(第六届国赛预赛第六题、第八届国赛预赛第四题)

设函数f(x)在闭区间[a,b]上具有连续导数, f(a) = A, f(b) = B.

求证:

承让:
$$\lim_{n \to \infty} n \left[ \int_a^b f(x) dx - \frac{b-a}{n} \sum_{k=1}^n f \left( a + \frac{k}{n} (b-a) \right) \right] = \frac{(a-b) (B-A)}{2}.$$
证明: 党 $x_k = a + \frac{k}{n} (b-a)$ ,  $k = 0, 1, 2, \dots, n$ .
$$n \left[ \int_a^b f(x) dx - \frac{b-a}{n} \sum_{k=1}^n f \left( a + \frac{k}{n} (b-a) \right) \right]$$

$$= n \left[ \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x) dx - \frac{b-a}{n} \sum_{k=1}^n f(x_k) \right]$$

$$= n \left[ \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) dx \right]$$

$$= n \left[ \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \frac{f(x) - f(x_k)}{x - x_k} (x - x_k) dx \right]$$

证明: 设
$$x_k = a + \frac{k}{n}(b-a)$$
,  $k = 0, 1, 2, \dots, n$ .

$$n \left[ \int_{a}^{b} f(x) dx - \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k}{n} (b-a)\right) \right]$$

$$=niggl[\sum_{k=1}^n\int_{x_{k-1}}^{x_k}\!\!f(x)\mathrm{d}x-rac{b-a}{n}\sum_{k=1}^nf(x_k)iggr]$$

$$=niggl[\sum_{k=1}^n\int_{x_{k-1}}^{x_k}(f(x)-f(x_k))\mathrm{d}xiggr]$$

$$= n \left[ \sum_{k=1}^{n} \int_{x_{k+1}}^{x_k} \frac{f(x) - f(x_k)}{x - x_k} (x - x_k) dx \right]$$

$$= n \left[ \sum_{k=1}^{n} \frac{f(\xi_k) - f(x_k)}{\xi_k - x_k} \int_{x_{k-1}}^{x_k} (x - x_k) dx \right]$$

$$=n\sum_{k=1}^n f'(\eta_k)\left(-rac{1}{2}
ight)(x_{k-1}-x_k)^{\,2}$$

$$=-rac{n}{2}\cdotrac{(b-a)}{n}\sum_{k=1}^n f'(\eta_k) \left(x_k-x_{k-1}
ight)$$

$$=rac{a-b}{2}\sum_{k=1}^{n}f'(\eta_{k})\left(x_{k}-x_{k-1}
ight)$$

$$\lim_{n o\infty} nigg[\int_a^b f(x)dx - rac{b-a}{n}\sum_{k=1}^n figg(a+rac{k}{n}\left(b-a
ight)igg)igg]$$

$$=\lim_{n o\infty}rac{a-b}{2}\sum_{k=1}^n f'(\eta_k)\,(x_k-x_{k-1})=rac{a-b}{2}\!\int_a^b\!f'(x)\mathrm{d}x=rac{(a-b)\,(B-A)}{2}$$