2018 年校赛试题答案

一．　填空题(每题 5 分，6 小题，共 30 分)

1. 答案：$\mathrm{e}-1$

解析：因为$\dfrac{n}{n+1}\cdot\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}=\dfrac{\mathrm{e}^{\frac{i}{n}}}{n+1}<\dfrac{\mathrm{e}^{\frac{i}{n}}}{n+\frac{1}{i}}<\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}$，

所以$\dfrac{n}{n+1}\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}<\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n+\frac{1}{i}}<\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}$.

因为$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}=\int_0^1\mathrm{e}^x\mathrm{d}x=\mathrm{e}-1$，

$\lim\limits_{n\to\infty}\dfrac{n}{n+1}\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}=\lim\limits_{n\to\infty}\dfrac{n}{n+1}\cdot\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n}=\mathrm{e}-1$，

所以由夹挤准则得$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{\mathrm{e}^{\frac{i}{n}}}{n+\frac{1}{i}}=\mathrm{e}-1$.

2. 答案：$r(\cos\theta+\sin\theta)=\mathrm{e}^{\frac{\pi}{2}}$

解析：以θ为参数，对数螺线的参数方程为$\begin{cases}x=r\cos\theta=\mathrm{e}^{\theta}\cos\theta\\ y=r\sin\theta=\mathrm{e}^{\theta}\sin\theta\end{cases}$. 因此，$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}=\dfrac{\mathrm{e}^{\theta}\sin\theta+\mathrm{e}^{\theta}\cos\theta}{\mathrm{e}^{\theta}\cos\theta-\mathrm{e}^{\theta}\sin\theta}=\dfrac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$，$\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{2}}=-1$.

切点坐标为$\left(0,\mathrm{e}^{\frac{\pi}{2}}\right)$，故切线的直角坐标方程为$y-\mathrm{e}^{\frac{\pi}{2}}=-x$，整理得

$x+y=\mathrm{e}^{\frac{\pi}{2}}$. 因此，极坐标方程为$r(\cos\theta+\sin\theta)=\mathrm{e}^{\frac{\pi}{2}}$.

3. 答案：$y=\sqrt{x+1}$

解析：由$yy''+y'^2=0$得$(yy')'=0$，所以$yy'=C$. 代入初值$y(0)=1$，

$y'(0)=\dfrac{1}{2}$，得$yy'=\dfrac{1}{2}$. 由$yy'=\dfrac{1}{2}$，得$2yy'=1$，即$(y^2)'=1$. 因此，

$y^2=x+C$. 代入初值$y(0)=1$，得$y^2=x+1$. 又因为$y(0)=1>0$，

所以$y=\sqrt{x+1}$.

4. 答案：$\ln 4-\ln 3$

解析：化定积分为二重积分

$$\int_0^1 \frac{x^3-x^2}{\ln x}\mathrm{d}x=\int_0^1 \mathrm{d}x\int_2^3 x^y\mathrm{d}y=\int_2^3 \mathrm{d}y\int_0^1 x^y\mathrm{d}x=\int_2^3 \frac{1}{y+1}\mathrm{d}y=\ln 4-\ln 3$$

5. 答案：7

解析：

$$\begin{aligned}\int_1^2 xf(x)\mathrm{d}x&=\int_1^0 xf(x)\mathrm{d}x+\int_0^2 xf(x)\mathrm{d}x=-\int_0^1 xf(x)\mathrm{d}x+\int_0^1 2uf(2u)\mathrm{d}(2u)\\&=7\int_0^1 xf(x)\mathrm{d}x\end{aligned}$$

6. 答案：0

解析：

$$\begin{aligned}a_{2n}&=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^{2m+1}x\cos 2nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}\cos^{2m+1}x\cos 2nx\,\mathrm{d}x\\&\xlongequal{x=\pi-t}\frac{2}{\pi}\int_{\pi}^{0}(-1)\cos^{2m+1}t\cos 2nt\cdot(-1)\,\mathrm{d}t=-\frac{2}{\pi}\int_0^{\pi}\cos^{2m+1}t\cos 2nt\,\mathrm{d}t=-a_{2n}\end{aligned}$$

二. （10 分）

解析：

$$\mathrm{e}^{(1+x)^{\frac{1}{x}}}-(1+x)^{\frac{\mathrm{e}}{x}}=\mathrm{e}^{(1+x)^{\frac{1}{x}}}-\mathrm{e}^{\frac{\mathrm{e}}{x}\ln(1+x)}$$

$$=(1+x)^{\frac{\mathrm{e}}{x}}\left[\mathrm{e}^{(1+x)^{\frac{1}{x}}-\frac{\mathrm{e}}{x}\ln(1+x)}-1\right]$$

$$\sim \mathrm{e}^{\mathrm{e}}\left[(1+x)^{\frac{1}{x}}-\frac{\mathrm{e}}{x}\ln(1+x)\right]$$

$$=\mathrm{e}^{\mathrm{e}}\left[e^{\frac{\ln(1+x)}{x}}-\frac{\mathrm{e}}{x}\ln(1+x)\right]$$

$$=\mathrm{e}^{\mathrm{e}+1}\left[\mathrm{e}^{\frac{\ln(1+x)}{x}-1}-\frac{\ln(1+x)}{x}\right]$$

$$\mathrm{e}^{\mathrm{e}+1}\left[\mathrm{e}^{\frac{\ln(1+x)}{x}-1}-\left(\frac{\ln(1+x)}{x}-1\right)-1\right]$$

$$\sim \mathrm{e}^{\mathrm{e}+1}\cdot\frac{1}{2!}\left[\frac{\ln(1+x)}{x}-1\right]^{2}$$

$$\sim \frac{\mathrm{e}^{\mathrm{e}+1}}{2}\left(-\frac{1}{2}x\right)^{2}$$

$$=\frac{\mathrm{e}^{\mathrm{e}+1}}{8}x^{2}$$

因此，$\lim\limits_{x\to 0}\dfrac{\mathrm{e}^{(1+x)^{\frac{1}{x}}}-(1+x)^{\frac{\mathrm{e}}{x}}}{x^{2}}=\dfrac{\mathrm{e}^{\mathrm{e}+1}}{8}$.

三. （10 分）

解析：由公式$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$，得

$\arctan\dfrac{1-2x}{1+2x}=\dfrac{\pi}{4}-\arctan 2x$. 而

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,\ x \in [-1, 1].$$

于是，当$x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$时，

$$\begin{aligned} f(x) &= \frac{\pi}{4} - 2x + \frac{(2x)^3}{3} - \frac{(2x)^5}{5} + \cdots + (-1)^n \frac{(2x)^{2n-1}}{2n-1} + \cdots \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{2n-1} x^{2n-1} \end{aligned}$$

因此，$\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = -f\left(\frac{1}{2}\right) + \frac{\pi}{4} = -\arctan\frac{1-1}{1+1} + \frac{\pi}{4} = \frac{\pi}{4}$.

四．（10 分）

证明：通过解微分方程$f(x) = \frac{b-x}{a} f'(x)$构造微分中值定理要用的函数. $\frac{f'(x)}{f(x)} = \frac{a}{b-x}$，$\frac{\mathrm{d}f(x)}{f(x)} = \frac{a\,\mathrm{d}x}{b-x}$，$\int \frac{\mathrm{d}f(x)}{f(x)} = \int \frac{a\,\mathrm{d}x}{b-x}$，

$\ln f(x) = -a\ln(b-x) + \ln C$，$f(x)(b-x)^a = C$.

设$F(x) = f(x)(b-x)^a$，则$F(x)$在$[a,b]$上连续，在(a,b)内连续，且$F(a) = f(a)(b-a)^a = 0$（$f(a)=0$），

$F(b) = f(b)(b-b)^a = 0$，由罗尔定理，存在$\xi \in (a,b)$，使得

$F'(\xi) = 0$，即$f'(\xi)(b-\xi)^a + f(\xi)a(b-\xi)^{a-1}(-1) = 0$，整理得

$f(\xi) = \frac{b-\xi}{a} f'(\xi)$，得证.

五．（10 分）

证明：因为$\dfrac{1}{n\ln n} \geqslant \displaystyle\int_{n}^{n+1} \dfrac{1}{x\ln x}\mathrm{d}x$，所以

$\displaystyle\sum_{n=2}^{k}\dfrac{1}{n\ln n} \geqslant \sum_{n=2}^{k}\int_{n}^{n+1}\dfrac{1}{x\ln x}dx = \int_{2}^{k+1}\dfrac{1}{x\ln x}dx$. 注意到

$\displaystyle\int_{2}^{k+1}\dfrac{1}{x\ln x}\mathrm{d}x = \int_{2}^{k+1}\dfrac{1}{\ln x}\mathrm{d}(\ln x) = \int_{\ln 2}^{\ln(k+1)}\dfrac{1}{u}\mathrm{d}u$，而反常积分

$\displaystyle\int_{\ln 2}^{+\infty}\dfrac{1}{u}\mathrm{d}u$ 是发散的，所以当$k\to\infty$时$\displaystyle\sum_{n=2}^{k}\dfrac{1}{n\ln n}$将无界，故$\displaystyle\sum_{n=2}^{\infty}\dfrac{1}{n\ln n}$

发散.

六．（15 分）

解析：(1)

$$A(u-A(u)) = A\left(u - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right)$$

$$= x\left(\frac{\partial u}{x} - \frac{\partial u}{\partial x} - x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial y\,\partial x}\right) + y\left(\frac{\partial u}{\partial y} - x\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial y} - y\frac{\partial^2 u}{\partial y^2}\right)$$

$$= -x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial y\,\partial x} - y^2\frac{\partial^2 u}{\partial y^2}$$

(2) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0 \Leftrightarrow A(A(u)-u) = 0$.

$$A(u) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right) + y\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right)$$

$$= x\left(\frac{\partial u}{\partial \xi}\frac{-y}{x^2} + \frac{\partial u}{\partial \eta}\right) + y\left(\frac{\partial u}{\partial \xi}\frac{1}{x} - \frac{\partial u}{\partial \eta}\right) = -\xi\frac{\partial u}{\partial \xi} + x\frac{\partial u}{\partial \eta} + \xi\frac{\partial u}{\partial \xi} - y\frac{\partial u}{\partial \eta}$$

$$= \eta\frac{\partial u}{\partial \eta}$$

$$A(A(u)-u) = A(A(u)) - A(u) = A\left(\eta\frac{\partial u}{\partial \eta}\right) - \eta\frac{\partial u}{\partial \eta},$$

$$A\left(\eta\frac{\partial u}{\partial\eta}\right)=x\frac{\partial}{\partial x}\left(\eta\frac{\partial u}{\partial\eta}\right)+y\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial\eta}\right)$$

$$=x\left(\eta\frac{\partial^2 u}{\partial\eta\partial\xi}\frac{-y}{x^2}+\left(\frac{\partial u}{\partial\eta}+\eta\frac{\partial^2 u}{\partial\eta^2}\right)\right)+y\left(\eta\frac{\partial^2 u}{\partial\eta\partial\xi}\frac{1}{x}-\left(\frac{\partial u}{\partial\eta}+\eta\frac{\partial^2 u}{\partial\eta^2}\right)\right)$$

$$=-\xi\eta\frac{\partial^2 u}{\partial\eta\partial\xi}+x\frac{\partial u}{\partial\eta}+x\eta\frac{\partial^2 u}{\partial\eta^2}+\xi\eta\frac{\partial^2 u}{\partial\eta\partial\xi}-y\frac{\partial u}{\partial\eta}-y\eta\frac{\partial^2 u}{\partial\eta^2}$$

$$=\eta\frac{\partial u}{\partial\eta}+\eta^2\frac{\partial^2 u}{\partial\eta^2}$$

$$A(A(u)-u)=A\left(\eta\frac{\partial u}{\partial\eta}\right)-\eta\frac{\partial u}{\partial\eta}=\eta^2\frac{\partial^2 u}{\partial\eta^2}.$$

因此，$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0\Leftrightarrow\eta^2\frac{\partial^2 u}{\partial\eta^2}=0$.

七．（15 分）

解析：(1) $\forall(x,y,z)\in\Sigma$，设$\overrightarrow{n^0}=\{\cos\alpha,\cos\beta,\cos\gamma\}$为点$(x,y,z)$处切平面指向椭球面外侧的单位法向量，因为

$\lambda(x,y,z)=x\cos\alpha+y\cos\beta+z\cos\gamma$，所以若取$\Sigma$的外侧，

$$I_1=\oiint_\Sigma(x\cos\alpha+y\cos\beta+z\cos\gamma)\mathrm{d}S=\oiint_\Sigma x\,\mathrm{d}y\,\mathrm{d}z+y\,\mathrm{d}z\,\mathrm{d}x+z\,\mathrm{d}x\,\mathrm{d}y$$

$$=3\iiint_V\mathrm{d}V=3\cdot\frac{4}{3}\pi abc=4\pi abc$$

(2) $\forall(x,y,z)\in\Sigma$，$\vec{n}=\left\{\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right\}$，

$$\lambda(x,y,z)=\{x,y,z\}\cdot\overrightarrow{n^0}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}=\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}},$$

若取Σ的外侧，则

$$
\begin{aligned}
I_2 &= \oiint\limits_{\Sigma} \sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\,\mathrm{d}S \\
&= \oiint\limits_{\Sigma} \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\mathrm{d}S \\
&= \oiint\limits_{\Sigma} \frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\frac{x}{a^2}\cdot\frac{x}{a^2}+\frac{y}{b^2}\cdot\frac{y}{b^2}+\frac{z}{c^2}\cdot\frac{z}{c^2}\right)\mathrm{d}S \\
&= \oiint\limits_{\Sigma} \frac{x}{a^2}\,\mathrm{d}y\,\mathrm{d}z+\frac{y}{b^2}\,\mathrm{d}z\,\mathrm{d}x+\frac{z}{c^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\iiint\limits_{V}\mathrm{d}V = \frac{4}{3}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)
\end{aligned}
$$