Week 1 Notes: MATA35

TA: Fergus Horrobin

1 Matrix Operations

1.1 Scalar Multiplication

Multiplying a matrix by a scalar is equivalent to multiplying each element by the same scalar. This means that $cA = c[a_{ij}] = [ca_{ij}]$ for some $c \in \mathbb{R}$. For example:

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 5 & 7 & 8 \\ 4 & 6 & 2 \end{bmatrix}$$

Then, if we want to find $3A$ we compute:

$$3A = 3\begin{bmatrix} 3 & 5 & 2 \\ 5 & 7 & 8 \\ 4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 3(3) & 3(5) & 3(2) \\ 3(5) & 3(7) & 3(8) \\ 3(4) & 3(6) & 3(2) \end{bmatrix} = \begin{bmatrix} 9 & 15 & 6 \\ 15 & 21 & 24 \\ 12 & 18 & 6 \end{bmatrix}$$

1.2 Transpositions

The transpose of a matrix A is the matrix A^T such that the rows are A are interchanged with the columns. Example:

$$A = \begin{bmatrix} 2 & 9 & 2 & 12 \\ 5 & 7 & 8 & 1 \end{bmatrix}$$

Then swapping the rows and columns of A we find that:

$$A^T = \begin{bmatrix} 2 & 5 \\ 9 & 7 \\ 2 & 8 \\ 12 & 1 \end{bmatrix}$$

Notice that A is a $2x4$ matrix whereas A^T is a $4x2$ matrix (the order of the dimensions changes).

There are also some properties of the transpose:

$$(A^T)^T = A$$
$$(A \pm B)^T = A^T + B^T$$
$$(kA)^T = kA^T$$
$$(A \cdot B)^T = B^T A^T$$

We can easily check any of these by trying them out with a matrix and some numbers (I will show you one here, I suggest you try and come up with an example for each of them to convince yourself that they are true). Let's try showing the 3rd one is true using the matrix we just took the transpose of.

According to the 3rd property, $(kA)^T = kA^t$. Let $k = 5$ and try to test it:

$$5A^T = \begin{bmatrix} 10 & 25 \\ 45 & 35 \\ 10 & 40 \\ 60 & 5 \end{bmatrix}$$

And we can see that:

$$(5A)^T = \begin{bmatrix} 10 & 45 & 10 & 60 \\ 25 & 35 & 40 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 25 \\ 45 & 35 \\ 10 & 40 \\ 60 & 5 \end{bmatrix} = 5A^T$$

So the identity has been shown to hold.

Finally, note that we call a matrix symmetric if it is a square matrix and $A^T = A$.

1.3 Addition

We can add two matrices together. The sum of matrices A and B, written as $A \pm B$ is given by:

$$(a \pm b)_{ij} = a_{ij} \pm b_{ij}$$

In other words, the sum of matrices is represented by the sum of their components.

For example, if we define:

$$A = \begin{bmatrix} 5 & 10 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 13 & 3 \\ 1 & 8 \end{bmatrix}$$

Then, we can calculate the sum $A + B$ as:

$$A + B = \begin{bmatrix} 5+13 & 10+3 \\ 2+1 & 3+8 \end{bmatrix} = \begin{bmatrix} 18 & 13 \\ 3 & 11 \end{bmatrix}$$

Notice that in order for the sum of two matrices to be defined they must be the same size. This means that if A_{nxm} and B_{kxl} then $n = k$ and $m = l$.

1.4 Matrix Multiplication

Finally, we can multiply matrices together. When multiplying two matrices together, we find the $ij-$th entry of the product $A \cdot B$ by taking the dot product of the $i-$th row of A and the $j-$th column of B.

Since the dot product is only defined if two vectors have the same length, this means that the length of a row vector of A must equal the length of a column vector of B. In other words, we must have that A_{mxl} and B_{lxn}. (Think about this for a second to make sure you understand this requirement).

Question: What will the dimension of the resultant matrix be?

Let's try an example, let:

$$A = \begin{bmatrix} 5 & 10 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 8 \end{bmatrix}$$

Then, compute $A \cdot B$:

$$A \cdot B = \begin{bmatrix} (5,10)\cdot(2,1) & (5,10)\cdot(3,8) \\ (2,3)\cdot(2,1) & (2,3)\cdot(3,8) \end{bmatrix} = \begin{bmatrix} 20 & 95 \\ 7 & 30 \end{bmatrix}$$

Some aspects of matrix multiplication are not directly analogous to real number multiplication, therefore we must be careful. Here are 3 common mistakes you must avoid making:

1. Matrix multiplication is not commutative. This means that in general:

$$A \cdot B \neq B \cdot A$$

2. The cancellation law fails for matrices. For example:

$$A \cdot B = A \cdot C \not\Rightarrow B = C$$

3. If $A \cdot B = 0$, it does not necessarily mean that $A = 0$ or $B = 0$.

I will not attempt to prove or demonstrate the above statements due to time constraints but I suggest you try to check examples of each one until you are familiar with these rules. There are some simple examples in the week 1 lecture slides.

2 Elementary Row Operations

There are 3 elementary row operations we can use to modify matrices. If we can obtain a matrix B from a matrix A by simply performing elementary row operations, we say that A and B are similar matrices.

The operations we can perform are:

1. Interchanging two rows: Swap one row with another row in the matrix.
2. Replacement: Replace a row with the sum of the row and some scalar multiple of another row.
3. Scaling: Multiply a row by a constant (not 0).

We are mainly interested in seeing examples of where this is used/useful.

2.1 REF and RREF

We will try to reduce matrices to REF or RREF. First, the definitions are:

REF:

1. Rows with all zero entries are at the bottom.
2. The first non-zero entry of a row is to the right of the first non-zero entry of the row above it.
3. Entries in columns below the leading entries are 0.

RREF:

1. 1, 2, 3 from above are still satisfied (ie it is REF).
2. The leading entries are all 1.
3. Each leading 1 is the only non-zero entry in the column.

Let's consider an example. Reduce the following matrix, A to RREF. Show which of the 3 elementary row operations you use in each step.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 8 & 2 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 2 & 2 & 0 \\ 1 & 8 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[R_1 = R_1/2]{R_2 = R_2 - R_1/2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 7 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2/7} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2/7 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3=7R_3/3]{R_3=R_3-2R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2/7 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that now it is in REF, let's continue to make it RREF.

$$\xrightarrow{R_1=R_1-R_2} \begin{bmatrix} 1 & 0 & -2/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2=R_2-2/7R_3]{R_1=R_1+2/7R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And now the matrix is in RREF. Note that the order of the steps is not important so if you try it yourself, your steps may look different. Also note that although he RREF is unique, the REF is not (for example, the second last step is also REF). We will now look at some simple examples for how to use RREF to solve equations involving matrices.

2.2 Inverse of a Matrix

We have not talked about division at all. Does matrix division exist? If A, B are matrices, can I write: A/B? The answer to this is no! Division is not defined for matrices. However, we still want to define a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$ where I is the identity matrix.

Note that if $c \in \mathbb{R}$ then, $c^{-1} = 1/c$ as we know. How do we define it for matrices?

The easiest way to find the inverse of a matrix (albeit a slightly tedious method) is to use row reduction. To do this, we setup an augmented matrix as follows to find the inverse, A^{-1} of A:

$$[A|I] \xrightarrow{\text{Reduce}} \left[I|A^{-1}\right]$$

This is never too complicated but can take a bit of time with larger matrices. Let's do a simple example with 2x2 matrix so you see how it works.

Suppose $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$. Find A^{-1}.

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2=R_2-2R1} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -7 & -2 & 1 \end{array}\right] \xrightarrow{R_2=-R_2/7} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right]$$
$$\xrightarrow{R_1=R_1-4R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1/7 & 4/7 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right] = \left[I|A^{-1}\right]$$

So we see that $A^{-1} = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix}$.

Exercise: Check that $AA^{-1} = A^{-1}A = I$ for this example.

2.3 Solving Systems of Equations

We can also use matrices to solve a system of algebraic equations. We will look at two ways to do this, first by row reducing the augmented matrix for the system, and then by matrix inversion.

Consider the following system of equations:

$$x + 4y = 5$$
$$2x + y = 4$$

We can write this equation in matrix form as:

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Let's determine values of x and y that satisfy this system of equations using each of the methods.

2.3.1 By Row Reduction

To solve by row reduction, we must setup and row reduce the augmented matrix as follows:

$$\left[\begin{array}{cc|c} 1 & 4 & 5 \\ 2 & 1 & 4 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & -7 & -6 \end{array}\right] \xrightarrow{R_2 = -R_2/7} \left[\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 6/7 \end{array}\right] \xrightarrow{R_1 = R_1 - 4R_2} \left[\begin{array}{cc|c} 1 & 0 & 11/7 \\ 0 & 1 & 6/7 \end{array}\right]$$

So the solution to this system is: $x = 11/7$ and $y = 6/7$. You could have found this by algebra as you have probably done in the past, but this method will prove more convenient for large systems of equations, particularly when finding solutions using a computer.

2.3.2 By the Inverse Matrix Method

The second way we can solve the system is by using the inverse matrix. Recall the matrix form of the system of equations. Then let:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Then we can write the system of equations as:

$$A\mathbf{x} = \mathbf{b}$$

Which looks like an algebraic equation which we could solve for $\mathbf{x}$ by dividing by A. Now remember that since A is a matrix, we cannot divide by it, however, we can multiply both side by A^{-1} as follows:

$$A\mathbf{x} = \mathbf{b} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

First note that matrix multiplication is non-commutative so be careful if you multiply the front of the LHS to also multiply in front of the RHS. Now we can use that $A^{-1}A = I$ and $I\mathbf{x} = \mathbf{x}$ to write:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

We already found the inverse of this matrix above, so we can calculate this easily as:

$$\mathbf{x} = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 11/7 \\ 6/7 \end{bmatrix}$$

Which is the same result as we found above by row reduction.