(2015 年预赛第六题)设$f(x,y)$在$x^2+y^2\leqslant 1$上有连续的二阶偏导数且$f_{xx}^2+2f_{xy}^2+f_{yy}^2\leqslant M$. 若$f(0,0)=0$，$f_x(0,0)=f_y(0,0)=0$，证明：$\left|\iint\limits_{x^2+y^2\leqslant 1} f(x,y)\mathrm{d}x\mathrm{d}y\right|\leqslant\frac{\pi\sqrt{M}}{4}$.

解析：由二元函数的一阶泰勒公式，得

$$
\begin{aligned}
f(x,y)=&f(0,0)+[f_x(0,0),f_y(0,0)]\begin{bmatrix}x\\y\end{bmatrix}+\\
&\frac{1}{2!}[x,y]\begin{bmatrix}f_{xx}(\theta x,\theta y) & f_{xy}(\theta x,\theta y)\\ f_{xy}(\theta x,\theta y) & f_{yy}(\theta x,\theta y)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\\
=&\frac{1}{2}[f_{xx}(\theta x,\theta y)x^2+2f_{xy}(\theta x,\theta y)xy+f_{yy}(\theta x,\theta y)y^2]
\end{aligned}
$$

$$
\begin{aligned}
|f(x,y)|=&\frac{1}{2}\left|\left[f_{xx}(\theta x,\theta y),\sqrt{2}f_{xy}(\theta x,\theta y),f_{yy}(\theta x,\theta y)\right]\begin{bmatrix}x^2\\ \sqrt{2}xy\\ y^2\end{bmatrix}\right|\\
\leqslant&\frac{1}{2}\sqrt{f_{xx}^2(\theta x,\theta y)+2f_{xy}^2(\theta x,\theta y)+f_{yy}^2(\theta x,\theta y)}\sqrt{x^4+2x^2y^2+y^4}\\
\leqslant&\frac{\sqrt{M}}{2}(x^2+y^2)
\end{aligned}
$$

因此，

$$
\begin{aligned}
\left|\iint\limits_{x^2+y^2\leqslant 1} f(x,y)\mathrm{d}x\mathrm{d}y\right|\leqslant&\iint\limits_{x^2+y^2\leqslant 1}|f(x,y)|\mathrm{d}x\mathrm{d}y\\
\leqslant&\iint\limits_{x^2+y^2\leqslant 1}\frac{\sqrt{M}}{2}(x^2+y^2)\mathrm{d}x\mathrm{d}y\\
=&\frac{\sqrt{M}}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r\\
=&\frac{\pi\sqrt{M}}{4}
\end{aligned}
$$