(2009 年预赛第一(1)题)计算 $\iint_D \frac{(x+y)\ln\left(1+\frac{y}{x}\right)}{\sqrt{1-x-y}}\mathrm{d}x\,\mathrm{d}y=$_______，

其中区域D由直线$x+y=1$与两坐标轴所围三角形区域.

解析：设坐标变换$\begin{cases}x=u\\x+y=v\end{cases}\Rightarrow\begin{cases}x=u\\y=v-u\end{cases}$，则

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left\|\begin{matrix}1&0\\-1&1\end{matrix}\right\|=1$. 区域$D$：$\begin{cases}0\leqslant x+y\leqslant 1\\x\geqslant 0\\y\geqslant 0\end{cases}\Rightarrow$区域$D_{uv}$：

$\begin{cases}0\leqslant v\leqslant 1\\u\geqslant 0\\v-u\geqslant 0\end{cases}\Leftrightarrow D_{uv}=\{(u,v)|0\leqslant u\leqslant 1,u\leqslant v\leqslant 1\}$.

$$\iint_D \frac{(x+y)\ln\left(1+\frac{y}{x}\right)}{\sqrt{1-x-y}}\mathrm{d}x\,\mathrm{d}y$$

$$=\iint_{D_{uv}}\frac{v\ln\frac{v}{u}}{\sqrt{1-v}}\mathrm{d}u\,\mathrm{d}v$$

$$=\iint_{D_{uv}}\frac{v\ln v}{\sqrt{1-v}}\mathrm{d}u\,\mathrm{d}v-\iint_{D_{uv}}\frac{v\ln u}{\sqrt{1-v}}\mathrm{d}u\,\mathrm{d}v$$

$$=\int_0^1\frac{v\ln v}{\sqrt{1-v}}\mathrm{d}v\int_0^v\mathrm{d}u-\int_0^1\frac{v}{\sqrt{1-v}}\mathrm{d}v\int_0^v\ln u\,\mathrm{d}u$$

$$=\int_0^1\frac{v^2\ln v}{\sqrt{1-v}}\mathrm{d}v-\int_0^1\frac{v^2(\ln v-1)}{\sqrt{1-v}}\mathrm{d}v$$

$$=\int_0^1\frac{v^2}{\sqrt{1-v}}\mathrm{d}v$$

$$\xlongequal[v=1-t^2]{\sqrt{1-v}=t}\int_1^0\frac{t^4-2t^2+1}{t}(-2t)\mathrm{d}t$$

$$=2\int_0^1(t^4-2t^2+1)\mathrm{d}t$$

$$=2\left(\frac{1}{5}-\frac{2}{3}+1\right)=\frac{16}{15}$$