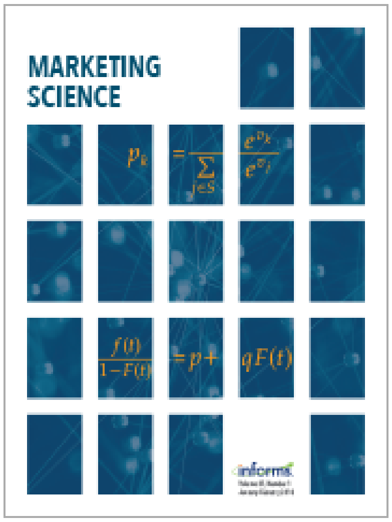

Marketing Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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informs
http://pubsonline.informs.org/journal/mksc

MARKETING SCIENCE
Vol. 40, No. 2, March–April 2021, pp. 241–260
ISSN 0732-2399 (print), ISSN 1526-548X (online)

Cryptocurrency Adoption with Speculative Price Bubbles

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Received: June 1, 2019
Revised: December 24, 2019; May 4, 2020
Accepted: May 28, 2020
Published Online in Articles in Advance: October 8, 2020

https://doi.org/10.1287/mksc.2020.1247

Abstract. We study product adoption in the context of a cryptocurrency market. Cryptocurrencies are subject to network effects and speculative investments, which are not part of standard models of product diffusion. To explore this unique setting, we marry models of stochastic bubbles and the standard model of product diffusion. A rational bubble is raised by speculative investors seeking short-term gains. We find that a bubble accelerates the adoption, which can help explain the fast diffusion of bitcoin. There are reinforcing interactions between the speculative investors and regular users of currency, which can make it easier to form a bubble (compared with a setting without regular users). Our findings suggest how bubbles may help to market products. We also provide conditions under which bubbles may unravel.

History: Yuxin Chen served as the senior editor and Zsolt Katona served as associate editor for this article.

Keywords: product adoption • diffusion • cryptocurrency • speculative bubbles • currency marketing

1. Introduction

Currencies have been utilized for millennia as mediums of exchange in economic transactions. Governments have historically marketed new currencies to help facilitate trade in their economies. The most recent major currency launched by a government institution was the euro in 1999. Recently, even nongovernment agents are able to market new currencies in digital formats. Since bitcoin emerged in 2009, there have been more than 1,600 cryptocurrencies introduced.[1] This research studies product diffusion in the context of a new cryptocurrency.

An important feature of a cryptocurrency is that it is particularly susceptible to speculative price bubbles, caused by investors who exploit currency exchange for speculative gains. Speculative beliefs of investors, very often detached from the fundamental value of a currency, can drastically affect the price that noninvestors, the users, pay to utilize the currency as a medium of exchange. Unlike a traditional currency, for which the government can manipulate the supply to combat bubbles, the supply of a cryptocurrency is typically set to follow a predetermined path, thereby making it more prone to price bubbles. A second important feature of a currency is its network externality. A user's benefit from adoption depends on the number of other users with whom to exchange. The symbiotic relationship between this network effect and speculative incentives implies adoption dynamics that are different than other new products (e.g. Bass 1969). In light of this relationship, we ask, *how does the presence of investors, who are interested in speculative returns, affect the diffusion of a new currency?* And *how does the adoption by users affect the creation of speculative price bubbles?*

These questions are connected to the recent and heated attention around bitcoin, especially in the years 2013 and 2017, which witnessed surges in bitcoin price. It is tempting to attribute the price surges to overoptimistic expectation of growth in bitcoin usage. But we entertain the reverse mechanism by asking whether the bubbles may have contributed to growth of bitcoin usage. Indeed, absent a formalized model, it is hard to answer this question. On one hand, speculative investors holding bitcoins may have weakened the transactional role of bitcoins, reducing users' benefit of adoption. On the other hand, a bubble could boost adoption by virtue of the network effects spurred by investors.

Our formalization starts by considering the demand for currency speculation by investors only. This benchmark illustrates conditions for which bubbles can sustain without users. Investors decide whether to participate in the currency market based on expectations about tomorrow's price of the currency. Their expectations are rational, meaning that they are consistent with the realization of tomorrow's price. Investors also rationally expect crashes—events where the price drops to zero. In other words, our analysis focuses on fully rational investors who seek out short-term gains. We refrain from adding behavioral factors (such as emotional responses or fears) to examine the most demanding environment for bubbles to form.

Without behavioral factors, the key requirement in the formation of bubbles is investor confidence (Weil 1987). A product can be valued above its fundamental value only if there is a minimum degree of confidence among agents that the currency will be no less valuable in the future, which, if fails to happen, implies a crash.

Next, we introduce currency adoption by users who demand currency as a medium of exchange and have no desire for speculative returns. The simultaneous presence of users and investors implies their interdependence on the equilibrium price path of the currency. Specifically, users must take into account the additional network externality implied by investors, who accept and provide liquidity for the currency. At the same time, investors take into account users' demand for currency as a medium of exchange as it affects the price tomorrow and subsequently the speculative return. Our model formally accounts for this interdependent demand and allows us to discern between the currency's *speculative value* and its *fundamental value*. The currency's fundamental value is defined as its price when no speculative investors participate in the market. Investors can drive up the price beyond this value, generating speculative value, in the form of a *price bubble*. A bubble may burst—dropping the price to the fundamental value.

We establish two results. First, the adoption of a currency is accelerated by the bubble generated by speculative investors. The literature on product diffusion has focused on products without financial or investment value (Bass 1969). Our model demonstrates that the price bubbles induce faster adoption rates than without investors. This result has implications for any government or nongovernment agent whose objective is to induce diffusion of a currency. Specifically, it suggests that opening up the currency to investors at its launch can help to achieve adoption goals. Indeed, it is worthwhile to note that the European Monetary Union initially launched the euro in 1999 only to investors, before rolling it out as a physical currency in 2002 (see Wikipedia 2020). Our theory provides a rationale for such a launch strategy.

Our second result states that the presence of users relaxes the conditions for a bubble to form and grow. Specifically, we show that the level of investor confidence needed to sustain a bubble decreases when there is more fundamental demand from users. Intuitively, the relaxation comes from an increased expectation in the currency's future fundamental value due to user adoption. What is interesting, however, is that even in the case where investing in the fundamental value provides zero expected return,[2] we show that such relaxation still holds. This result is due precisely to the reinforcing interplay between investors and users, which raises an investor's expected return beyond the return from investing in fundamental value.

These effects may help us understand the path of bitcoin prices, which is shown in Figure 1. Consider the steep ascent, around November 2013, when the Shared Coin service was offered publicly for free. This service offered anonymity in transactions by mixing cryptocurrency funds to obscure the trail back to a fund's original source (Athey et al. 2016). Arguably, this new service increased the potential user base for bitcoin. Through the lens of our model, the price volatility starting in late 2013 can be explained in a completely rational way by the relaxed confidence requirement to trigger a bubble. In other words, as Shared Coin attracted more users, the necessary confidence for bubbles had been relaxed, permitting easier conditions for speculative participation. By the end of 2015, the price stabilized around $250, which is greater than the price before Shared Coin's introduction. This, according to our model, would be reflective of the increased fundamental value of bitcoin as a medium of exchange.[3]

The above story is not without limits, however. A rational bubble can unravel, which would negate the participation of investors and, consequently, any user–investor interaction. Specifically, as the price in a bubble grows, investors' confidence (i.e., expectation of tomorrow's price) must also grow for the expected return to stay nonnegative. A bubble becomes unsustainable at some point if it requires a level of confidence so high that is infeasible given the possible demand and supply for currency tomorrow. As early-period investors rationally anticipate this lack of sustainability, they refuse to participate and the bubble unravels. We find that a key moderator for the existence of bubbles is the financial role of investors relative to their role in adoption.

Figure 1. (Color online) Bitcoin Price from April 2013 to May 2015

When investors have a large financial capacity to drive up today's price relative to the network externality they exert on adoption, bubbles are easier to unravel. In such cases, even those bubbles that do not unravel turn out to be likely to crash early on, before they grow to significant sizes. In fact, with model primitives realistic to the bitcoin's history, a bubble of any significance is almost impossible in a setting without users (or where investors exert little network externality on users).

Our work combines the product diffusion literature in marketing with the currency formation and asset bubble literature in macrofinance. In marketing, starting with Bass (1969), the product diffusion literature examines the path of consumer adoption following the introduction of an innovative product. Subsequent work augments the classic work of Bass (1969) to include imitators and influencers (Steffens and Murthy 1992, Van den Bulte and Joshi 2007). Much of that prior work implicitly assumes network externalities where the probability that a user adopts depends on the latest adopter population. Network externalities are a key property of currency and are an essential aspect of our main effects. There has also been great interest in marketing on the empirical evaluation of diffusion models and on network externalities (e.g., Bass et al. 1994, Bronnenberg and Mela 2004, Garber et al. 2004, Shriver 2015). However, neither the theory nor the empirical literature has explored the role of price bubbles.

Within the finance literature, there are several groundlaying theory works on asset bubbles. Blanchard (1979) first pointed out that bubbles do not necessarily indicate irrational behaviors—speculative bubbles followed by market crashes can be consistent with the assumption of rational expectations. Tirole (1982, 1985) showed that under rational expectation, short-term investors (instead of infinitely lived agents) are necessary for sustaining speculative detachment from fundamental values. As a result, our model features investors seeking short-term gains. Tirole (1982, 1985) considered only deterministic models. Weil (1987) extended the analysis to stochastic bubbles with the possibility of crashes. In doing so, he introduced the concept of *confidence*: a bubble can exist only if there is enough confidence that it will persist (not crash). Following his work, we focus on evaluating how the confidence level required for a bubble to exist is affected by the presence of user adoption.

Since then, there have been more studies exploring factors behind the formation of bubbles. While largely maintaining the assumption of rational expectations, these studies focus on various deviations from perfect markets in the form of market frictions. These include asymmetric information (Allen and Gorton 1993), the inability of arbitrageurs to synchronize selling strategies (Abreu and Brunnermeier 2003), risk aversion (Branch and Evans 2011), and portfolio constraints (Hugonnier 2012). Our objective is not to provide another explanation of bubbles, per se, but rather to understand the interplay between speculative incentives and product adoption, or, put more broadly, the mutual impact between investors and users.

2. The Market for Currency

The demand for currency in our model comes from two sources: *investors* and *users*. Investors demand currency on the expectation that they can sell it in the future for a speculative gain. Users demand currency to conduct transactions with others. Users can be interpreted as merchants and customers exchanging goods and services. We start by examining the demand for currency by investors only. By doing so, we illustrate how a currency bubble can survive from purely speculative demand. We then incorporate user demand for the full model to capture how these two sources of currency demand interact. Later, in Section 3, we derive equilibrium properties of the full model. We compare the bubbly and nonbubbly equilibria in the full model to analyze the effect of a bubble on adoption. We compare the bubbly equilibria in the full model and in the investor-only model to analyze the effect of user adoption on bubble formation.

Before detailing the model, it is useful to point out that we separate the roles of investors and users as a way to isolate the different mechanisms (speculative investment and transactional usage) and see how they interact. In reality, an agent may assume both roles. Our model does not forbid such a view, as long as the agent keeps separate accounts for transactional and investment purposes. Modeling the portfolio decision (how much to allocate across investments and transactions), however, is beyond the scope of this paper.

2.1. Investor Only

There are T discrete periods. In every period $t = 1, \ldots, T$, an investor is endowed with y dollars. She may invest in either the coins or an outside option. For simplicity, assume that the outside option provides an interest rate $r = 0$.[4] Let p_t denote the price of a coin in period t. A risk-neutral investor[5] decides whether to buy coins in period t based on the belief about the next period's price, $\mathbb{E}(p_{t+1})$. We parameterize the belief in the following way:

$$p_{t+1} = \begin{cases} L_t \omega_t & \text{with probability } \omega_t, \\ 0 & \text{with probability } 1 - \omega_t. \end{cases} \tag{1}$$

In the investor's belief, ω_t represents the probability that a positive price will sustain in $t + 1$, and $L_t > 0$ is the maximum possible price level in period $t + 1$. Both ω_t and L_t are determined by rational expectation, which we will make clear below when deriving the equilibrium. This specification of investor belief is motivated by a discretization of the more flexible beta distribution: $p_{t+1} \sim L_t \cdot \mathrm{Beta}(\alpha_t, 1/\alpha_t)$, where a larger α_t puts more probability on the larger values in $[0, L_t]$.[6]

The single parameter ω_t captures the level of *investor confidence* in the coin's price next period. A larger value of ω_t simultaneously implies a stronger belief that a crash will not happen tomorrow and a higher expected price tomorrow: $\mathbb{E}(p_{t+1}) = L_t\omega_t^2$. The notion of investor confidence is borrowed from Weil (1987), who shows how currency bubbles can emerge in an overlapping generations model similar to ours.[7]

Specifically, an investor in period t is willing to buy coins at price p_t only if the expected return satisfies $\frac{\mathbb{E}(p_{t+1})}{p_t} \geq 1$, or equivalently, $\frac{L_t}{p_t}{\omega_t}^2 \geq 1$. This describes the behavior of investors, who are price takers. Given a price p_t, if confidence ω_t is such that the expected return is larger than 1, then each investor's demand for coins is equal to $y/p_t > 0$. If the expected return is smaller than 1, the demand is zero.

We suppose that there is a supply of coins A_t for period t. In what follows, we assume that the supply is increasing over time (i.e., no coins are destroyed). Let $N > 0$ be the total mass of investors. Let $n_t < N$ be the mass of investors that buy coins in period t. The remaining $N - n_t$ do not buy. Then, for the market to clear, we require the condition

$$\frac{yn_t}{p_t} = A_t.$$

The highest possible price level for period t, L_{t-1}, happens when every investor demands coins, so

$$\frac{yN}{L_{t-1}} = A_t,$$

or simply $L_{t-1} = yN/A_t$. For any $p_t \in (0, L_{t-1})$, we must have some investors buying coins while some others do not buy (i.e., $0 < n_t < N$).[8] This requires investors to be indifferent between buying the coins and buying the outside good; the expected return must be 1:

$$\frac{L_t}{p_t}\omega_t^2 = 1. \tag{2}$$

This equality bears the same spirit as the arbitrage-free condition that underlies the efficient market hypothesis in finance.[9]

Throughout this paper, we employ the notion of *rational expectations* to compare outcomes across various settings. This requires investors to be correct about tomorrow's price in equilibrium. In other words, prices are realized exactly according to investors' beliefs in equilibrium. So, as long as the bubble is sustained (i.e., a crash has not happened), we have

$$p_{t+1} = L_t\omega_t,$$

which, together with equality in (2), implies the following recursive relationship:

$$\omega_{t+1} = \sqrt{\frac{A_{t+2}}{A_{t+1}}\omega_t}. \tag{3}$$

This equation characterizes the evolution of investor's confidence in equilibrium. One obvious confidence path is $\omega_t = 0$ for all t, which implies $p_t = 0$ for all t. This path actually corresponds to the nonbubbly equilibrium. In what follows, we shall focus on interior equilibria in which $\omega_t \in (0, 1)$, and therefore a crash remains a meaningful, but uncertain, possibility. What we have characterized so far, a *bubbly equilibrium*, signifies a realized event in which investors' demand sustains a positive price from period to period. The price path is supported by demand from investors, who rationally expect such a price path as well as the risk of crash. We formally define a bubbly equilibrium as follows.

Definition 1. Given y, N, and $\{A_t\}_{t=1}^T$, a *bubbly equilibrium* is a sequence $\{\omega_t, n_t, p_t\}_{t=1}^T$ with $\omega_t \in (0, 1)$ that satisfies the following conditions for $t = 1, \ldots, T$: (i) the market clears, $yn_t/p_t = A_t$; (ii) the expected return is equal to 1, $L_t\omega_t^2/p_t = 1$; and (iii) price p_{t+1} is rationally expected as $\omega_t L_t$.

It is important to note that a bubbly equilibrium is defined conditional on the bubble being sustained. When we say that $\{p_t\}_{t=1}^T$ is part of a bubbly equilibrium, we mean that these will be the realized prices as long as the bubble does not crash. It is understood that in every period t, there is probability $1 - \omega_{t-1}$ that the price ends up being 0 (the bubble bursts). When such an event happens, the prices and investor sizes will be zero thereafter.

Equation (3) says that a bubbly equilibrium, if it exists, is unique for any given ω_1 under our model specifications.[10] To ensure the existence of a bubbly equilibrium, we must specify two boundary conditions. First, for a finite T, we must assume that the investors in period $T + 1$ are guaranteed an "exit interest rate" equal to 1. This assumption effectively eliminates the "end-period effect" because the investors in the last period need not form a belief on tomorrow's price. We allow $T = \infty$ to capture the situation of no such exit interest rate. Second, we must keep the value of ω_t meaningful. If A_t is strictly increasing over time and we mechanically follow Equation (3), then ω_t is strictly increasing over time too, and may become larger than 1 in some period t, which would imply the unraveling of the candidate bubble backward from t. Given any T, we can ensure that $\omega_1, \ldots, \omega_T < 1$ by starting with a sufficiently small ω_1. In other words, there is an upper bound, $\Omega(T)$, such that the bubbly equilibrium exists for T iff $\omega_1 \le \Omega(T)$. It is not difficult to see that $\Omega(T)$ is decreasing in T. The following lemma characterizes $\Omega(\infty)$ under a specific supply path that we focus on later.

Lemma 1. *In the investor-only setting with $A_t = 1 - \beta^t$, $\beta \in (0, 1)$, a bubbly equilibrium exists for $T = \infty$ iff*

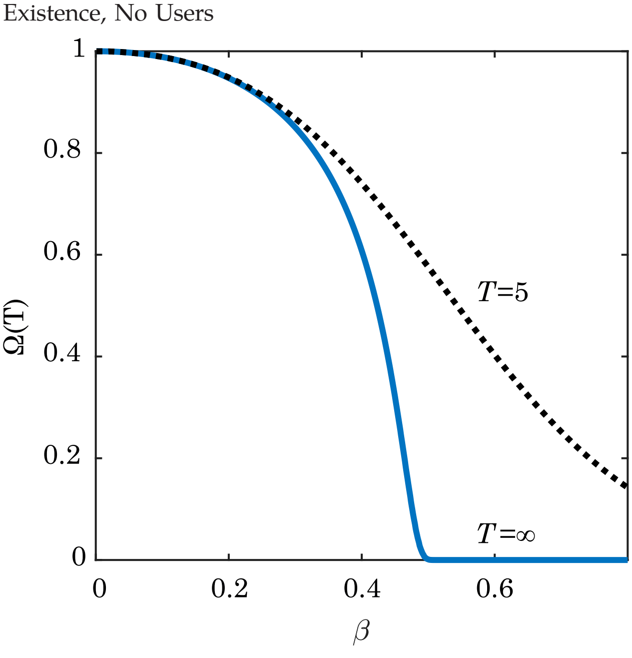
$$\omega_1 \le \left(1 - \beta^2\right) \prod_{k=3}^{\infty} \left(1 - \beta^k\right)^{2^{k-3}} \equiv \Omega(\infty).$$

For any $T < \infty$, the above inequality becomes a sufficient condition for a bubbly equilibrium to exist.

The intuition behind this upper bound is that a high ω_1 leads to a high price being rationally expected in the second period, which makes coins unattractive to the second-period investors. The same logic applies to later periods. If ω_1 is so high that the investors in a later period are unwilling to buy coins even with full confidence, then the bubble unravels.

In Figure 2, we numerically compute the upper bound $\Omega(T)$ for $T = \infty$ (and $T = 5$ for comparison). For reasonably large values of β, such as $\beta > 0.5$, this upper bound is virtually zero. In such cases, even if a bubbly equilibrium may exist, it is most likely to crash in the early periods. In fact, the proof of the lemma (in Appendix B) shows that $\Omega(\infty)$ is positive for $\beta < 0.5$ and zero for $\beta \ge 0.5$. As we will discuss later, the historical supply of bitcoins points to a relatively large β. So in the case of bitcoin, our model suggests that a bubble of any significance is unlikely without users present.

Figure 2. (Color online) Upper Bounds of ω_1 for Bubble Existence, No Users



The exposition so far has established that a speculative bubble can be raised on an otherwise valueless item, where both the prices and the crash of the bubble are essentially self-fulfilled beliefs of investors—completely detached from the fundamental value of the item (zero). In a bubble, both investor confidence and the coin price are increasing over time. Thus, we have the following mirror intuition: higher and higher confidence is required to sustain investors' demand in the face of increasing price, while simultaneously, the increasing price fulfills the higher and higher beliefs over time. This is the bubble environment we wish to also capture later in the extended model with user adoption.

In general, a level of confidence, as introduced by Weil (1987), is a requirement on investors. A level of confidence close to 1 requires all investors in the market to believe that a crash will not happen. In our model, as the growing price "pressures" the confidence to grow, the bubble puts a higher and higher requirement on investors. So far, we have assumed that it is possible for ω_t to approach 1 as long as it does not go beyond 1; however, we may also think of an a priori upper bound $\widetilde{\omega} < 1$ on confidence. In words, $\widetilde{\omega}$ measures the maximum level of confidence that investors can psychologically hold up (Weil 1987, Shiller 2000). Obviously, this $\widetilde{\omega}$ would put a more stringent condition on ω_1 than that in Lemma 1.

Finally, before we introduce users, it is useful to point out that we can normalize $N = 1$. A mass of $N > 1$ (or $N < 1$) investors can always be thought of as a single unit of investors with each investor holding a smaller (or larger, respectively) amount of dollars.

2.2. User Demand

We consider a mass of $M = 1$ potential users who desire coins purely as a medium of exchange. For exposition, we first consider the case where no investors are present. We employ a Bass-like adoption model. In each period, a user decides whether to carry the coins or alternative currencies (such as dollars). The benefit of carrying coins is the ability to conduct transactions with them, which is increasing in the number of other users. We also account for costs associated with using a cryptocurrency. For example, we can think of the costs as the opportunity cost of not using dollars and the time and effort to maintain the required software, as well as technological and privacy risks.

Let m_t denote the mass of users who adopt the coins in a given period t. Then a user i has the following utility for adoption in period $t+1$:

$$U_{i,t+1} = V(m_t) - c_i,$$

where $c_i \geq 0$ is the user's cost of using the coins in the period, and $V(m_t)$ is the benefit from using the coins in the period. For simplicity, we assume the specification $V(m_t) = m_t$ to capture the network effects inherent in currency markets. As in the Bass model, users' decisions in the current period are informed by the state of adoption in the previous period.[11]

We let the cost c_i be distributed among the $M = 1$ users according to a cumulative distribution function $G(\cdot)$. To obtain analytical solutions, we will make a linear specification:

$$G(c) = \begin{cases} \delta + \lambda c & \text{if } 0 \leq c < \frac{1-\delta}{\lambda}, \\ 1 & \text{otherwise,} \end{cases} \tag{4}$$

where δ is the set of consumers with zero adoption costs, and $\lambda > 0$ captures the density toward low adoption costs.[12] A higher value of λ indicates lower adoption costs among users. For the first period, $m_1 = \delta$ naturally represents the size of the initial installed base of users. For later periods, a user i adopts the currency in $t+1$ if and only if $c_i < V(m_t)$. Assuming $\lambda + \delta < 1$, we have the adoption dynamics satisfy[13]

$$m_{t+1} = \delta + \lambda m_t. \tag{5}$$

The adoption path $\{m_t\}_{t=1}^{\infty}$ will converge to some value $m^* \in (0, M)$. This convergent point is defined by the identity $m^* = \delta + \lambda m^*$, or, equivalently,

$$m^* = \frac{\delta}{1-\lambda}. \tag{6}$$

We refer to m^* as the *natural adoption ceiling*, as it refers to the highest level of adoption that evolves without the influence of investors.

Again, let $p_t > 0$ be the price of a coin in period t, and let A_t be the total supply of coins in existence in period t. If each user demands $x > 0$ dollars worth of coins to do exchanges, then the total demand for coins is given by $m_t x / p_t$. In order to match supply and demand for coins, we impose the market-clearing condition for users in the absence of investors:

$$\frac{x \cdot m_t}{p_t} = A_t.$$

Equation (5) and the market-clearing condition above imply a sequence of prices and adoption levels:

$$m_t = \frac{1-\lambda^t}{1-\lambda} \cdot \delta, \tag{7}$$

$$p_t = \frac{x}{A_t}\left(\frac{1-\lambda^t}{1-\lambda}\right) \cdot \delta. \tag{8}$$

We see A_t enters the price equation above. The supply of a cryptocurrency often closely follows a predefined rule. First, the cryptocurrency creator typically sets a supply ceiling, which we will denote by A_∞. Second, the cryptocurrency system controls its supply over time by constantly adjusting the difficulty of mining as well as regularly halving the output of mining. Consequently, we can model the cryptocurrency supply by $A_t = A_\infty(1 - \beta^t)$ for some $\beta \in (0, 1)$. In fact, the historical supply of bitcoin has followed this path very closely,[14] though the price of bitcoin has experienced large surges and falls. We note the size of A_∞ only has a nominal effect; a larger A_∞ simply inflates the coin price in every period. Hence, we can normalize $A_\infty = 1$, and Equation (8) becomes

$$p_t = \frac{x}{1-\beta^t}\left(\frac{1-\lambda^t}{1-\lambda}\right) \cdot \delta.$$

When $\beta = \lambda$, the above price for coins in the absence of speculative investors is constant over time:

$$p_t = \frac{x\delta}{1-\lambda}. \tag{9}$$

In what follows, we will focus on this case where the supply of coins is controlled so that the price resulting purely from user demand, in the absence of investors, is constant over time. The main reason to focus on $\beta = \lambda$ is to compare the settings with and without users on an equal footing—recall that in the setting without users (Section 2.1), the price absent of investors buying is also constant over time (at zero). It is instructive to consider a single, marginal investor. Investing in a constant p_t provides a return equal to the outside option, so a single investor who has insufficient assets to swing the price all by herself will find no strict incentives to invest. However, if $\beta < \lambda$, the p_t in Equation (8) is increasing over time, which gives the investor a strict incentive to invest. Consequently, this case bears an obvious advantage in terms of bubble formation over the setting without users. Similarly, the $\beta > \lambda$ case bears an obvious disadvantage in terms of bubble formation.

Hence, the condition $\beta = \lambda$ sets up a stage for us to cleanly identify the impact that users bring to the formation of a bubbly equilibrium, which we will focus on next. As in the setting without users, even though a single marginal investor finds no incentives to invest, collectively, investors can still "coordinate" to raise a bubble. Different from the setting without users, however, investors must account for how their actions will affect user adoption as well as how user adoption will affect coin prices.

2.3. Equilibrium with Users and Investors

We now bring together the two isolated settings described above (Section 2.1 and Section 2.2). We use a two-sided user market to conceptualize the interaction between users and investors.

There are two sides of users: shoppers and vendors, as illustrated in Figure 3. A shopper's benefit of adoption increases with the number of vendors and vice versa. At the same time, they both care about the number of investors who provide liquidity for coin exchange. We formalize a model for this two-sided market in Appendix A.3. The working of the model can be summarized as follows. Within a period, shoppers first decide whether to carry coins (instead of an alternative currency, such as dollars). A key factor in a shopper's decision is how many vendors accept coins. If a shopper decides to carry coins, he first exchanges his endowment $x > 0$ into coins via an investor (e.g., bank). The ease of exchange (i.e., liquidity) depends on the number of coin-accepting investors. Next, the shopper goes to coin-accepting vendors for shopping. After transacting with shoppers, the vendors either deposit the received coins or exchange them into alternative currencies, such as dollars.

We show, in the appendix, that the above adoption dynamics can be reduced to a formulation similar to that in Section 2.2. Specifically, let m_t be the mass of users (which tracks both the shoppers and vendors) and n_t be the mass of investors in period t. Without loss of generality, we can let $N = 1$, $M = 1$, and $x = 1$.[15] A user's benefit now depends on both the number of users and the number of investors, but potentially at different rates:[16]

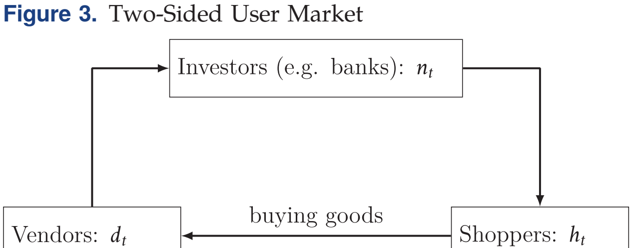
$$V(m_t + n_t) = m_t + \phi n_t,$$

where ϕ captures the differential rate of network effects that investors exert on adoption. In period $t+1$, a user with adoption cost c_i adopts the currency if $V(m_t + n_t) > c_i$, where c_i is distributed among users according to Equation (4). Here, we require $\delta + \lambda + \lambda\phi < 1$ so that the support of G covers $[0, M + \phi N]$. As a result, we have following adoption dynamics (compared with Equation (5)):

$$m_{t+1} = G\big(m_t + \phi n_t\big) = \delta + \lambda\big(m_t + \phi n_t\big). \tag{10}$$

Compared with Equation (5), the adoption dynamics above suggests that the presence of investors n_t accelerates adoption. However, the caveat is that for this

Figure 3. Two-Sided User Market



result to hold, a bubble must form so that n_t is nonzero. We will focus on the condition for bubble formation in Section 3.2, particularly in comparison with the no-user setting.

Next, consider an investor's belief and decision. As before, an investor holds y dollars and will purchase coins when he expects a nonnegative return.[17] In the no-user setting with Equation (1), we specify an investor's belief of tomorrow's price as a detachment from the coin's fundamental value (zero in that setting). We will make a similar specification here. However, different from before, the adoption of users implies that there may be a *positive fundamental value* of the currency. Particularly, in the event of a crash, the price will drop to not zero, but rather this positive value, supported by the presence of users. It represents the value of the coin coming purely from users' demand for transactions. A bubble manifests itself as a detached price from the fundamental value. Accordingly, we specify an investor's belief as follows:

$$p_{t+1} = \begin{cases} \omega_t L_t + S_t & \text{with probability } \omega_t, \\ S_t & \text{with probability } 1 - \omega_t. \end{cases}$$

As in Equation (1), a higher ω_t denotes a higher level confidence among investors. In the above, $S_t > 0$ denotes the fundamental value expected in t for period $t+1$. Under rational expectations, S_t equals the price in $t+1$ that can be supported even without any investor demand ($n_{t+1} = 0$). At the other extreme, L_t denotes the maximum price detachment from S_t. Under rational expectations, $L_t + S_t$ will be the price in $t+1$ if all investors decide to buy coins ($n_{t+1} = 1$). We derive expressions for S_t and L_t after defining the equilibrium.

Now we can define the bubbly equilibrium with users present. In the equilibrium, users and investors are tied together by (i) the fact that they pay the same price for the currency and (ii) that investors exhibit network effects to user adoption, albeit at a possibly different rate compared with the network effects created by users.

Definition 2. Given a set of model primitives $y, \lambda, \delta, \phi, T$, and $A_t = 1 - \lambda^t$, a sequence $\mathcal{B} \equiv \{p_t, \omega_t, m_t, n_t\}_{t=1}^{T}$ with $\omega_t \in (0,1)$ constitutes a *bubbly equilibrium* if Equation (10) and the following three conditions are satisfied for all t:

i. *Market clearing*. The total demand for coins by investors and users equals the supply of coins:

$$\frac{n_t \cdot y + m_t}{p_t} = A_t.$$

ii. *Rational expectation*. The belief about the next period's noncrash price is exactly realized:

$$p_{t+1} = L_t\omega_t + S_t, \ \forall t.$$

iii. *Arbitrage-free.* The expected return of investing in coins equals the outside good's return (normalized to 1):

$$\frac{\mathbb{E}(p_{t+1})}{p_t} = \frac{1}{p_t}\left(L_t\omega_t^2 + S_t\right) = 1.$$

As before, the bubbly equilibrium defined here focuses on cases where confidence is in the interior, so that a crash remains a meaningful, but uncertain, possibility. It is useful to point out that there is a nonbubbly equilibrium in which $\omega_t = 0$ for all t. In this case, no investors participate in the coin market, the price p_t tracks the fundamental value S_t in all periods, and, importantly, m_t follows the same path as in the user-only model (Section 2.2). The other extreme case where $\omega_t = 1$ for all periods, however, generally cannot constitute an equilibrium.[18]

Applying Definition 2, we see that the lowest possible price expected for period $t+1$, denoted by S_t, must satisfy the market clearing with no investors ($n_{t+1} = 0$):

$$\frac{m_{t+1} + 0}{S_t} = A_{t+1} \Rightarrow S_t = \frac{\delta + \lambda m_t + \lambda\phi n_t}{A_{t+1}}. \tag{11}$$

The currency's maximal speculative value, L_t, is derived from the maximal price $S_t + L_t$ that clears the market with full investor demand ($n_{t+1} = 1$):

$$\frac{m_{t+1} + y}{S_t + L_t} = A_{t+1} \Rightarrow L_t = \frac{y}{A_{t+1}}. \tag{12}$$

Intuitively, L_t depicts the maximum speculative value that is sustained through the demand from exactly $N = 1$ investors.

Before deriving the key properties of the bubbly equilibrium, it is useful to point out that the bubbly equilibrium is defined conditional on the bubble being sustained. When we say that $\{p_t\}_{t=1}^T$ is part of a bubbly equilibrium, we mean that these will be the realized prices as long as the bubble does not crash. In each period, there is a self-fulfilled probability ω_t that the bubble will crash tomorrow. If a crash is realized in a period t, we may assume that ω_t and n_t will be zero thereafter, and p_t will track the fundamental values.[19]

3. Equilibrium Properties

Our analysis of the bubbly equilibrium is moderated by two primitives of the model: y and ϕ. The interpretation of these two parameters is linked to the two mechanisms that drive the dynamics of our model: the financial side that sets price through the market-clearing condition and the adoption side that follows Equation (10). Given the normalizations in the model ($M = N = 1$ and $x = 1$), we can interpret y and ϕ, respectively, as the an investor's potential impact on the financial side and her potential impact on the adoption side, both relative to a user. So by varying y and ϕ, we decouple investors' roles on the financial side and the adoption side.

It seems reasonable to think that investors can (i) influence the financial side more than users, that is, $y \geq 1$, but (ii) influence the adoption less than users, that is, $\phi \leq 1$. At the very least, it seems reasonable to restrict attention to cases where $y \geq \phi$, which is what we will focus on below.

Though the cases where $y < \phi$ should be unlikely in reality, mathematically, Definition 2 of the bubbly equilibrium still applies to such cases. We will discuss the model behaviors in such cases toward the end of this section.

3.1. Special Case ($y = \phi$)

We first consider the special case where $y = \phi$, which leads to relatively straightforward algebra and most clearly exposes the intuition of two of our main results: (i) a bubble accelerates user adoption, and (ii) a bubble with users is easier to form than an investor-only bubble, in the sense that it puts demanding conditions on the investor's confidence. The second result can be reinterpreted to mean that a bubble with users is less likely to crash compared with an investor-only bubble, as we will show. We use the $y = \phi$ case to build some intuitions before we consider the general cases in Section 3.2.

Now suppose sequence $\mathcal{B}$ constitutes a bubble path (i.e., no crash happening). First, in period t under $A_t = 1 - \lambda^t$, the market-clearing condition gives us

$$p_t = \frac{m_t + n_t y}{1 - \lambda^t}.$$

By (11) and (12), we have

$$S_t = \frac{m_{t+1}}{1 - \lambda^{t+1}} = \frac{\delta + \lambda\left(m_t + yn_t\right)}{1 - \lambda^{t+1}}; \quad L_t = \frac{y}{1 - \lambda^{t+1}}.$$

Setting the expected return in this period t equal to 1 gives us another expression for price p_t:

$$p_t = L_t\omega_t^2 + S_t = \frac{y\omega_t^2}{1 - \lambda^{t+1}} + \frac{\delta + \lambda\left(m_t + yn_t\right)}{1 - \lambda^{t+1}}.$$

Equating these two expressions for p_t yields an expression for the masses of users and investors:

$$yn_t + m_t = \frac{1 - \lambda^t}{1 - \lambda}\left(y\omega_t^2 + \delta\right),$$

which leads us to an expression for next-period user mass:

$$m_{t+1} = \delta + \lambda\left(yn_t + m_t\right) = \frac{1 - \lambda^{t+1}}{1 - \lambda}\delta + \frac{\lambda - \lambda^{t+1}}{1 - \lambda}y\omega_t^2. \tag{13}$$

Note that $\omega_t^2 > 0$ implies $m_{t+1} > \frac{1-\lambda^{t+1}}{1-\lambda}\delta$, which can be compared with Equation (7), which describes the natural adoption without investors (which is also the user adoption path in the nonbubbly equilibrium). If we use superscript *USER* to indicate the setting without investors, then we have $m_{t+1} > m_{t+1}^{USER}$ for all $t = 1, 2, \ldots$. In words, the adoption with a bubble is faster than the nonbubbly adoption.

The bubble price in $t+1$ is rationally expected as

$$p_{t+1} = L_t\omega_t + S_t = \frac{y\omega_t + m_{t+1}}{1-\lambda^{t+1}},$$

which, together with the next-period market-clearing condition, $p_{t+1} = \frac{m_{t+1}+yn_{t+1}}{1-\lambda^{t+1}}$, implies a simple expression for n_{t+1}:

$$n_{t+1} = \omega_t.$$

This simple one-to-one relation between investor mass and confidence actually also holds in the more general case, as we will show. Intuitively, it says that investor confidence drives investor participation in the market. If confidence is zero, no investor will buy and the price will simply follow Equation (9) and maintain a constant level over time, which is actually the nonbubbly equilibrium.

Finally, to obtain ω_{t+1}, we set the expected return in period $t+1$ to 1:

$$p_{t+1} = L_{t+1}\omega_{t+1}^2 + S_{t+1}.$$

To express ω_{t+1} in terms of the variables in period t, we use

$$S_{t+1} = \frac{\delta + \lambda(m_{t+1} + yn_{t+1})}{1-\lambda^{t+2}}; \quad L_{t+1} = \frac{y}{1-\lambda^{t+2}},$$

together with the expressions for m_{t+1}, p_{t+1}, and n_{t+1} as we have derived just above. The algebra is a bit involved but it leads to a relatively simple expression for ω_{t+1}:

$$\begin{aligned}\omega_{t+1}^2 &= \omega_t^2 + \frac{1-\lambda}{1-\lambda^{t+1}}(\omega_t - \omega_t^2)\\ &= \omega_t - \frac{\lambda-\lambda^{t+1}}{1-\lambda^{t+1}}(\omega_t - \omega_t^2).\end{aligned}$$

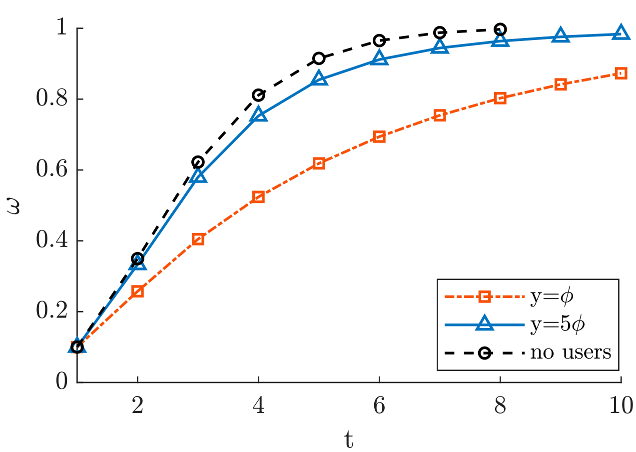
Note that $\omega_t - \omega_t^2 > 0$, so the first line above says $\omega_{t+1} > \omega_t$. In other words, investor confidence grows monotonically. On the other hand, the second line says $\omega_{t+1}^2 < \omega_t$ or $\omega_{t+1} < \sqrt{\omega_t}$. Compare this result with Equation (3) derived under the investor-only setting: $\omega_{t+1}^{INV} = \sqrt{\frac{A_{t+2}}{A_{t+1}}\omega_t^{INV}}$ (here, we use superscript *INV* to indicate the investor-only setting). We see that the investor confidence with users present need not grow as fast as when users are absent, implying that user presence relaxes the condition for bubble formation.

It turns out that for the general case $y \geq \phi$, the inequality $\omega_{t+1} < \sqrt{\omega_t}$ may not hold.[20] Intuitively, this is because of the following. Compared with the special case, $y \geq \phi$ assigns a smaller value for ϕ. The smaller impact that investors have on the user adoption, the less they can drive up tomorrow's fundamental price by buying today, which hurts the expected return. As a result, the confidence level has to grow faster to keep the expected return nonnegative. Nevertheless, for the general case, we will show that $\omega_{t+1} < \sqrt{\frac{A_{t+2}}{A_{t+1}}\omega_t}$ holds, so the investor confidence still grows at a slower rate compared with the no-user setting. A graphical comparison is made in Figure 4.

An alternative way to look at the slowed growth of ω_t is that if the highest required confidence levels are the same between the settings with and without users, $\omega_T = \omega_T^{INV}$, then we must have $\omega_t > \omega_t^{INV}$ for all $t = 1, 2, \ldots, T-1$. Because a crash happens in period t with probability $1-\omega_t$, these last inequalities indicate that the bubble with users present is less likely to burst (or more likely to sustain) than the bubble without users.

Finally, given $\omega_{t+1} > \omega_t$, we see from (13) that $m_{t+1} > m_t$ for all t. In words, the user mass monotonically increases over time. This seemingly regular result is by no means easily granted in our context—it is conceivable that the presence of speculative investors could distort the adoption process so that the latter becomes nonmonotonic. The intuition behind this "unusually regular" result derives from the assumption of rational expectation, which asserts that if there were a period in which the user mass declines, investors in the prior period would have correctly expected it. However, this anticipation of a decline in

Figure 4. (Color online) Examples of Confidence Paths



Notes. The figure displays the paths of confidence ω_t in three different bubbles. The first bubble assumes $y = \phi$, the second bubble assumes $y = 5\phi$, and the third bubble has no users present. The other primitives are same across the bubbles: $\phi = 0.5$, $\lambda = 0.6$, and $\delta = 0.1$.

users (which translates to a decline in fundamental value) would make the very bubble difficult to sustain. This same intuition holds for the general case too.

3.2. General Case

In this section, we consider the general case where $y \geq \phi$. First, we will extend the results derived in the special case (Section 3.1). Second, removing the assumption of $y = \phi$ decouples investors' roles on the financial side and adoption side, and consequently allows us to explore how the relative size of these two roles affects bubble formation.

The first lemma breaks down the various conditions in Definition 2 to allow the computation of the bubbly equilibrium in a recursive fashion.

Lemma 2. *With $y \geq \phi$, sequence $\{\omega_t, m_t, n_t\}_{t=1}^T$ constitutes a part of a bubbly equilibrium $\mathcal{B}$ iff for all t, we have $\omega_t, n_t, m_t \in (0, 1)$ such that the mass of users satisfies*

$$n_t = \frac{(\delta + y\omega_t^2)(1 - \lambda^t) - (1 - \lambda)m_t}{y(1 - \lambda^{t+1}) - \phi\lambda(1 - \lambda^t)}, \tag{14}$$

the mass of investors satisfies

$$m_{t+1} = \delta + \lambda m_t + \lambda\phi n_t, \tag{15}$$

and for $t < T$, the investor's beliefs satisfy

$$\omega_{t+1}^2 = \frac{(1 - \lambda)m_{t+1} + \omega_t(y - \phi\lambda - y\lambda^{t+2} + \phi\lambda^{t+2}) - \delta + \delta\lambda^{t+1}}{y(1 - \lambda^{t+1})}. \tag{16}$$

In addition, in any bubbly equilibrium, we have, for all t,

$$n_{t+1} = \omega_t.$$

The possibility to compute a bubbly equilibrium in a recursive fashion implies that the bubbly equilibrium, if it exists, is unique for any given ω_1.

Some intuitions can be gained from Equation (16), which describes the evolution of investor confidence. We can rewrite it as follows (recall m^* is the natural adoption ceiling given in Equation (6)):

$$\omega_t^2 = \frac{\delta\lambda^t + (1 - \lambda)(m_t - m^*) + \omega_{t-1}(y - \phi\lambda - y\lambda^{t+1} + \phi\lambda^{t+1})}{y(1 - \lambda^t)}.$$

There are three terms in the numerator, all of which may exert "pressure" for ω_t to grow, albeit driven by different factors. The first term, $\delta\lambda^t$, vanishes as $t \to +\infty$ but is positive for finite t. It captures the pressure from the increasing supply of coins. The growth of supply suppresses future prices, which must be compensated by increases in confidence in order to keep investors in the market. The eventual convergence to zero of $\delta\lambda^t$ comes from the prefixed ceiling of supply, which says that the growth of supply will eventually flatten out.

The second term, $(1 - \lambda)(m_t - m^*)$, captures the effect of user adoption. For early periods where $m_t < m^*$, the term is negative, so it actually relieves the pressure for confidence to increase. Intuitively, this is because the natural tendency for m_t is to grow toward m^*, which raises the coins' fundamental value and consequently helps keep investors in the market at lower confidence levels. However, in later periods where $m_t > m^*$, the natural tendency in user adoption is for m_t to revert back to m^* (which will happen if investors leave the market). Unlike the earlier periods, this downward tendency puts *more* pressure for confidence to grow.

The third term in the numerator is a multiple of ω_{t-1}. It basically captures the feature of rational expectations; a higher ω_{t-1} from yesterday implies a higher rationally expected price today, which in turn demands a higher expectation for tomorrow, ω_t, to sustain the bubble.

The next two propositions are based on Lemma 2 and provide bounds on the trajectory of ω_t over time. The first proposition delivers a main result in this paper: the presence of users relieves the pressure for ω_t to grow. In other words, the requirement for bubble formation is relaxed by the presence of users. The same result was derived in Section 3.1 for the special case $y = \phi$.

Proposition 1. *Suppose $y \geq \phi$ and $m_1 = \delta$. In a bubbly equilibrium, we have*

$$\omega_{t+1} < \sqrt{\frac{A_{t+2}}{A_{t+1}}}\omega_t,$$

for all periods $t \geq 1$.

Proposition 2. *Suppose $y \geq \phi$ and $m_1 = \delta$. In a bubbly equilibrium, we have*

$$\omega_{t+1} > \omega_t,$$

for all periods $t \geq 1$.

A quick intuition for how user presence makes a bubble "easier" is that the natural user adoption makes the coin's fundamental value increase over time, which gives investors more incentive to buy. Although this intuition may be true in general, it is not what underlies Proposition 1. What Proposition 1 shows is that, even if coin's fundamental value without bubbles is constant over time (see Equation (9)), bubble formation is still easier when users are present. The key intuition for this result comes from the interaction between users and investors. By buying the coins (and thus creating a bubble), investors can push adoption beyond the natural rate, which in turn raises the coins' value over time. It is this interaction, *rationally foreseen* by investors, that makes them more willing to participate in a bubble.[21]

We use the next proposition to formally address the existence of a bubbly equilibrium. However, from a more substantive point of view, it actually reinforces our main result delivered by Proposition 1.

Proposition 3. *Given a set of primitives y, λ, δ, ϕ, T, and $m_1 = \delta$,* (i) *with $y \geq \phi$, a bubbly equilibrium exists as long as ω_1 takes a value such that a bubbly equilibrium exists for the investor-only case;* (ii) *for $y = \phi$, a bubbly equilibrium exists for any $\omega_1 \in (0, 1)$; and* (iii) *for $y/\phi \to +\infty$, a bubbly equilibrium exists if and only if ω_1 takes a value such that a bubbly equilibrium exists for the investor-only case.*

The proposition tells us that, compared with the no-user setting, the condition for bubble existence is relaxed by the presence of users, and it is most relaxed when $y = \phi$ and least relaxed when $y \gg \phi$. The result suggests that bubbles become easier to form as y/ϕ decreases. Given the normalizations made in our model, a decrease in y can actually be represented by an increase in the number of potential users, M.[22] Hence, not only do we have the "discrete result" that bubble formation is easier when users are present, but we also have the "continuous result" that bubble formation becomes easier when *more* users are present.

We use the following exercise to further illustrate the effect of y/ϕ on bubble formation. Recall that in the investor-only case, the condition for equilibrium existence is that ω_1 is sufficiently small such that ω_t will not grow beyond 1 in some future period (Lemma 1). A similar requirement applies here. Specifically, for a given set of primitives, there is an upper bound for ω_1 such that a bubbly equilibrium exists for any $T = 1, 2, \ldots, \infty$ iff ω_1 is below this upper bound. Although this upper bound does not have an analytical expression, it can be computed numerically. Figure 5 displays this upper bound for ω_1 as a function of y/ϕ. The left plot assumes $\lambda = 0.4$, and the right plot assumes $\lambda = 0.6$.

There are two important observations: (i) the presence of users relaxes the bounds, and (ii) a larger ratio between y and ϕ tightens the bounds. These observations are consistent with what we have seen so far in Propositions 1–3.

Next, we turn our attention away from investor confidence to focus on user adoption. This leads us to another main result of this paper, which formalizes the insight that a bubble accelerates user adoption. The same result was derived in Section 3.1 for the special case $y = \phi$. To show this result in the general case requires the following lemma.

Lemma 3. *Suppose $y \geq \phi$. Along the path of any bubbly equilibrium with $m_1 < m^*$, we have*

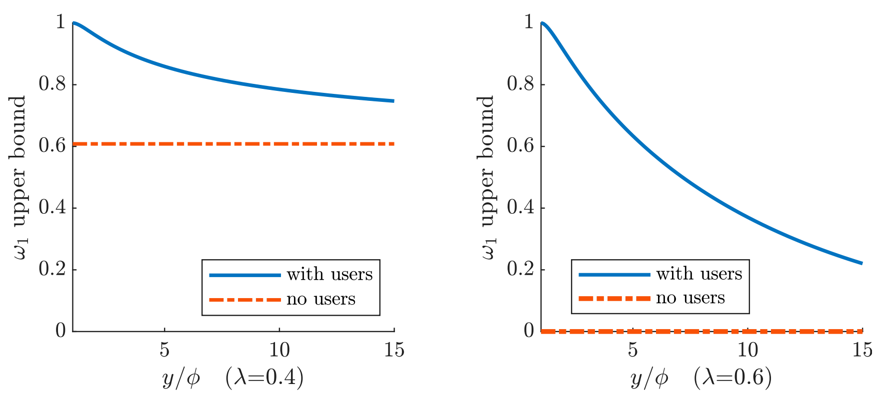
$$m_t < m_{t+1},$$

for all $t \geq 1$ on the bubbly path.

Like we discussed in the $y = \phi$ case, this seemingly regular result is by no means easily granted in our model—it is conceivable that the presence of speculative investors could distort the adoption process so that the latter becomes nonmonotonic. The intuition behind Lemma 3 has its roots in the assumption of rational expectation. The details of the intuition are the same as given in Section 3.1, so we shall not repeat them here.

With the monotonicity of m_t, we can move to show that m_t will eventually surpass m^*, the adoption ceiling without investors. Of course, if T is too small, m_t will not reach m^* before the end period T. So we need to evoke the existence of a bubbly equilibrium for large T.

Figure 5. (Color online) Upper Bounds of ω_1 for Bubble Existence



Notes. The plots show the upper bound of ω_1 that permits a bubbly equilibrium for $T = \infty$. The first-period user mass m_1 is set at δ. The specific values of δ and ϕ (or y) do not affect the curves shown here.

Proposition 4. *Suppose $y \geq \phi$, $m_1 < m^*$, and a bubbly equilibrium exits for $T = \infty$; then there exists t^* such that $t > t^*$ implies $m_t > m^*$.*

It is important to point out that our results on m_t are conditional on the sustenance of a bubble (as is our definition of bubbly equilibrium). When a bubble bursts, m_t will break away from its trajectory in a bubbly equilibrium. However, m_t will still stay above the levels of user sizes in natural adoption. To see this, suppose the bubble bursts in period s; then $m_{t+1} = \delta + \lambda m_t$ for all $t \geq s$. Because of the bubble, we have $m_s > m_s^{USER}$ (recall that superscript *USER* denotes the user-only setting). Hence, $m_t > m_t^{USER}$ holds for all $t \geq s$. As a result, though we have shown only that user adoption is faster within a bubble, the same holds true ex ante accounting for the possibility of a crash.

3.3. Improper Case

So far, we have restricted attention to $y \geq \phi$. The inequality asserts that investors hold a stronger influence on the financial side than the adoption side, compared with users. As said, mathematically, our equilibrium definition also applies to cases where $y < \phi$. Although we think such cases are unlikely in reality, they do offer some interesting intuitions.

With $y < \phi$, a bubbly equilibrium may exhibit behaviors that are irregular compared with those behaviors derived in Section 3.2. Particularly, neither ω_t nor m_t needs to be monotonically increasing.[23] Intuitively, this is because investors' participation can substantially accelerate adoption (because of a relatively large ϕ) without significantly raising today's price (because of a relatively small y). As a result, investors can easily raise the expected return of coins, which makes an increase in confidence sometimes unnecessary in sustaining a bubble.

More specifically, the monotonic growth of ω_t in Proposition 2 is the result of several forces. First, in a bubble with rationally expected prices, the price must increase over time. The growing price puts pressure on the confidence level to grow as well (in order to keep the expected return nonnegative and thus sustain the bubble). Second, the coin supply is increasing over time, which suppresses future prices. This suppression pressures the investor confidence to grow (in order to keep the expected return nonnegative). Third, user adoption gradually raises the fundamental value of coins, which may relieve the pressure for confidence to grow. When $y \geq \phi$, the first two forces overcome the third. However, when $y < \phi$, investors can substantially accelerate user adoption without raising today's price, which allows the third force to overcome the first two. When this happens, the current-period confidence level will be lower than that in the last period. Furthermore, such a decline in confidence can translate into a decline in user adoption, as it implies a decline in investor participation thanks to rational expectations.

4. Discussions and Conclusion

Our study is the first to look into the possible interactions between financial bubbles and product diffusion for new currency adoption. It best portrays currencies whose supplies evolve in a predictable fashion, such as cryptocurrencies. In that context, we explored conditions for when and how investors and users reinforce each other in sustaining speculative returns and adoption benefits, respectively. On one hand, users benefit from the network externality created by investors in a bubble, which then accelerates user adoption. On the other hand, investors expecting a more steady stream of users to buy currency do not need to rely as heavily on the confidence of their fellow investors to sustain a return. Our results point to the conclusion that a currency bubble is most likely to form (or less likely to crash once formed) when some event increases the base of potential users.

There were two major price surges in the bitcoin history before 2019. The first major surge happened around November 2013. Our conclusion above is consistent with our observations from the introduction about the impact of the Shared Coin service, which coincides with the timing of this price surge. The anonymity offered by the Shared Coin service expanded the potential user base. Under the lens of our model, this expansion can be interpreted as an increase in M (or an increase in y with our normalization), which therefore relaxes the level of the investor confidence required to form a bubble.

The surged price fell rapidly in early 2014. Although a bubble bursting could always be a self-fulfilled, random event (as permitted in our model), our theory can offer a more concrete event-based explanation, rooted in the user-side of the market. In February 2014, one of the largest bitcoin exchanges, Mt. Gox, filed for bankruptcy amid reports of 750,000 bitcoins stolen, which caused a major upset among users regarding bitcoin's security. Supposing that this event scraped away the potential user base for bitcoin, then a bubble would be harder to sustain in terms of investor confidence, thereby compelling the crash.

The second major price bubble happened in late 2017. Earlier in 2017, several countries—Japan, Russia, and Norway—announced their legitimization of the cryptocurrency, thereby increasing the potential user base and subsequently, according to our model, relaxing the level of investor confidence required for a bubble. The price plummeted in early 2018. Although, again, a crash can be a self-fulfilling event without triggering causes, several events may have advanced the downfall, including the anticipation of China's and South Korea's bans on the trading of bitcoins as well as hacks at several cryptocurrency exchanges (e.g., the Coincheck hack). Even though these exchanges were trading not bitcoins but other cryptocurrencies, the hacks may have made investors worry about the security of cryptocurrencies in general.

Taking our results a step further, they seem to suggest that any entity introducing a currency can take advantage of bubbles to induce currency adoption. Indeed, it is worthwhile to note that the European Monetary Union initially launched the euro in 1999 to investors only as an accounting currency, before it became the European Union's standard medium of exchange. Our theory provides a rationale for such a launch strategy, as it shows the synergy between users and investors. Lessons from our model may be more widely applied now that nongovernment entities can easily market their own currencies.

Acknowledgments
The authors thank Fabio Caldieraro, Hongxun Ruan, Yi Zhu, and the participants of the 14th Bass FORMS Conference (University of Texas at Dallas) for comments, as well as Irina Osipova for research assistance. This paper was circulated previously with the title "Cryptocurrency Adoption: The Role of Speculative Price Bubbles in Product Diffusion."

Appendix A
A.1. Continuous Beliefs
For our analytical results, we relied on a simplified discrete system of investor beliefs (see Equation (1)). Below, we consider a continuous version for this belief.

Let us specify $p_{t+1} \sim L_t \times \text{Beta}(\alpha_t, 1/\alpha_t)$. As Figure A.1 illustrates, the mass of this distribution moves away from 0 toward L_t as α_t increases from 0 to $+\infty$. The distribution family includes the uniform distribution when $\alpha_t = 1$. Also, Figure A.1 shows that the distribution family that we consider is inclusive with respect to mirror beliefs: the probability density function (p.d.f.) of $\text{Beta}(\alpha_t, 1/\alpha_t)$ is exactly the mirror image of the p.d.f. of $\text{Beta}(\alpha'_t, 1/\alpha'_t)$ with $\alpha'_t = 1/\alpha_t$.

The two-point specification in Equation (1) is an approximate discretization of the continuous family. To see this, suppose that we want to discretize the beta family to a two-point distribution as follows while preserving the mean as well as the variance:

$$p_{t+1} = \begin{cases} g_t \cdot L_t & \text{with probability } \omega_t, \\ 0 & \text{with probability } 1 - \omega_t. \end{cases} \tag{A.1}$$

Using the formulas for the mean and variance for a beta distribution, we have

$$g_t \omega_t = \frac{\alpha_t}{\alpha_t + 1/\alpha_t};$$
$$g_t^2 (1 - \omega_t)\omega_t = \frac{1}{(\alpha_t + 1/\alpha_t)^2(\alpha_t + 1/\alpha_t + 1)}.$$

These give us ω_t and g_t, both as functions of α_t. For ω_t, we have a closed-form expression:

$$\omega_t = \frac{\alpha_t^3 + \alpha_t^2 + \alpha_t}{\alpha_t^3 + \alpha_t^2 + \alpha_t + 1}.$$

It is seen that there is a one-to-one mapping from α_t to ω_t: $[0, +\infty) \to [0, 1]$. So, basically, α_t represents the investor confidence in the continuous-belief model, playing the same role as ω_t in the discrete-belief model.

Unfortunately there is no closed-form expression for g_t. However, we can trace the values of g_t and ω_t as we vary α_t, which gives us g_t as a function of ω_t. This function is plotted in Figure A.2. We see it is very close to the 45° line. So the discretized distribution (A.1) is closely approximated by the belief specification that we used in the main model (i.e., Equation (1)).

A.2. Bitcoin Supply
Government-backed currency is issued at a rate determined by the central bank, which takes into account the concurrent state and prospects of the economy. In the United States, the Federal Reserve System increases the monetary base by issuing dollars.

In the context of cryptocurrency, the bitcoin generation algorithm defines how currency will be created and at what rate. Particularly, the supply of bitcoins is controlled to follow a preset rule toward a preset ceiling. The result is a supply trajectory that has closely followed an exponential curve: $A_t = A_\infty(1 - \beta^t)$, where A_t is the supply in period t, A_∞ is the preset ceiling, and $\beta < 1$ captures the growth rate. Figure A.3 shows both the historical supply of bitcoins and the fitted exponential curve. The yearly β is estimated to be around 0.825.

Figure A.1. (Color online) Probability Density Function of $\text{Beta}(\alpha, 1/\alpha)$

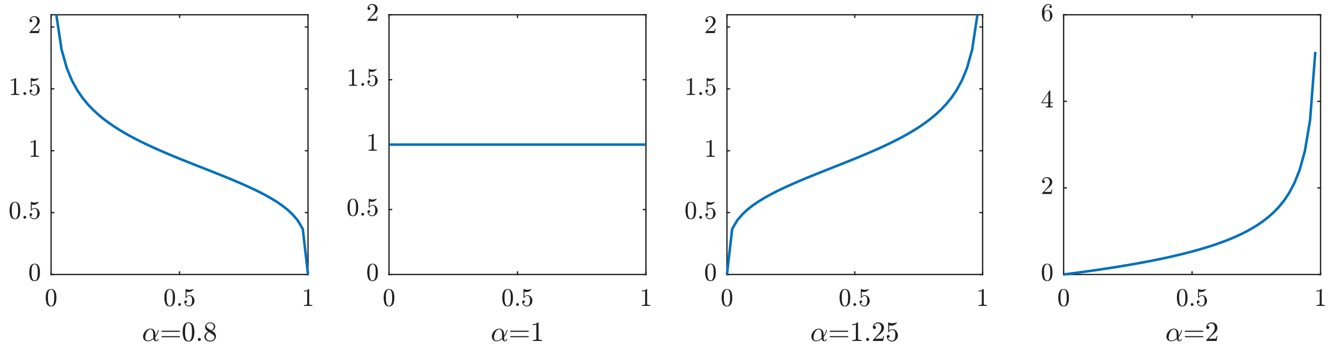


Figure A.2. Plot of $g_t(\omega_t)$

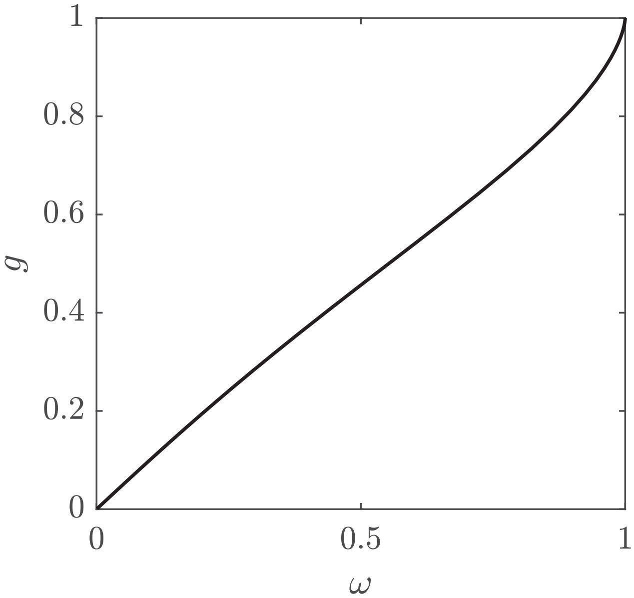
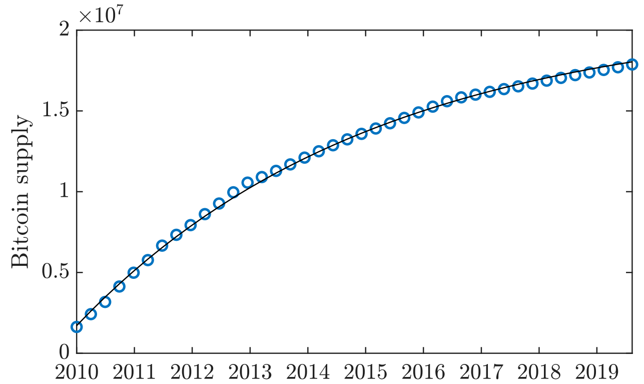


Figure A.3. (Color online) Bitcoin Historical Supply



Notes. The circles mark the quarterly supply of bitcoins (historical data). The solid curve shows $A_t = A_\infty(1-\beta^t)$ fitted to the circles. The quarterly $\beta \simeq 0.953$ and the yearly $\beta \simeq 0.825$.

The key cause behind this exponential supply curve is the control of mining. Bitcoins are supplied through mining. Miners are network computers that pack individual transaction records into blocks of transactions, which are then "chained" together by the bitcoin system to create an entire history of bitcoin transactions (hence, the term "blockchain"). Block creation is accompanied by the issuance of new bitcoins. The difficulty of packing blocks is adjusted by the bitcoin system so that regardless of the mining activities (e.g., the number of miners or the computational power of mining machines), blocks are created at a rate of six per hour. At the same time, the number of bitcoins issued per block is set to decrease geometrically, with a 50% reduction every 210,000 blocks, or approximately four years. The result is that the number of bitcoins in existence follows the exponential curve with a ceiling of 21 million (A_∞).

A.3. Two-Sided Market for Coins

In the main text, we rely on Equation (10) to describe the Bass-like adoption dynamic. Here, we present what can be seen as a microfoundation for Equation (10). It uses a two-sided market to describe coin transactions and exchange. The two sides are the two types of users—shoppers and vendors.

In each period t, we denote the mass of shoppers that carry coins to do transaction by $h_t \in [0, H]$; the mass of vendors that accept coins is denoted by $d_t \in [0, D]$. Here, H is the total mass of shoppers, who may or may not have adopted the coins. Similarly, D is the total mass of venders. We normalize $H = 1$. Typically, there are more shoppers than vendors, so $D \le 1$.

Figure 3 in the main text summarizes the working of the two-sided market. Within a period, shoppers first decide whether to carry coins (instead of an alternative currency such as dollars). A key factor in a shopper's decision is how many vendors accept coins, an issue that we will take into account below when specifying the shopper's value from adoption (see Figure A.4 for an illustration). If a shopper decides to carry coins, he exchanges his unit endowment ($x = 1$) into coins. He can do this exchange via a coin-investing bank or individual investor. The ease of exchange, or the liquidity of coins, depends on the number of coin-accepting investors, an issue that we will also account for when specifying the shopper's value from adoption.

Next, the shopper goes to one of the coin-accepting vendors for shopping. After transacting with shoppers, the vendors either deposit the received coins or exchange them into alternative currencies, such as dollars. Again, the ease of deposition and exchange depends on with the mass of coin-accepting investors.

In accordance with the above description of the two-sided market, we specify a shopper's value of adopting coins in period $t+1$, denoted by V^h_{t+1}, as a function of the fraction of coin-accepting vendors and the fraction of coin-accepting investors. As in the Bass model, we assume that users are myopically adjusting their adoption decisions. Specifically,

$$V^h_{t+1} = d_t/D + \phi n_t/N = d_t/D + \phi n_t,$$

where parameter ϕ allows the user's utility to put different weights on the ease of exchange (liquidity) and on the ease of transactions. Similarly for the vendors, we specify

$$V^d_{t+1} = h_t/H + \phi n_t/N = h_t + \phi n_t.$$

Under the same adoption cost distribution as specified in Equation (4) for both shoppers and vendors, the following adoption dynamics are implied:

$$\begin{aligned} h_{t+1} &= \delta + \lambda\big(d_t/D + \phi n_t\big), \\ d_{t+1}/D &= \delta + \lambda\big(h_t + \phi n_t\big). \end{aligned} \tag{A.2}$$

Intuitively, these two equations imply that in the market of coins, (i) the mass of shoppers tomorrow increases with the mass of vendors and the mass of investors today, and (ii) the mass of vendors tomorrow increases with the mass of shoppers and the mass of investors today. Taking the difference between these two adoption equations, we have

$$h_{t+1} - d_{t+1}/D = -\lambda(h_t - d_t/D). \tag{A.3}$$

It is natural to set $h_1 = \delta$ and $d_1 = \delta D$ (which can be thought of as the consequences of $d_0 = 0, h_0 = 0$, and $n_0 = 0$). As a result, Equation (A.3) implies that for all t,

$$h_t = d_t/D.$$

This equality says that the sizes of shoppers and vendors grow in parallel. This result should be intuitive: as more vendors accept coins, more shoppers are willing to carry coins; conversely, as more shoppers carry coins, more vendors find it compelling to accept coins. Now Equation (A.2) can be written as

$$h_{t+1} = \delta + \lambda\left(h_t + \phi n_t\right),$$

which is equivalent to Equation (10), if we think of each shopper here as a user in the main model. Because the sizes of shoppers and vendors grow in parallel, tracking the number of shoppers also tracks the number of vendors.

A.4. Extensions on Adoption Values

A.4.1. Nonlinear Adoption Benefit We consider an extension where the network effects on user adoption are nonlinear. Specifically, suppose the benefits that a user derives from adoption are given by

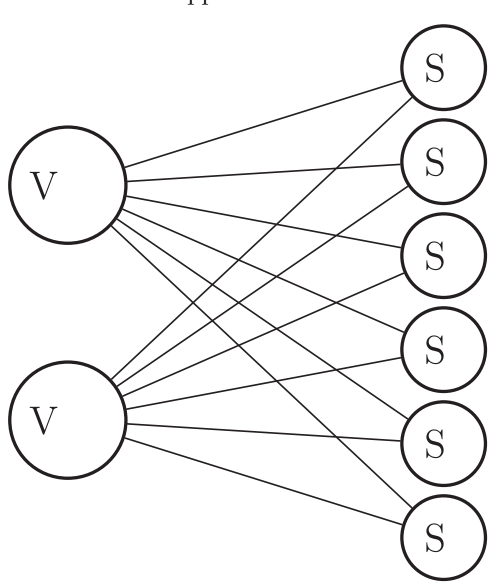
$$V(m_t, n_t) = m_t + \eta m_t^2 + \phi n_t.$$

Intuitively, the nonlinear term creates a snowball effect where adoption itself accelerates the adoption rate. Following the line of thought in our benchmark model, this additional accelerating effect should further relax the condition for bubble formation.

This seems to be indeed the case. Figure A.5 displays the upper bound of ω_1 that permits a bubbly equilibrium, as a function of y/ϕ. The figure extends the comparison we made in Figure 5. The top curve in Figure A.5 displays the case with the snowball effect. The middle curve displays the case without the snowball effect (i.e., the model considered in Section 2.3). The bottom curve displays the case without users (i.e., the model considered in Section 2.1).

Higher upper bounds indicate more relaxed conditions for bubble formation. Hence, we see that the snowball effect relaxes the condition for bubble formation. This result is not specific to the model primitives used in Figure A.5. It holds for all of the many sets of primitives that we have examined.

Figure A.4. Vendor–Shopper Network



Notes. This plot illustrates the interaction between vendors (V) and shoppers (S) when there are four times the shoppers compared with vendors. When one of the shoppers decides to carry coins, each vendor sees one-eighth of its customers want to do transactions in coins. When one of the vendors decides to accept coins, each shopper sees half of his or her vendors accept coins.

A.4.2. Investors Valuing Liquidity We consider an extension where investors value liquidity. Specifically, in addition to demanding an expected return equal to that of the outside good, investors demand an extra return to compensate the relative low liquidity of coins (compared with the outside good, such as short-term government bonds). Accordingly, we specify the nonarbitrage condition as

$$\frac{L_t}{p_t}{\omega_t}^2 = 1 + \iota \cdot (1 - n_t).$$

Parameter ι calibrates investors' value for liquidity. The higher ι is, the more they need to be compensated for low liquidity. The liquidity increases with n_t, the mass of investors, and reaches its highest level when all investors are trading coins ($n_t = 1$).

From the above nonarbitrage condition, we see that, compared with $\iota = 0$, any positive ι requires a higher ω_t. Hence, intuitively, when investors care about liquidity, the confidence level must grow faster, which makes a rational bubble more difficult to sustain. Note that this intuition applies to both the settings with and without users. So the question is how the two settings compare in terms of bubble formation, when both have investors demanding extra return for liquidity.

Figure A.5. (Color online) ω_1 Upper Bounds for Bubble Existence, Nonlinear Adoption

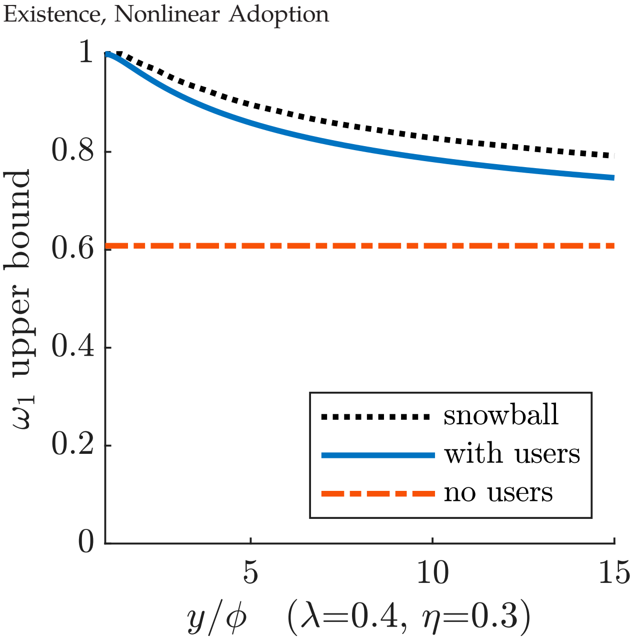
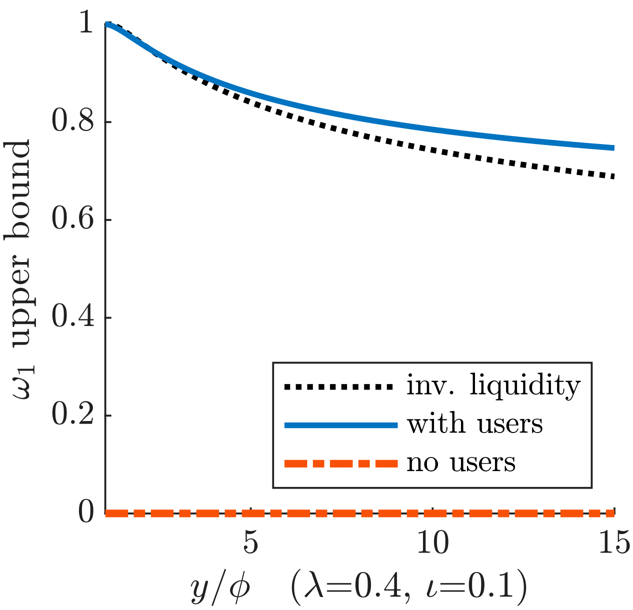


Figure A.6. (Color online) ω_1 Upper Bounds for Bubble Existence, Investor (Inv.) Valuing Liquidity



To answer this question, Figure A.6 displays the upper bound of ω_1 that permits a bubbly equilibrium, as a function of y/ϕ. The figure extends the comparison we made in Figure 5. The top curve in Figure A.6 displays the case where $\iota = 0$ (which reduces the model to the same as in Section 2.3). The middle curve displays the case with $\iota = 0.1$. The bottom curve displays the case again with $\iota = 0.1$ but without users.

We see that for the setting with users, moving from $\iota = 0$ to $\iota = 0.1$ indeed tightens the upper bounds. However, for the setting without users, $\iota = 0.1$ effectively makes a rational bubble impossible (the bottom curve overlaps with zero). Hence, bubbles are easier to form in the setting with users compared with the setting without users, which is consistent with our findings in Section 3. This result is not specific to the model primitives used in Figure A.6. It holds for all of the many sets of primitives that we have examined.

Appendix B. Proofs

Proof of Lemma 1. By Equation (3) and $A_t = 1 - \beta^t$, we have

$$\omega_2^2 = \frac{1-\beta^3}{1-\beta^2}\omega_1, \quad \omega_3^4 = \left(\frac{1-\beta^4}{1-\beta^3}\right)^2 \omega_2^2, \quad \omega_4^8 = \left(\frac{1-\beta^5}{1-\beta^4}\right)^4 \omega_3^4, \ \ldots.$$

Multiplying these equations, we have, for any $t \geq 3$,

$$\omega_t^{2^{t-1}} = \frac{1}{1-\beta^2}\cdot\frac{1}{1-\beta^3}\cdot\frac{1}{(1-\beta^4)^2}\cdots\frac{(1-\beta^{t+1})^{2^{t-2}}}{(1-\beta^t)^{2^{t-3}}}\cdot\omega_1 = \omega_1\cdot\frac{(1-\beta^{t+1})^{2^{t-2}}}{1-\beta^2}\cdot\prod_{k=3}^{t}\frac{1}{(1-\beta^k)^{2^{k-3}}}.$$

Let

$$B_k := (1-\beta^k)^{2^{k-3}}.$$

Note $0 \leq B_k \leq 1$. We have

$$\omega_t^{2^{t-1}} = \omega_1 \cdot \frac{B_{t+1}}{1-\beta^2}\cdot\frac{1}{\prod_{k=3}^{t} B_k}. \tag{B.1}$$

First we show the sufficiency of $\omega_1 \leq \Omega(\infty)$ for existence. If the condition in the lemma holds, then we must have, for any $t \geq 3$,

$$\omega_1 \leq (1-\beta^2)\prod_{k=3}^{t} B_k,$$

which, together with (B.1), implies

$$\omega_t^{2^{t-1}} \leq (1-\beta^{t+1})^{2^{t-2}} \quad \Leftrightarrow \quad \omega_t \leq \sqrt{1-\beta^{t+1}} < 1.$$

The case with $t = 2$ can be directly verified, under which the condition in the lemma translates to $\omega_1 \leq 1-\beta^2$ and $\omega_2 \leq \sqrt{1-\beta^3}$.

Next, we turn to the necessity of $\omega_1 \leq \Omega(\infty)$ when $T = \infty$. This part of the proof relies on first understanding the limit of B_k. Note $\log B_k = \log(1-\beta^k)/2^{3-k}$. Using L'Hôpital's rule,

$$\lim_{k\to\infty}\log B_k = \lim_{k\to\infty}\frac{\frac{-1}{1-\beta^k}\log(\beta)\beta^k}{-\log(2)2^{3-k}} = \frac{\log(\beta)}{2^3\log(2)}\lim_{k\to\infty}(2\beta)^k.$$

This result implies that (i) B_k converges to 1 if $\beta < 0.5$, (ii) to $e^{-1/8}$ if $\beta = 0.5$, and (iii) to 0 if $\beta > 0.5$. Thus, we know $\prod_{k=3}^{\infty} B_k$ equals 0 if $\beta \geq 0.5$. As to what happens for $\beta < 0.5$, we can use the ratio test on the series $\log B_k$:

$$\lim_{k\to\infty}\frac{\log B_{k+1}}{\log B_k} = 2\lim_{k\to\infty}\frac{\frac{1}{1-\beta^{k+1}}\log(\beta)\beta^{k+1}}{\frac{1}{1-\beta^k}\log(\beta)\beta^k} = 2\beta,$$

which says $\log B_k$ is a convergent series, and thus $\prod_{k=3}^{\infty} B_k$ is a positive finite number, for $\beta < 0.5$.

Case (i) ($\beta < 0.5$). Taking limit of (B.1), we have

$$\lim_{t\to\infty}\omega_t^{2^{t-1}} = \omega_1\cdot\frac{1}{1-\beta^2}\cdot\frac{1}{\prod_{k=3}^{\infty} B_k}.$$

Thus, if the condition in the lemma fails, then we have $\lim_{t\to\infty}\omega_t^{2^{t-1}} > 1$, which implies that there exists some t such that $\omega_t > 1$, thereby violating Definition 1.

Case (ii) ($\beta = 0.5$). The condition in the lemma amounts to $\omega_1 \leq 0$. With $\omega_1 > 0$, Equation (B.1) implies

$$\lim_{t\to\infty}\omega_t^{2^{t-1}} = \omega_1\cdot\frac{e^{-1/8}}{1-\beta^2}\cdot\frac{1}{0} = +\infty.$$

Therefore, we can find t such that $\omega_t > 1$, violating Definition 1.

Case (iii) ($\beta > 0.5$). The condition in the lemma amounts to $\omega_1 \le 0$. Note that

$$\prod_{k=3}^{t} B_k < \prod_{k=3}^{t}\left(1-\beta^t\right)^{2^{k-3}} = \left(1-\beta^t\right)^{2^{t-2}-1}.$$

Therefore, Equation (B.1) implies

$$\omega_t^{2^{t-1}} > \omega_1 \cdot \frac{1-\beta^t}{1-\beta^2} \cdot \left(\frac{1-\beta^{t+1}}{1-\beta^t}\right)^{2^{t-2}}.$$

One can show that the exponential term on the right side goes to $+\infty$ under $\beta > 0.5$ by using L'Hôpital's rule on the its logarithmic expression. Thus, with $\omega_1 > 0$, we again can find t such that $\omega_t > 1$. □

Proof of Lemma 2. We derive recursive formulae of (14), (15), and (16) under coin supply $A_t = 1-\lambda^t$. Now suppose $\mathcal{B}$ constitutes a bubble path (i.e., no crash happening). First, using the market-clearing condition, we get

$$p_t = \frac{yn_t + m_t}{1-\lambda^t}.$$

by Equations (11) and (12), we have

$$S_t = \frac{\delta + \lambda m_t + \lambda\phi n_t}{1-\lambda^{t+1}}, \quad L_t = \frac{y}{1-\lambda^{t+1}}.$$

Setting the expected return in this period t equal to 1 gives us another expression for p_t:

$$p_t = L_t\omega_t^2 + S_t = \frac{y}{1-\lambda^{t+1}}\omega_t^2 + \frac{\delta + \lambda m_t + \lambda\phi n_t}{1-\lambda^{t+1}}.$$

Taking the two equations for p_t together yields

$$\frac{y\omega_t^2}{1-\lambda^{t+1}} + \frac{\delta + \lambda m_t + \lambda\phi n_t}{1-\lambda^{t+1}} = \frac{yn_t + m_t}{1-\lambda^t},$$

which implies

$$n_t = \frac{\left(\delta + y\omega_t^2\right)\left(1-\lambda^t\right) - (1-\lambda)m_t}{y\left(1-\lambda^{t+1}\right) - \phi\lambda\left(1-\lambda^t\right)}.$$

The next-period bubble price is rationally expected as

$$p_{t+1} = L_t\omega_t + S_t = \frac{m_{t+1} + y\omega_t}{1-\lambda^{t+1}}, \tag{B.2}$$

where the next-period mass of users is given by (10)

$$m_{t+1} = \delta + \lambda m_t + \lambda\phi n_t.$$

Market clearing in period $t+1$ gives us

$$n_{t+1} = \frac{p_{t+1}A_{t+1} - m_{t+1}}{y} = \omega_t.$$

Next, we can compute S_{t+1} and L_{t+1} as

$$S_{t+1} = \frac{\delta + \lambda m_{t+1} + \lambda\phi n_{t+1}}{1-\lambda^{t+2}}, \quad L_{t+1} = \frac{y}{1-\lambda^{t+2}}. \tag{B.3}$$

Setting the expected return in $t+1$ to 1, we have the expression for ω_{t+1} as follows:

$$\omega_{t+1}^2 = \frac{p_{t+1} - S_{t+1}}{L_{t+1}}.$$

Using (B.2) and (B.3), we can show that

$$\omega_{t+1}^2 = \frac{(1-\lambda)m_{t+1} + \omega_t\left(y - \phi\lambda - y\lambda^{t+2} + \phi\lambda^{t+2}\right) - \delta + \delta\lambda^{t+1}}{y\left(1-\lambda^{t+1}\right)},$$

which completes our proof. □

Proof of Proposition 1. Before establishing the result with $m_1 = \delta$, we show the inductive argument

$$\omega_t^2 < \frac{1-\lambda^{t+1}}{1-\lambda^t}\omega_{t-1} \Rightarrow \omega_{t+1}^2 < \frac{1-\lambda^{t+2}}{1-\lambda^{t+1}}\omega_t.$$

Fix a t, and suppose that

$$\omega_t^2 < \frac{A_{t+1}}{A_t}\omega_{t-1} = \frac{1-\lambda^{t+1}}{1-\lambda^t}\omega_{t-1}.$$

By Lemma 2, which says $n_t = \omega_{t-1}$, we have

$$n_t > \omega_t^2\frac{1-\lambda^t}{1-\lambda^{t+1}}.$$

The above inequality, together with (14) in Lemma 2, gives us

$$\frac{\left(\delta + y\omega_t^2\right)\left(1-\lambda^t\right) - (1-\lambda)m_t}{y\left(1-\lambda^{t+1}\right) - \phi\lambda\left(1-\lambda^t\right)} > \omega_t^2\frac{1-\lambda^t}{1-\lambda^{t+1}},$$

which can be reduced to an upper bound on m_t:

$$m_t < \frac{\left(1-\lambda^t\right)\left[\delta\left(1-\lambda^{t+1}\right) + \omega_t^2\phi\left(\lambda - \lambda^{t+1}\right)\right]}{(1-\lambda)\left(1-\lambda^{t+1}\right)}.$$

One can show that (16) in Lemma 2 can be reduced to the following, if we substitute m_{t+1} with m_t using (15) and (14):

$$\begin{aligned}\omega_{t+1}^2 = &\frac{(1-\lambda)\left(y^2\omega_t^2 + y\delta + (y-\phi)\lambda m_t\right)}{y\left(y - \phi\lambda + \phi\lambda^{t+1} - y\lambda^{t+1}\right)} \\ &+ \frac{(1-\lambda)(1-\omega_t)\omega_t}{1-\lambda^{t+1}} - \frac{\delta - \lambda\omega_t(y-\phi)}{y}.\end{aligned} \tag{B.4}$$

Applying the upper bound of m_t to the above expression produces an upper bound for ω_{t+1}^2 and, consequently, an upper bound for $\omega_{t+1}^2 - \frac{1-\lambda^{t+2}}{1-\lambda^{t+1}}\omega_t$. One can show the last upper bound reduces to

$$\omega_{t+1}^2 - \frac{1-\lambda^{t+2}}{1-\lambda^{t+1}}\omega_t < -\frac{\lambda\phi\omega_t\left[1-\lambda^{t+1} - \omega_t\left(1-\lambda^t\right)\right]}{y\left(1-\lambda^{t+1}\right)}.$$

Because $1-\lambda^{t+1} - \omega_t(1-\lambda^t) \ge 1-\lambda^{t+1} - (1-\lambda^t) > 0$, we have

$$\omega_{t+1}^2 - \frac{1-\lambda^{t+2}}{1-\lambda^{t+1}}\omega_t < 0,$$

which is what we want to prove.

Now suppose $m_1 = \delta$. By (B.4) with $t = 1$, we can show that

$$\omega_2^2 - \omega_1 \cdot \frac{1-\lambda^3}{1-\lambda^2} = \frac{\omega_1^2 \lambda\phi}{(1+\lambda)\big[y + \lambda(y-\phi)\big]} - \frac{\omega_1 \lambda \phi}{y} \le 0,$$

with the equality holding only at $\omega_1 = 0$. □

Proof of Proposition 2. By way of induction, we fix a $t \ge 2$, assume that

$$\omega_t > \omega_{t-1},$$

and then show $\omega_{t+1} > \omega_t$. Under the above assumption, Lemma 2 says $n_t = \omega_{t-1}$, which gives us

$$\omega_t > n_t.$$

Now, substituting n_t in the above with (14) in Lemma 2, we get

$$m_t > \frac{\delta(1-\lambda^t) + \lambda\phi\omega_t(1-\lambda^t) - y\omega_t(1-\omega_t+\omega_t\lambda^t - \lambda^{t+1})}{1-\lambda}.$$

Applying this inequality to (B.4) produces a lower bound for ω_{t+1}^2, which can be shown to reduce to

$$\omega_{t+1}^2 > \omega_t^2 + \frac{(1-\lambda)(1-\omega_t)\omega_t}{1-\lambda^{t+1}},$$

which implies that

$$\omega_{t+1} > \omega_t.$$

Now we let $t = 2$ and show that $\omega_2 > \omega_1$. Assuming that $m_1 = \delta$, we can show by (B.4) with $t = 1$ that

$$\begin{aligned}\omega_2^2 - \omega_1^2 = \omega_1 \cdot &\frac{y(1+\lambda+\lambda^2) - \lambda\phi(1+\lambda)}{y(1+\lambda)}\\ &- \omega_1^2 \cdot \frac{y + (2\lambda+\lambda^2)(y-\phi)}{(y+\lambda y - \lambda\phi)(1+\lambda)}.\end{aligned}$$

The right-hand side in the above equation is quadratic and concave in ω_1. Therefore, can check the endpoint cases $\omega_1 = 0$ and $\omega_1 = 1$. The first case makes the above right side zero. For the second case, the right-hand side reduces to

$$\frac{\lambda^2(y-\phi)^2}{y(y+\lambda y - \lambda\phi)} \ge 0.$$

Hence, we have $\omega_2 > \omega_1$ for any $\omega_1 \in (0,1)$. □

Proof of Proposition 3. For part (i), to prove the existence, we need to show that $n_t, m_t, \omega_t \in (0,1)$ under the recursive characterization in Lemma 2.

Propositions 1 and 2 rely on Lemma 2 to show that ω_t increases over time but at a slower rate than ω_t^{INV}, where the superscript *INV* denotes the investor-only case and satisfies the condition in Lemma 1. Hence, if the primitives are such that $\omega_t^{INV} \in (0,1)$ for all t, then we must have $\omega_t \in (0,1)$ for all t.

As to the mass of investors, with $m_1 = \delta$, we have

$$n_1 = \frac{(\delta + y\omega_1^2)(1-\lambda) - (1-\lambda)\delta}{y(1-\lambda^2) - \phi\lambda(1-\lambda)} = \frac{y\omega_1^2}{y + (y-\phi)\lambda},$$

which guarantees that $n_1 \in (0,1)$. For $t \ge 1$, we have $n_{t+1} = \omega_t$, so we have $n_{t+1} \in (0,1)$ as well.

As to the mass of users, it evolves by

$$m_{t+1} = \delta + \lambda m_t + \lambda\phi n_t.$$

Recall the restriction on parameters that $\delta + \lambda + \lambda\phi < 1$, so we have $m_t \in (0,1) \Rightarrow m_{t+1} \in (0,1)$. As a result, with $m_1 = \delta \in (0,1)$, we have $m_t \in (0,1)$ for all t.

For part (ii), for the special case $y = \phi$, we have, from (B.4)

$$\omega_{t+1}^2 = \omega_t - \frac{\lambda - \lambda^{t+1}}{1-\lambda^{t+1}}(\omega_t - \omega_t^2),$$

which implies that $\omega_{t+1}^2 < \omega_t$ as well as $\omega_{t+1} > \omega_t$. This guarantees us that $\omega_t \in (0,1)$ for any t, as long as $\omega_1 \in (0,1)$.

For part (iii), finally, consider $y \to +\infty$. We have

$$\begin{aligned}\omega_{t+1}^2 &\to \frac{(1-\lambda)\omega_t^2}{1-\lambda^{t+1}} + \frac{(1-\lambda)(1-\omega_t)\omega_t}{1-\lambda^{t+1}} - \delta + \lambda\omega_t\\ &= \frac{1-\lambda^{t+2}}{1-\lambda^{t+1}}\omega_t.\end{aligned}$$

This relation is the same as in the investor-only case of Lemma 1. □

Proof of Lemma 3. Equations (14) and (15) tell us that

$$m_{t+1} = \delta + \lambda m_t + \lambda\phi\frac{(\delta + y\omega_t^2)(1-\lambda^t) - (1-\lambda)m_t}{y(1-\lambda^{t+1}) - \phi\lambda(1-\lambda^t)}.$$

Hence, for all t,

$$m_t < m_{t+1} \Leftrightarrow m_t < y\frac{\delta(1-\lambda^{t+1}) + \omega_t^2\phi(\lambda - \lambda^{t+1})}{(1-\lambda)(y - y\lambda^{t+1} + \phi\lambda^{t+1})}.$$

Now, fix any t. Given $m_{t-1} < m_t$, we have

$$m_{t-1} < y\frac{\delta(1-\lambda^t) + \omega_{t-1}^2\phi(\lambda-\lambda^t)}{(1-\lambda)(y - y\lambda^t + \phi\lambda^t)}. \tag{B.5}$$

By contradiction, suppose that $m_t \ge m_{t+1}$. Then

$$m_t \ge y\frac{\delta(1-\lambda^{t+1}) + \omega_t^2\phi(\lambda - \lambda^{t+1})}{(1-\lambda)(y - y\lambda^{t+1} + \phi\lambda^{t+1})}, \tag{B.6}$$

by (14) in Lemma 2, which implies

$$\omega_{t-1} = n_t \le \frac{y\omega_t^2(1-\lambda^t) - \delta\lambda^t}{y - (y-\phi)\lambda^{t+1}}. \tag{B.7}$$

The equality above also comes from Lemma 2.

Next, using both (14) and (15), we have

$$m_t = \delta + \lambda m_{t-1} + \lambda\phi\frac{(\delta + y\omega_{t-1}^2)(1-\lambda^{t-1}) - (1-\lambda)m_{t-1}}{y(1-\lambda^t) - \phi\lambda(1-\lambda^{t-1})}.$$

Substituting m_{t-1} in the above with inequality (B.5) gives

$$m_t < y\frac{\delta(1-\lambda^t) + \omega_{t-1}^2\phi(\lambda-\lambda^t)}{(1-\lambda)(y - y\lambda^t + \phi\lambda^t)},$$

which, under (B.7), implies that

$$m_t < y\frac{\delta(1-\lambda^t) + \phi(\lambda-\lambda^t)\frac{y\omega_t^2(1-\lambda^t)-\delta\lambda^t}{y-(y-\phi)\lambda^{t+1}}}{(1-\lambda)(y - y\lambda^t + \phi\lambda^t)}.$$

We reach a contradiction if we can show that the left-hand side of the above inequality is smaller than the left-hand side of (B.6). It can be shown that the difference between the two left-hand sides amounts to

$$-\frac{y\phi\lambda^t(1-\lambda^t)[\delta + \lambda\phi\omega_t^2 + (1-\lambda)y\omega_t^2]}{(1-\lambda)[y-(y-\phi)\lambda^t][y-(y-\phi)\lambda^{t+1}]} < 0.$$

So we indeed reach a contradiction, which proves the inductive claim

$$m_{t-1} < m_t \Rightarrow m_t < m_{t+1}.$$

Finally, let us look at the first period. We will show that $m_1 < m^*$ implies $m_1 < m_2$. By Lemma 2, we have

$$m_2 = \delta + \lambda m_1 + \lambda\phi\frac{(\delta + y\omega_1^2)(1-\lambda) - (1-\lambda)m_1}{y(1-\lambda^2) - \phi\lambda(1-\lambda)}.$$

With the above expression for m_2, it can be shown than

$$m_1 < m_2 \Leftrightarrow m_1 < \underbrace{\frac{y(\delta + \lambda\delta + \lambda\phi\omega^2)}{y - \lambda^2(y-\phi)}}_{\Xi}.$$

If $m^* < \Xi$, then $m_1 < \Xi$ and we are done. If $m^* \geq \Xi$, however, then

$$\delta\lambda + \lambda y\omega_1^2 - y\omega_1^2 \geq 0,$$

which can be shown to imply

$$\delta + y\omega_1^2 \leq \Xi.$$

However, for the bubbly equilibrium to exist, we must have $n_1 > 0$. Using (14) in Lemma 2, we see that $n_1 > 0$ implies

$$m_1 < \delta + y\omega_1^2.$$

As a result, we have $m_1 < \Xi$, and again we are done. □

Proof of Proposition 4. By Lemma 3, we know that m_t is monotonically increasing, which, together with the fact that m_t is bounded above by 1, implies that m_t converges to a limit. We denote this limit as $\bar{m}$. Similarly, we know that ω_t converges and denote the limit by $\bar{\omega}$.

Using (14) and (15) in Lemma 2, we have

$$\bar{m} = \delta + \lambda\bar{m} + \lambda\phi\frac{(\delta + y\bar{\omega}^2) - (1-\lambda)\bar{m}}{y - \phi\lambda},$$

which has the unique solution

$$\bar{m} = \frac{\delta + \lambda\phi\bar{\omega}^2}{1-\lambda}.$$

Using (16) in Lemma 2, we have

$$\bar{\omega}^2 = \frac{(1-\lambda)\bar{m} + \bar{\omega}(y - \phi\lambda) - \delta}{y} = \frac{\lambda\phi\bar{\omega}^2 + \bar{\omega}(y-\phi\lambda)}{y},$$

which has two solutions, either $\bar{\omega} = 0$ or $\bar{\omega} = 1$. As long as $\omega_1 > 0$, it must be that $\bar{\omega} = 1$. Hence,

$$\bar{m} = \frac{\delta + \lambda\phi}{1-\lambda} = m^* + \frac{\lambda\phi}{1-\lambda},$$

which completes our proof. □

Endnotes

[1] Even Facebook plans to launch a cryptocurrency, Libra, in the near future (see Horwitz and Olson 2019).

[2] This is achieved by focusing on the case where the positive effect of user adoption on the fundamental value is exactly offset by the negative effect of the growth in the cryptocurrency supply. Keeping the fundamental value constant allows us to compare the settings with and without users on an equal footing (the fundamental value in the no-user setting is constant at zero).

[3] Our model tells a similar story to explain another major price spike of bitcoin in late 2017. Earlier in 2017, several countries—Japan, Russia, and Norway—announced their legitimization of the cryptocurrency, thereby increasing the potential user base and, subsequently, according to our model, relaxing the confidence required for a bubble.

[4] It is not difficult to generalize our model to $r > 0$ by introducing a growth rate on the investor population or the dollar holding of each investor. Results will remain the same qualitatively, albeit with more complex algebra.

[5] Risk neutrality of all agents is assumed throughout the model. This assumption simplifies the analysis, though we acknowledge that assuming alternative risk preferences could affect the results.

[6] In Appendix A.1, we show a one-to-one mapping between $\omega_t \in (0,1)$ and $\alpha_t \in (0,+\infty)$. To obtain analytical solutions, we make use of the discrete version in (1) throughout the main text.

[7] In Weil (1987), however, the confidence is fixed constant. As a consequence, the speculative price does not surge up, as we often observe in real bubbles. We let confidence be time varying to more closely capture price surges.

[8] Under rational expectation, one can show that $p_t = 0$ implies either a crash has happened or the prices in all periods are 0 (the nonbubbly equilibrium). On the other hand, $p_t = L_{t-1}$ implies the expected return in t is less than 1 so $n_t = 0$, which contradicts $p_t > 0$.

[9] The best known application of this condition is perhaps the arbitrage pricing theory, introduced by Ross (1976). Also see Roll and Ross (1980) and a more recent discussion in Malkiel (2003).

[10] Other equilibria may exist under alternative specifications of investor belief (1), for example, $p_{t+1} = \omega_t^2 L_t$ or $\sqrt{\omega_t}L_t$ with probability ω_t. We thank an anonymous referee for pointing this out.

[11] There are a few empirical studies in marketing that extend adoption models to forward-looking consumers, including Song and Chintagunta (2003) and Nair (2007).

[12] We acknowledge that adoption costs, in reality, may decline over time because of progresses in adoption and infrastructure. If so, adoption would accelerate further than shown here. For simplicity, therefore, we keep the adoption costs exogenous while allowing the benefits to be endogenous.

[13] In practice, holding a cryptocurrency poses risks associated with technological or legal disruptions. For example, some cryptocurrencies have experienced hacks of cryptoexchanges and trading bans by governments. Although our main model abstracts away from such risks, we discuss the impacts of these disruptions in Section 4.

[14] In Appendix, Section A.2, we discuss in more detail how our supply model reflects the institutional realities associated with bitcoin.

[15] To see this, first suppose $N = 2$. We can transform it into $N = 1$ by thinking of each investor in the model as representing two actual investors. We also need to double y, the dollar holding of each investor in the model. A similar transformation applies to $M \neq 1$. As to x, notice that doubling the values for both x and y will only have a nominal effect: the prices in all periods double. Hence, between x and y, we can normalize one of them to 1.

[16] In Appendix A, Section A.4.1, we consider an extension allowing users to exert nonlinear network effects.

[17] In Appendix A, Section A.4.2, we consider an extension where investors may also value liquidity (as users do) in addition to expected return.

[18] This can be seen from Proposition 2 extended to include $\omega_t = 1$. The same proof will show that $\omega_{t+1} \geq \omega_t$ with $\omega_{t+1} = \omega_t$ with the equality holding only when $y = \phi$. Hence, aside from the $y = \phi$ case, $\omega_t = 1$ would imply $\omega_{t+1} > 1$, a contradiction.

[19] For such a postcrash state to be sustainable, investors must have no incentive to start buying coins again. We verify this as follows. Let t be the period when the crash has happened. Then, $p_t = S_{t-1} = m_t/A_t$ and $p_{t+1} = m_{t+1}/A_{t+1}$. Because the bubble sped up adoption, we have $m_t > \frac{1-\lambda^t}{1-\lambda}\delta$. So $\frac{m_{t+1}}{m_t} = \frac{\delta+\lambda m_t}{m_t} < \frac{1-\lambda^{t+1}}{1-\lambda^t}$. As a result, $\frac{p_{t+1}}{p_t} < 1$.

[20] Some readers might notice that ω_t here evolves independently of m_t. This result is also special to the $y = \phi$ condition.

[21] Another way to see the intuition here is to note if we let m_t follow its natural path by setting $\phi = 0$, Equation (16) reduces to (3). Hence, the natural user adoption by itself cannot produce Proposition 1.

[22] To see this, consider the case where there are $M = 2$ units of users. We normalize M to 1 by treating every two actual users as one user in the model. However, this normalization means that each user in the model holds $x = 2$ units of endowment. So we further normalize this endowment to $x = 1$ by deflating dollars by a factor of two. A consequence of this deflation is that investor endowment, y, needs to be halved as well.

[23] For an example, $y = 0.25$, $\phi = 1$, $\lambda = \delta = 0.25$, and $\omega_1 = 0.5$.

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