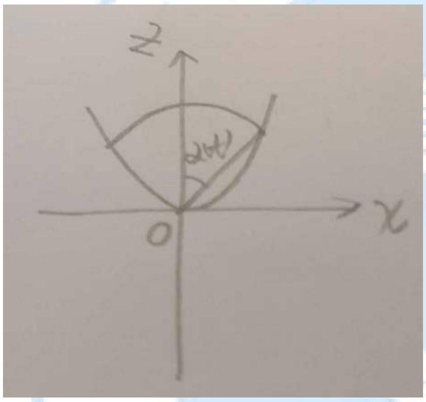
(2012 年预赛第六题)设$f(x)$为连续函数，$t>0$. 区域Ω是由抛物面$z=x^2+y^2$和球面$x^2+y^2+z^2=t^2\ (t>0)$所围起来的部分. 定义三重积分$F(t)=\iiint_\Omega f(x^2+y^2+z^2)\mathrm{d}\Omega$，求$F(t)$的导数$F'(t)$.

解析：设C是抛物面与球面的交线$\begin{cases} z=x^2+y^2 \\ x^2+y^2+z^2=t^2 \end{cases}$，则$z^2+z-t^2=0$，解得$z=\dfrac{-1\pm\sqrt{1+4t^2}}{2}$，其中$z=\dfrac{-1-\sqrt{1+4t^2}}{2}$(不合题意，舍去). Ω的截面如下图所示，其中



$\alpha(t)=\arccos\dfrac{z}{t}=\arccos\dfrac{\sqrt{1+4t^2}-1}{2t}$,

由$z=x^2+y^2$得，$\rho\cos\varphi=\rho^2\sin^2\varphi$，$\rho=\dfrac{\cos\varphi}{\sin^2\varphi}$.

$$F(t)=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\alpha(t)}\sin\varphi\,\mathrm{d}\varphi\int_0^t\rho^2 f(\rho^2)\mathrm{d}\rho+\int_0^{2\pi}\mathrm{d}\theta\int_{\alpha(t)}^{\frac{\pi}{2}}\sin\varphi\,\mathrm{d}\varphi\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}\rho^2 f(\rho^2)\mathrm{d}\rho$$
$$=2\pi\left[\left(\int_0^{\alpha(t)}\sin\varphi\,\mathrm{d}\varphi\right)\left(\int_0^t\rho^2 f(\rho^2)\mathrm{d}\rho\right)+\int_{\alpha(t)}^{\frac{\pi}{2}}\sin\varphi\left(\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}\rho^2 f(\rho^2)\mathrm{d}\rho\right)\mathrm{d}\varphi\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\int_0^{\alpha(t)}\sin\varphi\,\mathrm{d}\varphi\right)\left(\int_0^t\rho^2 f(\rho^2)\mathrm{d}\rho\right)\right]$$

$$=\sin\alpha(t)\cdot\alpha'(t)\left(\int_0^t\rho^2 f(\rho^2)\mathrm{d}\rho\right)+\left(\int_0^{\alpha(t)}\sin\varphi\,\mathrm{d}\varphi\right)t^2 f(t^2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\alpha(t)}^{\frac{\pi}{2}}\sin\varphi\left(\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}\rho^2 f(\rho^2)\mathrm{d}\rho\right)\mathrm{d}\varphi\right]=-\sin\alpha(t)\left(\int_0^{\frac{\cos\alpha(t)}{\sin^2\alpha(t)}}\rho^2 f(\rho^2)\mathrm{d}\rho\right)\cdot\alpha'(t)$$

$$\frac{\cos\alpha(t)}{\sin^2\alpha(t)}=\frac{\frac{\sqrt{1+4t^2}-1}{2t}}{1-\left(\frac{\sqrt{1+4t^2}-1}{2t}\right)^2}=\frac{\frac{\sqrt{1+4t^2}-1}{2t}}{1-\frac{1+4t^2+1-2\sqrt{1+4t^2}}{4t^2}}$$

$$=\frac{\sqrt{1+4t^2}-1}{2t}\cdot\frac{4t^2}{2\left(\sqrt{1+4t^2}-1\right)}=t$$

所以

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}t}&=2\pi\left[\sin\alpha(t)\cdot\alpha'(t)\left(\int_0^t\rho^2 f(\rho^2)\mathrm{d}\rho\right)+\left(\int_0^{\alpha(t)}\sin\varphi\,\mathrm{d}\varphi\right)t^2 f(t^2)\right.\\&\qquad\left.-\sin\alpha(t)\cdot\alpha'(t)\left(\int_0^t\rho^2 f(\rho^2)\mathrm{d}\rho\right)\right]\\&=2\pi\left(\int_0^{\alpha(t)}\sin\varphi\,\mathrm{d}\varphi\right)t^2 f(t^2)\\&=2\pi(1-\cos\alpha(t))t^2 f(t^2)\\&=2\pi\left(1-\frac{\sqrt{1+4t^2}-1}{2t}\right)t^2 f(t^2)\\&=\pi\left(2t+1-\sqrt{1+4t^2}\right)tf(t^2)\end{aligned}$$