(2012 年预赛第二题)计算$\int_0^{+\infty} \mathrm{e}^{-2x}|\sin x|\mathrm{d}x$.

解析：对于$k=1,2,\cdots$，我们有

$$\int_{(k-1)\pi}^{k\pi} \mathrm{e}^{-2x}|\sin x|\mathrm{d}x \xlongequal{t=x-(k-1)\pi} \int_0^{\pi} \mathrm{e}^{-2t-2(k-1)\pi}\sin t\,\mathrm{d}t = \mathrm{e}^{-2(k-1)\pi}\int_0^{\pi} \mathrm{e}^{-2t}\sin t\,\mathrm{d}t$$

$$\int_0^{\pi} \mathrm{e}^{-2x}\sin x\,\mathrm{d}x$$
$$= \int_0^{\pi}\left(-\frac{1}{2}\sin x\right)\mathrm{d}(\mathrm{e}^{-2x}) = \left(-\frac{1}{2}e^{-2x}\sin x\right)\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\mathrm{e}^{-2x}\cos x\,\mathrm{d}x$$
$$= \frac{1}{2}\int_0^{\pi}\left(-\frac{1}{2}\cos x\right)\mathrm{d}(\mathrm{e}^{-2x}) = \frac{1}{2}\left[\left(-\frac{1}{2}\mathrm{e}^{-2x}\cos x\right)\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi}\mathrm{e}^{-2x}\sin x\,\mathrm{d}x\right]$$
$$= \frac{1}{4}(1+\mathrm{e}^{-2\pi}) - \frac{1}{4}\int_0^{\pi}\mathrm{e}^{-2x}\sin x\,\mathrm{d}x$$

解得$\int_0^{\pi}\mathrm{e}^{-2x}\sin x\,\mathrm{d}x = \frac{1}{5}(1+\mathrm{e}^{-2\pi})$. 因此，

$$\int_0^{n\pi}\mathrm{e}^{-2x}|\sin x|\mathrm{d}x = \sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\mathrm{e}^{-2x}|\sin x|\mathrm{d}x = \frac{1}{5}(1+\mathrm{e}^{-2\pi})\sum_{k=1}^{n}(\mathrm{e}^{-2\pi})^{k-1}$$
$$= \frac{1}{5}(1+\mathrm{e}^{-2\pi})\frac{1-\mathrm{e}^{-2n\pi}}{1-\mathrm{e}^{-2\pi}}$$

所以$\lim\limits_{n\to\infty}\int_0^{n\pi}\mathrm{e}^{-2x}|\sin x|\mathrm{d}x = \frac{1+\mathrm{e}^{-2\pi}}{5(1-\mathrm{e}^{-2\pi})}$.

设$n(x) = \left[\frac{x}{\pi}\right]$，则$n(x)\pi \leqslant x < [n(x)+1]\pi$，且

$\lim\limits_{x\to+\infty} n(x) = +\infty$. 因此

$\int_0^{n(x)\pi}\mathrm{e}^{-2t}|\sin t|\mathrm{d}t \leqslant \int_0^{x}\mathrm{e}^{-2t}|\sin t|\mathrm{d}t < \int_0^{[n(x)+1]\pi}\mathrm{e}^{-2t}|\sin t|\mathrm{d}t$，由夹挤

准则可知，$\int_0^{+\infty}\mathrm{e}^{-2x}|\sin x|\mathrm{d}x = \lim\limits_{x\to+\infty}\int_0^{x}\mathrm{e}^{-2t}|\sin t|\mathrm{d}t = \frac{1+\mathrm{e}^{-2\pi}}{5(1-\mathrm{e}^{-2\pi})}$.