（2010 年预赛第五题）设l是过原点、方向为(α,β,γ)（其中$\alpha^2+\beta^2+\gamma^2=1$）的直线，均匀椭球$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$（其中$0<c<b<a$，密度为1）绕$l$旋转.

(1) 求其转动惯量；

(2) 求其转动惯量关于方向(α,β,γ)的最大值和最小值.

解析：(1)设旋转轴l的方向向量为$\vec{l}=(\alpha,\beta,\gamma)$，椭球内任意一点$P(x,y,z)$的向径为$\vec{r}=(x,y,z)$，则$\vec{r}$在$\vec{l}$上的投影为

$\vec{r}\cdot\vec{l}=\alpha x+\beta y+\gamma z$，于是点$P$到旋转轴$l$的距离的平方为

$$\begin{aligned}d^2&=|\vec{r}|^2-\left|\vec{r}\cdot\vec{l}\right|^2\\&=x^2+y^2+z^2-(\alpha x+\beta y+\gamma z)^2\\&=(1-\alpha^2)x^2+(1-\beta^2)y^2+(1-\gamma^2)z^2-2(\alpha\beta xy+\beta\gamma yz+\alpha\gamma xz)\end{aligned}$$

$$\begin{aligned}J_l&=\iiint_\Omega d^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\&=\iiint_\Omega[(1-\alpha^2)x^2+(1-\beta^2)y^2+(1-\gamma^2)z^2-2(\alpha\beta xy+\beta\gamma yz+\alpha\gamma xz)]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

因为Ω关于坐标x对称，被积函数xy关于变量x是奇函数，所以$\iiint_\Omega xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0$. 同理可得，$\iiint_\Omega yz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0$，

$\iiint_\Omega xz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0$.

$$\begin{aligned}\iiint_\Omega x^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z&=\int_{-a}^{a}x^2\mathrm{d}x\iint_{\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1-\frac{x^2}{a^2}}\mathrm{d}y\,\mathrm{d}z=\int_{-a}^{a}\pi bc\left(1-\frac{x^2}{a^2}\right)x^2\mathrm{d}x\\&=2\pi bc\int_0^a\left(x^2-\frac{x^4}{a^2}\right)\mathrm{d}x=\frac{4\pi a^3bc}{15}\end{aligned}$$

同理可得，$\iiint\limits_{\Omega} y^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\dfrac{4\pi b^3ac}{15}$，$\iiint\limits_{\Omega} z^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\dfrac{4\pi c^3ab}{15}$.

因此，$J_l=\dfrac{4\pi abc}{15}[(1-\alpha^2)a^2+(1-\beta^2)b^2+(1-\gamma^2)c^2]$.

其中，计算$\iiint\limits_{\Omega} x^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$也可以采用椭球坐标系

$$\begin{cases}x=a\rho\sin\varphi\cos\theta\\ y=b\rho\sin\varphi\sin\theta\\ z=c\rho\cos\varphi\end{cases}\Rightarrow\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right|=abc\rho^2\sin\varphi，\text{所以}$$

$$\begin{aligned}\iiint\limits_{\Omega} x^2\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1 a^2\rho^2\sin^2\varphi\cos^2\theta\cdot abc\rho^2\sin\varphi\,\mathrm{d}\rho\\&=a^3bc\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta\int_0^{\pi}\sin^3\varphi\,\mathrm{d}\varphi\int_0^1\rho^4\mathrm{d}\rho\\&=\pi a^3bc\cdot\frac{4}{3}\cdot\frac{1}{5}=\frac{4\pi a^3bc}{15}\end{aligned}$$

(2) 目标函数为$f(\alpha,\beta,\gamma)=(1-\alpha^2)a^2+(1-\beta^2)b^2+(1-\gamma^2)c^2$，

构造拉格朗日函数为

$$L(\alpha,\beta,\gamma,\lambda)=(1-\alpha^2)a^2+(1-\beta^2)b^2+(1-\gamma^2)c^2+\lambda(\alpha^2+\beta^2+\gamma^2-1)$$

则$\begin{cases}L_x'=2\alpha(\lambda-a^2)=0\\ L_y'=2\beta(\lambda-b^2)=0\\ L_z'=2\gamma(\lambda-c^2)=0\\ L_\lambda'=\alpha^2+\beta^2+\gamma^2-1=0\end{cases}$，解得$Q_1(\pm1,0,0)$，$Q_2(0,\pm1,0)$，

$Q_3(0,0,\pm1)$. 比较可知，$J_{\max}=\dfrac{4\pi abc}{15}(a^2+b^2)$，

$J_{\max}=\dfrac{4\pi abc}{15}(b^2+c^2)$.