（第六届国赛预赛第六题、第八届国赛预赛第四题）

设函数$f(x)$在闭区间$[a,b]$上具有连续导数，$f(a)=A,f(b)=B$.

求证：

$$\lim_{n\to\infty} n\left[\int_a^b f(x)dx-\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{k}{n}(b-a)\right)\right]=\frac{(a-b)(B-A)}{2}.$$

证明：设$x_k=a+\frac{k}{n}(b-a)$，$k=0,1,2,\cdots,n$.

$$\begin{aligned}
&n\left[\int_a^b f(x)\mathrm{d}x-\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{k}{n}(b-a)\right)\right]\\
&=n\left[\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} f(x)\mathrm{d}x-\frac{b-a}{n}\sum_{k=1}^{n} f(x_k)\right]\\
&=n\left[\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}(f(x)-f(x_k))\mathrm{d}x\right]\\
&=n\left[\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\frac{f(x)-f(x_k)}{x-x_k}(x-x_k)\mathrm{d}x\right]\\
&=n\left[\sum_{k=1}^{n}\frac{f(\xi_k)-f(x_k)}{\xi_k-x_k}\int_{x_{k-1}}^{x_k}(x-x_k)\mathrm{d}x\right]\\
&=n\sum_{k=1}^{n} f'(\eta_k)\left(-\frac{1}{2}\right)(x_{k-1}-x_k)^2\\
&=-\frac{n}{2}\cdot\frac{(b-a)}{n}\sum_{k=1}^{n} f'(\eta_k)(x_k-x_{k-1})\\
&=\frac{a-b}{2}\sum_{k=1}^{n} f'(\eta_k)(x_k-x_{k-1})
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty} n\left[\int_a^b f(x)dx-\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{k}{n}(b-a)\right)\right]\\
&=\lim_{n\to\infty}\frac{a-b}{2}\sum_{k=1}^{n} f'(\eta_k)(x_k-x_{k-1})=\frac{a-b}{2}\int_a^b f'(x)\mathrm{d}x=\frac{(a-b)(B-A)}{2}
\end{aligned}$$