Psychological Reports, 1991, 68, 855-858.

COEFFICIENT ALPHA FOR A PRINCIPAL COMPONENT AND THE KAISER-GUTTMAN RULE [1]

HENRY F. KAISER [2]

Educational Testing Service,
Berkeley, California

Summary.—The formula for coefficient alpha for a principal component, first derived in 1957 by Kaiser, is developed. Its use in the Kaiser-Guttman Rule for the number of components is discussed, both in theory and in practice, with Hotelling's (1933) original correlation matrix.

More than 33 years ago, in the fall of 1957, as a fresh assistant professor of education at the University of Illinois, I derived the formula for Cronbach's (1951) coefficient alpha for a principal component (Kaiser, 1957). For many reasons—none of them very good—I did not publish this development formally, although I mentioned the particular result that coefficient alpha for a principal component would be positive if and only if its associated eigenvalue was greater than one (Kaiser, 1960). This provoked the far and away most popular Rule for the "number of components" (Kaiser, 1960, 1986): accept only those principal components with associated eigenvalues greater than one.

Consider a linear composite x_s:

$$x_s = \sum_{j=1}^{p} w_{sj} z_j \qquad [1]$$

where z_j is element j of the linear composite and w_{sj} is the weight for z_j to yield its portion of x_s. Alpha (s) is given most generally by

$$\text{alpha}(s) = \left(\frac{p}{p-1}\right)\left(1 - \frac{\sum_{j=1}^{p} w_{sj}^2 s_j^2}{s_x^2}\right) \qquad [2]$$

where p is the number of elements in the composite, s_j^2 is the variance of element j of the composite, and where s_x^2 is the variance of the composite x_s.

The conventional scaling of component analysis establishes all p of the element variances s_j^2 equal to one. The variance s_x^2 of the composite x_s is also

---

[1]The research reported in this paper was supported in part by the Office of the Dean, Graduate School of Education, University of California, Berkeley.
[2]Correspondence should be addressed to Professor Henry F. Kaiser, Graduate School of Education, University of California, Berkeley, Berkeley, California 94720.

taken to be one. Then alpha(s) for a composite x_s which is now a component x_s, given this conventional scaling, is simply

$$\text{alpha}(s) = \left(\frac{p}{p-1}\right)\left(1 - \sum_{j=1}^{p} w_{sj}^2\right) \qquad [3]$$

the general formula for alpha of component x_s.

Thus we need to evaluate the expression in the final parenthesis to use [3] to find alpha for principal component x_s. Then, shifting to matrix algebra, the fundamental postulate of component analysis says that $z = Ax$ where z is the p vector-attribute of the p elements z_j, where A, of order $p \times q$, $p \geq q$, is the primary component pattern matrix, and x is the q vector-component of the q components x_s. Then from $z = Ax$,

$$x = (A'A)^{-1}A'z\ , \qquad [4]$$

or then letting $W' = (A'A)^{-1}A'$,

$$x = W'z\ . \qquad [5]$$

But, in particular, for *principal* components,

$$A'A = M^2 \qquad [6]$$

where M^2 is the $q \times q$ *diagonal* matrix of eigenvalues. That is, A is orthogonal by columns for principal components, with column sums-of-squares equal to eigenvalues. [Notationally, I designate an eigenvalue as a squared quantity, as they often represent variances; also, it is convenient to have singular values (= eigenvalues$^{1/2}$) without superscripts.]

From [4], [5], and [6],

$$W' = (A'A)^{-1}A'\ , \qquad [7]$$

$$W' = M^{-2}A'\ . \qquad [8]$$

Now we have the diagonal matrix of alpha for all q principal components

$$\text{diag(alpha)} = \left(\frac{p}{p-1}\right)(I - \text{diag}\ W'W) \qquad [9]$$

using the diagonal matrix of row sums-of-squares of W', which clearly is diag $W'W$.

Then, remembering [8],

$$\text{diag}\ W'W = \text{diag}\ (M^{-2}(A'A)M^{-2}) \qquad [10]$$

$$= \text{diag}\ (M^{-2}M^2M^{-2}) \qquad [11]$$

$$= \text{diag}\ (M^{-2}) \qquad [12]$$

from which

$$\text{diag(alpha)} = \left(\frac{p}{p-1}\right)(I - \text{diag}\, M^{-2}) \tag{13}$$

For an individual principal component x_s:

$$\text{alpha(s)} = \left(\frac{p}{p-1}\right)\left(1 - \frac{1}{m_s^2}\right) \tag{14}$$

as it most often has been written.

Clearly, only when $m_s^2 > 1$ will alpha(s) > 0. When alpha (s) ≤ 0, its principal component must surely be rejected (Cronbach, 1951). And, remembering that coefficient alpha is a lower bound to a composite's reliability (Guttman, 1945), any positive alpha may underlie a composite with substantial reliability and should be accepted.

In any case, the so-called Kaiser-Guttman Rule (as so dubbed by Raymond B. Cattell), suggested in the preceding paragraph, is used ubiquitously in the Real World (Kaiser, 1986) for establishing the "number of components." And while occasionally it disagrees with the best subjective judgment of factor-analytic grandmasters, it appears usually to give a result for the number of components which is in the Right Ballpark. Lewis R. Goldberg (personal communication, 1991) tells me that the Kaiser-Guttman Rule may overfactor when p is large and that it may underfactor when p is small. In general, then, the Kaiser-Guttman q is perhaps too strongly correlated with its p. But it is almost always the default option at computer centers around the world, resulting it its being used hundreds of thousands of times since 1956, when it first appeared in my doctoral dissertation (Kaiser, 1956) without the presently derived coefficient alpha rationale.

Finally, I wish I had had the foresight over the last 33 years, to jot down every one of the many times, now forgotten, this Rule has "popped out" in seemingly unrelated contexts. For example, one recalled is my coefficient gamma (Kaiser, 1968):

$$\text{gamma(s)} = \frac{m_s^2 - 1}{p - 1},$$

a measure of "average" intercorrelation, and clearly *another rationale for the* Kaiser-Guttman Rule.

Several anonymous referees have suggested I prepare a small numerical example of the application of the Kaiser-Guttman Rule.

Consider the correlation matrix of order $p = 4$, taken from Hotelling's (1933) basic paper on principal components:

$$\begin{bmatrix} 1.000 & 0.698 & 0.264 & 0.081 \\ 0.698 & 1.000 & -0.061 & 0.092 \\ 0.264 & -0.061 & 1.000 & 0.594 \\ 0.081 & 0.092 & 0.594 & 1.000 \end{bmatrix}$$

The eigenvalues of this matrix are: $m_1^2 = 1.846$, $m_2^2 = 1.465$, $m_3^2 = .521$, $m_4^2 = .167$. From Formula [14], the associated coefficients alpha are: alpha(1) = .611, alpha(2) = .423, alpha(3) = −1.226, alpha(4) = −6.651. Thus, following the Kaiser-Guttman Rule, we retain the initial $q = 2$ principal components and censor the final two.

REFERENCES

CRONBACH, L. J. (1951) Coefficient alpha and the internal structure of tests. *Psychometrika*, 16, 297-334.

GUTTMAN, L. (1945) A basis for analyzing test-retest reliability. *Psychometrika*, 10, 255-282.

HOTELLING, H. (1933) Analysis of a complex of statistical variables into principal components. *Journal of Educational Psychology*, 24, 417-441, 498-520.

KAISER, H. F. (1956) The varimax method of factor analysis. Unpublished doctoral dissertation, School of Education, Univer. of California, Berkeley.

KAISER, H. F. (1957) Alpha-reliability of components and factors. (Unpublished manuscript, Bureau of Educational Research, College of Education, Univer. of Illinois)

KAISER, H. F. (1960) The application of electronic computers to factor analysis. *Educational and Psychological Measurement*, 20, 141-151.

KAISER, H. F. (1968) A measure of the average intercorrelation. *Educational and Psychological Measurement*, 28, 245-247.

KAISER, H. F. (1986) The application of electronic computers to factor analysis. *Current Contents*, 40, 18.

Accepted April 8, 1991.