Chapter 3

正则表达式

3.1 正则表达式

- 有穷自动机
  - 通过机器装置描述正则语言
  - 用计算机编写相应算法, 易于实现
- 正则表达式
  - 通过表达式描述正则语言, 代数表示方法, 使用方便
  - 应用广泛
    - `grep` 工具 (Global Regular Expression and Print)
    - Emacs / Vim 文本编辑器
    - `lex` / `flex` 词法分析器
    - 各种程序设计语言 Python / Perl / Haskell / ···

3.1.1 语言的运算

设 L 和 M 是两个语言, 那么

并 (Union) $$L \cup M = \{ w \mid w \in L \text{ 或 } w \in M \}$$

连接 (Concatenation) $$L \cdot M = \{ w \mid w = xy,\ x \in L \text{ 且 } y \in M \}$$

幂 (Power) $$L^0 = \{\varepsilon\}$$
$$L^1 = L$$
$$L^n = L^{n-1} \cdot L$$

克林闭包 (Kleene Closure) $\qquad L^* = \bigcup_{i=0}^{\infty} L^i$

例 1. 若有语言 $L = \{0, 11\}$ 和 $M = \{\varepsilon, 001\}$, 那么

$$
\begin{aligned}
L \cup M &= & L^0 &= \\
LM &= & L^1 &= \\
ML &= & L^2 &=
\end{aligned}
$$

例 2. 对于空语言 $\varnothing$

$$
\begin{aligned}
&\varnothing^0 = \{\varepsilon\} \\
\forall n \geq 1, \quad &\varnothing^n = \varnothing \\
&\varnothing^* = \{\varepsilon\}
\end{aligned}
$$

四则运算表达式的递归定义:

1. 任何数都是四则运算表达式;
2. 如果 a 和 b 是四则运算表达式, 那么

   $$a + b,\ a - b,\ a \times b,\ a \div b \text{ 和 } (a)$$

   都是四则运算表达式.

3.1.2 正则表达式的递归定义

定义. *如果 Σ 为字母表, 则 Σ 上的正则表达式 (Regular Expression) 递归定义为:*

1. $\varnothing$ 是一个正则表达式, 表示空语言;
   $\boldsymbol{\varepsilon}$ 是一个正则表达式, 表示语言 $\{\varepsilon\}$;
   $\forall a \in \Sigma$, $\mathbf{a}$ 是一个正则表达式, 表示语言 $\{a\}$;
2. 如果正则表达式 $\mathbf{r}$ 和 $\mathbf{s}$ 分别表示语言 R 和 S, 那么

   $$\mathbf{r} + \mathbf{s},\ \mathbf{rs},\ \mathbf{r}^* \text{ 和 } (\mathbf{r})$$

   都是正则表达式, 分别表示语言

   $$R \cup S,\ R \cdot S,\ R^* \text{ 和 } R.$$

此外正闭包定义为 $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$, 显然 $\mathbf{r}^* = \mathbf{r}^+ + \boldsymbol{\varepsilon}$.

3.1.3 运算符的优先级

正则表达式中三种运算以及括号的优先级:

1. 首先,“括号”优先级最高;
2. 其次,“星”运算: $\mathbf{r}^*$;
3. 然后,“连接”运算: $\mathbf{rs}$, $\mathbf{r}\cdot\mathbf{s}$;
4. 最后,“加”最低: $\mathbf{r}+\mathbf{s}$, $\mathbf{r}\cup\mathbf{s}$;

例 3.

$$\begin{aligned}\mathbf{1}+\mathbf{01}^* &= \mathbf{1}+\big(\mathbf{0}(\mathbf{1}^*)\big)\\ &\neq \mathbf{1}+(\mathbf{01})^*\\ &\neq (\mathbf{1}+\mathbf{01})^*\\ &\neq (\mathbf{1}+\mathbf{0})\mathbf{1}^*\end{aligned}$$

3.1.4 正则表达式示例

例 4.

E	$\mathbf{L}(E)$
$\mathbf{a}+\mathbf{b}$	$\mathbf{L}(\mathbf{a})\cup\mathbf{L}(\mathbf{b})=\{a\}\cup\{b\}=\{a,b\}$
$\mathbf{bb}$	$\mathbf{L}(\mathbf{b})\cdot\mathbf{L}(\mathbf{b})=\{b\}\cdot\{b\}=\{bb\}$
$(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b})$	$\{a,b\}\{a,b\}=\{aa,ab,ba,bb\}$
$(\mathbf{a}+\mathbf{b})^*(\mathbf{a}+\mathbf{bb})$	$\{a,b\}^*\{a,bb\}=\{a,b\}^*\{a\}\cup\{a,b\}^*\{bb\}=$ $\left\{w\in\{a,b\}^* \,\middle|\, w \text{ 仅以 } a \text{ 或 } bb \text{ 结尾.}\right\}$
$\mathbf{1}+(\mathbf{01})^*$	$\{1,\varepsilon,01,0101,010101,\dots\}$
$(\mathbf{0}+\mathbf{1})^*\mathbf{01}(\mathbf{0}+\mathbf{1})^*$	$\left\{x01y \,\middle|\, x,y\in\{0,1\}^*\right\}$

例 5. 给出正则表达式 $(\mathbf{aa})^*(\mathbf{bb})^*\mathbf{b}$ 定义的语言.

$$\begin{aligned}\mathbf{L}\big((\mathbf{aa})^*(\mathbf{bb})^*\mathbf{b}\big) &= \mathbf{L}\big((\mathbf{aa})^*\big)\cdot\mathbf{L}\big((\mathbf{bb})^*\big)\cdot\mathbf{L}\big(\mathbf{b}\big)\\ &= \big(\{a\}\cdot\{a\}\big)^*\cdot\big(\{b\}\cdot\{b\}\big)^*\cdot\{b\}\end{aligned}$$

$$\begin{aligned}
&= \{a^2\}^* \cdot \{b^2\}^* \cdot \{b\} \\
&= \{\, a^{2n} \mid n \ge 0 \,\} \cdot \{\, b^{2n} \mid n \ge 0 \,\} \cdot \{b\} \\
&= \{\, a^{2n} b^{2m+1} \mid n \ge 0, m \ge 0 \,\}
\end{aligned}$$

例 6. Design regular expression for $L = \{w \mid w$ consists of 0's and 1's, and the third symbol from the right end is 1.$\}$

$$\mathbf{(0+1)^* 1(0+1)(0+1)}$$

例 7. Design regular expression for $L = \{w \mid w \in \{0,1\}^*$ and w has no pair of consecutive 0's.$\}$

$$\mathbf{1^*(011^*)^*(0+\varepsilon)} \text{ 或 } \mathbf{(1+01)^*(0+\varepsilon)}$$

课堂练习.

Give regular expressions for each of the following languages over $\Sigma = \{0,1\}$.

(1) All strings containing the substring 000.

(2) All strings *not* containing the substring 000.

例. Design regular expression for $L = \{\, w \mid w \in \{0,1\}^*$ and w contains $01 \,\}$.

例. Write a regular expression for $L = \{\, w \in \{0,1\}^* \mid 0$ and 1 alternate in $w \,\}$.

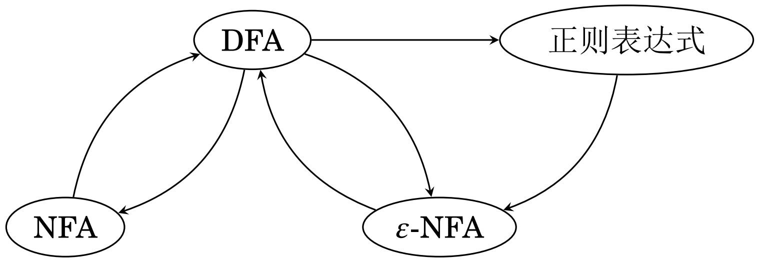
例. Find a regular expression for the set $\{\, a^n b^m \mid (n+m)$ is odd $\,\}$.

例. Give regular expression for the complement of $L = \{\, a^n b^m \mid n \ge 3, m \le 4 \,\}$.

例. Write a regular expression for the set of all C real numbers.

3.2 有穷自动机和正则表达式

DFA, NFA, ε-NFA 和正则表达式的等价性



3.2.1 由 DFA 到正则表达式, 递归表达式法

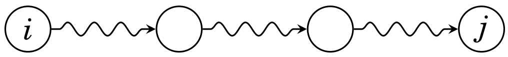
定理 3. *若 $L = \mathbf{L}(A)$ 是某 DFA A 的语言, 那么存在正则表达式 R 满足 $L = \mathbf{L}(R)$.*

证明: 设 DFA A 的状态共有 n 个, 对 DFA A 的状态编号, 令 1 为开始状态, 即

$$A = \big(\{1,2,\dots,n\}, \Sigma, \delta, 1, F\big),$$

设正则表达式 $R_{ij}^{(k)}$ 表示从 i 到 j 但中间节点不超过 k 全部路径的字符串集:

$$R_{ij}^{(k)} = \left\{ x \mid \hat{\delta}(i,x) = j, x\text{经过的状态除两端外都不超过}k \right\}.$$



也就是说, 正则表达式 $R_{ij}^{(k)}$ 是所有那样的字符串的集合, 它能够使有穷自动机从状态 i 到达状态 j, 但中间的路径上, 不经过编号高于 k 的任何状态.

如果 1 是开始结点, 对每个属于终态 F 的结点 j, 正则表达式 $R_{1j}^{(n)}$ 都是这个自动机所识别语言的一部分. 那么与 $A = \big(\{1,2,\dots,n\},\Sigma,\delta,1,F\big)$ 等价的正则表达式为

$$\bigcup_{j\in F} R_{1j}^{(n)}$$

且递归式为

$$R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}\big(R_{kk}^{(k-1)}\big)^* R_{kj}^{(k-1)}$$

$$R_{ij}^{(0)} = \begin{cases} \left\{ a \mid \delta(q_i,a) = q_j \right\} & i \neq j \\ \left\{ a \mid \delta(q_i,a) = q_j \right\} \cup \{\varepsilon\} & i = j \end{cases}$$

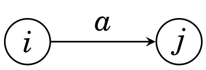
下面对 k 归纳, 证明可用以上递归式求得 $R_{ij}^{(k)}$.

归纳基础: 当 $i \neq j$, $k = 0$ 时, 即 i 到 j 没经过任何中间节点

- 没有 i 到 j 的状态转移

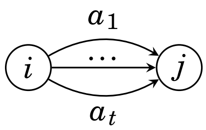
$$R_{ij}^{(0)} = \varnothing$$

- 有一个 i 到 j 的状态转移



$$R_{ij}^{(0)} = \mathbf{a}$$

- 有多个 i 到 j 的状态转移



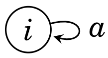
$$R_{ij}^{(0)} = \mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_t$$

归纳基础 (续): 当 $i = j, k = 0$ 时, 即从 i 到自身没经过任何中间节点

- 状态 i 没有到自己的转移

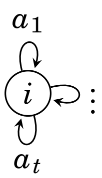
$$R_{ii}^{(0)} = \boldsymbol{\varepsilon}$$

- 状态 i 有一个到自身的转移



$$R_{ii}^{(0)} = \mathbf{a} + \boldsymbol{\varepsilon}$$

- 状态 i 有多个到自身的转移

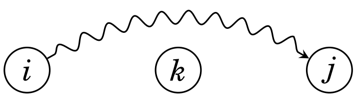


$$R_{ii}^{(0)} = \mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_t + \boldsymbol{\varepsilon}$$

归纳假设: 假设已知 $R_{ij}^{(k-1)}, R_{ik}^{(k-1)}, R_{kk}^{(k-1)}$ 和 $R_{kj}^{(k-1)}$.

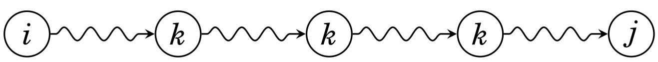
归纳递推: 那么 $R_{ij}^{(k)}$ 中全部路径, 可用节点 k 分为两部分

- 从 i 到 j 不经过 k 的



$$R_{ij}^{(k)} = R_{ij}^{(k-1)}$$

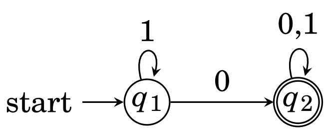
- 从 i 到 j 经过 k 的



$$R_{ij}^{(k)} = R_{ik}^{(k-1)}\left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}$$

因此 $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}\left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}$. □

例 8. 将如图 DFA 转换为正则表达式.



- 计算 $R_{ij}^{(0)}$

$R_{ij}^{(k)}$	$k=0$
$R_{11}^{(0)}$	$\boldsymbol{\varepsilon}+\mathbf{1}$
$R_{12}^{(0)}$	$\mathbf{0}$
$R_{21}^{(0)}$	$\varnothing$
$R_{22}^{(0)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}$

- 计算 $R_{ij}^{(1)}=R_{ij}^{(0)}+R_{i1}^{(0)}\left(R_{11}^{(0)}\right)^{*}R_{1j}^{(0)}$

$R_{ij}^{(k)}$	$k=0$
$R_{11}^{(0)}$	$\boldsymbol{\varepsilon}+\mathbf{1}$
$R_{12}^{(0)}$	$\mathbf{0}$
$R_{21}^{(0)}$	$\varnothing$
$R_{22}^{(0)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}$

$R_{ij}^{(k)}$	$k=1$
$R_{11}^{(1)}$	$(\boldsymbol{\varepsilon}+\mathbf{1})+(\boldsymbol{\varepsilon}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{1})^{*}(\boldsymbol{\varepsilon}+\mathbf{1})$
$R_{12}^{(1)}$	$\mathbf{0}+(\boldsymbol{\varepsilon}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{1})^{*}\mathbf{0}$
$R_{21}^{(1)}$	$\varnothing+\varnothing(\boldsymbol{\varepsilon}+\mathbf{1})^{*}(\boldsymbol{\varepsilon}+\mathbf{1})$
$R_{22}^{(1)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}+\varnothing(\boldsymbol{\varepsilon}+\mathbf{1})^{*}\mathbf{0}$

- 几个基本的化简规则

  如果 **r** 和 **s** 是两个正则表达式

$$
\begin{aligned}
(\boldsymbol{\varepsilon}+\mathbf{r})^{*} &= \mathbf{r}^{*} \\
(\boldsymbol{\varepsilon}+\mathbf{r})\mathbf{r}^{*} &= \mathbf{r}^{*} \\
\mathbf{r}+\mathbf{r}\mathbf{s}^{*} &= \mathbf{r}\mathbf{s}^{*} \\
\varnothing\mathbf{r} = \mathbf{r}\varnothing &= \varnothing && \text{零元} \\
\varnothing+\mathbf{r} = \mathbf{r}+\varnothing &= \mathbf{r} && \text{单位元}
\end{aligned}
$$

- 化简 $R_{ij}^{(1)}$

$R_{ij}^{(k)}$	$k=1$	化简
$R_{11}^{(1)}$	$(\boldsymbol{\varepsilon}+\mathbf{1})+(\boldsymbol{\varepsilon}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{1})^{*}(\boldsymbol{\varepsilon}+\mathbf{1})$	$\mathbf{1}^{*}$
$R_{12}^{(1)}$	$\mathbf{0}+(\boldsymbol{\varepsilon}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{1})^{*}\mathbf{0}$	$\mathbf{1}^{*}\mathbf{0}$
$R_{21}^{(1)}$	$\varnothing+\varnothing(\boldsymbol{\varepsilon}+\mathbf{1})^{*}(\boldsymbol{\varepsilon}+\mathbf{1})$	$\varnothing$
$R_{22}^{(1)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}+\varnothing(\boldsymbol{\varepsilon}+\mathbf{1})^{*}\mathbf{0}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}$

- 计算 $R_{ij}^{(2)}=R_{ij}^{(1)}+R_{i2}^{(1)}\left(R_{22}^{(1)}\right)^{*}R_{2j}^{(1)}$

$R_{ij}^{(k)}$	$k=1$	$R_{ij}^{(k)}$	$k=2$
$R_{11}^{(1)}$	$\mathbf{1}^*$	$R_{11}^{(2)}$	$\mathbf{1}^*+\mathbf{1}^*\mathbf{0}(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*\varnothing$
$R_{12}^{(1)}$	$\mathbf{1}^*\mathbf{0}$	$R_{12}^{(2)}$	$\mathbf{1}^*\mathbf{0}+\mathbf{1}^*\mathbf{0}(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})$
$R_{21}^{(1)}$	$\varnothing$	$R_{21}^{(2)}$	$\varnothing+(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*\varnothing$
$R_{22}^{(1)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}$	$R_{22}^{(2)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}+(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})$

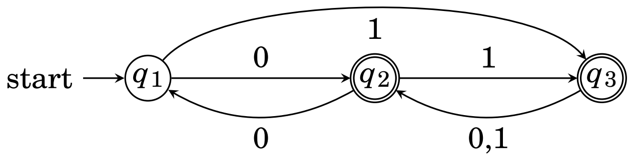
- 化简 $R_{ij}^{(2)}$

$R_{ij}^{(k)}$	$k=2$	化简
$R_{11}^{(2)}$	$\mathbf{1}^*+\mathbf{1}^*\mathbf{0}(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*\varnothing$	$\mathbf{1}^*$
$R_{12}^{(2)}$	$\mathbf{1}^*\mathbf{0}+\mathbf{1}^*\mathbf{0}(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})$	$\mathbf{1}^*\mathbf{0}(\mathbf{0}+\mathbf{1})^*$
$R_{21}^{(2)}$	$\varnothing+(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*\varnothing$	$\varnothing$
$R_{22}^{(2)}$	$\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1}+(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})^*(\boldsymbol{\varepsilon}+\mathbf{0}+\mathbf{1})$	$(\mathbf{0}+\mathbf{1})^*$

- 因只有 q_2 是接受状态, 所以该 DFA 正则表达式为

$$R_{12}^{(2)}=\mathbf{1}^*\mathbf{0}(\mathbf{0}+\mathbf{1})^*.$$

例 9. 将如图 DFA 转换为正则表达式.



	$k=0$	$k=1$	$k=2$
$R_{11}^{(k)}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$(\mathbf{00})^*$
$R_{12}^{(k)}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}(\mathbf{00})^*$
$R_{13}^{(k)}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}^*\mathbf{1}$
$R_{21}^{(k)}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}(\mathbf{00})^*$
$R_{22}^{(k)}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}+\mathbf{00}$	$(\mathbf{00})^*$
$R_{23}^{(k)}$	$\mathbf{1}$	$\mathbf{1}+\mathbf{01}$	$\mathbf{0}^*\mathbf{1}$
$R_{31}^{(k)}$	$\varnothing$	$\varnothing$	$(\mathbf{0}+\mathbf{1})(\mathbf{00})^*\mathbf{0}$
$R_{32}^{(k)}$	$\mathbf{0}+\mathbf{1}$	$\mathbf{0}+\mathbf{1}$	$(\mathbf{0}+\mathbf{1})(\mathbf{00})^*$
$R_{33}^{(k)}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}+(\mathbf{0}+\mathbf{1})\mathbf{0}^*\mathbf{1}$

仅状态 2 和 3 是接受状态:

$$\begin{aligned}R_{12}^{(3)}&=R_{12}^{(2)}+R_{13}^{(2)}\big(R_{33}^{(2)}\big)^*R_{32}^{(2)}\\&=\mathbf{0}(\mathbf{00})^*+\mathbf{0}^*\mathbf{1}\big(\boldsymbol{\varepsilon}+(\mathbf{0}+\mathbf{1})\mathbf{0}^*\mathbf{1}\big)^*(\mathbf{0}+\mathbf{1})(\mathbf{00})^*\end{aligned}$$

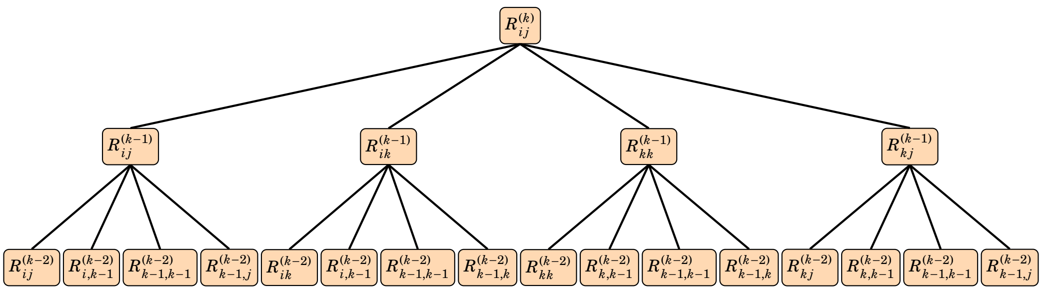
$$
\begin{aligned}
&= \mathbf{0(00)^* + 0^*1\big((0+1)0^*1\big)^*(0+1)(00)^*} \\
R_{13}^{(3)} &= R_{13}^{(2)} + R_{13}^{(2)}\big(R_{33}^{(2)}\big)^* R_{33}^{(2)} \\
&= \mathbf{0^*1 + 0^*1\big(\boldsymbol{\varepsilon} + (0+1)0^*1\big)^*\big(\boldsymbol{\varepsilon} + (0+1)0^*1\big)} \\
&= \mathbf{0^*1\big((0+1)0^*1\big)^*}
\end{aligned}
$$

则 DFA 的正则表达式为:

$$R_{12}^{(3)} + R_{13}^{(3)} = \mathbf{0^*1\big((0+1)0^*1\big)^*\big(\boldsymbol{\varepsilon} + (0+1)(00)^*\big) + 0(00)^*}.$$

分治 (Divide and Conquer) – 普遍且实用的递归求解方式

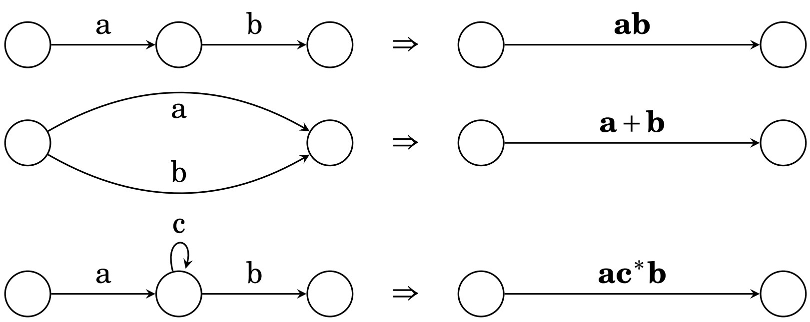
1. 将问题实例分解为子问题实例 – divide step
2. 子问题实例可递归解决
3. 将子问题实例合并可得到原问题实例 – conquer step



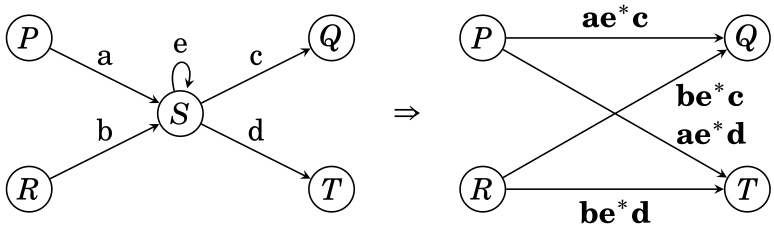
3.2.2 由 DFA 到正则表达式, 状态消除法

从有穷自动机中删除状态, 并使用新的路径替换被删除的路径, 在新路径上设计新的正则表达式, 产生一个等价的“自动机”.

- 从 DFA 中逐个删除状态
- 用标记了正则表达式的新路径替换被删掉的路径
- 保持“自动机”等价.

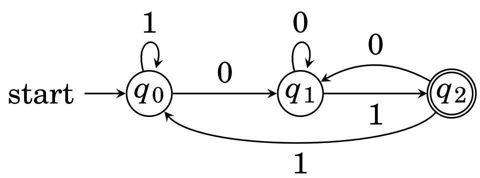


- 更一般的情况如图, 若要删除状态 S, 需添加相应路径

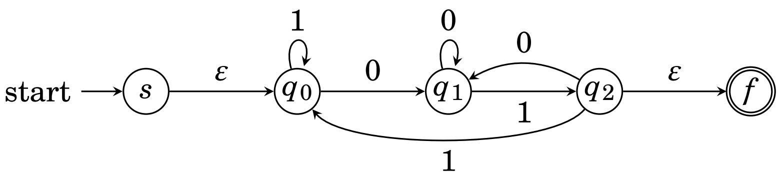


要为被删除的状态 S 的每个“入”和“出”路径的组合, 补一条等价的新路径, 结点上的循环用闭包表示. 保持新路径与被删掉的路径集合等价.

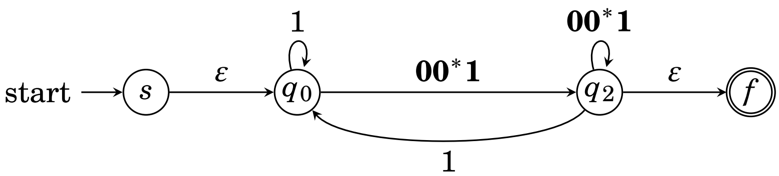
例 10. 利用状态消除法, 设计下图自动机的正则表达式.



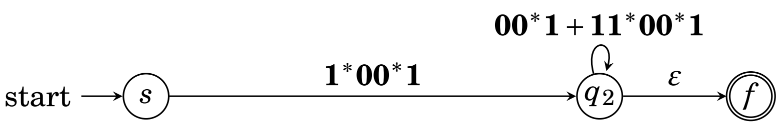
1. 利用空转移, 添加新的开始 s 和结束状态 f:



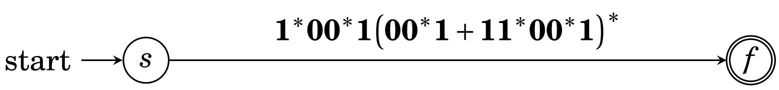
2. 消除状态 q_1, 添加路径 $q_0 \to q_2$ 和 $q_2 \to q_2$:



3. 消除状态 q_0, 添加路径 $s \to q_2$ 和 $q_2 \to q_2$:



4. 消除状态 q_2, 添加路径 $s \to f$:



5. 因此该自动机的正则表达式为

$$\mathbf{1^*00^*1(00^*1+11^*00^*1)^*}.$$

3.2.3 由正则表达式到 ε-NFA

定理 4. 正则表达式定义的语言, 都可被有穷自动机识别.

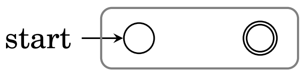
由正则表达式构造 ε-NFA(比定理 4 更严格的命题)

任何正则表达式 $\mathbf{r}$, 都存在等价的 ε-NFA A, 即 $\mathbf{L}(A) = \mathbf{L}(\mathbf{r})$, 并且 A 满足:

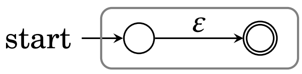
1. 仅有一个接收状态;
2. 没有进入开始状态的边;
3. 没有离开接受状态的边.

证明: 归纳基础:

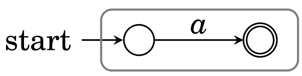
1. 对于 $\varnothing$, 有 ε-NFA:



2. 对于 $\boldsymbol{\varepsilon}$, 有 ε-NFA:



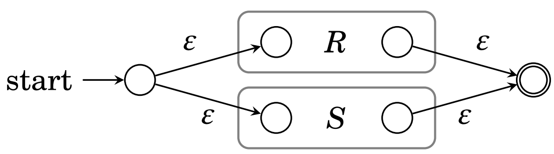
3. $\forall a \in \Sigma$, 对于 $\mathbf{a}$, 有 ε-NFA:



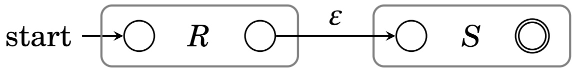
归纳递推: 假设正则表达式 $\mathbf{r}$ 和 $\mathbf{s}$ 的 ε-NFA 分别为 R 和 S

那么 $\mathbf{r} + \mathbf{s}$, $\mathbf{rs}$ 和 $\mathbf{r}^*$, 可由 R 和 S 分别构造如下:

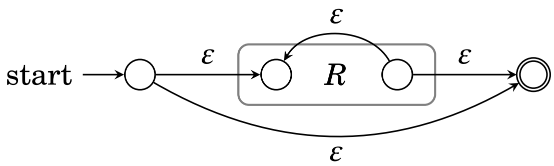
1. 对于 $\mathbf{r} + \mathbf{s}$, 有 ε-NFA:



2. 对于 $\mathbf{rs}$, 有 ε-NFA:

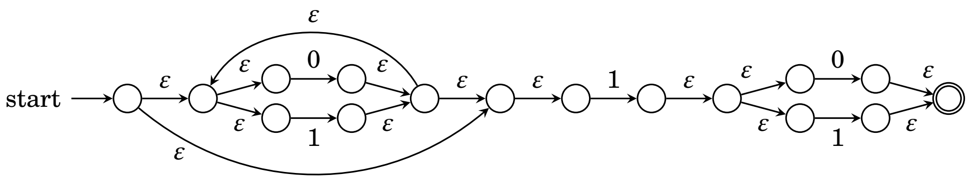


3. 对于 $\mathbf{r}^*$, 有 ε-NFA:



因此任何结构的正则表达式, 都有等价的 ε-NFA. □

例 11. 正则表达式 $(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})$ 构造为 ε-NFA.



思考题

正则表达式到 ε-NFA 构造方法中的 3 个限制条件, 都有必要吗?

3.3 正则表达式的代数定律

3.3.1 基本的代数定律

定义. 含有变量的两个正则表达式, 如果以任意语言替换其变量, 二者所表示的语言仍然相同, 则称这两个正则表达式等价. 在这样的意义下, 正则表达式满足一些代数定律.

- 并运算

$$(L+M)+N=L+(M+N) \qquad \text{结合律}$$
$$L+M=M+L \qquad \text{交换律}$$
$$L+L=L \qquad \text{幂等律}$$
$$\varnothing+L=L+\varnothing=L \qquad \text{单位元}\varnothing$$

- 连接运算

$$(LM)N=L(MN) \qquad \text{结合律}$$

$$\boldsymbol{\varepsilon} L = L\boldsymbol{\varepsilon} = L \qquad \text{单位元}\ \boldsymbol{\varepsilon}$$
$$\varnothing L = L\varnothing = \varnothing \qquad \text{零元}\ \varnothing$$
$$LM \neq ML$$

- 分配率

$$L(M+N) = LM + LN \qquad \text{左分配律}$$
$$(M+N)L = ML + NL \qquad \text{右分配律}$$

- 闭包运算

$$(L^*)^* = L^*$$
$$\varnothing^* = \boldsymbol{\varepsilon}$$
$$\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon}$$
$$L^* = L^+ + \boldsymbol{\varepsilon}$$
$$(\boldsymbol{\varepsilon} + L)^* = L^*$$

3.3.2 发现与验证代数定律

检验方法

要判断表达式 E 和 F 是否等价, 其中变量为 $L_1, \ldots, L_n$:

1. 将变量替换为具体表达式, 得正则表达式 $\mathbf{r}$ 和 $\mathbf{s}$, 例如替换 L_i 为 $\mathbf{a}_i$;
2. 判断 $\mathbf{L}(\mathbf{r}) \stackrel{?}{=} \mathbf{L}(\mathbf{s})$, 如果相等则 $E = F$, 否则 $E \neq F$.

例 12. 判断 $(L+M)^* = (L^*M^*)^*$.

1. 将 L 和 M 替换为 $\mathbf{a}$ 和 $\mathbf{b}$;
2. $(\mathbf{a}+\mathbf{b})^* \stackrel{?}{=} (\mathbf{a}^*\mathbf{b}^*)^*$;
3. 因为 $\mathbf{L}\left((\mathbf{a}+\mathbf{b})^*\right) = \mathbf{L}\left((\mathbf{a}^*\mathbf{b}^*)^*\right)$;
4. 所以 $(L+M)^* = (L^*M^*)^*$.

例 13. 判断 $L + ML = (L+M)L$.

1. 将 L 和 M 替换为 $\mathbf{a}$ 和 $\mathbf{b}$;

2. 判断 $\mathbf{a}+\mathbf{ba} \stackrel{?}{=} (\mathbf{a}+\mathbf{b})\mathbf{a}$;

3. 因为 $aa \notin \mathbf{a}+\mathbf{ba}$ 而 $aa \in (\mathbf{a}+\mathbf{b})\mathbf{a}$;

4. 所以 $\mathbf{a}+\mathbf{ba} \neq (\mathbf{a}+\mathbf{b})\mathbf{a}$;

5. 即 $L+ML \neq (L+M)L$.

注意

这种方法仅限于判断正则表达式, 否则可能会发生错误.

例 14. 若用此方法判断 $L\cap M\cap N \stackrel{?}{=} L\cap M$, 以 $\mathbf{a},\mathbf{b},\mathbf{c}$ 替换 L,M,N, 有

$$\{a\}\cap\{b\}\cap\{c\} = \varnothing = \{a\}\cap\{b\},$$

而显然

$$L\cap M\cap N \neq L\cap M.$$

例.

- $(L+M)^* = (L^*M^*)^*$
- $(\varepsilon+L)^* = L^*$
- $L^* \stackrel{?}{=} L^*L^*$

  由 $\mathbf{a}^* = \mathbf{a}^*\mathbf{a}^*$ 得 $L^* = L^*L^*$ 成立.
- $(L+M)^*M \stackrel{?}{=} (L^*M)^*$

  替换得 $(\mathbf{a}+\mathbf{b})^*\mathbf{b} \stackrel{?}{=} (\mathbf{a}^*\mathbf{b})^*$, 因为 $\boldsymbol{\varepsilon} \notin (\mathbf{a}+\mathbf{b})^*b$ 且 $\boldsymbol{\varepsilon} \in (\mathbf{a}^*\mathbf{b})^*$, 所以不相等.
- $(R+S)^* \stackrel{?}{=} R^*+S^*$
- $(RS+R)^*R \stackrel{?}{=} R(SR+R)^*$
- $(RS+R)^*RS \stackrel{?}{=} (RR^*S)^*$
- $(R+S)^*S \stackrel{?}{=} (R^*S)^*$

3.4 练习题

1. Give regular expressions for each of the following languages over the alphabet $\{0,1\}$.

   (1) Strings that end with the suffix $0^5 = 00000$

(2) All strings containing the substring 000.

(3) All strings *not* containing the substring 000.

(4) Every string except 000.

(5) All strings w such that in every prefix of w, the number of 0s and 1s differ by at most 1.

(6) All strings w such that in every prefix of w, the number of 0s and 1s differ by at most 2.

(7) All strings that start with 00 and contain at least one 1.

(8) All strings that start with 01 and contain at least two 0s.

(9) All strings that start with 1 and contain at least two 0s.

(10) All strings containing at least two 0s and at least one 1.

(11) All strings in which the substring 000 appears an even number of times. (For example, 0001000 and 10000 are in this language, but 000110000 and 00100000 are not.)

(12) Strings in which every occurrence of the substring 00 appears before every occurrence of the substring 11.

(13) Strings in which the number of 0s and the number of 1s differ by a multiple of 3.

(14) Strings that contain an even number of 1s and an odd number of 0s.

(15) Strings that represent a number divisible by 5 in binary.

2. [Execise 3.1.1] Write regular expressions for the following languages:

   (a) The set of strings over alphabet $\{a, b, c\}$ containing at least one a and at least one b.

   (b) The set of strings of 0's and 1's whose tenth symbol from the right end is 1.

   (c) The set of strings of 0's and 1's with at most one pair of consecutive 1's.

3. [!Execise 3.1.2] Write regular expressions for the following languages:

   (a) The set of all strings of 0's and 1's such that every pair of adjacent 0's appears before any pair of adjacent 1's.

   (b) The set of strings of 0's and 1's whose number of 0's is divisible by five.

4. [!!Execise 3.1.3] Write regular expressions for the following languages:

   (a) The set of all strings of 0's and 1's not containing 101 as a substring.

   (b) The set of all strings with an equal number of 0's and 1's, such that no prefix has two more 0's than 1's, nor two more 1's than 0's.

(c) The set of all strings of 0's and 1's whose number of 0's is divisible by five and whose number of 1's is even.

(d) The set of all strings of 0's and 1's whose number of 0's is even and whose number of 1's is even.

5. [!Execise 3.1.4] Give English descriptions of the languages of the following regular expressions:

    (a) $(1+\varepsilon)(00^*1)^*0^*$

    (b) $(0^*1^*)^*000(0+1)^*$

    (c) $(0+10)^*1^*$

6. [!Execise 3.1.4] $\emptyset$ and $\{\varepsilon\}$ are only two languages whose closure is finite.

7. [Exercise 3.2.1]: Here is a transition table for DFA:

	0	1
$\rightarrow q_1$	q_2	q_1
q_2	q_3	q_1
$*q_3$	q_3	q_2

    (a) Give all the regular expressions $R_{ij}^{(0)}$. *Note*: Think of state q_i as if it were the state with integer number i.

    (b) Give all the regular expressions $R_{ij}^{(1)}$. Try to simplify the expressions as much as possible.

    (c) Give all the regular expressions $R_{ij}^{(2)}$. Try to simplify the expressions as much as possible.

    (d) Give a regular expression for the language of the automaton.

    (e) Construct the transition diagram for the DFA and give a regular expressions for its language by eliminating state q_2.

8. [Exercise 3.2.2]: Repeat Exercise 3.2.1 for the following DFA:

    Note that solutions to parts (a), (b) and (e) are *not* available for this exercise.

	0	1
$\rightarrow q_1$	q_2	q_3
q_2	q_1	q_3
$*q_3$	q_2	q_1

9. [Exercise 3.2.3]: Convert the following DFA to a regular expression, using the state-elimination technique of Section 3.2.2.

	0	1
$\rightarrow{}^*p$	s	p
q	p	s
r	r	q
s	q	r

10. [Exercise 3.2.4]: Convert the following regular expressions to NFA's with ε-transitions.

    (a) 01^*.

    (b) $(0+1)01$.

    (c) $00(0+1)^*$.

11. [Exercise 3.2.5]: Eliminate ε-transitions from your ε-NFA's of Exercise 3.2.4. A solution to part (a) appears in the book's Web pages.