(2015 年预赛第六题)设f(x,y)在 $x^2+y^2 \le 1$ 上有连续的二阶偏导数 且 $f_{xx}^2+2f_{xy}^2 \le M$. 若f(0,0)=0, $f_x(0,0)=f_y(0,0)=0$,证

明:
$$\left| \iint_{x^2+y^2\leqslant 1} f(x,y) \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant \frac{\pi \sqrt{M}}{4}.$$

解析:由二元函数的一阶泰勒公式,得

$$egin{aligned} f(x,y) &= f(0,0) + \left[f_x(0,0),f_y(0,0)
ight] egin{bmatrix} x \ y \end{bmatrix} + \ &rac{1}{2!} \left[x,y
ight] egin{bmatrix} f_{xx}(heta x, heta y) & f_{xy}(heta x, heta y) \ f_{yy}(heta x, heta y) \end{bmatrix} egin{bmatrix} x \ y \end{bmatrix} \ &= rac{1}{2} \left[f_{xx}(heta x, heta y) x^2 + 2f_{xy}(heta x, heta y) xy + f_{yy}(heta x, heta y) y^2
ight] \end{aligned}$$

$$egin{align} |f(x,y)| &= rac{1}{2} igg| igg[f_{xx}(heta x, heta y), \sqrt{2} \, f_{xy}(heta x, heta y), f_{yy}(heta x, heta y) igg] igg[rac{x^2}{\sqrt{2} \, xy} igg] igg| \ &\leqslant rac{1}{2} \sqrt{f_{xx}^2(heta x, heta y) + 2f_{xy}^2(heta x, heta y) + f_{yy}^2(heta x, heta y)} \sqrt{x^4 + 2x^2 y^2 + y^4} \ &\leqslant rac{\sqrt{M}}{2} (x^2 + y^2) \end{aligned}$$

因此,

$$\left| \iint_{x^2 + y^2 \leqslant 1} f(x, y) \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant \iint_{x^2 + y^2 \leqslant 1} |f(x, y)| \, \mathrm{d}x \, \mathrm{d}y$$

$$\leqslant \iint_{x^2 + y^2 \leqslant 1} \frac{\sqrt{M}}{2} (x^2 + y^2) \, \mathrm{d}x \, \mathrm{d}y$$

$$= \frac{\sqrt{M}}{2} \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r^3 \, \mathrm{d}r$$

$$= \frac{\pi \sqrt{M}}{4}$$