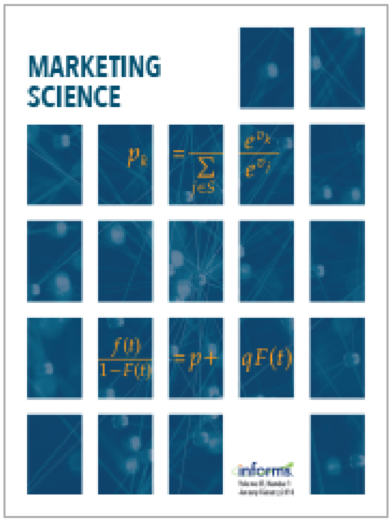


Marketing Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

The Additive Risk Model for Purchase Timing

P. B. Seetharaman,

MARKETING SCIENCE
Vol. 23, No. 2, Spring 2004, pp. 234–242
ISSN 0732-2399 | EISSN 1526-548X | 04 | 2302 | 0234

DOI 10.1287/mksc.1030.0021

The Additive Risk Model for Purchase Timing

P. B. Seetharaman
John M. Olin School of Business, Washington University,
Campus Box 1133, One Brookings Drive, St. Louis, Missouri 63130-4899, seethu@olin.wustl.edu

This paper proposes the *Additive Risk Model* (ARM), first used by Aalen (1980), to explain households' interpurchase times. Unlike the Proportional Hazard Model (PHM), first proposed by Cox (1972), the ARM incorporates the effects of covariates on the individual hazard function in an *additive* (as opposed to *multiplicative*) manner. While a large number of previous studies on interpurchase timing have dealt with the question of correctly specifying the parametric distribution for interpurchase times, no study has explicitly investigated the question of correctly specifying the effects of covariates in the model. This study looks at this issue.

We propose an ARM that is suitable for purchase-timing data, and compare its empirical performance to that of the PHM and the Accelerated Failure Time Model (AFTM) using scanner panel data on laundry detergents, paper towels, and toilet tissue. We find that the ARM not only estimates and validates the observed interpurchase times better than existing models, but also recovers a time-varying price elasticity and shows a high degree of robustness in the estimated covariate effects to alternative parametric specifications of the baseline hazard. The estimates of covariate parameters under the PHM, on the other hand, are highly sensitive to alternative parametric specifications of the baseline hazard.

Key words: purchase timing models; additive risk model; proportional hazard model; accelerated failure time model; log-logistic hazard
History: This paper was received March 25, 2002, and was with the authors 1 month for 2 revisions; processed by Pradeep Chintagunta.

1. Introduction

There exists a sizable body of empirical research that deals with the estimation of *purchase-timing* decisions of households, using statistical models. These studies characterize households' temporal decisions of *when* to buy a particular product over time. A number of these studies use panel data to estimate the parameters of the purchase-timing model (for a recent study, see Seetharaman and Chintagunta 2003). Early empirical studies, mostly in the sixties and seventies, modeled households' interpurchase times using statistical distributions such as Erlang-2, negative binomial, etc. (see, for example, Chatfield and Goodhardt 1973). Later studies, mostly in the eighties, modeled households' interpurchase times using econometric models—such as the logistic regression model—in order to account for the effects of marketing variables (such as price, advertising, etc.). A statistical model that is well suited to capture both of these features of purchase-timing data, i.e., intrinsic temporal purchase patterns (captured in the early statistical models such as the NBD) and the effects of marketing variables (captured in the later econometric models such as the Logit), is the *Proportional Hazard Model*, first proposed by Cox (1972) and used in an extensive range of marketing applications over the past decade (see, for example, Jain and Vilcassim 1991, Helsen and Schmittlein 1993, Wedel et al. 1995, Chintagunta 1998).

In the Proportional Hazard Model (PHM henceforth), the construct of interest is a household's instantaneous probability of making a purchase in a product category, conditional on the elapsed time since the household's previous purchase in the product category. This conditional probability, also called the *hazard function*, is multiplicatively decomposed into two components: One, the *baseline hazard* captures the household's intrinsic temporal purchase pattern; two, the *covariate function* captures the influence of marketing variables. In other words, the baseline hazard characterizes the distribution of the household's interpurchase times after controlling for the effects of marketing variables. In the absence of marketing variables, suitable assumptions on the baseline hazard and unobserved heterogeneity across households in the PHM's parameters yield statistical purchase-timing models—such as the Erlang, NBD, CNBD, etc.—that have been extensively used to describe purchase-timing data for over three decades (Chatfield and Goodhardt 1973, Schmittlein and Morrison 1983, Wheat and Morrison 1990). Given its ability to handle time-varying covariates and nest traditional purchase-timing models as special cases, the PHM has become the most popular model to analyze interpurchase times of households.

In the past, researchers working with the PHM have tested a variety of parametric specifications for the baseline hazard. Some commonly used distributions are exponential, Erlang-2, Weibull, log-logistic, Gompertz, etc. Flexible baseline hazard specifications that permit a variety of different shapes for the baseline hazard have also been used. One such flexible specification is the *Quadratic Box-Cox specification*, used to analyze interpurchase times by Jain and Vilcassim (1991). Another flexible specification is the *Expo-Power specification*, recently proposed by Saha and Hilton (1997) to study equipment failure times, and used in a purchase timing context by Seetharaman and Chintagunta (2003). The weekly nature of households' shopping trips, coupled with the fact that not all shopping trips involve purchase of the product, has made the discrete-time version of the PHM (also called "Grouped PHM") more suitable for purchase-timing data than the continuous-time version (for an application of the discrete-time PHM, see Helsen and Schmittlein 1993).

Despite extensive work on testing alternative specifications of the baseline hazard, no empirical study on purchase timing has tested alternative specifications of the covariate function. For example, the plausibility of the PHM's assumption that marketing variables influence purchase timing by acting in a *multiplicative* manner on the baseline hazard has never been empirically tested using data. An alternative specification of covariate effects is the Accelerated Failure Time Model (AFTM henceforth), in which marketing variables are allowed to directly influence the shape parameter of the baseline hazard (Chintagunta 1998). A third possibility could be that marketing variables act on the baseline hazard in an *additive* manner (instead of a *proportional* manner as in the PHM, and a *shape-shifting* manner as in the AFTM). Such an assumption yields a purchase-timing model called the *Additive Risk Model*[1] (ARM henceforth), first proposed by Aalen (1980) and shown to be preferable to the PHM in several biological and epidemiology problems (Breslow and Day 1980, Pocock et al. 1982, Buckley 1984, Pierce and Preston 1984, Breslow 1986, Thomas 1986). The ARM also handles time-varying covariates and nests traditional purchase-timing models—such as NBD, Erlang-2, etc.—as special cases. This study investigates whether the ARM is suitable for purchase-timing data and compares its empirical performance to that of the widely used PHM using scanner panel data on three product categories: laundry detergents, paper towels, and toilet tissue. As an additional benchmark, we also estimate the AFTM on the available purchase data and compare its performance to that of the ARM and PHM.

[1] Also called the *Additive Hazard Model* or the *Linear Hazard Model*.

Our empirical findings are as follows: First, the ARM outperforms the PHM, which in turn outperforms the AFTM, in explaining observed purchases and in predicting holdout purchases of households; second, the estimated baseline hazards are quite similar between the ARM and PHM; third, the computed price elasticity is U-shaped over time under the ARM while it is time constant and severely understated (i.e., close to zero) under the PHM; fourth, the estimated parameters of the covariate function are very robust to alternative parametric specifications of the baseline hazard under the ARM, while they are highly sensitive to alternative baseline hazard specifications under the PHM.

Given the wide coverage given to the PHM in academic papers and text-books—such as Kalbfleisch and Prentice 1980, Lancaster 1990 etc.—the paucity of coverage for the ARM in the academic literature is striking. We found only eight published papers on the ARM from the 1990s in the academic literature on theoretical and applied statistics (Greenwood and Wefelmeyer 1990, 1991; Huffer and McKeague 1991; Henderson and Milner 1991; Sasieni 1992; Henderson and Oman 1993; Lin and Ying 1994; McKeague and Sasieni 1994). This is in sharp contrast to about a hundred papers that we found on the PHM over the same period of time. Given the excellent empirical performance of the ARM in the three datasets used in this study, we hope that this paper spurs more research interest on the ARM among purchase-timing researchers and encourages its adoption in other application areas as well.

The rest of the paper is organized as follows. In the next section, we propose the ARM to study purchase timing, develop likelihood-based procedures to estimate the parameters of the ARM, and contrast the proposed ARM with the PHM and AFTM. In §3, we present our empirical findings. In §4, we conclude with a summary.

2. Model Formulation and Estimation

Aalen's (1980) ARM specifies a household's instantaneous probability (also called the household's *hazard function*) of making a purchase in a product category, conditional on the elapsed time (t) since the household's previous purchase in the product category is as follows:[2]

$$h_i(t, X_t) = h_i(t) + \psi_i(X_t), \tag{1}$$

where $h_i(t, X_t)$ stands for household i's *hazard function* at time t, X_t is a *row-vector of covariates* (product-specific marketing variables such as price, display,

[2] From this point onwards, we will use the canonical notation t to refer to *time since previous purchase*, and not to *calendar time*. *Calendar time* will be represented using the notation t_j.

and feature, and household-specific variables such as product inventory) facing household i at time t, $h_i(t)$ stands for household i's *baseline hazard* at time t, and $\psi_i(X_t)$ stands for household i's *covariate function* at time t. In this additive model, the baseline hazard represents the probability distribution characterizing the household's interpurchase times, and the covariate function shifts this baseline hazard up or down depending on the values of the covariates. The following functional form is typically used to represent the covariate function:

$$\psi_i(X_t) = e^{X_t \beta_i}, \tag{2}$$

where β_i stands for a column-vector of parameters corresponding to the covariates contained in X_t. The choice of this functional form is dictated by the fact that the hazard function must always be nonnegative. This yields the following form of the ARM.

$$h(t, X_t) = h(t) + e^{X_t \beta}, \tag{3}$$

where the subscript i has been dropped for nota-tional ease.

One can also write the hazard function as follows:

$$h(t, X_t) = \frac{f(t, X_t)}{1 - F(t, X_t)}, \tag{4}$$

where $f(t, X_t)$ stands for the *probability density function* (pdf) corresponding to the household i's *hazard function* at time t, and $F(t, X_t)$ stands for the corresponding *cumulative distribution function* (cdf). This equation can be understood as follows: While $f(t, X_t)$ stands for the household's (unconditional) probability of buying the product at time t, $1 - F(t, X_t)$ stands for the probability that the household has not bought the product until time t. Combining Equations (3) and (4) yields

$$\frac{f(t, X_t)}{1 - F(t, X_t)} = h(t) + e^{X_t \beta}, \tag{5}$$

where $1 - F(t, X_t)$ is also called the *survivor function*, i.e., the probability that the household "survives" a purchase until time t, represented by $S(t, X_t)$. Therefore, an alternative way of writing Equation (5) is as follows:

$$f(t, X_t) = \left[h(t) + e^{X_t \beta}\right] * S(t, X_t). \tag{6}$$

In order to simplify Equation (5) into a more "estimation-friendly" form, we can rewrite it as follows:

$$\frac{dF(t, X_t)}{1 - F(t, X_t)} = \left[h(t) + e^{X_t \beta}\right] * dt, \tag{7}$$

which is a first-order differential equation that can be solved as shown below.

$$\int_0^{F(t,X)} \frac{dF(u, X_u)}{1 - F(u, X_u)} = \int_0^t \left[h(u) + e^{X_u \beta}\right] * du, \tag{8}$$

where the lower limit of integration (i.e., 0) corresponds to the time of the household's previous purchase in the product category. Solving this yields

$$F(t, X_t) = 1 - e^{-\int_0^t [h(u) + e^{X_u \beta}]\, du}, \tag{9}$$

which is equivalent to

$$S(t, X_t) = e^{-\int_0^t [h(u) + e^{X_u \beta}]\, du}. \tag{10}$$

Substituting from Equation (10) in Equation (6) yields the following estimable version of the continuous-time ARM.

$$f(t, X_t) = \left[h(t) + e^{X_t \beta}\right] * e^{-\int_0^t [h(u) + e^{X_u \beta}]\, du}, \tag{11}$$

where $f(t, X_t)$ stands for the probability density associated with the purchase event that occurs at time t and covariate vector X_t. There are a number of alternative parametric specifications that one can use for $h(t)$, such as exponential, Erlang-2, Weibull, log-logistic, log-normal, Gompertz, Raleigh, inverse-Gaussian, etc. Estimation of the parameters of the ARM at the individual level can proceed as follows: Suppose a household provides n interpurchase-time observations in the product category, say, $(t_1, t_2, t_3, \ldots, t_n)$. Each of these interpurchase times is associated with a covariate vector, yielding a stacked set of covariate vectors given by $(X_1, X_2, X_3, \ldots, X_n)$. The following likelihood function can be maximized to estimate the parameters of the ARM at the individual level.

$$L = \prod_{j=1}^{n} f(t_j - t_{j-1}, X_j), \tag{12}$$

where n stands for the total number of purchases made by the household in the product category, t_j stands for the calendar time associated with purchase j, X_j stands for the covariate vector at calendar time t_j, and $f(\cdot)$ is given by Equation (11). It is clear from Equation (12) that this procedure computes the likelihood function based on a household's purchase observations only. However, a household undertakes shopping trips at discrete points of time, often weekly, which makes the household's purchase timing in a particular product category conditional on the household's discrete-time interval of shopping. To accommodate this, we rewrite Equation (10) in discrete time as follows:

$$S(t, X_t) = e^{\int_0^t h(u)\, du - \sum_{u=1}^{t} e^{X_u \beta}}, \tag{13}$$

where u is discrete time, measured in the time interval of shopping trips (usually weeks). In this formulation, the integral of Equation (10) is replaced in part by a summation which is consistent with the empirical fact that marketing variables associated with a product category (contained in X_t) stay constant within a given week, but vary from one week to another. The household's probability of purchasing the product in discrete time t (since the previous purchase), also called the *discrete-time hazard*, is then given by

$$\begin{aligned}\Pr(t, X_t) &= 1 - \frac{S(t, X_t)}{S(t-1, X_{t-1})}\\ &= 1 - \frac{e^{-\int_0^t h(u)\,du - \sum_{u=1}^{t} e^{X_u\beta}}}{e^{-\int_0^{t-1} h(u)\,du - \sum_{u=1}^{t-1} e^{X_u\beta}}}\\ &= 1 - e^{-\int_{t-1}^{t} h(u)\,du - e^{X_t\beta}}, \qquad (14)\end{aligned}$$

where $\Pr(t, X_t)$ stands for the probability that the purchase event will occur at discrete time t and covariate vector X_t. $\Pr(t, X_t)$ is also called the discrete-time hazard (as opposed to $h(t, X_t)$, which is called the continuous-time hazard). The following likelihood function can be maximized to estimate the parameters of the discrete-time ARM at the individual level.

$$L = \prod_{\nu=1}^{T} \Pr(\nu, X_\nu)^{\delta_\nu} * [1 - \Pr(\nu, X_\nu)]^{1-\delta_\nu}, \qquad (15)$$

where δ_ν is an indicator variable that takes the value 1 if the product is purchased by the household at shopping trip ν and 0 otherwise, and $\Pr(\nu, X_\nu)$ is the household's probability of purchasing the product at shopping trip ν and is given by Equation (14). The discrete-time ARM, also called the *grouped ARM*, has been used by Huffer and McKeague (1991) to analyze cancer mortality times among Japanese atomic bomb survivors. The discrete-time ARM is preferable to the continuous-time ARM because it explicitly accounts for nonpurchases. This is the model that we propose in this study. As regards the baseline hazard, $h(t)$, we use the log-logistic specification, which has been shown by Seetharaman and Chintagunta (2002) to be empirically superior to alternative specifications for purchase-timing data. The log-logistic hazard is given below.

$$h(t) = \frac{\gamma\alpha(\gamma t)^{\alpha-1}}{1 + (\gamma t)^{\alpha}}, \qquad S(t) = \frac{1}{1 + (\gamma t)^{\alpha}}, \qquad (16)$$

where $\gamma, \alpha > 0$. This baseline hazard can be monotonically decreasing or inverted U-shaped (with a turning point at $t = (\alpha - 1)^{1/\alpha}/\gamma$).

Last but not least, in order to avoid pooling bias in the parameter estimates of the baseline hazard and the covariate function, we allow the parameters of the model to follow a multivariate discrete distribution across households, whose locations and masses are estimated using the available data (as in Heckman and Singer 1984). Given below is the likelihood equation for such a heterogeneous ARM.

$$L = \prod_{h=1}^{H} \sum_{s=1}^{S} \pi_s * \left(\prod_{j=1}^{T_h} \Pr_s(\nu, X_\nu)^{\delta_\nu} * [1 - \Pr_s(\nu, X_\nu)]^{1-\delta_\nu} \right), \qquad (17)$$

where π_s stands for the probability mass associated with support s, T_h stands for the number of shopping trips for household h, and $\Pr_s(\cdot)$ stands for the probability of the discrete-time ARM associated with support s and is given by Equation (14).

This completes our exposition of the proposed ARM for purchase-timing data. The likelihood equation, given in Equation (17), can be maximized using gradient-based techniques to obtain maximum-likelihood estimates of model parameters. Estimation involves the a priori specification of the number of supports S. While the optimal value of S is not directly estimable, we start with the homogeneous specification (i.e., $S = 1$) and keep adding supports until model fit stops improving (in terms of the *Schwarz Bayesian Criterion*). Next, we discuss two alternative models of purchase timing, which can be used as benchmarks for the proposed model.

Cox's (1972) Proportional Hazard Model

Cox's (1972) PHM specifies the household's *hazard function* as follows:

$$h(t, X_t) = h(t) * e^{X_t\beta}. \qquad (18)$$

In other words, the PHM says that the effect of the covariate function, $e^{X_t\beta}$, on the baseline hazard, $h(t)$, is *multiplicative* rather than additive. This yields the following discrete-time PHM model:

$$\Pr_s(t, X_t) = 1 - e^{-e^{X_t\beta_s}\int_{t-1}^{t} h_s(u)\,du}, \qquad (19)$$

which has been estimated on purchase-timing data by Helsen and Schmittlein (1993) and, more recently, Seetharaman and Chintagunta (2003), and serves as the benchmark model for the discrete-time, parametric ARM of Equation (14).

Prentice and Kalbfleisch's (1979) Accelerated Failure Time Model

Prentice and Kalbfleisch's (1979) AFTM directly specifies one of the parameters of the *baseline hazard function* to be a linear function of covariates. The log-logistic version of this model has been used by Chintagunta (1998) to model households' purchase-timing decisions, and is reproduced below.

$$\begin{aligned} h(t) &= \frac{[\gamma_o + X_t\gamma_1]\,\alpha([\gamma_o + X_t\gamma_1]t)^{\alpha-1}}{1 + ([\gamma_o + X_t\gamma_1]t)^{\alpha}},\\ S(t) &= \frac{1}{1 + ([\gamma_o + X_t\gamma_1]t)^{\alpha}}, \end{aligned} \qquad (20)$$

where γ_o is an intercept parameter and γ_1 is a vector of parameters capturing the influence of marketing variables on the baseline hazard. In other words, the AFTM is obtained from the log-logistic hazard of Equation (16) by allowing the parameter γ to be a linear function of marketing variables. We use this model as an additional benchmark for our proposed ARM.

3. Empirical Results

We employ IRI's scanner panel data on household purchases in a metropolitan market in a large U.S. city. For our analysis, we pick three product categories: laundry detergents, paper towels, and toilet tissue. We believe that the "weak separability" assumption, i.e., the assumption that households' purchase decisions in a product category can be analyzed independently compared to their purchase decisions in other product categories in the store, is more reasonable for these product categories than for other product categories such as soup or soft drinks. This is because there are no obvious substitutes or complements for detergents or paper towels or toilet tissue, and these products can be considered necessities for most households. These datasets cover a period of two years from June 1991 to June 1993 and contain shopping trip information on 494 panelists across four different stores in an urban market. For each product category, the dataset contains information on marketing variables—price, in-store displays, and newspaper feature advertisements—at the stock-keeping-unit (SKU) level for each store/week.

Choosing households that bought at the two largest stores in the market (that collectively account for 90% of all shopping trips in the database) yields 488 households. From these households, we pick a random sample of 300 households making a total of 39,276 shopping trips at the two largest stores. For those shopping trips when a household does not purchase a particular product category, we compute marketing variables as share-weighted average values across all SKUs in the product category, where shares are household specific and computed using the observed purchases of the household over the study period. Such share weighting has precedence in the marketing literature on category purchase models (see, for example, Chib et al. 2002). Descriptive statistics pertaining to the marketing variables in the three product categories are provided in Table 1.

Table 1 Descriptive Statistics

Product	Price ($/RP)	Display	Feature	No. of purchases
Detergents	1.0588	0.1201	0.0756	3,136
Towels	0.8028	0.1469	0.1359	4,437
Tissue	0.3253	0.1671	0.1822	5,475

Note. Number of households = 300; number of shopping trips = 39,276.

Detergents appear to be less displayed at the store and less advertised in the newspaper compared to paper towels and toilet tissue. Households make the largest number of purchases, and therefore have shortest interpurchase times, in the toilet tissue category. We plot the empirical distributions of observed interpurchase times in Figure 1. The frequency distributions appear to have a mode close to one week (i.e., seven days) and are skewed to the left. Interestingly, the histograms show peaks in multiples of seven days. This is consistent with the empirical observation that households undertake shopping trips in weekly intervals (Kahn and Schmittlein 1989). The purpose of estimating the proposed ARM is to *estimate* the density functions associated with these histograms *after accounting for the effects of marketing variables*.

Given in Table 2A are the results of fit comparisons between the proposed ARM and the benchmark models. In all three product categories, the proposed ARM is the best-fitting model, while the AFTM is the worst-fitting model. (*Note*: The three models have identical numbers of parameters.) Using a holdout sample of 63 households (whose purchase data are not used in the estimation), we compute validation

Figure 1 Observed Interpurchase Times

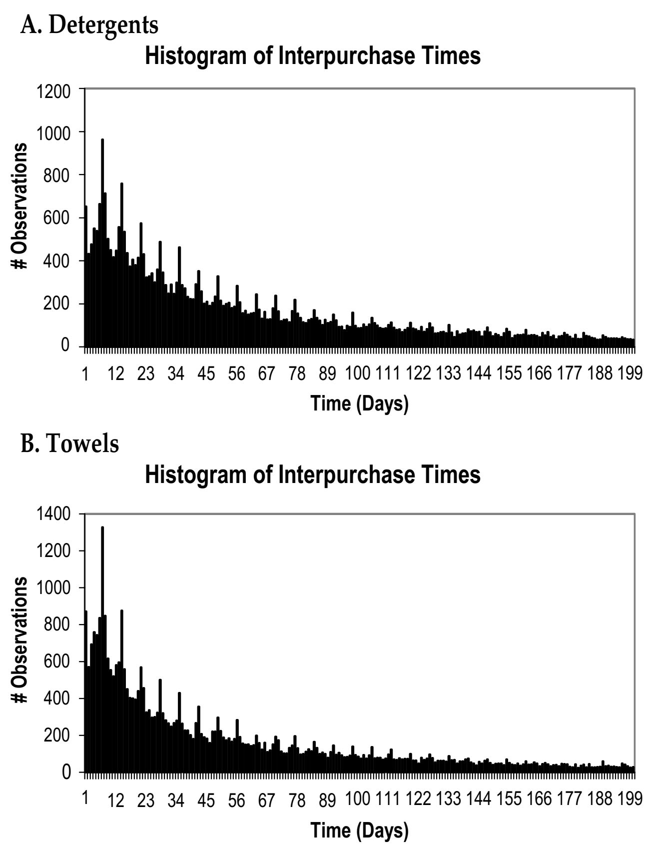


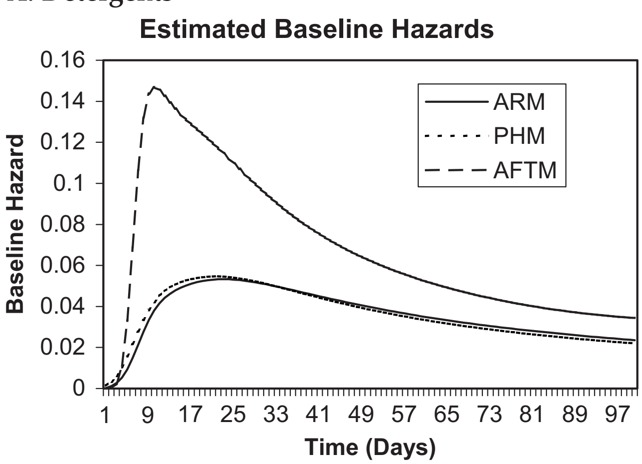
Table 2 Fit Results (Log-Likelihoods)

	ARM	PHM	AFTM
A. In-Sample			
Category			
Detergents	−7,891	−8,428	−9,628
Towels	−9,235	−9,791	−11,827
Tissue	−10,319	−10,934	−13,816
B. Out-of-Sample			
Parameter			
Detergents	−1,541	−1,636	−1,924
Towels	−1,982	−2,073	−2,454
Tissue	−1,890	−2,011	−2,499

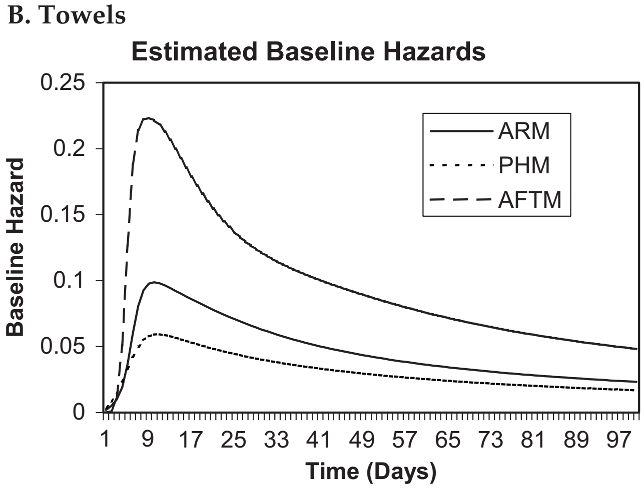
log-likelihoods based on the estimated parameters. The results of such a validation comparison are given in Table 2B. Again, in all three product categories, the proposed ARM is observed to validate the holdout data better than the benchmark models. This establishes that the proposed ARM is preferable to existing models in terms of fitting and predicting observed purchase-timing outcomes.

Figure 2 Baseline Hazards

A. Detergents



B. Towels



We plot the estimated log-logistic baseline hazard, i.e., $1 - S(t)/S(t-1)$, in Figure 2. The baseline hazard is observed to be nonmonotonic (i.e., inverted U-shaped) in both categories, peaking at about 10 days for towels under all three models. In detergents, however, the baseline hazard is observed to peak at about 20 days under the ARM and PHM, and at about 10 days under the AFTM. Overall, the shapes of the estimated baseline hazards are quite similar between model specifications, which suggests that one may not seriously misestimate the baseline hazard if one employed the less-predictive PHM or AFTM instead of the ARM. Our finding of a nonmonotonic baseline hazard is consistent with previous studies on purchase timing (for a review of previous studies, see Seetharaman and Chintagunta 2002).

Given in Table 3 are the parameter estimates based on the three models for laundry detergents and paper towels (because the results from toilet tissue are qualitatively very similar, they have been suppressed for readability). One would expect the estimated display and feature parameters to have positive signs; i.e., if the product category is on display or feature, the household's expected likelihood of purchasing the product would be higher. This is true not only under the proposed ARM, but also under the PHM and AFTM. One would expect the estimated price parameter to have a negative sign, i.e., the higher the price of the product, the lower the household's expected likelihood of purchasing the product. This is true of all heterogeneity supports for the ARM and AFTM, but true of only one support (for both product categories) for the PHM. This suggests that one may misestimate the directional effects of price if one employed the PHM instead of the ARM. These findings can be interpreted in the light of findings in O'Neill (1986) that the use of the PHM when the true model is ARM can cause asymptotic bias in the estimated parameters of the covariate function.

In order to understand the substantive effects of the estimated price parameter, we compute a *price elasticity* measure based on an assumption of a 20% price cut from the regular price. Because the price elasticity measure is time dependent, we investigate the temporal pattern of price elasticity between the PHM, ARM, and AFTM. The results, presented in Figure 3, are quite striking. The price elasticity follows a U-shaped temporal pattern under the ARM, reaching a minimum at 15 days and 10 days, respectively, for detergents and towels, and mildly increasing thereafter. In contrast, the price elasticity measure is time constant under the PHM. More importantly, the price elasticity measure is severely understated, and close to zero, under the PHM. The AFTM, despite its inferior fit compared to the PHM, does a better job at recovering the price elasticity measure in that

Table 3 Estimation Results—Parameter Estimates and Standard Errors

Parameter	ARM	PHM	AFTM*
A. Detergents			
γ	0.05 (0.01), 0.02 (0.00), 0.12 (0.01)	0.02 (0.00), 0.04 (0.00), 0.11 (0.00)	0.00 (0.00), 0.03 (0.00), 4.54 (0.08)
α	2.91 (0.10), 1.88 (0.08), 4.44 (0.10)	1.51 (0.07), 3.25 (0.07), 3.41 (0.10)	4.02 (0.08), 8.70 (0.11), 5.87 (0.12)
Price	−4.76 (0.18), −5.66 (0.23), −2.17 (0.27)	0.61 (0.23), −0.28 (0.11), 0.46 (0.14)	−0.10 (0.03), −2.14 (0.15), −0.30 (0.06)
Display	3.47 (0.18), 1.22 (0.29), 1.70 (0.30)	2.61 (0.17), 0.85 (0.12), 1.58 (0.14)	0.57 (0.06), 2.02 (0.10), 0.42 (0.04)
Feature	0.98 (0.17), 1.35 (0.19), 1.18 (0.28)	2.08 (0.15), 1.46 (0.12), 1.11 (0.17)	1.33 (0.10), 1.79 (0.13), 0.41 (0.03)
Inventory	−0.09 (0.01), −0.03 (0.01), −0.01 (0.00)	−0.02 (0.01), −0.12 (0.01), −0.01 (0.00)	−0.03 (0.01), −0.03 (0.01), −0.01 (0.00)
Prob.'s	0.46, 0.44, 0.10	0.49, 0.34, 0.17	0.60, 0.24, 0.16
LL	−7,891	−8,428	−9,628
B. Towels			
γ	0.15 (0.02), 0.01 (0.00), 0.06 (0.02)	0.14 (0.04), 0.02 (0.00), 0.04 (0.01)	0.04 (0.01), 0.11 (0.02), 0.70 (0.04)
α	5.80 (0.18), 1.33 (0.03), 2.30 (0.05)	3.49 (0.10), 1.39 (0.08), 1.74 (0.08)	6.54 (0.11), 4.22 (0.08), 4.11 (0.09)
Price	−3.09 (0.24), −7.97 (0.21), −5.90 (0.04)	0.86 (0.39), −0.51 (0.22), 0.70 (0.15)	−0.26 (0.11), −0.62 (0.12), −1.23 (0.12)
Display	1.95 (0.18), 0.77 (0.18), 2.70 (0.09)	1.08 (0.15), 1.05 (0.09), 2.08 (0.10)	0.33 (0.04), 0.71 (0.06), 1.22 (0.07)
Feature	0.30 (0.16), 2.72 (0.19), 2.46 (0.09)	0.47 (0.19), 3.24 (0.10), 2.20 (0.10)	0.21 (0.03), 0.82 (0.06), 1.79 (0.09)
Inventory	−0.06 (0.02), −0.08 (0.01), −0.04 (0.01)	−0.02 (0.01), −0.31 (0.01), −0.06 (0.01)	0.00 (0.00), −0.03 (0.01), −0.11 (0.01)
Prob.'s	0.16, 0.33, 0.51	0.18, 0.40, 0.43	0.18, 0.38, 0.44
LL	−9,235	−9,791	−11,827

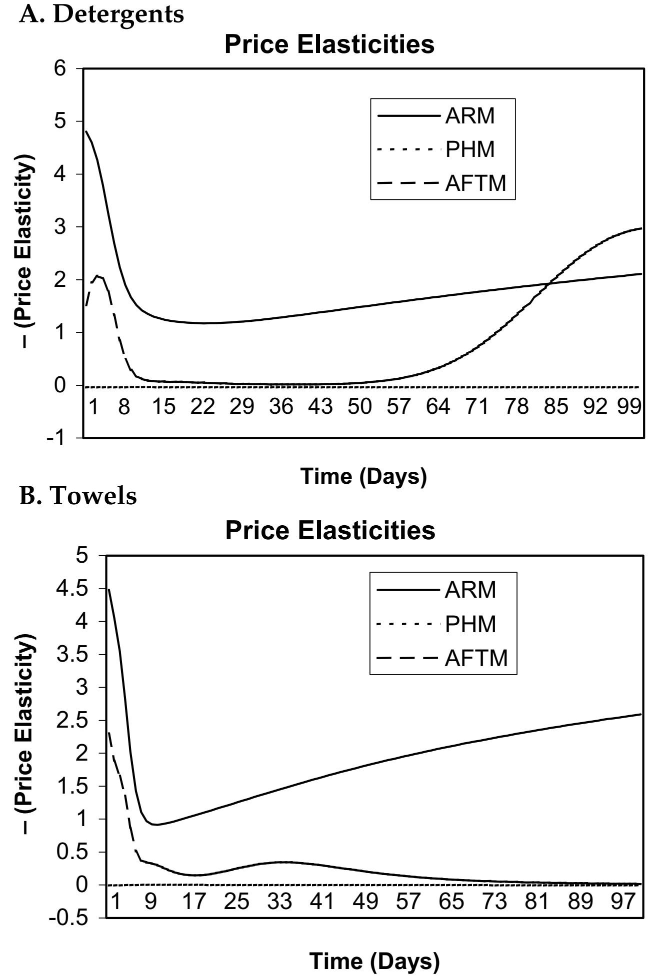
*The estimate of γ is reported at the average values of price, display, and feature in the product category.

it is both time varying (as under the ARM) and not as understated as under the PHM. Our explanation for the nonmonotonicity in the estimated price elasticity measure over time under the ARM (and AFTM) is the following: As time elapses since product purchase, the household runs out of product inventory at home, and it becomes less necessary for the product to offer a huge price cut in order to induce the household to purchase it. This leads to a decrease in the household's price elasticity over time. Simultaneously, as time elapses since product purchase, the product category becomes less "salient" in the consumer's mind and it takes a "shelf-talker" such as a price cut to get the consumer to remember and buy the product. These two countervailing forces on the household's price elasticity may be responsible for the observed nonmonotonicity in Figure 3, with the first force dominating in the first few weeks, and the second force dominating in later weeks. An alternative explanation for the estimated nonmonotonicity could be the effects of heterogeneous baseline hazards across households in the market, which makes aggregate price elasticities appear to be nonmonotonic even though the disaggregate price elasticities truly are not.[3] To investigate whether this aggregation explanation is at work, we look at the support-specific price elasticities themselves. These turn out to be nonmonotonic as well, which rules out the aggregation explanation.

In order to investigate the sensitivity of the estimated marketing mix parameters to the specification

Figure 3 Price Elasticities (Based on a 20% Price Cut)

A. Detergents



[3] The authors thank an anonymous reviewer for proposing this alternative explanation.

Table 4 Testing Alternative Specifications of Baseline Hazard for Detergents

Parameter	Exponential	Erlang-2	Weibull	Log-logistic	Expo-power
I. ARM					
γ	0.04 (0.00)	0.06 (0.00)	0.06 (0.00)	0.07 (0.00)	0.04 (0.00)
α	NA	NA	0.78 (0.05)	2.31 (0.10)	1.14 (0.20)
θ	NA	NA	NA	NA	−0.01 (0.00)
Price	−4.54 (0.09)	−4.27 (0.08)	−4.66 (0.09)	−4.72 (0.08)	−4.67 (0.06)
Display	1.81 (0.10)	1.65 (0.10)	1.87 (0.11)	1.86 (0.11)	1.83 (0.07)
Feature	0.87 (0.10)	0.88 (0.09)	0.86 (0.10)	0.89 (0.10)	0.91 (0.07)
Inventory	−0.03 (0.00)	−0.03 (0.00)	−0.03 (0.00)	−0.04 (0.00)	−0.04 (0.00)
LL	−9,185	−9,387	−9,119	−8,965	−9,021
II. PHM					
γ	0.10 (0.00)	0.25 (0.00)	0.14 (0.00)	0.07 (0.00)	0.09 (0.00)
α	NA	NA	0.84 (0.05)	2.23 (0.10)	1.06 (0.20)
θ	NA	NA	NA	NA	−0.01 (0.00)
Price	−0.71 (0.07)	−1.40 (0.06)	−0.61 (0.07)	0.11 (0.09)	−0.55 (0.06)
Display	1.48 (0.07)	1.33 (0.07)	1.42 (0.07)	1.65 (0.07)	1.41 (0.07)
Feature	1.29 (0.07)	1.14 (0.07)	1.35 (0.07)	1.65 (0.07)	1.40 (0.07)
Inventory	−0.02 (0.00)	−0.02 (0.00)	−0.02 (0.00)	−0.03 (0.00)	−0.02 (0.00)
LL	−9,344	−9,487	−9,268	−9,188	−9,153

of the baseline hazard, we re-estimate the proposed ARM and the PHM under four alternative parametric specifications of the baseline hazard, i.e., exponential, Erlang-2, Weibull, and expo-power. The results of this exercise for detergents are given in Table 4. We find that the estimated effects of marketing variables under the ARM are remarkably robust to alternative parametric specifications of the baseline hazard. Unlike the ARM, the estimated parameters of the covariate function in the PHM, however, are highly sensitive to the choice of the baseline hazard specification. To the extent that optimal pricing of the product depends on the estimated price elasticity, the tenuousness of the estimated price elasticity under the PHM would lead to poor pricing prescriptions for the retailer. This empirical finding is important to document given the widespread use of the PHM on purchase-timing data.

4. Conclusions

We adopt the Additive Risk Model (ARM), first proposed by Aalen (1980), to analyze purchase-timing data. We compare the empirical performance of the proposed ARM with that of the Proportional Hazard Model (PHM) and the Accelerated Failure Time Model (AFTM), using scanner panel data on laundry detergents, paper towels, and toilet tissue. Our main findings are as follows: First, the proposed ARM fits and validates observed purchase outcomes better than the PHM and AFTM; second, the shape of the estimated baseline hazard is almost the same under the ARM, PHM, and AFTM; third, the estimated price elasticity under the ARM (and under the AFTM) is time varying, while the price elasticity under the PHM is constant over time and severely understated in magnitude (i.e., close to zero); fourth, the estimated price parameter of the ARM is robust, while the estimated price parameter of the PHM is highly sensitive to alternative parametric specifications of the baseline hazard.

5. Issues for Future Research

While we have demonstrated the superior empirical performance of the ARM compared to the widely used PHM (as well as the relatively underused AFTM) in the context of interpurchase times, we encourage researchers working on other types of event times—such as time to employment, time to job turnover, etc.—to also estimate and investigate the empirical performance of the ARM. There are a number of important areas for future research:[4]

(1) It will be useful to investigate alternative functional forms for the covariate function within the ARM.

(2) It will be important to study whether a mixture of proportional, additive, and accelerated formulations of covariate effects is warranted in datasets that include several covariates.

(3) Investigating the analytical differences between the ARM and PHM is important to obtain a deeper theoretical understanding of the two model formulations. This would require careful analysis of the mathematical and statistical properties of the two models.

(4) It is discomforting that the best-fitting PHM specification, i.e., the log-logistic, recovers incorrectly signed estimates of the price parameter for all three categories studied in this paper (even though the

[4] The authors thank the area editor and a reviewer for alerting us to these issues.

other baseline hazard specifications do not). While this clearly delineates the robustness of the price parameter estimates under the ARM to alternative baseline hazard specifications, it still begs the question of why estimates of the PHM are so tenuous. Investigating the statistical properties of the log-logistic PHM, and possibly the PHM in general, using numerical simulations, will throw light on this issue.

Acknowledgments

The authors thank Andrei Strijnev for setting up the datasets used in this study.

References

Aalen, O. O. 1980. A model for nonparametric regression analysis of counting processes. *Mathematical Statistics and Probability Theory*. *Lecture Notes in Statistics*, Vol. 2. Springer-Verlag, New York, 1–25.

Breslow, N. E. 1986. Cohort analysis in epidemiology. A. C. Atkinson, S. C. Fienberg, eds. *A Celebration of Statistics: The ISI Centenary Volume*. Springer-Verlag, New York, 109–143.

Breslow, N. E., N. E. Day. 1980. *Statistical Models in Cancer Research*. IARC, Lyon, France.

Buckley, J. D. 1984. Additive and multiplicative models for survival rates. *Biometrics* **40** 51–62.

Chatfield, C. K., G. J. Goodhardt. 1973. A consumer purchasing model with Erlang interpurchase times. *J. Amer. Statist. Association* **63**(4) 828–835.

Chib, S., P. B. Seetharaman, A. Strijnev. 2002. Analysis of multi-category purchase incidence decisions using IRI market basket data. *Advances in Econometrics*, Vol. 16. Elsevier Science Publishers, Oxford, U.K., 57–92.

Chintagunta, P. K. 1998. Inertia and variety seeking in a model of brand purchase timing. *Marketing Sci.* **17**(3) 253–270.

Cox, D. R. 1972. Regression models and life tables. *J. Roy. Statist. Soc. B* **34** 187–220.

Greenwood, P. E., W. Wefelmeyer. 1990. Efficiency of estimators for partially specified filtered models. *Stochastic Process Appl.* **35**(1) 43–53.

Greenwood, P. E., W. Wefelmeyer. 1991. Efficient estimating equations for nonparametric filtered models. N. U. Prabhu, I. V. Basawa, eds. *Statistical Inference in Stochastic Processes*, Vol. 1. Marcel Dekker, New York, 107–141.

Heckman, J., B. Singer. 1984. A method for minimizing the impact of distributional assumptions in econometric models for duration data. *Econometrica* **52**(2) 271–318.

Helsen, K., D. C. Schmittlein. 1993. Analyzing duration times in marketing: Evidence for the effectiveness of hazard rate models. *Marketing Sci.* **11**(4) 395–414.

Henderson, R., A. Milner. 1991. Aalen plots under proportional hazards. *Appl. Statist.* **40** 401–410.

Henderson, R., P. Oman. 1993. Influence in linear hazard models. *Scandinavian J. Statist.* **20** 195–212.

Huffer, F. W., I. W. McKeague. 1991. Weighted least squares estimation for Aalen's additive risk model. *J. Amer. Statist. Association* **86**(413) 114–129.

Jain, D. C., N. J. Vilcassim. 1991. Investigating household purchase timing decisions: A conditional hazard function approach. *Marketing Sci.* **10**(1) 1–23.

Kahn, B. E., D. C. Schmittlein. 1989. Shopping trip behavior: An empirical investigation. *Marketing Lett.* **1**(1) 55–69.

Kalbfleisch, J. D., R. L. Prentice. 1980. *The Statistical Analysis of Failure Time Data*. Wiley, New York.

Lancaster, T. 1990. *The Econometric Analysis of Transition Data*. Cambridge University Press, Cambridge, U.K.

Lin, D. Y., Z. Ying. 1994. Semiparametric analysis of the additive risk model. *Biometrika* **81**(1) 61–71.

McKeague, I. W., P. D. Sasieni. 1994. A partly parametric additive risk model. *Biometrika* **81**(3) 501–514.

O'Neill, T. J. 1986. Inconsistency of the misspecified proportional hazards model. *Statist. Probab. Lett.* **4**(5) 219–222.

Pierce, D. A., D. L. Preston. 1984. Hazard function modeling for dose-response analysis of cancer incidence in A-bomb survivor data. R. L. Prentice, D. J. Thompson, eds. *Atomic Bomb Survivor Data: Utilization and Analysis*. SIAM, Philadelphia, PA, 51–66.

Pocock, S. J., S. M. Gore, G. R. Kerr. 1982. Long-term survival analysis: The curability of breast cancer. *Statist. Medicine* **1** 93–104.

Prentice, R. L., J. D. Kalbfleisch. 1979. Hazard rate models with covariates. *Biometrics* **35** 25–39.

Saha, A., L. Hilton. 1997. Expo-power: A flexible hazard function for duration data models. *Econom. Lett.* **54** 227–233.

Sasieni, P. D. 1992. Information bounds for the additive and multiplicative intensity models. J. P. Klein, P. K. Goel, eds. *Survival Analysis: State of the Art*. Kluwer, Dordrecht, The Netherlands, 249–265.

Schmittlein, D. C., D. G. Morrison. 1983. Prediction of future random events with the condensed negative binomial distribution. *J. Amer. Statist. Association* **78** 449–456.

Seetharaman, P. B., P. K. Chintagunta. 2003. The proportional hazard model for purchase timing: A comparison of alternative specifications. *J. Bus. Econom. Statist.* **21**(3) 1–15.

Thomas, D. C. 1986. Use of auxiliary information in fitting non-proportional hazards models. S. H. Moolgavkar, R. L. Prentice, eds. *Modern Statistical Methods in Chronic Disease Epidemiology*. Wiley, New York, 197–210.

Wedel, M., W. A. Kamakura, W. S. Desarbo, F. T. Hofstede. 1995. Implications for asymmetry, nonproportionality and heterogeneity in brand switching from piece-wise exponential mixture hazard models. *J. Marketing Res.* **32**(4) 457–462.

Wheat, R. D., D. G. Morrison. 1990. Assessing purchase timing models: Whether or not is preferable to when. *Marketing Sci.* **9**(2) 162–170.