(2013 年预赛第四题)设$|f(x)|\leqslant\pi$，$f'(x)\geqslant m>0\,(a\leqslant x\leqslant b)$，证明：$\left|\int_a^b \sin f(x)\mathrm{d}x\right|\leqslant\dfrac{2}{m}$.

证明：

1）若$f(x)$恒正，即$0<f(x)\leqslant\pi$，则

$$\begin{aligned}\left|\int_a^b \sin f(x)\mathrm{d}x\right| &= \int_a^b \sin f(x)\mathrm{d}x = \int_a^b \frac{f'(x)\sin f(x)\mathrm{d}x}{f'(x)} \\ &\leqslant \frac{1}{m}\int_a^b \sin f(x)\mathrm{d}(f(x)) \xlongequal{u=f(x)} \frac{1}{m}\int_{f(a)}^{f(b)} \sin u\,\mathrm{d}u \\ &\leqslant \frac{1}{m}\int_0^{\pi} \sin u\,\mathrm{d}u = \frac{2}{m}\end{aligned}$$

2）若$f(x)$恒负，即$-\pi\leqslant f(x)<0$，则

$$\begin{aligned}\left|\int_a^b \sin f(x)\mathrm{d}x\right| &= \int_a^b [-\sin f(x)]\mathrm{d}x = \int_a^b \left[\frac{-f'(x)\sin f(x)}{f'(x)}\right]\mathrm{d}x \\ &\leqslant \frac{1}{m}\int_a^b [-\sin f(x)]\mathrm{d}(f(x)) \xlongequal{u=f(x)} \frac{1}{m}\int_{f(a)}^{f(b)} (-\sin u)\mathrm{d}u \\ &\leqslant \frac{1}{m}\int_{-\pi}^{0} (-\sin u)\mathrm{d}u = \frac{2}{m}\end{aligned}$$

3）若存在$x_0\in[a,b]$，使得$f(x_0)=0$，则

$$\begin{aligned}\left|\int_a^b \sin f(x)\mathrm{d}x\right| &= \left|\int_a^{x_0} \sin f(x)\mathrm{d}x + \int_{x_0}^b \sin f(x)\mathrm{d}x\right| \\ &\leqslant \max\left\{\int_a^{x_0} [-\sin f(x)]\mathrm{d}x, \int_{x_0}^b \sin f(x)\mathrm{d}x\right\} \\ &\leqslant \max\left\{\frac{1}{m}\int_{f(a)}^{0} (-\sin u)\mathrm{d}u, \frac{1}{m}\int_0^{f(b)} \sin u\mathrm{d}u\right\} \\ &\leqslant \frac{2}{m}\end{aligned}$$

综上，$\left|\int_a^b \sin f(x)\mathrm{d}x\right|\leqslant\dfrac{2}{m}$.