances using ν_C^* rather than ν_0 produces lower utility, due to discrete bidding.

Table 2 **Numerical Examples for Discrete Bid Shilling**

	Distribution 1: Right-tailed						Distribution 2: Close to Symmetric					
B_n's Aggressiveness	Unaggressive ($\gamma = 3$)		Neutral ($\gamma = 1$)		Aggressive ($\gamma = 0.4$)		Unaggressive ($\gamma = 3$)		Neutral ($\gamma = 1$)		Aggressive ($\gamma = 0.4$)	
I	500	100	500	100	500	100	500	100	500	100	500	100
ν_0	2000	2000	2000	2000	2000	2000	4000	4000	4000	4000	4000	4000
ν_C^*	4000	4000	4000	4000	4000	4000	6000	6000	6000	6000	6000	6000
s_t^*	3500	3900	3500	3900	3500	3900	5500	5900	5500	5900	5500	5900
s_0	3000	3800	3000	3800	3000	3800	5000	5800	5000	5800	5000	5800
n (*shill*)	2 to 8	none	2 to 40	none	2 to 40	29 to 40	2 to 16	none	2 to 40	none	2 to 40	46+
n (*not shill*)	9 to 40	2 to 40	none	2 to 40	none	2 to 28	17 to 40	2 to 40	none	2 to 40	none	2 to 45
$U(\pi_S)\|_{\text{not shill},s_0}$	1000	1800	1000	1800	1000	1800	1000	1800	1000	1800	1000	1800
$(n)E[U(\pi_S)]\|_{\text{shill},s_0}$	(5) 1178.16	(5) 1792.51	(5) 1242.53	(5) 1793.68	(5) 1249.87	(5) 1793.69	(10) 1155.48	(5) 1793.88	(5) 1242.41	(5) 1794.13	(5) 1243.03	(10) 1794.23
″	(7) 1096.90	(10) 1788.28	(10) 1245.98	(10) 1793.79	(10) 1309.18	(12) 1794.42	(15) 1040.93	(10) 1791.68	(10) 1244.66	(10) 1794.15	(10) 1260.24	(25) 1795.65
″	(11) 867.35	(15) 1783.07	(15) 1251.37	(15) 1794.01	(15) 1368.3	(25) 1798.52	(20) 891.60	(15) 1789.20	(15) 1246.94	(15) 1794.23	(15) 1291.01	(40) 1798.63
″	(14) 676.61	(20) 1776.71	(2) 1259.05	(20) 1794.32	(20) 1409.24	(40) 1806.6	(30) 572.80	(20) 1786.40	(20) 1248.87	(20) 1794.35	(20) 1324.97	(60) 1804.54
s_0	2500	3700	2500	3700	2500	3700	4500	5700	4500	5700	4500	5700
n (*shill*)	2 to 12	2 to 17	2 to 40	2 to 40	2 to 40	2 to 40	2 to 25	2 to 29	2 to 40	2 to 40	2 to 40	2 to 40
n (*not shill*)	13 to 40	18 to 40	none	none	none	none	26 to 40	29 to 40	none	none	none	none
$U(\pi_S)\|_{\text{not shill},s_0}$	500	1700	500	1700	500	1700	500	1700	500	1700	500	1700
$(n)E[U(\pi_S)]\|_{\text{shill},s_0}$	(7) 852.37	(5) 1711.97	(5) 1068.26	(5) 1713.25	(5) 1082.67	(5) 1713.26	(10) 960.78	(10) 1711.60	(5) 1058.57	(5) 1714.05	(5) 1060.01	(5) 1714.04
″	(11) 567.83	(12) 1705.76	(10) 1072.98	(10) 1713.37	(10) 1144.97	(10) 1713.71	(20) 676.05	(25) 1703.33	(10) 1060.80	(10) 1714.08	(10) 1079.52	(10) 1714.16
″	(15) 339.31	(20) 1695.93	(15) 1080.93	(15) 1713.60	(15) 1186.6	(15) 1714.73	(30) 382.86	(35) 1695.89	(15) 1062.71	(15) 1714.16	(15) 1108.98	(15) 1714.46
″	(20) 183.89	(30) 1678.38	(20) 1087.88	(20) 1713.91	(20) 1209.87	(20) 1716.29	(35) 280.59	(40) 1691.48	(20) 1064.84	(20) 1714.27	(20) 1137.27	(20) 1714.94

Note 1: $\underline{\nu} = \$1,000$; $\bar{\nu} = \$10,000$

Note 2: Distribution 1 is based Beta(2, 4.16); Distribution 2 is based on Beta(2, 2.46).

Note 3: Computations performed for n from 2 to 40, except in one case.

optimal to shill if $n = 5$, since $E[U(\pi_S) \mid s_0] = \$1{,}178.16$, or if $n = 7$, since $E[U(\pi_S) \mid s_0] = \$1{,}096.90$, but not if $n = 11$, where $E[U(\pi_S) \mid s_0] = \$867.35$, or if $n = 14$, where $E[U(\pi_S) \mid s_0] = \$676.61$. The benefits of shilling are greater when $I = \$500$ than when $I = \$100$.

The propositions do not address the *magnitude* of utility comparisons in these examples, which cannot be generalized to all auctions. It is useful, however, to discuss some observations from these example auctions. For instance, the impact of B_n's relative aggressiveness on $E[U(\pi_S)]$ with reserves is, on average, considerably larger than the impact of identifying the optimal reserve. This suggests it is worthwhile to include aggressiveness in discrete bid models and for sellers to try to estimate aggressiveness.

The improvement ratio from using v_D^* instead of v_C^* is often 1.2 or 1.3 in Table 1, and reaches 3.24. This suggests that "returns" from using a discrete model in discrete bid auctions, instead of relying on continuous bid results, often exceed 20%. The magnitude of these increases is often quite small. However, the continuous bid models, which have received much attention in the auction literature, also generate values of $E[U(\pi_S)]$ only slightly higher than the "null" model. Thus, the discrete bid models do make a substantive difference in the seller's welfare when judged by the same standards that have been applied to existing continuous auction models. For these examples, furthermore, the difference $(E[U(\pi_S)]_C - E[U(\pi_S)]_N)$ consistently gets smaller as n increases, whereas $(E[U(\pi_S)]_D - E[U(\pi_S)]_C)$ either increases or stays about the same.

Whether these differences in $E[U(\pi_S)]$ are useful depends on the seller. In many stock and bond operations, new approaches that increase annual returns by even a few hundredths of a percent are considered quite valuable. The returns from using the optimal shilling strategy are more substantial than for reserves. For shilling, there is no simple and obvious "null" model analogous to setting $v_D = v_0$. However, Table 2 does show that shilling optimally can increase $E[U(\pi_S)]$ substantially, be this to shill or accept s_0.

6. Discussion

We have shown that discrete bid rules and variations in relative bidder aggressiveness that can occur with discrete bidding substantially alter sellers' optimal reserve and shilling strategies compared with continuous bidding. Furthermore, the decision of whether to use a reserve or shill and the level of B_n's relative aggressiveness can have a large impact on $E[U(\pi_S)]$ with discrete bidding. Auction practitioners, particularly sellers, can use the models we propose to gain insights into the nature of these strategy and utility changes, to predict when they occur, and determine how optimal strategies change directionally with other auction factors. Although sellers sometimes may have difficulty estimating all of these factors, such as aggressiveness, our results indicate that they do need to be concerned about aggressiveness and try to gather relevant information.

Relaxing the Covert Shilling Assumption. Before the auction, all bidders could believe the seller might shill, definitely will shill, or could have varying beliefs. During the auction even trusting bidders might detect clumsy shilling. Bidders suspicious of shilling might refuse to attend auctions where they believe it might occur, or attend but cease bidding if they detect shilling, or reduce their maximum bid to below its "trusting" level. As a last resort, they might try to estimate s_t^* and immediately bid that amount if shilling is optimal in an attempt to save one increment in the final price since the seller will not bid $s_t^* + I$.

However, bidders will encounter informational and welfare obstacles to implementing these responses to shilling, which reduces their usefulness. First, refusing to attend, or ceasing bidding, denies them the chance to earn a positive expected auction surplus, instead of zero. Second, shilling sometimes makes bidders better off, and bidders cannot determine this for a particular auction without knowing the seller's v_0 and utility function, which affect s_t^* and whether the seller shills or accepts s_0. Yet bidders typically do not have either information, and their prior beliefs about it may be in error. The number of bidders also affects their optimal response, since bidders are better off in many shilling auctions when n increases beyond a point, and the seller should not shill. Third, bidders who suspect shilling can reduce their bids, whereas bidders with no such suspicions will not. Here, the suspicious bidders risk reducing bids excessively, and consequently losing to bona fide bidders rather than shills, while they still would have earned positive surplus with a higher bid.

Occasionally, a shilling seller can obtain information on bidders' valuations from indiscreet bidder behavior unrelated to bid amounts (such as telling a companion "I really want that teacup—I own the matching saucer"). The possibility of shilling implies that in auctions where covert shilling allows sellers (but not bidders) to gain, bidders have an extra incentive to keep their valuations private and not inadvertently to signal valuation information with indiscreet behavior. Neither our model nor any model that assumes bidders' valuations are private covers such behavior.

Auctions with a Common Value Component. Our reserve model does not apply to discrete bid auctions where bidders have common or "affiliated" bidder valuations (Milgrom and Weber 1982). For such bidders, but not for IPV bidders, optimal continuous reserves can depend on n (see also McAfee and McMillan 1987, p. 722). We expect that optimal shilling in such auctions will also differ from our IPV results. Covert shill bids will tend to increase these bidders' valuations, which increase as others bid higher. This can make bidders worse off. Sellers in such auctions may be more likely to shill if they observe that some peoples' bids depend on what others are bidding, which reveals them as common value bidders. On the other hand, cases may still arise when sellers accept s_0 and do not shill, which makes the bidders better off with shilling than reserves. Consequently, the overall impact of discrete bidding on optimal reserve and shilling strategies, and participants' welfare, when valuations have a common value component, is not immediately clear.

Bid Probability that Depends on *n*. Our model assumes B_n's bid probability $\wp(\cdot)$ depends only on j, the number of other remaining bidders, and not on n, the number of bidders participating at the auction's beginning. Furthermore, our model assumes, as do most models with IPV bidders, that bidders need not know n. It is possible that some bidders' $\wp(\cdot)$ might depend on n as well as j. For example, in a crowded auction room B_n might become more aggressive, or intimidated and less aggressive. Our model does not cover this situation.

Auctioneer's Cooperation and Commission. Auctioneers can volunteer different levels of cooperation with shilling, such as forgiving or collecting commissions on property bought with shill bids, or stopping auctions if they detect shilling. Shilling becomes less profitable to sellers as this cooperation decreases. However, even the most cooperative auctioneer's optimal shilling strategy might differ from the seller's if the auctioneer receives a commission only for sold property. In this case the auctioneer's optimal reserve may differ from the seller's and is likely to be less. Thus, for both reserves and shilling, future research may wish to examine equilibria as sellers and commission auctioneers negotiate strategies in discrete bid auctions.

Internet Auctions. Our model's assumptions must be modified to reflect rules and practices peculiar to Internet auctions. Many such services display e-mail addresses for all bidders and report how many times each bids. After bidding closes each bidder's maximum bid is often displayed. To shill covertly and not be detected afterwards, an Internet seller needs a larger set of confederates to submit shill bids from various e-mail addresses. Shills who regularly submit bids within seconds of each other, to obtain the correct shilling "foot," could also be revealed by this public information. Thus, a model of covert Internet shilling could not allow this practice. However, we do not expect that changing the model to reflect Internet bidding would change our model's basic results regarding the optimal point at which to stop shilling, whether to shill, the impact of aggressiveness, and the impact on sellers' and bidders' welfare. Internet bidders can also choose to be extremely aggressive by allowing the Internet service to automatically raise bids up to a specified maximum, an option available in many live auctions.

Internet auctions usually indicate that the current bid is below the reserve rather than having an electronic auctioneer raise bids. However, our model's basic results for discrete reserves should still apply to such Internet auctions, especially the dependence on aggressiveness and n.

Internet auction increments, typically between 1% and 5% of a current bid, are smaller than in most live auctions. One explanation is that because most Internet auctions take several days and participants can bid at any time, smaller increments do not increase bidders' time costs as they would in live auctions (Rothkopf and Harstad 1994b). Furthermore, many Internet auctions feature standardized items available in large quantities, such as baseball cards, toys, or coins. These usually trade within small price ranges, requiring smaller increments to make a market (just as stock shares have for years traded with very small increments of \$1/32, which is 0.0625% of the price of a \$50 share). By contrast, articles in Internet art auctions are often unique and trade over a much larger price range, and many such auctions do use larger increments. Further research may examine how uncertainty or fluctuation in objects' values affect the increments used in auctions that sell them.[12]

Appendix

This appendix defines the bid probability function $\wp(\cdot)$, and describes its properties. It also contains sketches of the proofs of Lemmas 1–6, Propositions 1–4, and a formal proof of Proposition 5. The complete proofs are contained in a technical appendix for this article, located on the *Marketing Science* website, at ⟨http://mktsci.pubs.informs.org⟩. It is also available from the authors.

All variables and functions are real valued.

Assumption 1. *$f(v)/(1\text{-}F(v))$ is strictly monotonically increasing in v* (see footnote 4 for discussion).

Definition 1. *The set of permitted discrete bids is $\Theta = \{s_a, s_a + I, \ldots, s_a + KI\}$, with integer $K \geq 2$, and $s_a + KI \leq \bar{v}$.*

Definition and Properties of $\wp(\cdot)$

Here, we define $\wp(\cdot)$ formally, based on the discussion of bidding aggressiveness in the text (§2). The "bid probability" is the probability that B_n obtains the current bid when j other bidders also remain. The bid probability for B_n is denoted by $\wp(j)|_{\text{Aggressive}}$, or $\wp(j)|_{\text{Neutral}}$, or $\wp(j)|_{\text{Unaggressive}}$, depending on whether B_n is relatively aggressive, neutral, or unaggressive. Note that $\wp(\cdot)$ is a function of j and some other parameter. The latter directly represents aggressiveness in any functional form appropriate for $\wp(\cdot)$.

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We require that $\wp(\cdot)$ satisfy the following three properties: (a) For any given j, $\wp(j)|_{\text{Aggressive}} > \wp(j)|_{\text{Neutral}} > \wp(j)|_{\text{Unaggressive}}$, with the inequalities being strict; and that $\wp(j)$ strictly increases with aggressiveness, regardless of the type of aggressiveness. Note $\wp(j)|_{\text{Neutral}} = 1/(j + 1)$. (b) $(j + 1)\wp(j)$ is nondecreasing in j for an aggressive B_n, nonincreasing in j for an unaggressive B_n, and constant in j for a neutral B_n, for all j corresponding to all n. This sufficient condition ensures that for each bidder, $\wp(j) - 1/(j+1)$ is *always* either > 0, or < 0, or $= 0$, for all j. Also, note that different bidders are allowed to have different types of aggressiveness. (c) $\wp(j)$ is not a constant in j for aggressiveness, neutrality, or unaggressiveness, for all j. It follows from properties (a) and (c) that $\wp(j)$ is a monotonic function of the parameter directly representing aggressiveness, for a given j. For example, γ is that parameter in the numerical examples in the text's §5. Because the proofs do not require that parameter explicitly, it does not appear as an argument in $\wp(\cdot)$.

Property (a) follows from the definition that higher aggressiveness increases B_n's bid probability, $\wp(j)$, ceteris paribus. The justification for property (c) has been discussed in the text (§2). Property (b) ensures that B_n has consistent aggressiveness, and does not "switch" from one type of aggressiveness to another as j changes. The technical appendix contains the proof that violating property (b) contradicts property (a). Please note that property (b) implies that our model does not describe auctions where individual bidders' aggressiveness can "switch."

Next, we state all lemmas together for ease of reference when reading the proofs.

Lemma 1. *$(n F^{n-1}(v)/(F^n(v) - F^n(v - I))$ strictly increases as n increases and is unbounded.*

Lemma 2. *$(n F^{n-1}(v - I))/(F^n(v) - F^n(v - I))$ strictly decreases as n increases and has a minimum > 0.*

Lemma 3. *$U(v - I - v_0)/(U(v - v_0) - U(v - I - v_0))$ strictly increases as v increases.*

Lemma 4. *$U(v - I - v_0)/(U(v - v_0) - U(v - I - v_0))$ increases as the seller's absolute risk aversion increases.*

Lemma 5. *$(1 - F(v))/(F(v) - F(v - I))$ strictly decreases as v increases.*

Lemma 6. *(a) $P(v_{(n),only})$ decreases as n increases if $\xi \leq 0$, provided*

$$n > \left[\left|\log \frac{F(s_0 - I)}{F(s_0)}\right|\right]^{-1};$$

and is not necessarily monotonic if $\xi > 0$.

(b) $P(v_{(n),only})$ lies strictly between 0 and 1.

Sketch of Proofs of Lemmas. For Lemmas 1 and 2, we rewrite the expressions using the term A, where $A = F(v)/F(v - I)$, and then apply real analysis and calculus to get the results. The proof of Lemma 3 compares how the numerator and the denominator change with v. The proof of Lemma 4 uses the definition of absolute risk aversion (from Pratt 1964, Theorem 1). The proof of Lemma 5 is shown by differentiating w.r.t. v and invoking Assumption 1. Note that Assumption 1 thus becomes a limiting assumption for when

Lemma 5 is true. As noted in footnote 4, this assumption is satisfied by several distributions and has been widely used in the literature. For Lemma 6a, we first enumerate the probabilities of the three mutually exclusive and exhaustive ways that bidding can stop at s_0. This allows us to derive the expression for $P(v_{(n),only} \mid s_0)$ for each of the three kinds of aggressiveness. The proof then examines separate subcases for each kind of dependence between ξ and n and for each kind of aggressiveness. The proof also requires that the three properties of $\wp$, as discussed earlier, hold. The proof of Lemma 6b shows that each of the three possible outcomes always has a strictly positive chance of occurring.

Sketch of Proof of Proposition 1. We show v^*_D exists and is unique because the gain and loss functions from raising the reserve, which are q and r in text Equations (1) and (2), always cross, and at only one point, over the range of the reserve. To prove the remainder of Proposition 1a–c, we start with a relatively aggressive B_n, so that $\xi > 0$. To show that v^*_D increases with n, we must examine separate subcases where ξ is nondecreasing in n versus strictly increasing in n. We then show for each subcase that as n increases, q and r can remain equal only if the reserve is eventually increased by a bidding increment and that as n continues to increase, v^*_D eventually reaches $\bar{v} - I$. We do this by isolating the impact of ξ on the equality $q = r$, and show that when $\xi > 0$ and n is increased, the least upper bound at which $q = r$ is $\bar{v} - I$. We use parallel reasoning to show that if B_n is relatively unaggressive ($\xi < 0$), then v^*_D decreases with n down to v_0.

To show that v^*_D increases with v_0, we show that the gain q is strictly monotonically increasing, and the loss r is strictly monotonically decreasing, w.r.t. v_0. Consequently the reserve when $q = r$ will increase with v_0. To show that v^*_D increases with B_n's relative aggressiveness for a given n, we use the fact that the gain q increases with aggressiveness, whereas the loss r does not depend on aggressiveness. To show that v^*_D approaches v^*_C as I approaches 0 for all ξ, we use real analysis and compare the result with v^*_C as derived by Matthews (1980). To show that v^*_D decreases as the seller becomes more risk averse, but only for $\xi \leq 0$, we show how making the seller more risk averse lowers the reserve at which $q = r$, provided $\xi \leq 0$.

Sketch of Proof of Proposition 2. To show that $E[U(\pi_S)]$ at v^*_D increases as B_n's relative aggressiveness decreases, we first formulate $E[U(\pi_S)]$ in a discrete auction with a reserve. This has not been done in the extant literature. We show that a discrete bid auction with a reserve has three mutually exclusive and exhaustive outcomes:

(i) The item does not sell. This happens if and only if $v_{(n)} < v^*_D$, with probability $P_I = F^n(v^*_D)$. This implies that $\pi_S = 0$.

(ii) The item can sell at v^*_D, in three mutually exclusive and exhaustive ways:

(a) Only B_n's value is $\geq v^*_D$. With "correct foot" bidding for the reserve, the item sells at v^*_D. The probability of this event is: $P_{II,a} = \binom{n}{1} [1 - F(v^*_D)] F^{n-1}(v^*_D)$.

(b) j ($j \geq 2$) bidders have $v^*_D \leq v_i < v^*_D + I$, but none has $v_i \geq v^*_D + I$.

$$P_{II,b} = \sum_{j=2}^{n} \binom{n}{j} [F(v^*_D + I) - F(v^*_D)]^j F^{n-j}(v^*_D).$$

(c) Only B_n's value is $\geq v^*_D + I$, and at least one of the $(n - 1)$ remaining bidders has $v^*_D \leq v_i < v^*_D + I$, and B_n has bid at v^*_D.

$$P_{II,c} = \sum_{j=1}^{n-1} n[1 - F(v^*_D + I)] \left(\frac{1}{j+1} + \xi\right) \binom{n-1}{j} [F(v^*_D + I) - F(v^*_D)]^j F^{n-1-j}(v^*_D).$$

(iii) The item sells for above the reserve because of competitive bidding. This happens in three ways for each bid level $v^*_D + kI \leq \bar{v}$, for $k = 1, \ldots, M$; where M is the number of possible bid levels above v^*_D.

(a) Only B_n's value is $\geq v^*_D + (k + 1)I$, and at least one of the $(n-1)$ remaining bidders has $v^*_D + kI \leq v_i < v^*_D + (k + 1)I$, and B_n has bid at $v^*_D + kI$.

$$P_{III,a} = \sum_{j=1}^{n-1} n[1 - F(v^*_D + (k + 1)I)] \left(\frac{1}{j+1} + \xi\right) \binom{n-1}{j} [F(v^*_D + (k + 1)I) - F(v^*_D + kI)]^j F^{n-1-j}(v^*_D + kI).$$

(b) Only B_n's value is $\geq v^*_D + kI$, and at least one of the $(n - 1)$ remaining bidders has $v^*_D + (k - 1)I \leq v_i < v^*_D + kI$, and B_n does not have bid at $v^*_D + (k - 1)$I, and so must bid $v^*_D + kI$.

$$P_{III,b} = \sum_{j=1}^{n-1} n[1 - F(v^*_D + kI)] \left(\frac{j}{j+1} - \xi\right) \binom{n-1}{j} [F(v^*_D + kI) - F(v^*_D + (k - 1)I)]^j F^{n-1-j}(v^*_D + (k - 1)I).$$

(c) Two or more bidders have $v^*_D + kI \leq v_i < v^*_D + (k + 1)I$, but none has $v_i \geq v^*_D + (k + 1)I$.

$$P_{III,c} = \sum_{j=2}^{n} \binom{n}{j} [F(v^*_D + (k + 1)I) - F(v^*_D + kI)]^j F^{n-j}(v^*_D + kI).$$

Hence, $E[U(\pi_s)] = U(v_0 - v_0) \cdot P_I + U(v^*_D - v_0) \cdot [P_{II,a} + P_{II,b} + P_{II,c}] + \sum_{k=1}^{M} U(v^*_D + kI - v_0) \cdot [P_{III,a} + P_{III,b} + P_{III,c}]$, with M satisfying $v^*_D + MI \leq \bar{v}$, and $\bar{v} - (v^*_D + MI) < I$.

We then show that only some of these outcomes have probabilities of occurring that depend on ξ. We differentiate these outcomes' probabilities w.r.t. ξ, and, after some rearrangement and simplification, get the result. Next, we show that decreasing ξ increases the set of possible auctions where $E[U(\pi_S)]$ is higher with discrete than continuous bidding for a given n. This depends on the argument that although $E[U(\pi_S)]$ does not depend on ξ in a continuous bid auction, it increases as ξ decreases when bidding is discrete.

Sketch of Proof of Proposition 3. To show that the optimal final shill bid, s^*_l, exists, we first derive the seller's expected utility from submitting a specific set of shill bids, provided bidding stops at s_0. To do so in the most general manner, we define a shill "run" as one or more shill bids placed consecutively by shill(s) without waiting for a response from bona fide bidders. Two shill runs must be separated by a bona fide bid. We then define a sequence of shill runs by $s_1, s_2, \ldots$, where the discrete bid level at which the ith shill run ends is denoted s_i (noting $s_i < \bar{v} - I$). Thus, $s_i \geq s_{i-1} + 2I$, $i =$

1,2, ... The expected utility $E[U(\pi_S) \mid s_0]$ from a series of shill runs ending at s_t is:

$$E(U(\pi_{s_1,\ldots,s_t} \mid s_0)) = U(s_t + I - \nu_0) \cdot P(\nu_{(n),only} \mid s_0) \cdot P(\nu_{(n)} \geq s_1 + I \mid s_0, \nu_{(n),only}) \cdot \prod_{j=2}^{j=t} P(\nu_{(n)} \geq s_j + I \mid \nu_{(n)} \geq s_{j-1} + I).$$

Using the properties of order statistics, we then estimate the probability that all of these shill bids will succeed. We also show that this probability does not depend on whether the intermediate shill bids are placed successively or separately. By inspection, we show that s_t^* does not depend on s_0, or ξ, or n. We show that s_t^* increases with v_0, and decreases as the seller becomes more risk averse, using real analysis and calculus. Next we show that $s_t^* + I$ can be higher or lower than v_D^*, and that this depends on ξ and n. This proof combines two results already proven—that s_t^* does not depend on ξ or n, but that v_D^* does (shown in Proposition 1). This also allows us to show that only when $\xi = 0$ does $s_t^* + I = v_D^*$ for all n. The derivation of $s_t^* + I$ shows that it equals v_C^* if the latter bid lies on a discrete bid level.

Sketch of Proof of Proposition 4. We show that sometimes the seller's expected utility from shilling to s_t^*, which is the optimal shilling strategy shown in Proposition 3, is less than the seller earns for certain by accepting s_0. The comparison between these two expected utilities depends on s_0, because that is the certainty revenue the seller risks if he shills, and because $P(v_{(n),only} \mid s_0)$, which raises the chances that shilling to s_t^* succeeds, depends on s_0. It depends on n and ξ because $P(v_{(n),only} \mid s_0)$ decreases with n, but only if $\xi \leq 0$.

Proof of Proposition 5a: Note that the information demands on the seller regarding ξ are the same in both the optimal shilling and reserve strategies. Compared with using a reserve, the shilling seller has more information when he sets the lowest acceptable bid because he knows s_0, which he does not know when setting the reserve. This allows the shilling seller (ex-ante) to enumerate each bidder's valuation with lower variance, compared with $F(v)$ used in a reserve auction. This happens because $\{S_0 = s_0\}$ refines bidders' valuations to $\{s_0 \leq v_{(n)} \leq \bar{v}\}$, $\{s_0 - I \leq v_{(n-1)} < s_0 + I\}$, and $\{\underline{v} \leq \{v_{(1)}, \ldots, v_{(n-2)}\} < s_0 + I\}$. Further, ex-ante, the set of possible decisions the seller can make—to accept or reject bids—is identical in both auctions. Also note that the bidders' ex-ante set of decisions is the same with both reserves and shilling (bidders don't change their bidding strategy during the auction, as stated in § 3). Hence, the seller can never be worse off by using the optimal shilling strategy compared with the optimal reserve strategy (appealing to Blackwell's Theorem on Comparison of Experiments 1951).

We now show that auctions exist where, ex-ante, the seller is strictly better off with the optimal shilling strategy than with the optimal reserve strategy. By Blackwell's Theorem, $E[U(\pi_S)]$ is strictly higher in the shilling auction when (i) $s_t^* + I \neq v_D^*$, or (ii) $s_t^* + I = v_D^*$ and it is optimal for the seller to not shill for at least one $s_0 < s_t^*$ (by Proposition 4 such auctions exist). In (i), shilling and reserve strategies yield different optimal lowest acceptable bids and hence different expected utilities for the seller. In (ii), optimal shilling is strictly dominating for the seller because the seller accepts at least one s_0 with shilling (based on more information) that he rejects with reserve, and he accepts that s_0 because the seller's expected utility is greater from accepting s_0 rather than shilling up to s_t^*. Now, the expected utility for the latter is the same as that from setting $v_D = v_D^*$ in the optimal reserve scenario, since $v_D^* = s_t^* + I$. Consequently, the seller's expected utility is strictly higher with optimal shilling. If $s_t^* + I = v_D^*$ and the seller shills for all $s_0 < s_t^*$, then the seller is as well off as in a reserve auction. Q. E. D.

Proof of Proposition 5b. (i) When $\xi > 0$, consider shilling auctions with $s_t^* + I < v_D^*$ (per Proposition 3c). The probability the item sells in these auctions is strictly greater with shilling than with reserves. Also, ex-ante, each bona fide bidder's expected surplus is greater in such shilling auctions, because (i) with probability > 0, the property would sell at some bid $< v_D^*$ in shilling auction, and (ii) P(any bidder's value $\geq s_t^* + I$) $> P$(bidder's value $\geq v_D^*$), since $s_t^* + I < v_D^*$. (ii) When $\xi \leq 0$, consider shilling auctions with $s_t^* + I = v_D^*$. Using Proposition 4 and the arguments we have just stated in the proof of Proposition 5a, we conclude that auctions exist in which bidders are strictly better off (unless seller shills for all $s_0 \leq s_t^*$,in which case the bidders are as well off as in reserve auction).

By Proposition 5a, the seller is at least as well off with shilling. Pareto dominance follows. Q. E. D.

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