Implementing the MM algorithm for a linear model

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Data preparation

```
## first, we load the air quality data

# load the air quality data
load("Data/Airq_numeric.Rdata")

# set to dataframe
dfAirQ <- data.frame(Airq)

# select dependent variable of air quality
Yair = dfAirQ$airq

# select all other variables as independent variables
Xair = dfAirQ[,-1]

# scale the independent variables, and add an intercept to these
XairScaled <- scale(Xair)
XairIntercept <- cbind(intercept = 1, XairScaled)


## second, we follow a similar procedure for the advertising data

# load the advertising data
load("Data/Advertising.Rdata")

# set the advertising to dataframe
dfAdv <- data.frame(Advertising)

# select dependent variable of sales
YAdv = dfAdv$Sales

# select independent variable - for this example we use TV
XAdv = data.frame(dfAdv$TV)

# scale the independent variables, and add an intercept to these
XAdvscaled <- scale(XAdv)
XAdvintercept <- cbind(intercept = 1, XAdvscaled)
colnames(XAdvintercept) <- c("intercept", "TV")
```

Compare the linear model with MM to the standard lm() function

```r
# set seed to ensure stability of results
set.seed(0)

# set e small
e <- 0.0001

# calculate the model using the MM algorithm
modelMM <- calcModelMM(XairIntercept, Yair, e)

# calc the model using the standard R library
modelTest <- lm(airq ~ XairScaled,dfAirQ)

# set notation of numbers
options(scipen = 12)


# Compare the sum of squared errors
ResiMM <- modelMM$RSS
ResiTest <- sum(resid(modelTest)^2)
dfResi <- dfCompare(ResiMM, ResiTest, c("RSS with MM", "RS with lm()"))
dfResi
```

```
##   RSS with MM RS with lm()
## 1     14071.6     14058.45
```

```r
# Compare R^2
RsquaredMM <- modelMM$Rsquared
RsquaredStandard <- summary(modelTest)$r.squared
dfRsquared <-dfCompare(RsquaredMM, RsquaredStandard, c("R^2 with MM", "R^2 with lm()"))
dfRsquared
```

```
##   R^2 with MM R^2 with lm()
## 1   0.3823472     0.3829221
```

```r
# Compare the beta's
BetaMM <- modelMM$Beta
BetaStandard <- as.double(modelTest$coefficients)
dfBetaCompare <- dfCompare(BetaMM, BetaStandard, c("Beta with MM", "Beta with lm()"))
dfBetaCompare
```

```
##           Beta with MM Beta with lm()
## intercept   104.692508     104.700000
## vala          5.603005       4.090287
## rain          3.410146       3.381518
## coasyes     -15.285871     -15.566666
## dens         -2.962245      -3.035229
## medi          5.375968       6.930640
```

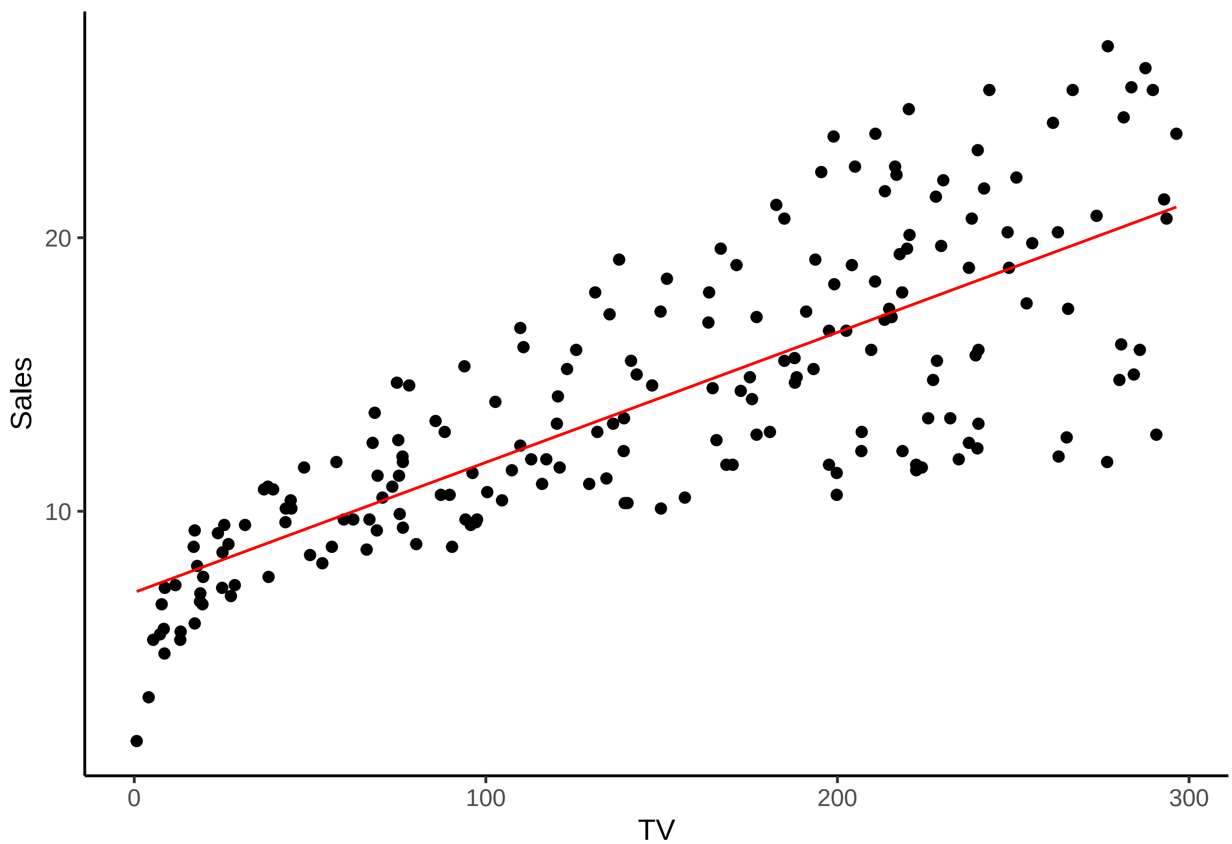
A different example: Bivariate regression with advertising data

```r
# Est. MM model
modelMMadv <- calcModelMM(XAdvintercept, YAdv, e)

# create df to show estimated results
Yestdf <- data.frame(cbind(modelMMadv$Yest, dfAdv$TV))
colnames(Yestdf) <- c("Yest", "TV")

# create plot that shows the line estimated by the MM function
ModelPlot <- ggplot(data = dfAdv, aes(x=TV, y=Sales)) +
  geom_point() +
  geom_line(data = Yestdf, aes(x=TV, y=Yest), color='red') +
  theme_classic()


ModelPlot
```



Appendix: Helper functions that generated the model

Below are all the functions we used for implementing the linear model with the MM algorithm.

```r
# calcRSS
# Calculates the residual squared errors for a multiple regression of the form Y = XBeta + e
#
# Parameters:
#   X: Dataframe of n x p (n = observations, p = independent variables)
#   Y: Dataframe of n x 1 dependent variables (n = observations)
#   Beta: p x 1 double with coefficients of said model
#
# Output:
#   ESquared: float, residual squared errors
#

calcRSS <- function(X, Y, Beta){

  # set the dataframes to matrices
  mX = as.matrix(X)
  mY = as.matrix(Y)
  mBeta = as.matrix(Beta)


  # calculate the errors
  mE <-  mY - mX %*% mBeta

  # get errors squared
  ESquared <- t(mE) %*% mE

  # return the residual sum of squared errors
  return(ESquared)

}

# calcLargestEigen
# Calculates the largest eigenvalue of an matrix of independent variables
#
# Parameters:
#   X: Dataframe of n x p (n = observations, p = independent variables)
#
# Output:
#   LargestEigenval: float, largest eigenvalue of said matrix
#

calcLargestEigen <- function(X){

  # get matrix of squared X
  mX = as.matrix(X)
  XSquared <- t(mX) %*% mX

  # get the eigenvalues of X squared
  EigenValXSquared <- eigen(XSquared)$values

  # from these eigenvalues, get the largest one
  LargestEigenVal <- max(EigenValXSquared, na.rm = TRUE)

```

```
  return(LargestEigenVal)

}

#CalcBetaK
# Calculates the kth beta in an MM algorithm
#
# Parameters:
#   prevBeta: double, k-1th beta
#   Lambda: double, largest eigenvalue of X (independent variables) squared
#   X: Dataframe of n x p (n = observations, p = independent variables)
#   Y: Dataframe of n x 1 dependent variables (n = observations)
#
# Output:
#   BetaK; matrix of new set of coefficients

calcBetaK <- function(prevBeta,Lambda, X, Y){

  # get matrix of squared X
  mX = as.matrix(X)
  XSquared <- t(mX) %*% mX

  # turn Y into matrix
  mY = as.matrix(Y)

  # calculate the Kth
  BetaK = prevBeta - ((1/Lambda) *  XSquared %*% prevBeta) + ((1/Lambda) * t(mX) %*% mY )

  return(BetaK)



}

# CalcStepScore
# Calculates the % improvement between the k-1th and kth set of beta's
#
# Parameters:
#   prevBeta: double, k-1th beta
#   currbeta: double, kth beta
#   X: Dataframe of n x p (n = observations, p = independent variables)
#   Y: Dataframe of n x 1 dependent variables (n = observations)
#
# Output:
#   StepScore; double, % improvement between the RSS of the two sets of beta's

calcStepScore <- function(X,Y, prevBeta, currBeta){

  # difference in RSS between previous and current set of beta's
  diffRSS <- (calcRSS(X,Y,prevBeta) - calcRSS(X,Y,currBeta))

  # divide difference with previous score to get % change
  StepScore <- diffRSS /calcRSS(X,Y,prevBeta)
```

```
  return(StepScore)

}

# getB0
# Set the initial set of Beta's, randomly, between 0 and 1
#
# Parameters:
#   X: Dataframe of n x p (n = observations, p = independent variables)
#
# Output:
#   B0: double, set of Beta's between 0 and 1

getB0 <- function(X){

  # determine the number of independent variables, generate as many random beta's
  nIndVar = ncol(X)
  Beta0 <- runif(nIndVar, min=0, max=1)

  return(Beta0)

}

# calcYest
# Calculates the predicted Y, based on the X and est. Beta's of a linear model
#
# Parameters:
#   X: Dataframe of n x p (n = observations, p = independent variables)
#   BetaEst: Estimated Beta's, px1 vector,
#
# Output:
#   Yestdf: Dataframe, predicted Y
#

calcYest <- function(X,BetaEst){

  # turn X and Beta's (est.) into matrix
  mBetaEst <- as.matrix(BetaEst)
  mX <- as.matrix(X)

  # multiply X with Beta (est.) to get predicted Y
  Yest <- mX %*% mBetaEst

  # turn into dataframe
  Yestdf <- as.data.frame(Yest)
  colnames(Yestdf) <- c("Yest")

  return(Yestdf)

}

# calcRsquared
# Calculates the r-squared
```

```r
#
# Parameters:
#   Y: dataframe, the true dependent variable
#   Yest: dataframe, the predicted dependent variable
#
# Output:
#   Rsquared: double, the Rsquared for a linear model

calcRsquared <- function(Y, Yest){

  # standardize Y, and Yest (mean of 0)
  standardY = Y - mean(Y)
  standardYest = Yest - mean(Yest$Yest)

  # turn into matrix to perform multiplication
  mY <- as.matrix(standardY)
  mYest <- as.matrix(standardYest)


  # calculate Rsquared
  numerator <- (t(mY) %*% mYest)^2
  denominator <- (t(mY) %*% Y) %*% (t(mYest) %*% mYest)
  Rsquared <- (numerator/denominator)

  return(Rsquared)


}


# calcModelMM
# Calculates a linear model, using the majorization in minimization (MM) algorithm
#
# Parameters:
#   X: Dataframe of n x p (n = observations, p = independent variables)
#   Y: Dataframe of n x 1 dependent variables (n = observations)
#
#
# Output:
#   model: dataframe, with the following attributes
#       - Beta: dataframe, the calculated Beta's
#       - RSS: double, Sum of squared residuals
#       - Yest: dataframe, the predicted Y
#       - Rsquared: double, R^2 for the predicted Y
#       - Residuals: dataframe, Y - Yest.
#


calcModelMM <- function(X,Y,e){

  # set the previous beta to initial, random beta's
  prevBeta <- getB0(X)
```

```
  # get largest eigenvalue for the square of independent variables
  Lambda <- calcLargestEigen(X)

  # set initial stepscore to 0, k to 1.
  StepScore <- 0
  k <- 1


  # run while, either if k is equal to 1, or the improvement between k-1th and kth set of beta's is sma
  while (k == 1 | StepScore > e ){

    # step to next k
    k <- k + 1

    # calculate beta's for this k
    BetaK <- calcBetaK(prevBeta, Lambda, X,Y)

    # new stepscore, % difference in RSS between new Beta's and previous beta's
    StepScore <- calcStepScore(X,Y,prevBeta,BetaK)

    # assign current beta's to prevBeta variable for next iteration
    prevBeta <- BetaK

  }


  # calculate several attributes of the linear model, put in dataframes or doubles
  BetaFinal <- data.frame(BetaK)
  RSSBetaK <- calcRSS(X,Y, BetaK)
  Yest <- calcYest(X, BetaFinal)
  Rsquared <- calcRsquared(Y, Yest)
  Resi <- data.frame(residuals = Y - Yest$Yest)

  # add these attributes together as a list to make it easily accessible
  results <- list(Beta = BetaFinal, RSS = RSSBetaK, Yest = Yest, Rsquared = Rsquared, Residuals = Resi)


  return(results)

}
```