Week 2 Notes: MATA35

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1 Systems of Linear Equations

Last week we briefly saw some of the methods that we may use to solve a system of linear equations using matrices and matrix operations. We would now like to expand on some of the theory surrounding systems of linear equations and try to understand how they are represented, both mathematially and geometrically.

To begin, let's define a linear equation of n variables. This is any equation in the form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

Then we can generalize to a system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

Then, consider that geometrically, each of the equations represents a line in n dimensional space. Therefore, we will have 3 possible cases when considering the existence of a solution.

1. A single, unique solution.
2. No solutions.
3. Infinitely many solutions.

We will not prove this but will examine it graphically.

2 Matrix Notation

We will find it convenient to use matrices to manipulate the systems of equations using matrices. Notice how we can write the equations in the form:

$$AX = B$$

Where A is an $m \times n$ matrix, and X and B are $m \times 1$ matrices.

Notice that we may construct the matrix directly from the system of equations. You should convince yourself that the general system of equations I defined above can be written equally correctly as:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_m \end{bmatrix}$$

Then consider the following definitions:

1. Two linear systems are equivalent if they have the same solution set.
2. The solution sets of two systems are the same if their augmented matrices are row equivalent.

So as a corollary of these statements, combined with what we learned previously about row reduction, notice that if we row reduce the augmented matrix, the resultant matrix is row equivalent to our system. So we introduce two methods of solving these types of equations: Gaussian Elimination and Guass-Jordan Elimination.

3 Gaussian Elimination

Let us work out an example of using Gaussian Elimination. We will consider the system:

$$5x + 7y = 2$$
$$3x + y = 4$$

Which can be written in an augmented matrix as:

$$[A|B] = \left[\begin{array}{cc|c} 5 & 7 & 2 \\ 3 & 1 & 4 \end{array}\right] \xrightarrow[R_1 = R_1/5]{R_2 = R_2 - 3/5R1} \left[\begin{array}{cc|c} 1 & 7/5 & 2/5 \\ 0 & -16/5 & 14/5 \end{array}\right]$$

So then we can use back-substitution to determine the solution. Start with the bottom row:

$$-16/5y = 14/5 \implies y = -14/16 = -7/8$$

Then use the result to substitute for y in the 1st row and continue solving.

$$x + 7/5 \times -7/8 = 2/5 \implies x = 2/5 + 49/40 = 13/8$$

Using Gaussian elimination combines some of the techniques you previously knew while using matrix arithmetic to simplify some of the calculations.

4 Gauss Jordan Elimination

Consider the system:

$$2x - y + 2z = 10$$
$$x - 2y + z = 8$$
$$3x - y + 2z = 11$$

Solve using Gauss-Jordan elimination. To begin, we setup the augmented matrix. Then we will row reduce to RREF and we will be able to easily read off the solution.

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 2 & 10 \\ 1 & -2 & 1 & 8 \\ 3 & -1 & 2 & 11 \end{array}\right] \xrightarrow{R_1 = R_2; R_2 = R1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 2 & -1 & 2 & 10 \\ 3 & -1 & 2 & 11 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 3 & 0 & -6 \\ 0 & 5 & -1 & -13 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 5R_2]{R_2 = R_2/3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow[R_1 = R_1 + 2R_2]{R_3 = -R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Then looking at the RREF, we can easily read off the solution as: $x = 1$, $y = -2$ and $z = 3$.