(2010 年预赛第一(2)题) 求 $\lim\limits_{x\to\infty} \mathrm{e}^{-x}\left(1+\dfrac{1}{x}\right)^{x^2}$.

解析：

$$\lim_{x\to\infty} \mathrm{e}^{-x}\left(1+\frac{1}{x}\right)^{x^2} = \lim_{x\to\infty} \mathrm{e}^{-x}\mathrm{e}^{x^2\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty} \mathrm{e}^{x^2\ln\left(1+\frac{1}{x}\right)-x} = \mathrm{e}^{\lim\limits_{x\to\infty}\left[x^2\ln\left(1+\frac{1}{x}\right)-x\right]}$$

$$\lim_{x\to\infty}\left[x^2\ln\left(1+\frac{1}{x}\right)-x\right]$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}}{\frac{1}{x^2}}$$

$$\xlongequal{t=\frac{1}{x}} \lim_{t\to 0}\frac{\ln(1+t)-t}{t^2}$$

$$= \lim_{t\to 0}\frac{t-\frac{t^2}{2}+o(t^2)-t}{t^2}$$

$$= -\frac{1}{2}$$

因此，$\lim\limits_{x\to\infty} \mathrm{e}^{-x}\left(1+\dfrac{1}{x}\right)^{x^2} = \mathrm{e}^{-\frac{1}{2}}$.