Week 4 Notes: MATA35

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1 Assignment 1 Review

Overall, people seemed to have a pretty good grasp of the content for assignment 1. Most of the errors were simply in the calculations (be careful when row reducing!) However, I found that people had particular difficulty with one of the more conceptual questions and I think it is quite important so I would like to quickly go over it.

1.1 Example

The question stated: Consider two matrices A and B which are of sizes such that the operation AB is defined.

a) Is $A^T B^T$ necessarily defined?

b) Is $B^T A^T$ necessarily defined?

Let's start by setting up the problem with two general matrices. Let $A_{m\times n}$, then $B_{n\times l}$. Note that the inner indices must be the same (n as I have defined them) because the operation AB is defined.

Now consider what happens when we transpose. Transposing simply reverses the order of the dimension (the column number bewcomes row number and vice versa). Therefore:

$$(A^T)_{n\times m}$$

$$(B^T)_{l\times n}$$

We can see from this that if we try to setup $A^T B^T$ we get:

$$(A^T)_{n\times m}(B^T)_{l\times n}$$

Since the middle numbers are not the same in general (but they could be in the specia case $l = m$) the operation is not necessarily defined.

Now consider $B^T A^T$. Repeating the same steps, we see:

$$(B^T)_{l\times n}(A^T)_{n\times m}$$

But this time, since the inner numbers are the same, n, this operation is guaranteed to be defined. As a side note, we could also know this from one of the identities of transposes. We know that: $(AB)^T = B^T A^T$ and since AB defined and the transpose is always defined, $(AB)^T$ must be defined which means $B^T A^T$ is defined.

2 Quiz 1 Review

On quiz 1 I gave you a question where I asked you to solve a system using Gauss-Jordan Elimination but the system had more unknowns than equations. I want to look at how we solve and present a solution for such a system. First off, it is important to recognize intuitively what this means. Since we have more unknowns than we have equations, we call the system overdetermined, and you should be able to convince yourself that there are an infinite number of possible solutions for such a system. This means we will need to introduce free parameters. Let's see how to do this now. I will use the problem from one of my versions of Quiz 1 as an example.

2.1 Example

Use Gauss-Jordan Elimination to solve the following linear system.

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 3x_4 = 2 \\ 2x_1 + 4x_2 + 4x_3 + 6x_4 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 7x_4 = 4 \end{cases}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 2 & 4 & 4 & 6 & 4 \\ 3 & 6 & 6 & 7 & 4 \end{array}\right] \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow[R_2=-R_2/2]{R_2 \iff R_3} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1=R_1-3R_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Which is now in RREF. Then let $x_2 = t$ and $x_3 = u$ be free parameters. Then the solutions are:

$$x_1 = -1 - 2t - 2u$$
$$x_2 = t$$
$$x_3 = u$$
$$x_4 = 1$$

For any choice of $t, u \in \mathrm{R}$

3 Eigenvalues and Eigenvectors

We want to learn how to find eigenvalues and the associated eigenvectors for a given matrix. This will come in handy for a range of problem, in particular, in the sciences we see eigenvalue problems appear quite frequently in the analysis of differential equations, such as the ones the govern population dynamics or the flow of blood through your arteries.

Let's start with the general eigenvalue equation. Let A be a square matrix, v be a column matrix and be a scalar. $\vec{0}$ represents the zero column matrix. Then an eigenvalue equation has the form:

$$Av = \lambda v \to (A - \lambda I)v = \vec{0}$$

We claim that the equation above has solutions iff $\det(A - \lambda I) = 0$. Then the eigenvalues of the equation are the roots of the determinant. This now gives us a way to find the eigenvalues. Once we have the eigenvalues, we can construct the eigenvectors by finding the vector v that satisfies the expression for each λ.

At this point none of this is particularly obvious. However, in a course like this, where we are not too concerned with a rigorous analysis of the theory, the methods are best learned by taking an example.

3.1 Example

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

First we solve $\det(A - \lambda I) = 0$. This gives us:

$$\det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(2-\lambda)(1-\lambda) - 1\right] - (1-\lambda)$$
$$= (1-\lambda)\left[(2-\lambda)(1-\lambda) - 2\right] = (1-\lambda)\left[2 - 3\lambda + \lambda^2 - 2\right] = (1-\lambda)(\lambda)(\lambda - 3) = 0$$

So we see here from the factored characteristic equation that there are three eigenvalues: $\lambda = 1$, $\lambda = 0$ and $\lambda = 3$. A good check is to veriy that $tr(A) = \Sigma\lambda_i$. We see that $tr(A) = 1 + 2 + 1 = 4$ and $\Sigma\lambda_i = 3 + 0 + 1 = 4$ so this agrees.

Now that we have the three eigenvalues, we need to solve the equation for the eigenvectors. We will do this one at at time. You can solve the equation using any of the methods we learned so far in the course (Gaussian elimination, Gauss-Joran etc.)

Let's start with $\lambda_1 = 0$. Then $(A - \lambda_1 I)\vec{v}_1 = 0$ becomes:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After row reducing this becomes:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in RREF. Then proceeding as in the example from Q1. Let $z = t$. Then:

$$x = t$$
$$y = t$$
$$z = t$$

Or, written as a matrix:

$$\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For any choice of $t \in \mathbb{R}$. We can perform the same analysis for $_2 = 3$ and $_3 = 1$. We will find that: *(exercise)*

$$\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For any choice of $r, s \in \mathbb{R}$. Additional questions to consider:

1) Are these orthogonal? If so, show this explicitly.

2) Do they form a basis for $\mathbb{R}^3$? Should they?

3) If yes in (2) are they an orthonormal basis? If not, can we make them one?