University of Zurich[UZH]

New Approach to the Circle Hough Transform for Detecting Cherenkov Rings in the LHC*b* RICH detector

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Abstract

This thesis explores possible algorithms for the detection of circles of Cherenkov photons which are produced by particles travelling through a RICH detectors in the LHC*b* experiment. These circles can be detected with the help of a Hough transform. There is a short introduction to the linear Hough transform (which detects straight lines) to give a rough idea about what the Hough transform can do. Then there is a brief look at the circle Hough transform (which detects circles). This part is split in 3 sections, discussing 1D, 2D and 3D Hough transforms. The 1D Hough transform can be used when the center is given and the radius needs to be found. The 2D Hough transform looks for the position of circle centers ((x, y) coordinate) for a given radius. The 3D Hough transform is used when neither center nor radius are known. This is the case for the data from LHC*b* where there are data points (photons from several circles and noise hits) and the algorithm has to find all the circles.

Apart from the standard Hough transform with a 3D accumulator space this thesis develops a new approach for a Hough transform. This approach works on the basis that each circle is defined by 3 points. Given 3 points one can calculate the circumcenter (so the center of the circle we are intersted in) and the radius of the circumcircle.

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1 Introduction

1.1 LHC - Large Hadron Collider

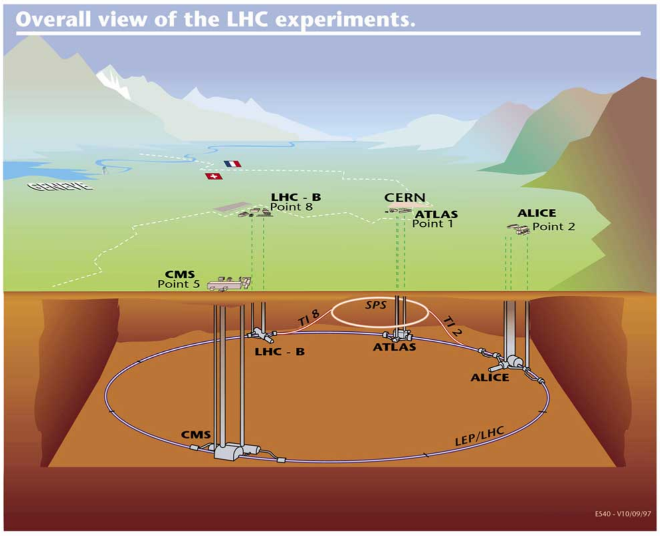


FIG. 1.1 – The LHC ring with its 4 experiments: ATLAS, CMS, ALICE and LHC*b*

The Large Hadron Collider (LHC) is the largest and highest-energy particle accelerator in the world colliding protons on protons at beam energies of up to 16.5 TeV and lead ions at beam energies up to 5.5 GeV/nucleon. It was built by the European Organisation for Nuclear Research from 1998 to 2008. It aims to test the predictions of different theories in high-energy particle physics, in particular for the search of the Higgs boson (which has been confirmed last year) and signs for new physics beyond the Standard Model of particle physics. The LHC lies in a tunnel 27 km in circumference and up to 100 m below the surface of the French-Swiss border near Geneva. The LHC was built in collaboration with over 10000 scientists and engineers from over 100 countries. The accelerator has been running with a center of mass energy $\sqrt{s} = 13$ TeV since 20 May 2015.

The LHC consists of 4 large experiments [1–4]:

ATLAS/CMS

- The two multi-purpose experiments at the LHC probing $p - p$ and heavy ions for direct searches of new particles.

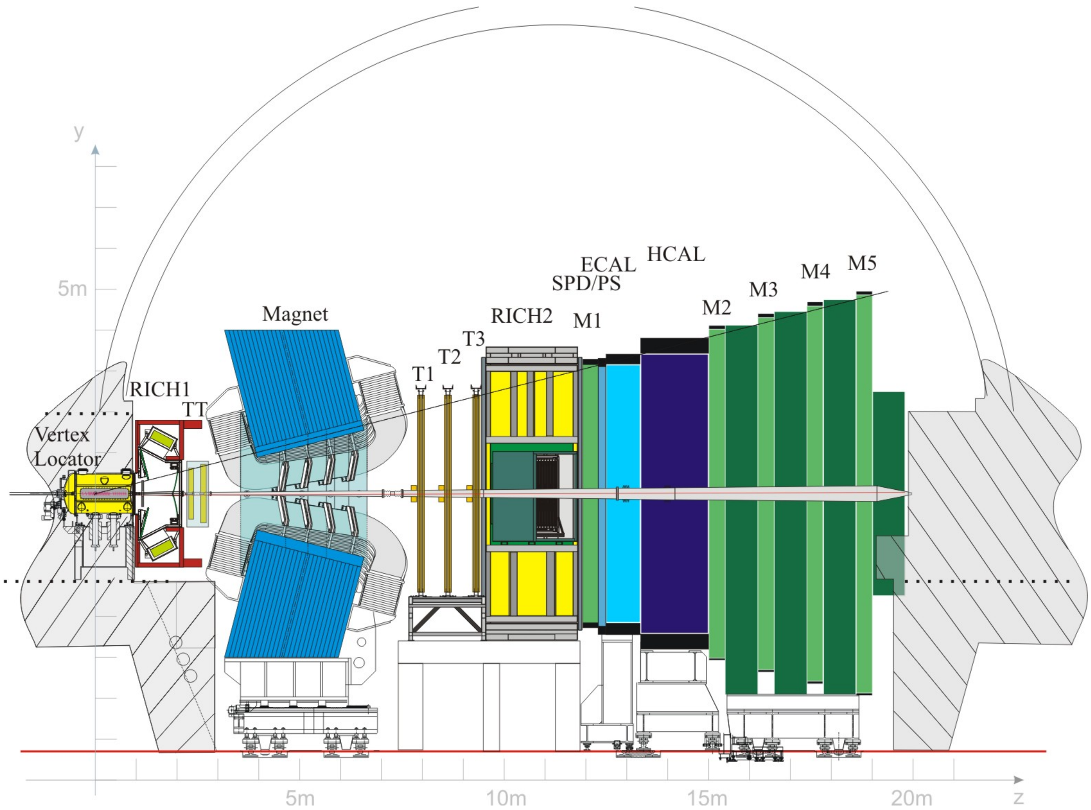


FIG. 1.2 – LHC*b* Detector: The beams collide inside the Vertex Locator. The RICH1 is positioned before the tracking station (TT) and the magnet. RICH2 is set up after the magnet and before the silicon trackers (T1-T3) and the muon and calorimeter (M1-M5)

ALICE

- ALICE (A Large Ion Collider Experiment) is a general-purpose, heavy ion detector at the CERN LHC which focuses on QCD, the strong interaction sector of the Standard Model e.g. for evidence for quark-gluon plasma.

LHC*b*

- LHC*b* is testing the Standard Model by confronting predictions with its precise measurements in CP violation and rare decays of particles containing *b* and *c* quarks.

1.2 LHC*b*

LHC*b* is one of the four big experiments conducted at the LHC (ATLAS, CMS and ALICE being the other 3). The main goal of this experiment is the study of decays of particles containing b and $\bar{b}$ quarks (B-Mesons). During collisions these particles are produced mostly at small polar angles with respect to the beam axis. This is reflected in the design of the LHC*b* detector which is a foward arm spectrometer 20 meters long with subdetectors arranged along the beam pipe as shown in figure 1.2.

A quick overview of the detector parts [5].

VELO The VErtex LOcator surrounds the region where the beams collide and $b/\bar{b}$ are produced. The VELO measures the distance between the $p-p$ collision point and the point where the B particles decay. B particles are short-lived (decaying after typically 1 cm thus the B particles are not measured directly but inferred from the separation of these two points and the properties of their decay products.

RICH The RICH detectors are built for particle identification in particular to distinguish charged kaons from pions. One detector on each side of the magnet is used to cover different momentum ranges. RICH detectors work by measuring emissions of Cherenkov radiation which is emitted if a particle travels faster than the speed of light through a certain medium (often compared to breaking the sound barrier). The emission angle depends on the speed of the particle, so knowing the speed and the momentum (from the curvature from the track induced by the magnet) the mass of the particle can be inferred.

Magnet A particle normally moves in a straight line but entering a magnetic field causes the path of charged particles to curve according to the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + (v \times \mathbf{B})\right)$$

thus allowing to determine the charge sign of the particle. Also the track curvature can be used to infer the momentum of the particle.

Tracking System The tracking system is based on 4 planar tracking stations. It is used to determine the momentum of charged particles by measuring the bending of the trajectory in the magnetic field. In the silicon detector a passing particle generates an electron-hole pair which deposits a charge on the silicon strips. In the gas-filled tubes a particle ionises the gas molucules which deposit charges on a wire.

Calorimeters They are designed to stop particles and measure their energy lost. The design of the stations is sandwich like. One metal plate and one scintillator plate. Interactions in the metal plate cause a secondary shower of charged particles which induce a scintillation light in the scintillator plate. The energy lost is proportional to the amount of light emmitted. Calometry is also the main way of identifying particles with no charge (e.g. photons, neutrons).

Muon system Muons play an important role in many analyses. There are 5 planar stations at the end of the detector. The total area covered by these stations is about 435 m^2. The goal of the absorber is to stop all particles except muons who still can pass. The muons get detected in the gas chambers where their trajectory is measured.

1.2.1 Particle identification

An important requirement at LHC*b* is particle identification. This is handled by CALO, Muon and RICH sub-detectors. The Calorimeters beside measuring energies and positions of electrons, photons and hadrons also provide identification of said particles e.g. by measuring the shape of the induced showers. The Muon system identifies muons to a very high level of purity which is essential for many J/Ψ's in their final states.

Hadron identification is very important for decays where the final states of interest are purely hadronic. The LHC*b* RICH system provides this, covering a momentum range of approximately 1–100 GeV. It is composed of two detectors. One positioned upstream of the dipole magnet and the other one positioned downstream of the dipole magnet. The optics is arranged similarly in both sub-detectors: spherical focusing mirrors project the Cherenkov photons onto a series of flag mirrors which then reflect them onto a series of photon detector arrays, located outisde the detector acceptance [6]. +Hadron identification, i.e. separation of kaons, pions and protons, is very important for many analyses. The LHC*b* RICH system provides this, covering a momentum range of approximately 1–100 GeV. It is composed of two detectors. One is positioned upstream of the dipole magnet and the other one positioned downstream of the dipole magnet. The optics is arranged similarly in both sub-detectors: spherical focusing mirrors project the Cherenkov photons onto a series of flat mirrors which then reflect them onto an array of photon detector, located outisde the detector acceptance [6].

2 Theory

2.1 Cherenkov radiation

The speed of light in vacuum, **c**, is a universal physical constant. According to Einstein's special theory of relativity, c is the maximum speed at which all matter (or information) in the universe can travel. The speed at which light propagates in a medium, however, can be significantly less can c.

Cherenkov radiation results when a charged particle travels through a dielectric medium with a speed greater than the speed of light through said medium. The velocity that must be exceeded is the phase velocity (v_{Phase} or short v_{P}) and not the group velocity ($v_{\text{Group}} = \frac{\partial\omega}{\partial k}$).

$$v_{\text{P}} = \frac{\lambda}{T} \quad \text{or} \quad \frac{\omega}{k}$$

where λ is the wavelength of the light and T the period and ω the angular frequency and the wavenumber k.

As a charged particle travels through the medium, it disrupts the local electromagnetic field. If the particle travels slower than the speed of light then the disturbance elastically relaxes to the mechinal equilibrium as the particle passes. However, if the particle travels faster than the speed of light, the limited response speed of the medium means that a disturbance is left in the wake of the particle, and the energy in this disturbance radiates as a coherent shockwave.

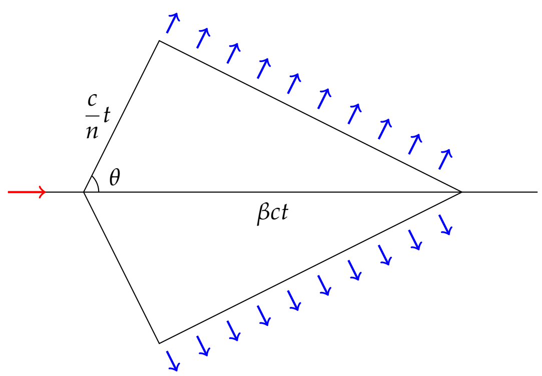


FIG. 2.1 – Cherenkov radiation where θ is equal to $\cos\theta = \frac{\frac{c}{n}t}{\beta ct}$

$$x_p = v_p \cdot t = \beta c t$$
$$x_{\text{em}} = v_{\text{em}} \cdot t = \frac{c}{n} t$$
$$\cos\theta = \frac{x_{\text{p}}}{x_{\text{em}}} = \frac{\frac{c}{n} t}{\beta c t} = \frac{1}{n\beta}$$

which is independent of time.

2.2 RICH detector

Particle identification is a fundamental requirement for many analyses at the LHC*b* experiment. The LHC*b* experiment is unique at the LHC in the sense that it uses RICH detectors for hadronic particle identification therefore much better than e.g. ATLAS or CMS. Using three different radiators the RICH detectors cover a wide range of momentum (1-100 GeV/c).

Both RICH-1 and RICH-2 are located in low magnetic field regions to keep the tracks straight while they pass through the radiators. They both also have a tilted spherical focusing primary mirror and a secondary flat mirror to limit the length of the detector along the beam pipe. The spherical focusing mirrors use the property that photons generated at a fixed angle with respect to the particle trajectory are then focused onto a ring in the photon detector plane (via the secondary flat mirror). The photon detectors are hybrid photon detectors (HPD) which have a resolution of 2.5 mm $\times$ 2.5 mm. The HPD are used to measure the spatial position of emitted Cherenkov photons. The HPD is a vacuum photon detector in which a photoelectron, released from the conversion in a photocathode of an incident photon, is accelerated by and applied voltage onto a silicon detector[3]. The total area of the two detector planes in the RICH-1 detector is 1302 mm $\times$555 mm each and 710 mm $\times$1477 mm for RICH-2.

The RICH-1 in front of the magnet covers a lower momentum range from 1-60 GeV/c. It is composed of 5 cm thick aerogel tiles arranged around the beam pipe. The aerogel with $n = 1.03$ is suited for the lowest momentum tracks. Directly behind the aerogel is circa 1 m of C_4F_{10} ($n = 1.0014$) which covers the intermediate region of momentum. For the highest momentum tracks, gaseous CF_4 ($n = 1.0005$) is used in the RICH-2 [7].

There is a strong correlation between the polar angle and momentum of the particles. Particles with a larger polar angle tend to have lower momentum. That is why RICH-1 with the aerogel is located before the dipole magnet so tracks with low momentum will be covered before they swept out of the acceptance by the magnet.

2.3 Hough transform

The Hough transform [9] is a feature extraction technique used in image analysis, computer vision and digital image processing.

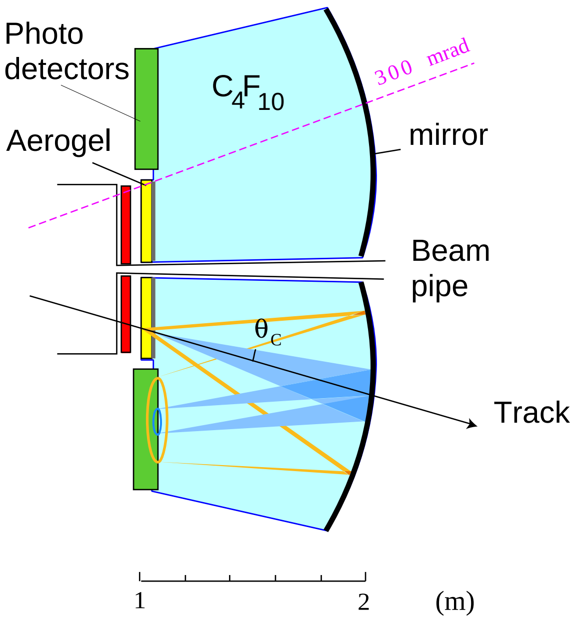


FIG. 2.2 – RICH-1 detector [8].

The purpose is to find imperfect instances of objects within a certain class of shapes by a voting procedure. This voting procedure is carried out in a parameter space from which object candidates are obtained as local maxima in a so called accummlator space that is explicitly constructed by the algorithm for computing the Hough transform.

Initially the Hough transform was concerned with finding straight lines [10] but has been extended to identifying positions of more complicated shapes, such as circles and ellipses.

2.3.1 Linear Hough transform

A linear function is often defined as:

$$f(x) = m \cdot x + b$$

where m is the slope of the line and b the intercept. For the Hough transform however, this representation is not ideal. For a vertical line m would go to infinity which gives us an unbound transform space for m. For this reason Duda and Hart suggested the r-θ parametrization [9].

$$r = x \cos\theta + y \sin\theta \tag{2.1}$$

where r is the distance from the origin to the closest point on the line and θ is the angle between the x-axis and the line connecting the origin with that closest point.

This means given a single point in the (x, y) plane, the set of all lines going through this point form a sinusoidal curve in r-θ space. Another point that lies on the same straight

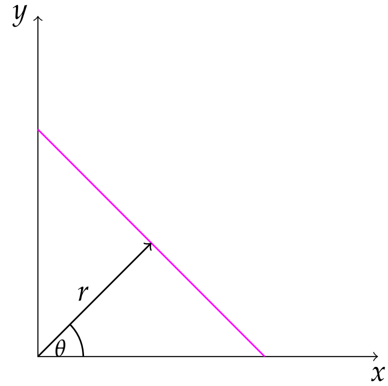


FIG. 2.3 – r-θ parametrisation

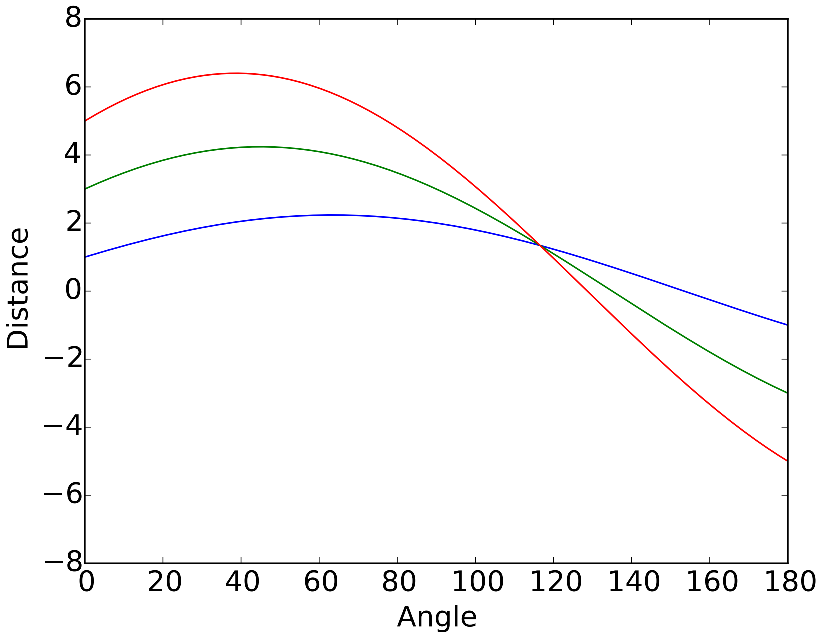


FIG. 2.4 – Example of a linear HT parameter space. Each line is representing a point of the line. For each point several lines are drawn for different angles according to equation 2.1 so this plot draws the perpendicular distances from these lines to the origin. When different lines intersect it means that these 3 Points lie on a line with the parameters given by the plot.

line in the (x, y) plane will produce a sinusoidal curve that intersects with the other at (r-θ) and so do all the points lying on the same straight line.

2.3.2 Circle Hough transform

For this thesis we are interested in circle detection so we need to adapt the linear Hough transform in order to find circles. In a two dimensional space (x, y), a circle can be described by:

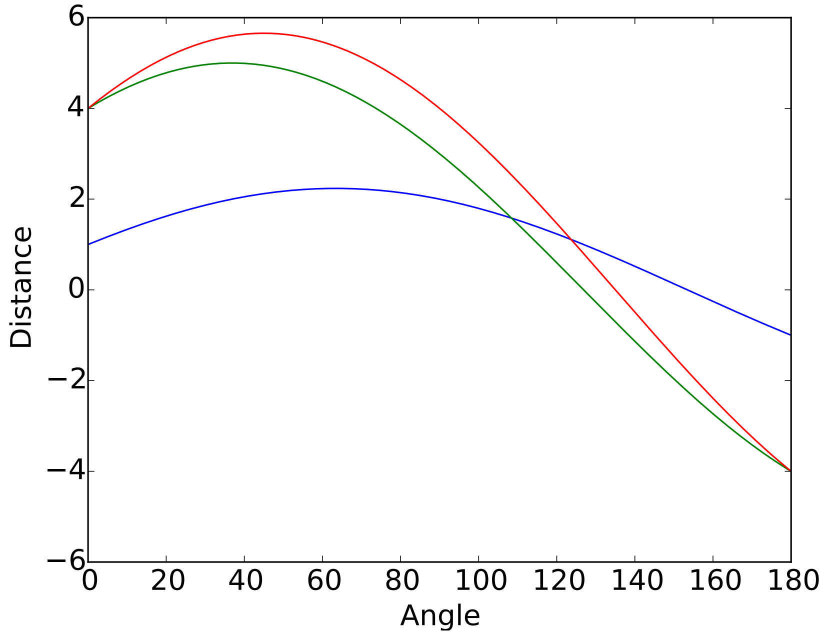


FIG. 2.5 – Example of a linear HT where points don't lie on a line. Since any two points can form a line there are still intersections but never more than 2.

$$(x - c_x)^2 + (y - c_y)^2 = r^2 \tag{2.2}$$

Where (c_x, c_y) is the center of the circle and r the radius. The possible parameters for the parameters space are now c_x, c_y and r. This means if we know the center of the circle the parameter space is one-dimensional and if we know the radius of the circle the parameter space is two-dimensional and of course if we know nothing the parameter space is three-dimensional.

2.4 Datasets

Two Monte-Carlo data sets were used for this thesis. One was a training data set of 250 simulated events and the test data on which the performance of the algorithm was tested, consisting of 10'000 simulated events with a total of 49'979 circles across all events. The properties of these events (radius, number of points per circle, number of circles per event) are taken from both simulation of LHC*b* events and real data taken from the experiment.

The number of points per circle is determined by the photoelecton yield N_{pe}. It is measured for both *normal* events and *ideal* events. The normal event is representative of nominal RICH running conditions during LHC*b* physics detector operations. The ideal event is a special event type with very low background hits [11].

	N_{pe} from data		N_{pe} from simulation	
Radiator	tagged $D^0 \to K^-\pi^+$	pp$\to$ pp $\mu^+\mu^-$	Calculated N_{pe}	true N_{pe}
Aerogel	5.0 ± 3.0	4.3 ± 0.9	8.0 ± 0.6	6.8 ± 0.3
C_4F_{10}	20.4 ± 0.1	24.5 ± 0.3	28.3 ± 0.6	29.5 ± 0.5
CF_4	15.8 ± 0.1	17.6 ± 0.2	22.7 ± 0.6	23.3 ± 0.5

TAB. 2.1 – Comparison of photoelectron yields (N_{pe}) determined from $D^* \to D^0\pi^+$ decays in simulation and data, and pp $\to$ pp $\mu^+\mu^-$ events in data, using the selections and methods described in the text [11]

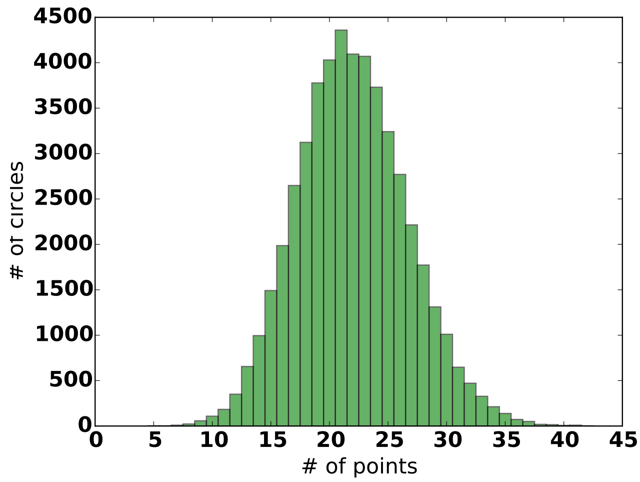


FIG. 2.6 – Number of points per circle (N_{pe}) for the 10'000 simulated events in the test data sample

Also to get an idea what the average size of a Cherenkov ring is can be seen in [12].

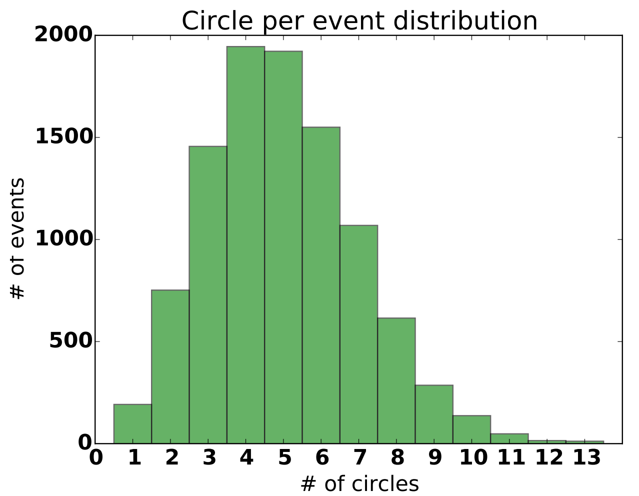


FIG. 2.7 – Number of circles per event in the test data sample

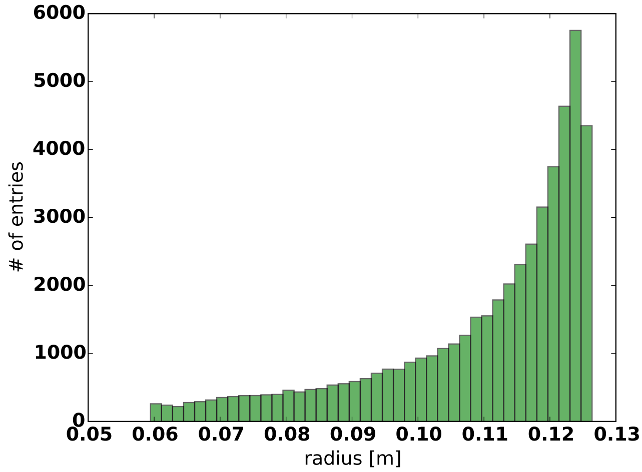


FIG. 2.8 – The radius distribution of the test data events.

Rings in RICH-1 have radiuses generally smaller than 0.15 m (given they are from the C_4F_{10} radiator; rings from the aerogel have slightly bigger radiuses – around 0.20 m). All generated rings in the test data have a radius that is smaller than 0.15 m.

2.5 Existing algorithm

There is already an algorithm in place in LHC*b* [12]. The disadvantage of the existing algorithm is due to the fact that it uses tracks to reconstruct the circles. All charged tracks that are reconstructed by the tracking system are reflected at the RICH mirrors in order to give a center for a Cherenkov ring. In order to match tracks with rings each ring is matched with the track that has its position closest to the center of the ring. This means that this method uses information from outside the RICH detector to construct the circles.

3 Methods

3.1 Conventional Hough transforms

In the following subsections we discuss the conventional Hough transforms for the case of a one, two and three dimensional parameter space. These methods were mainly considered to get an idea of what was possible with the conventional Hough transform. The conventional Hough transforms use the accumulator space which is essentially a histogram. As such it depends on the size of the binning. The smaller the binning the higher the accuracy for the parameter (either r or (x, y) coordinate) and conversely if the binning is larger the parameters will only roughly match the real ring. The execution time depends directly on the binning; the finer the binning the longer the execution time and vice versa.

For closer study the method of choice was the combinatorial triplet Hough transform discussed in depth in section 3.2.

3.1.1 1D: Known center - find radius

In this case the center(s) of the circle(s) is/are known so only the radius is missing. For the radius an array is defined with a minimum value and increasing stepsize to the maximum possible radius. For the example studied here the minimum was chosen to be 0, the maximum radius to be 1 and the stepsize equal to 0.001. The following scoring function $\eta(r)$ was used to calculate the distance of a data point (x, y) from the given center (c_x, c_y) and radius r:

$$\eta(r) = (c_x - x)^2 + (c_y - y) - r^2 \tag{3.1}$$

If $\eta(r) = 0$ means that the particular point (x, y) would lie on the circle of radius r with center (c_x, c_y). A Gaussian distribution 3.2 is used to have a well defined value for such point. For a bigger σ a point (x, y) that is not exactly on the circle will still contribute to the total score. The smaller the σ the tighter the range is in which a point will be considered lying on a circle or not.

$$w(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\eta^2}{2\sigma^2}\right) \tag{3.2}$$

The value from equation 3.2 is added to the bin in the radius histogram which corresponds to the r used to calculate the weight.

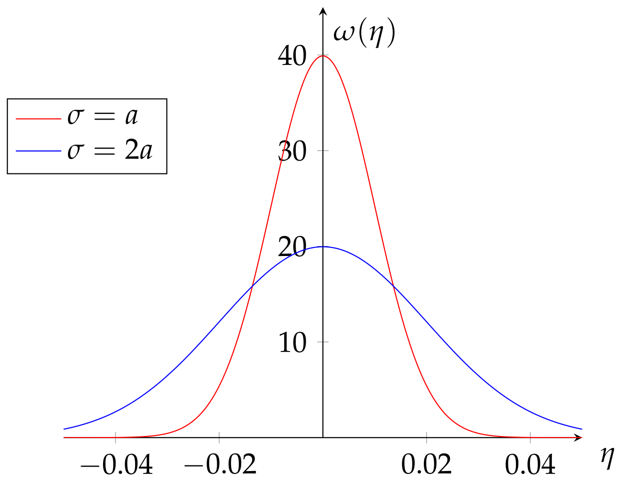


FIG. 3.1 – Using the probability density function of the normal distribution to calculate the score of a point in order to have a well defined maximum if a point lies directly on the circle and $\eta(r) = 0$. Increasing the σ increases the width of the function and by extension increases the score of points who are not lying directly on the circle

Equation 3.2 is of course just the known coordinates of the circle equation with c_x, c_y being the center of the circle, x, y are the data points and r is the radius. If a lot of the data points have the same distance r from the circle center there will be a high score for this particular radius. The index for the highest score can then be used to find the corresponding radius. The pseudo code for this Hough transform is shown in Code Snippet 3.1. An example of a resulting radius histogram is shown in figure 3.2

```
DMENSION = 1001
r= linspace(0,1,DIMENSION)
for c in centers:
  scores = zeros(DIMENSION)
  for x,y in allPoints:
    s = 2*BIN_WIDTH
    eta = (c_x-x)**2 + (c_y-y)**2 - r**2
    scores += 1. / ( sqrt( 2 * Pi ) * s )
               * exp( -( eta ** 2 )
                / ( 2 * s ** 2 ) )
  index = max(scores)
  circle = {}
  circle['center'] = c
  circle['radius'] = r[index]
```

Code Snippet 3.1 – Pseudo code for the 1D Hough transform. r is an array of length 1001 so η will also be an array of length 1001. Scores is where the score for each iteration is stored. For each point the score is computed and added to the scores array and at the end the index with the highest score is the index we need to get the radius

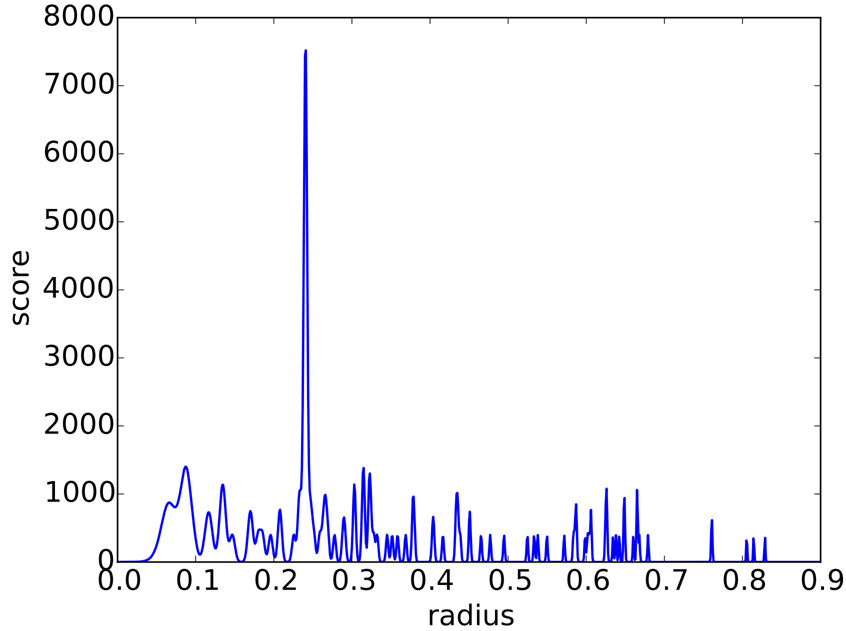


FIG. 3.2 – A graph visualisation of a 1D radius histogram. The peak between 0.2 and 0.3 has by far the highest score and the bin position of the peak is the radius candidate for a given circle center (c_x, c_y)

Complexity

The complexity of this algorithm is of $\mathcal{O}(n)$ where n is the number of bins in the radius array.

3.1.2 2D: Known radius - find center

In this case the radius is known and the x and y coordinates of the center (c_x, c_y) are unknown. Now, the accumulator is 2 dimensional. The range of that space is the dimension of the detection plane which for this thesis is $[-0.5, 0.5]$. The size of the bins is chosen to be 0.001 since the known radius given from the test data was of the same order. So if the detection plane was in reality 1 m by 1 m the binning of the accumulator in each dimension is 1 mm. As in the one dimensional case we use the scoring function 3.1 in combination with the weight function 3.2.

Complexity

The complexity of this algorithm is $\mathcal{O}(n \times m)$ where n and m are the number of bins for one coordinate respectively of the histogram. The calculation of the weight has to be done for each data point of the 2D histogram. So in a 1000×1000 histogram with 400 data points we calculate 400'000'000 times the weight of a grid point. Reducing the

```
DIMENSION = 1001
xbins = linspace(-0.5,0.5,DIMENSION)
ybins = linspace(-0.5,0.5,DIMENSION)
x, y = broadcast_arrays( xbins[..., newaxis],
                         ybins[newaxis,...] )

for r in Radiuses:
  weights = zeros( (DIMENSION,DIMENSION) )
  for xd,yd in allPoints:
    s = 2*BIN_WIDTH
    eta = (xd-x)**2 + (yd-y)**2 - r**2
    weights += 1. / ( sqrt( 2 * pi ) * s )
               * exp( -( eta ** 2 )
               / ( 2 * s ** 2 ) )
  i, j = argmax(weights)
  removeUsedPoints()
  circle['Center'] = (xbins[i], ybins[j])
  circle['Radius'] = r
```

Code Snippet 3.2 – Pseudo code for the 2D Hough transform. xbins and ybins are arrays of length 1001. Here we use array broadcasting in order to avoid for loops and the weights can be evaluated in one line. This means that the x and y variables have dimension (1001,1001) but they don't take up that much memory. The x variable for example just broadcasts its value from the first row down to all the other rows and for y it broadcasts the first column to all the other columns. The variable weights is a 1001 by 1001 matrix. Again the entry with the highest score is the candidate for a possible circle center and if found is stored in a final variable called circle.

dimensions of the histogram weakens the accuracy of the whole algorithm but can speed up the calculations considerably. With a 1000×1000 histogram the resolution in each space dimension is 1 mm. The RICH Technical Design Report states the resolution of the HPD is 2.5 mm $\times$ 2.5 mm.

The need (not entirely true – see below) to calculate the weight for each grid point and data point means that there is a loop over data points and two loops for the x and y coordinate of the grid. To improve upon that there is the possibility of array broadcasting.

Array broadcasting

Consider following one dimensional arrays where x is a 1D histogram binning entries from 1 to 4 and same for y.

$$x = [1, 2, 3, 4]$$
$$y = [1, 2, 3, 4]$$

Now all combinations between an element of x and y represent a 2D grid $((1, 1), (1, 2), ...)$. So to iterate through all those grid points one would have to create 2 for-loops iterating through x and y as shown in below pseudo code snippet:

```
  def noBroadcast():
  a = np.random.randn(100)
  b = np.random.randn(100)
  for x in a:
    for y in b:
      print (1-x)^2 + (2-y)^2 - 9
```

This is not only slow but also doesn't look too nice if there are even more loops. Broadcasting now turns the one dimensional arrays of length n into two n by n matrices

$$x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

And with this the loops can be omitted:

```
def withBroadcast():
  a = np.random.randn(100)
  b = np.random.randn(100)
  x,y = np.broadcast_arrays(a[...,np.newaxis],
                            b[np.newaxis,...])
  print (1-x)^2 + (2-y)^2 - 9
```

In this case this prints a 4 by 4 array with the function evualted for each combination of entries of x and y

$$\begin{bmatrix} -8 & -9 & -8 & -5 \\ -7 & -8 & -7 & -4 \\ -4 & -5 & -4 & -1 \\ 1 & 0 & 1 & 4 \end{bmatrix}$$

A runtime comparison shows

```
In [3]: %timeit withBroadcast()
10000 loops, best of 3: 76.8 us per loop

In [4]: %timeit noBroadcast()
100 loops, best of 3: 7.99 ms per loop
```

So the version with broadcasting is 100 times faster than the double loop. And the memory consumption is moderate since the broadcasted entries aren't new memory locations but just refer to the initial array.

Optimizations

It was mentioned before that for each data point the weight for the whole grid has to be calculated. This is not true. In the 2D case each grid point is a potential center for a circle only if it is not further a away and a threshold radius R_T, so if a grid point is further away than this threshold radius this calculation could be skipped. This could probably be done even smarter with the use of a sub grid so only points in the surrounding sub grids are considered of being possible centers and not the whole space.

Simple example: 2 circles without background

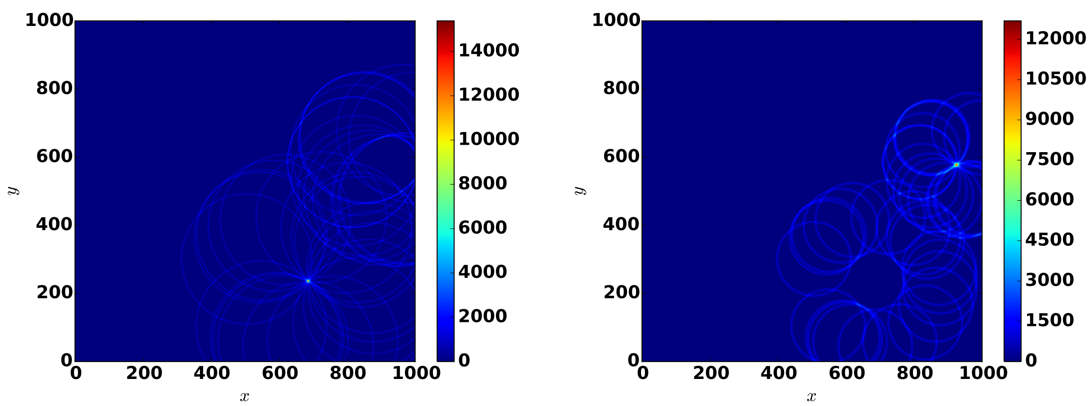


FIG. 3.3 – (Left) 2D weight matrix in the first iteration of the Hough transform algorithm. (Right) Second iteration of the 2D weight matrix of the Hough transform. Points that satisfied the condition being less than a certain ϵ away from the radius found in the first iteration are removed leaving (hopefully) only points available that belong to the second circles

3.1.3 3D: All parameters unknown

In this case all that is known are the data points and the algorithm has to retrieve both the center and the radius of the circles. The accumulator space is now in three dimensions, two for the coordinates of the center and one for the radius. Similar to the 2D case, array broadcasting (see code snippet 3.3) is used to speed up the calculations of the weights. Furthermore, the algorithm has to decide itself whether or not all circles have been found since there unlike in the previous two cases, there isn't any information availabe about the circles so a condition has to be set to decide when there are no more circles.

For deciding whether or not the algorithm has found all rings, a simple score threshold is used that whenever the highest score of the weight matrix is less than the threshold the algorithm stops and it is assumed that all circles have been found. The pseudo code for calculating the weights and the threshold check is shown in 3.4. In the 3D case the algorithm has to remove data points that have contributed to the highest score. If not done the algorithm will loop endlessly because always the same score will be found. Once the highest score has been found the center coordinates and radius are extracted from the histogram and with these the algorithm calculates for every data point, if this data point lies on that circle given the extracted center and radius. If the data point lies within two times the bin width of the circle, it will be removed (see 3.5).

```
xbins = np.linspace(-0.5,0.5,DIMENSION)
ybins = np.linspace(-0.5,0.5,DIMENSION)
rbins = np.linspace(0,0.5, R_DIMENSION)

x,y,r = np.broadcast_arrays(\
            xbins[np.newaxis,...,np.newaxis],\
            ybins[np.newaxis,np.newaxis,...],\
            rbins[...,np.newaxis,np.newaxis])
```

Code Snippet 3.3 – Broadcasting of the 3 arrays x, y, r. With this 3 for-loops can be avoided improving speed and readability of the code.

As before a scoring function is used but this time the scoring function is of the form $\eta(x, y, r)$ and each point in weights then stands for the score of the x, y, r entries and their respective value.

$$\eta(x, y, r) = (x - d_x)^2 + (y - d_y) - r^2 \tag{3.3}$$

where d_x and d_y are the data points.

$$w(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\eta^2}{2\sigma^2}\right) \tag{3.4}$$

Complexity

The complexity of this algorithm is of order $\mathcal{O}(m \times n \times r)$. As in the case the 1D and 2D Hough transform the accuracy directly depends on the binning. For the 3D Hough transform to match the resolution of the HPD of the LHCb RICH detector a binning of 400 makes sense if the detector dimensions are 1 m×1 m.

```
while True:
  weights = np.zeros(\
         (R_DIMENSION, DIMENSION, DIMENSION))

  for x0,y0 in data['allPoints']:
    s = 0.001
    eta = (x-x0)**2 + (y-y0)**2 - r**2
    weights += 1./( sqrt( 2 * sconst.pi ) * s )*\
                     np.exp( -( eta ** 2 ) / \
                     ( 2 * s ** 2 ) )
  index = np.argmax( weights )
  rr,ii,jj = np.unravel_index( index,
              (R_DIMENSION, DIMENSION, DIMENSION))
  score = weights[rr][ii][jj]
  if score < THRESHOLD:
    break
```

Code Snippet 3.4 – The while loop works as long as the found score is higher than `THRESHOLD`. If the score is lower than the threshold the loop breaks and the function returns the results that have been found.

```
circle['center'] = (xbins[ii], ybins[jj])
circle['radius'] = rbins[rr]
circles.append(circle)

used_xy += [tup for tup in data['allPoints'] if
    abs( ( tup[0] - circle['center'][0] ) ** 2 +
         ( tup[1] - circle['center'][1] ) ** 2 -
         circle['radius'] ** 2 ) < 2 * 0.001]
data['allPoints'][:] =
            [tup for tup in data['allPoints'] if
    abs( ( tup[0] - circle['center'][0] ) ** 2 +
         ( tup[1] - circle['center'][1] ) ** 2 -
         circle['radius'] ** 2 ) >= 2 * 0.001]
```

Code Snippet 3.5 – In order to avoid finding the same circle over and over again the algorithm has to remove points that belong to a ring. If a data point lies within two times the bin width of a circle the algorithm considers that data point to be part of that circle and removes that data point from the list.

Optimisations

As for the 2D HT introducing a sub grid can be introduced for the x, y plane so only grid points in the vicinity of a data point are used for calculating the score.

3.2 Combinatorial triplet Hough transform

A circle is uniquely defined by 3 points and its radius and center can be calculated from the coordinates of these points. If there are 15 points lying on the same circle there are 455 possible combinations of triplets According to the binomial distribution.

$$\binom{N}{3} = \frac{N!}{k!(N-k)!}$$

Calculating the center and radius for these 455 triplets should result in the same center (x, y) and same radius r for all the triplets (floating point inaccuracy not considered).

Having one background hit in addition to the 15 circle hits increases the number of triplets number to 560. The triplets with points solely consisting of points on the circle still have the same center and radius but the new combinations that now include a background hit will vary and it is unlikely that any two triplets that include the background point will have the same center and radius. Here is an overview of the algorithm studied in this thesis.

1. Build all possible triples of points given the data points
2. For all triplets calculate the center and the radius of the potential circle
3. Remove all triplets that yield a radius bigger than a certain threshold.
4. Create a histogram with the distribution of r for the remaining triplets. Peaks in this histogram hint to a circle.
5. Scan the radius histogram for peaks and look at the 2D histogram of (x, y) for the triplets belonging to a given peak in the r distribution. If there is also a peak in the (x, y) histogram the set of the points of the triplets lie on a circle with a radius and center given by the peaks in the r and (x, y) histograms.

3.2.1 Generating the triplets

To generate the triplets the built-in function `itertools.combinations()` of python is used. The input is an iterable, in our case a list of tuples (each tuple is the x and y coordinate of a data point) which is used to create all possible combinations of triples (of said tuples).

3.2.2 Calculating the Circle given 3 points

Let (A, B, C) be a triplet of points in a 2D plane and a, b, c the lengths of the sides opposite to the respective corner. The semiperimeter is defined as

$$s = \frac{a+b+c}{2}. \tag{3.5}$$

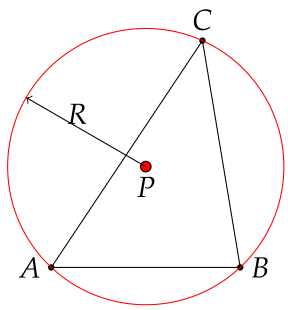


FIG. 3.4 – The circumradius (R) and the circumcenter (P) of a circle defined by three points (A, B, C).

Using this we can calculate the radius R of the circumcircle of the triangle $\overline{ABC}$:

$$R = \frac{abc}{4\sqrt{s(a+b-s)(a+c-s)(b+c-s)}} \tag{3.6}$$

We have $\lambda_1, \lambda_2, \lambda_3$ as the barycenteric coordinates of the circumcenter:

$$\lambda_1 = a^2 \cdot (b^2 + c^2 - a^2) \tag{3.7}$$

$$\lambda_2 = b^2 \cdot (a^2 + c^2 - b^2) \tag{3.8}$$

$$\lambda_3 = c^2 \cdot (a^2 + b^2 - c^2) \tag{3.9}$$

Multiplying a matrix consisting of the column vectors of A, B, C with a column vector of $\lambda_1, \lambda_2, \lambda_3$ and dividing the resulting vector by the sum of the barycentric coordinates (for normalization) yields the coordinate of the circumcenter of the triangle $\overline{ABC}$

$$\begin{pmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \boldsymbol{P'} \tag{3.10}$$

$$\frac{\boldsymbol{P'}}{\lambda_1 + \lambda_2 + \lambda_3} = \boldsymbol{P} \tag{3.11}$$

3.2.3 Finding the radius and center of a circle

Once R and **P** are known these values are stored as a pair in a tuple. A list holds then all tuples of $(R, \mathbf{P})$ values for all the triplets.

The next step is binning this data for R. This is done in a way that the center data of a tuple is associated with the R bin of the tuple. This allows the algorithm when looking for a peak in the R histogram to access the center data for this R value so essentially it splits the problem first in a 1D Hough transform where the algorithm looks for a peak in the radius distribution and with a given radius the algorithm can then search the (x, y) space for a peak and thus determine the center.

Finding the radius

Once the radius histogram is created the algorithm looks for the index where the maximum value of the histogram is stored. The algorithm checks first if the sum of the entry at the index and the left and right neighbour exceeds the radius threshold which is a tuning parameter. If the value exceeds the threshold the algorithm extracts the center data from the index and its left and right neighbour respectively.

Finding the center

Having a list of (x, y) points and one radius the algorithm now looks for a peak in the (x, y) plane. If the maximum found exceeds the center threshold then radius and the center coordinates are saved as a circle candidate.

3.2.4 Combinatorics

The main drawback of this method is that the combinatorics increase with a high number N of data points as $\binom{N}{3}$. For example, for 200 data points the number of triplets is

$$\binom{200}{3} = 1'313'400$$

and for 300 data points it is

$$\binom{300}{3} = 4'455'100.$$

The execution time of the algorithm is roughly of the order of $\mathcal{O}(N^3)$ which can be easily seen when taking the upper bound of $\binom{N}{3} \leq \frac{N^3}{3!}$ (see figure 3.5).

Optimisation

The algorithm needs a threshold on the minimum height of the peak in the (x, y) and r histograms in order to decide if a candidate is a circle or not. The threshold is defined as such that the minimum number of points on a circle has to be $\binom{T}{3}$ in order to be considered as a circle candidate.

But not only the number of triplets generated is a speed bump for the execution time but also the time it takes to create the triplets scales with $\frac{N^3}{3!}$. If there was a way to reduce not only the number of triplets created but also the time needed to create them would speed up the algorithm considerably.

This leads to the following idea: to split the original data set randomly into two lists. For each of these lists all possible combinations of triplets is generated separately and the two lists are then merged in one single list. In Figure 3.6 the difference in combinatorics between $\binom{N}{3}$ and $2\binom{\frac{N}{2}}{3}$ can be seen.

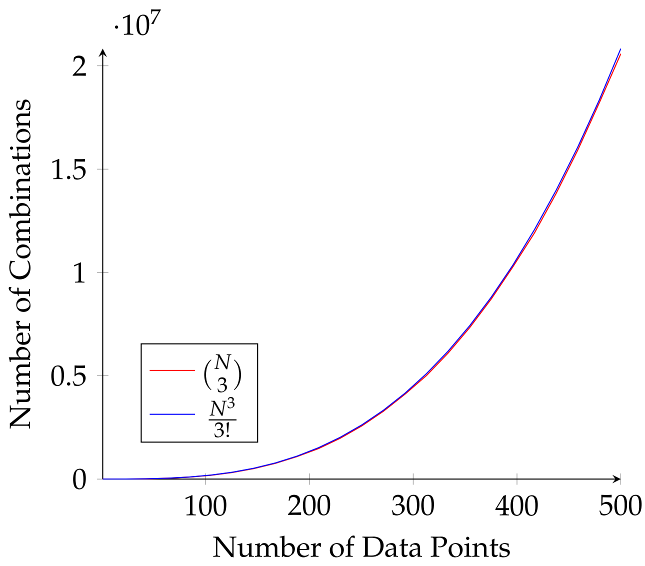


FIG. 3.5 – Scaling of the algorithm. Binomial Growth with $\binom{N}{3}$ compared with the approximation $\frac{N^3}{3!}$ where N is the number of data points. So the algorithm runs in the order of $\mathcal{O}(N^3)$

The problem with this approach is the possible loss of information and efficiency, since the algorithm has a threshold that defines how many entries a bin in the (x, y) and r histograms must have in order to be accepted as a candidate circle. This threshold should be high enough that triplets that contain noise points do not contribute significantly to the circle candidates but low enough that real circles with a low number of points can still be found.

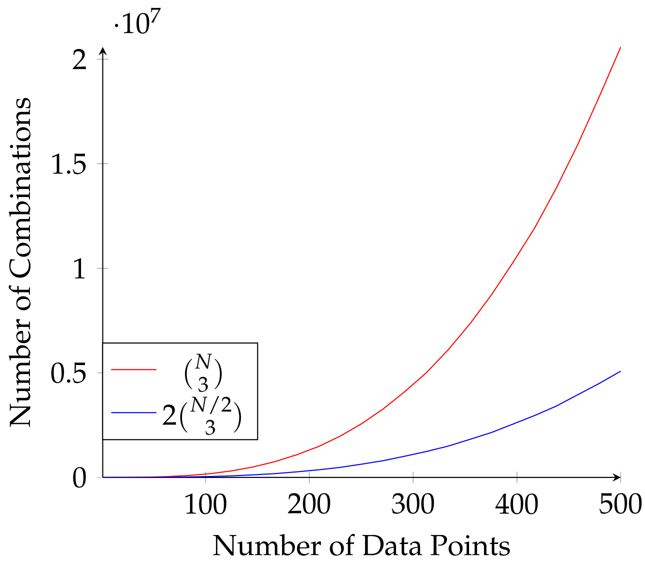


FIG. 3.6 – Number of combinations with $\binom{N}{3}$ compared to the number of combinations generated from $\binom{N/2}{3}$

So the question is what is probability of splitting points that belong to a specific circle in such way that at least one list has enough points that their combinations can reach the threshold e.g. if the threshold is 120 hits in the histograms this corresponds to 10 points ($\binom{10}{3} = 120$) having at least the number of points needed so the number of their triplets will be enough to pass the threshold. As soon as a circle has 20 points this becomes moot as always one list will have more than 10 points (see Figure 3.7).

In Figure 3.7 it can be seen that even if we split the lists there is a more than 50% chance that we lose no information once a circle has 13 points.

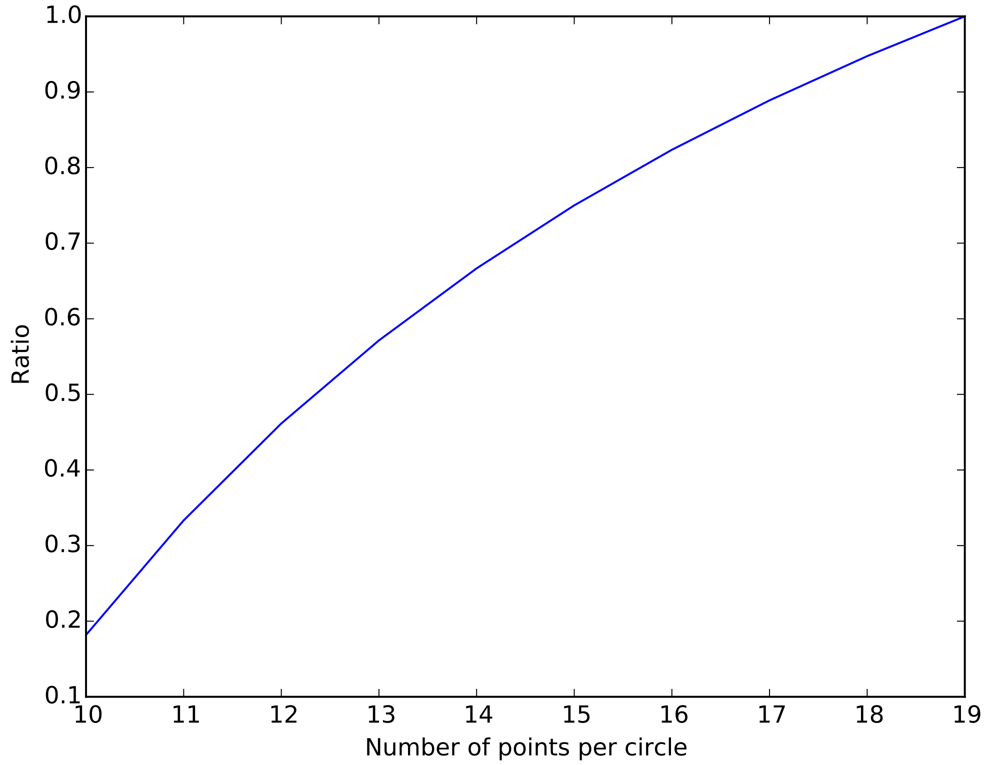


FIG. 3.7 – The probability when splitting randomly a list of x points into two that one list has more than 10 points.

Bin size dependency

In the conventional Hough transform the weight function is calculated with the bin centers of the accumlator space. The larger the binning of the accumulator is, the less accurate the calculations become. This is a main factor for the execution time of the algorithm. The advantage of this combinatorial approach lies in the fact that the parameters can be caculated accurately up to floating point precision and a histogram is only needed group the results. A fine binning only marginally affects the execution time of the algorithm since the only operation done in the histogram is finding the maximas and this is only done a few times (stops when the found maximum is lower than the ring finding threshold).

3.2.5 Possible optimisation: average radius of random circles in a unit square

An interesting property of calculating the radius of triplets generated from points that are distributed uniformely in the unit square is that they always obey a certain shape.

To prove this the expected area of a triangle formed by three points randomly chosen from the unit square[1] has to be calculated. Let $A = (x_A, y_A)$, $B = (x_B, y_B)$, $C = (x_C, y_C)$ be the vertices of the random triangle T. We consider the case where $y_A > y_B > y_C$ which takes $\frac{1}{6}$ of the total "Volume". Fix y_A, y_B, y_C for the moment and we can write.

$$y_B = (1-t)y_A + ty_C, \qquad 0 \leq t \leq 1.$$

The side AC of T intersects the horizontal level $y = y_B$ at the point $S = (s, y_B)$ with

$$s = s(x_A, x_C, t) = (1-t)x_A + tx_C \tag{3.12}$$

The area X of T is then given by

$$X = \frac{1}{2}|x_B - s|(y_C - y_A)$$

We now start integrating with respect to our six variables. The innermost integral is with respect to x_B and gives

$$\begin{aligned} X_1 &:= \int_0^1 X \mathrm{d}x_B = \frac{1}{2}(y_C - y_A)\left(\int_0^s (s - x_B)dx_B + \int_s^1 (x_B - s)dx_B\right) \\ &= \frac{1}{4}(y_C - y_A)(1 + 2s + 2s^2) \end{aligned}$$

Next we integrate over y_B:

$$X_2 := \frac{1}{4}\int_0^1 \int_{y_A}^1 (y_C - y_A)^2\, dy_C\, dy_A \times \int_0^1 \int_0^1 \int_0^1 (1 - 2s + 2s^2)\, dt\, dx_C\, dx_A$$

This gives

$$X_3 = \frac{1}{4} \cdot \frac{1}{12} \cdot \frac{11}{18} = \frac{11}{6 \cdot 144}$$

But generalizing our assumption at the beginning $y_A < y_B < y_C$ we multiply this result by 6 and obtain then $\frac{11}{144}$.

From [14] follows that the average area of a triangle in a unit circle is $\frac{3}{2\pi}$ so dividing this area by the area of the unit circle gives the expected area covered by a triangle in an arbitrary circle

[1] This proof is taken from [13]

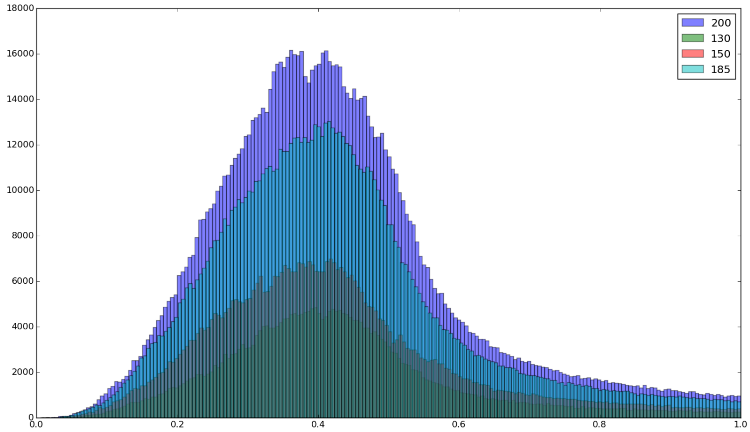


FIG. 3.8 – Radius distribution for uniformly generated background. The expected value of $R \approx 0.4$ is quite well represented for differently sized data sets

$$\frac{\text{Expected area of a triangle in a unit circle}}{\text{Area of the unit circle}} = \frac{\frac{3}{2\pi}}{\underbrace{\pi r^2}_{r=1}} = \frac{3}{2\pi^2}$$

So finally the average area of a random circle can be obtained by dividing the average area of a triangle from the unit square by expected area covered by a triangle in an arbitrary circle so essentially the average area of a random circle within the unit square:

$$\frac{\frac{11}{144}}{\frac{3}{2\pi^2}} = \frac{11\pi^2}{216}$$

This should be equal to πR^2 where R is the average radius of a random circle in the unit cube

$$R\pi^2 = \frac{11\pi^2}{216}$$
$$R = \sqrt{\frac{11\pi}{216}} = 0.399986$$

Generating a random background and then plotting the radius histogram for this data set shows that there is indeed a peak around $R \approx 0.4$.

4 Results

In this section results for the conventional 1D, 2D, 3D Hough transform and the combinatorial triplet Hough transform are presented. 1D and 2D Hough transform were not pursued in depth. They are presented here as they offer a nice way of understanding how each extra dimension expands the algorithm and also shows the flaws with each added dimension.

For the 1D, 2D and 3D Hough transform very simple data was used. There were no physical constraints when generating the circles and the number of points per circle and the distribution of the radiuses doesn't reflect the real data obtained by LHC*b*.

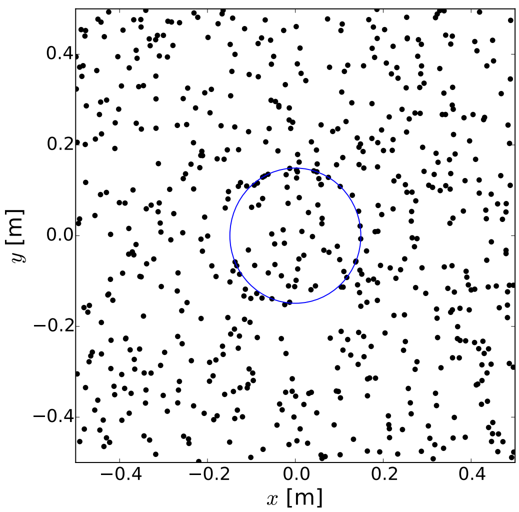
For these conventional Hough transforms following data sets were tested

- 1 circle and 600 background hits
- 2 circles and 0 background hits
- 5 circles and 30 background hits
- 6 circles and 200 background hits (results only shown for 2D and 3D)

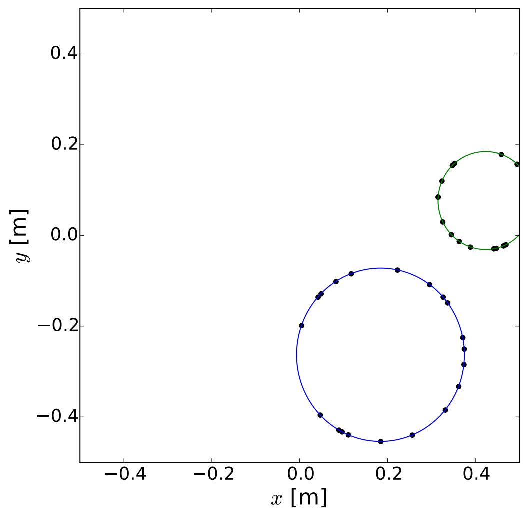
The points per circle ranges from 17–31 in all the circles from the data sets and the background is uniformly distributed in the unit square. The event with 1 circle and 600 background hits is meant to test how robust the algorithm is with a lot of background hits. The second test case is meant to test if the algorithm can handle two circle objects, the third is a mix between several circles and background and the last event is a special case where two circles lie very close to each other and for the 2D and 3D case where the center is unknown these algorithms can encounter problems with detecting the right circles. In Figure 4.1 the real circles plotted from simulation parameters for the different events.

4.1 1D Hough transform results

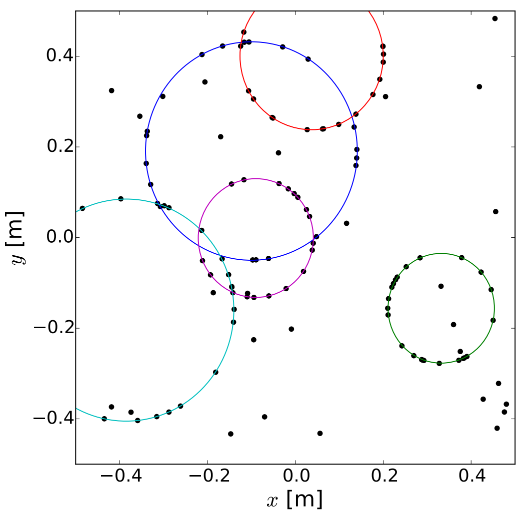
There are two different plots: for each set the radius score is shown in Figures 4.4 and 4.3 and the resulting circles in Figure 4.2. The radius score is a 1D plot of the weight function $w(\eta)$ 3.2. The highest peak indicating the maximum score and its location telling the value of the radius. The resulting circle plot is the center (which was known) and the extracted radius combined, drawing the resulting circle. The numerical values from the high scores and the second highest score are shown in Table 4.1. The algorithm has no problems to find any of the circles even with background. But this was expected as the algorithm only needs to search in one dimension and makes the whole process very easy. The fourth event was also tested for the 1D Hough transform but not shown in this thesis because they didn't add anything that is not already covered with the other 3 events.



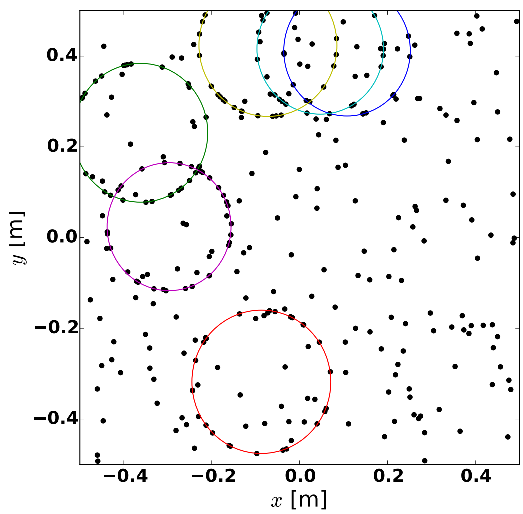
(A) 1 circle, 600 background hits.



(B) 2 circles, 0 background hits

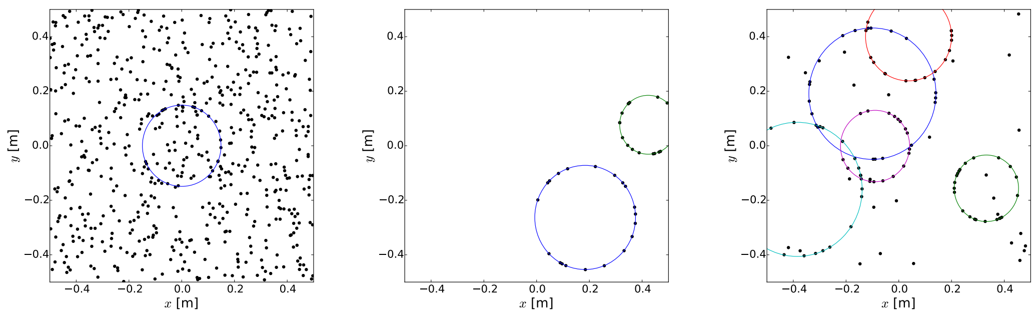


(C) 5 circles, 30 background hits



(D) 6 circles, 200 background hits

FIG. 4.1 – These are the circles as generated by the simulation.



(A) 1 circle, 600 background hits. (B) 2 circles, 0 background hits (C) 5 circles, 30 background hits

FIG. 4.2 – Circles found by the 1D Hough transform. The circle in Figure 4.2a has its center in the origin so the algorithm did find the circle

TAB. 4.1 – Radius scores for the different events. There is always a big difference between the highest score which determines the circle and the second highest score which is noise. In the example with 1 circle and 600 background the second highest score is relatively high which is due to the many background hits that coincidentally have the same center as the real circle but with a different radius than the real one

Event	Highest score	2nd highest score
1 circle 600 background hits	8′546	3′798
2 circle 0 background hits	7′904	828
	6′373	857
5 circle 30 background hits	7′519	1′401
	9′085	1′476
	6′758	1′341
	7′381	848
	7′410	1′498

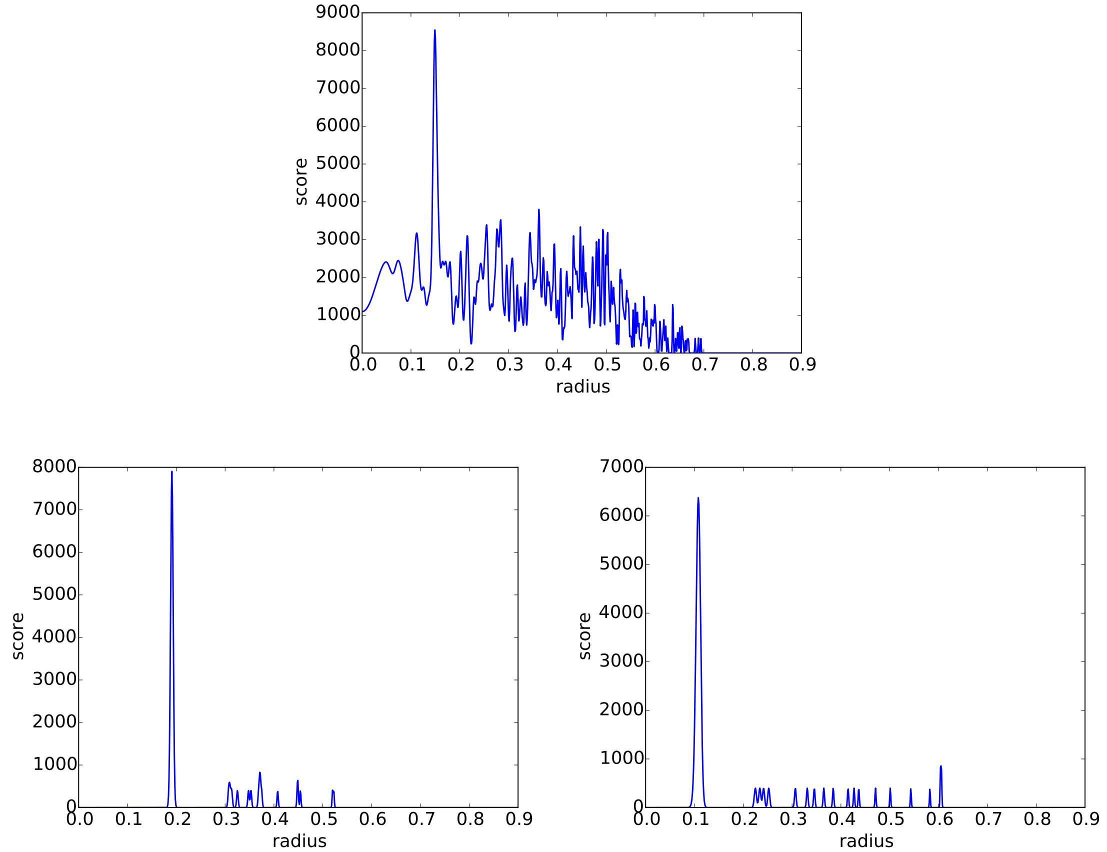


FIG. 4.3 – Radius scores for 1 circle and 600 background (top) and 2 circles with 0 background hits (bottom). For the event with only 1 circle and 600 backgrounds the scores from background hits for a given radius are higher than in the other events with less noise but the correct radius has still a distinct peak.

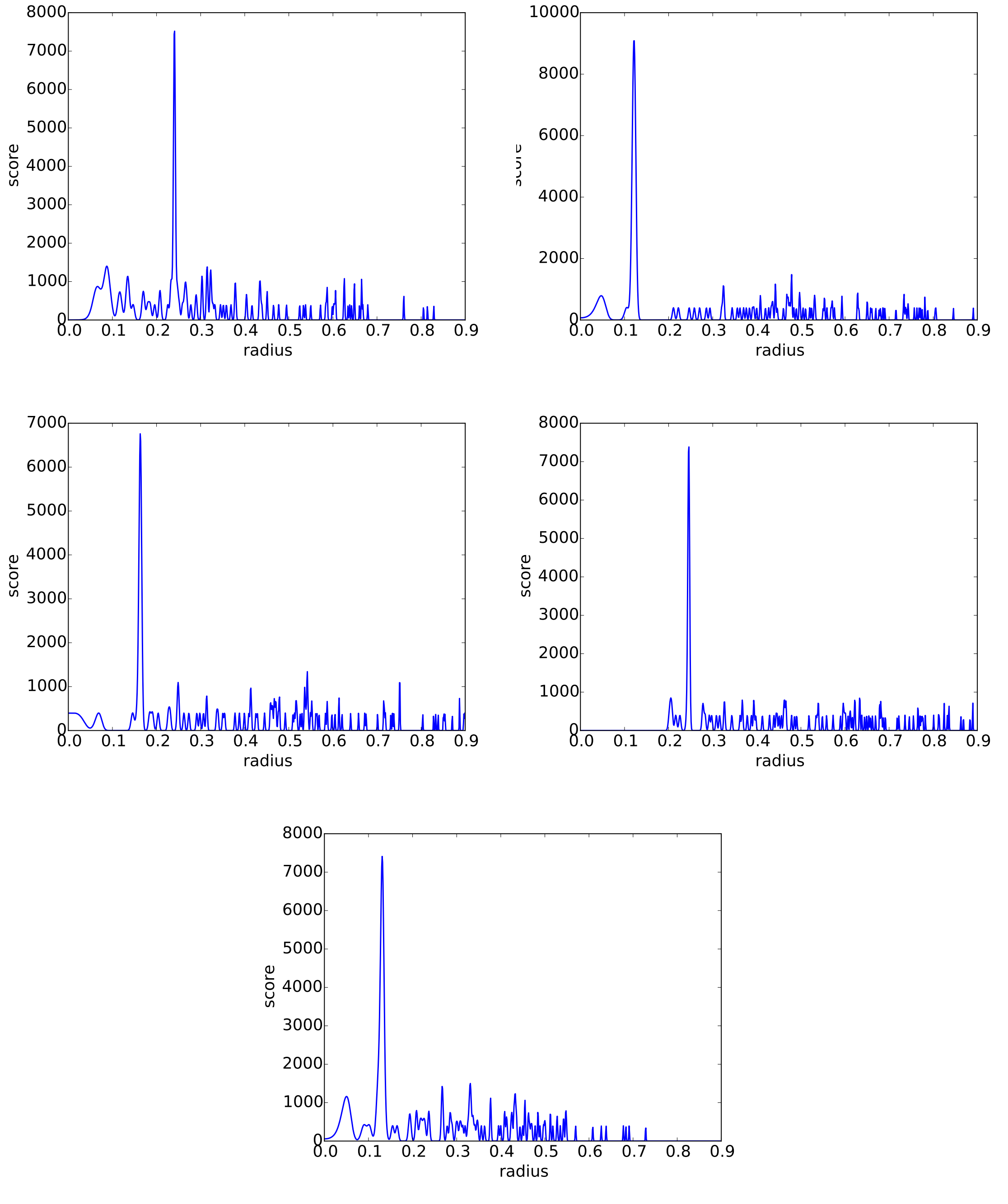


FIG. 4.4 – Radius scores for the 1D Hough transform for the 5 circles with 0 background.

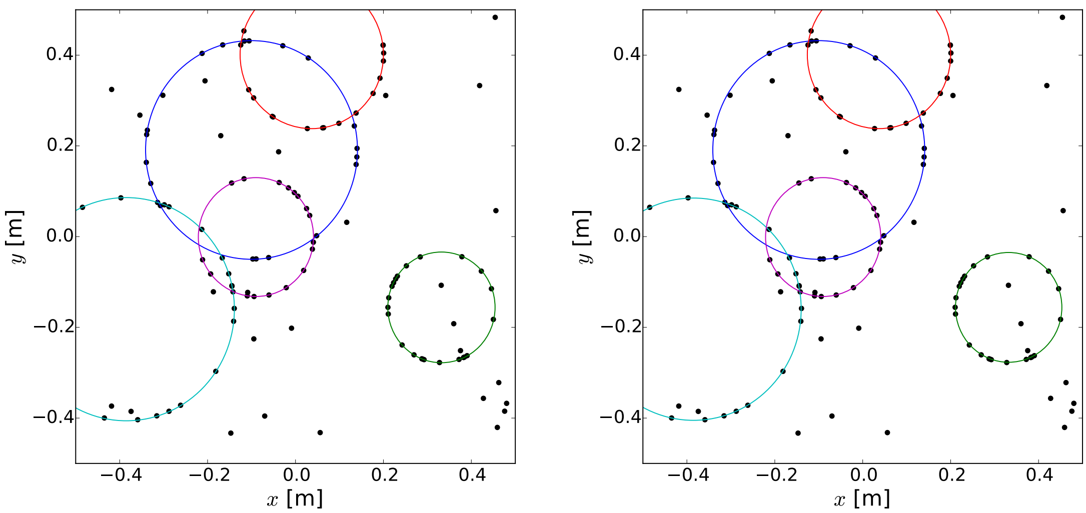


FIG. 4.5 – The result for 5 circles with 30 background hits. On the left the result obtained by the 1D Hough transform algorithm while the circles from the simulated data is on the right

4.2 2D Hough transform results

The 2D Hough transform uses again the weight function 3.2 to search for peaks with high score in the (x, y) for known radiuses. This means instead of a 1D score it is now a two dimensional plane where the highscore has to be found. Figure 4.9 shows how a slice out of this plane looks like (similar to the 1D radius histogram).

The same events as for the 1D Hough transform were investigated to make a comparison about reliability. Additionally the results for 6 circles and 200 background hits are shown in this section since the algorithm had issues with this event to detect all the circles initially.

The reason for the wrong detection of the cirle is quite simple. Two of the circles have very similar radiuses. The algorithm looks first for the magenta circle (because that happens to be the way the centers are arranged in the list) and since so many points of the yellow circle lie so close together it is easy for the magenta circle (who has a similar radius) to get a high score with these points, a higher score than it would get with its proper points. If the yellow points were more evenly distributed on the circle or if there were more magenta points this probably wouldn't happen but that is something that can't be controlled.

In Figure 4.6 and 4.7 the circles found by the algorithm and in Table 4.2 the scores for the circle candidate and the second highest score are shown (see Figure 4.1 on page 37 for the results from the data).

Theoretically it is also possible that the yellow circle gets fitted to the magenta points but since there are still enough yellow points left after the removal of the points assigned to the magenta circle they still have the highest score with their own points and the magenta circle goes unnoticed.

Also, if the yellow circle would be checked before magenta then the algorithm would find the proper circles as well. So it actually can depend on the order in which the circles are searched.

4.2.1 2D Hough transform, 2 circles, 0 background hits

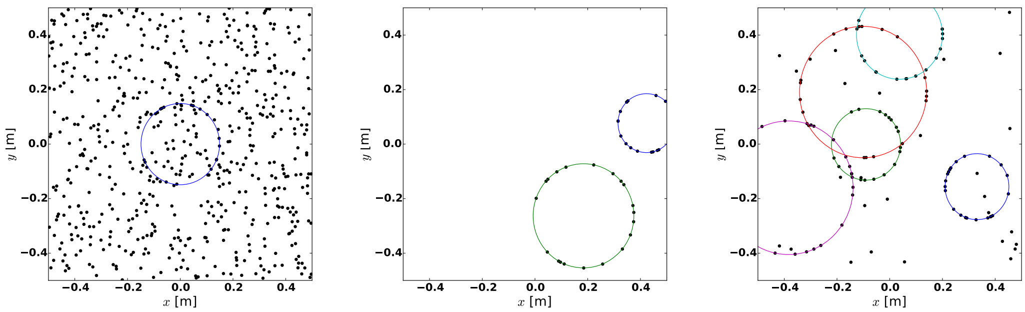
Two circle objects have to be found in this event with zero background. This poses no problem unless the two circles have the same radius then without removing points that have been used for one circle it will always find the circle with the higher score. Introducing a mechanism that removes points that were assigned to a circle solves this problem.

4.2.2 2D Hough transform, 5 circles, 30 background hits

This event added more circles but also some background hits. The algorithm had no problems finding all the circles. In Figure 4.10 the score matrix

Tab. 4.2 – Radius scores for the different events. There is always a big difference between the highest score which determines the circle and the second highest score which is noise. In the example with 1 circle and 600 background the second highest score is relatively high which is due to the many background hits that coincidentally have the same center as the real circle but with a different radius than the real one. For the event with 6 circles and 200 backgrounds sometimes the second highest score is extremely close to the highscore. This happens mainly when two circles have a similar radius and thus the second highest score is not just noise but actually another circle.

Event	Highest score	2nd highest score
1 circle 600 background hits	8720	5778
2 circles 0 background hits	7904	2154
	6373	2275
5 circles 30 background hits	7488	4489
	9095	3717
	6892	2387
	6498	2373
	6721	1953
6 circles 200 background hits	11869	5583
	10466	8687
	8762	5790
	6872	5203
	10155	3595
	4764	3759



(A) 1 circle, 600 background hits. The circle is centered at 0.0/0.0.

(B) 2 circles, 0 background hits

(C) 5 circles, 30 background hits

FIG. 4.6 – The first three events were solved correctly by 2D Hough transform

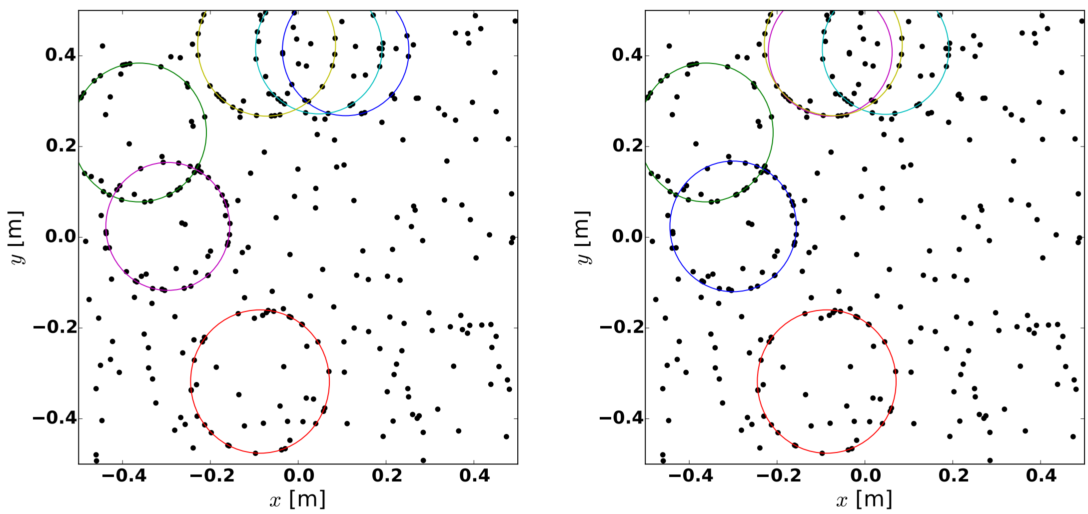


FIG. 4.7 – The rings generated by the simulation on the right and the rings found by the algorithm on the left. The magenta colored circle in the left image is incorrect. It replaced the yellow circle (taken from the simulated data on the left)

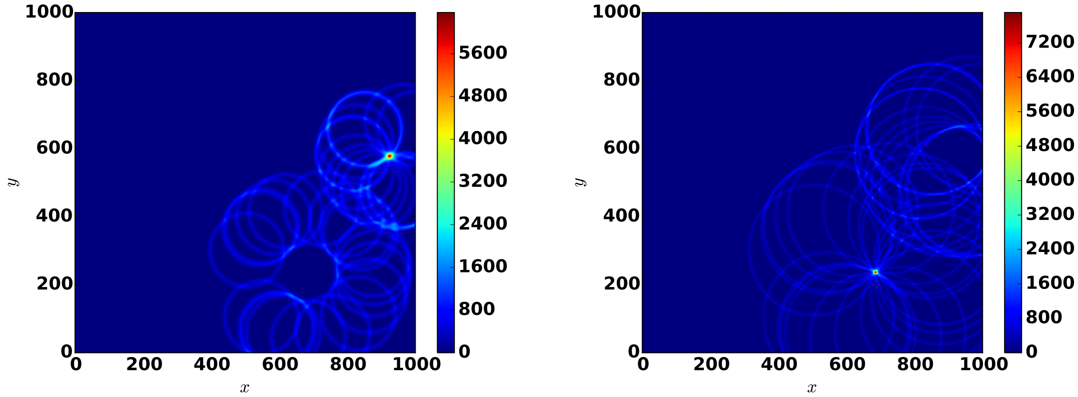


FIG. 4.8 – Center score for the 2D Hough transform for 2 circles with 0 background.

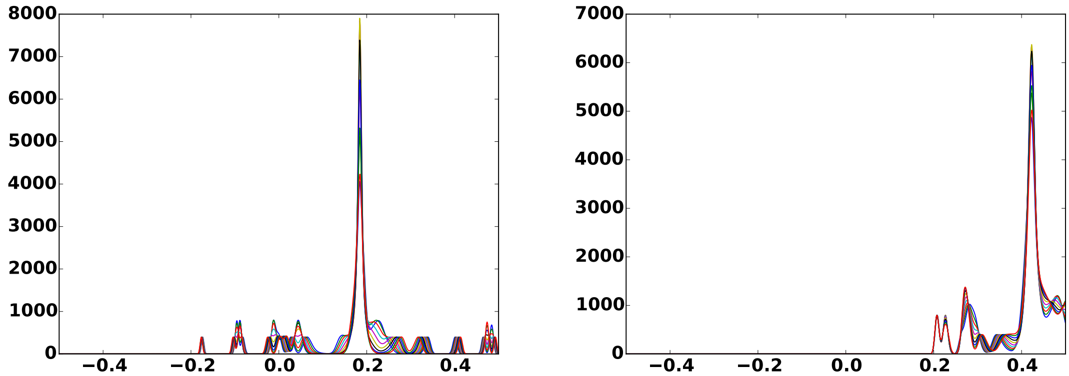


FIG. 4.9 – Each line in the figures is a horizontal slice around the maximum values out of the 2D histogram for the 2 circle event. These 2D histograms are very similar to the radius histogram from the 1D HT but now instead of the weights for the radius which is 1 dimensional the weights for x, y are calculated which are 2 dimensional

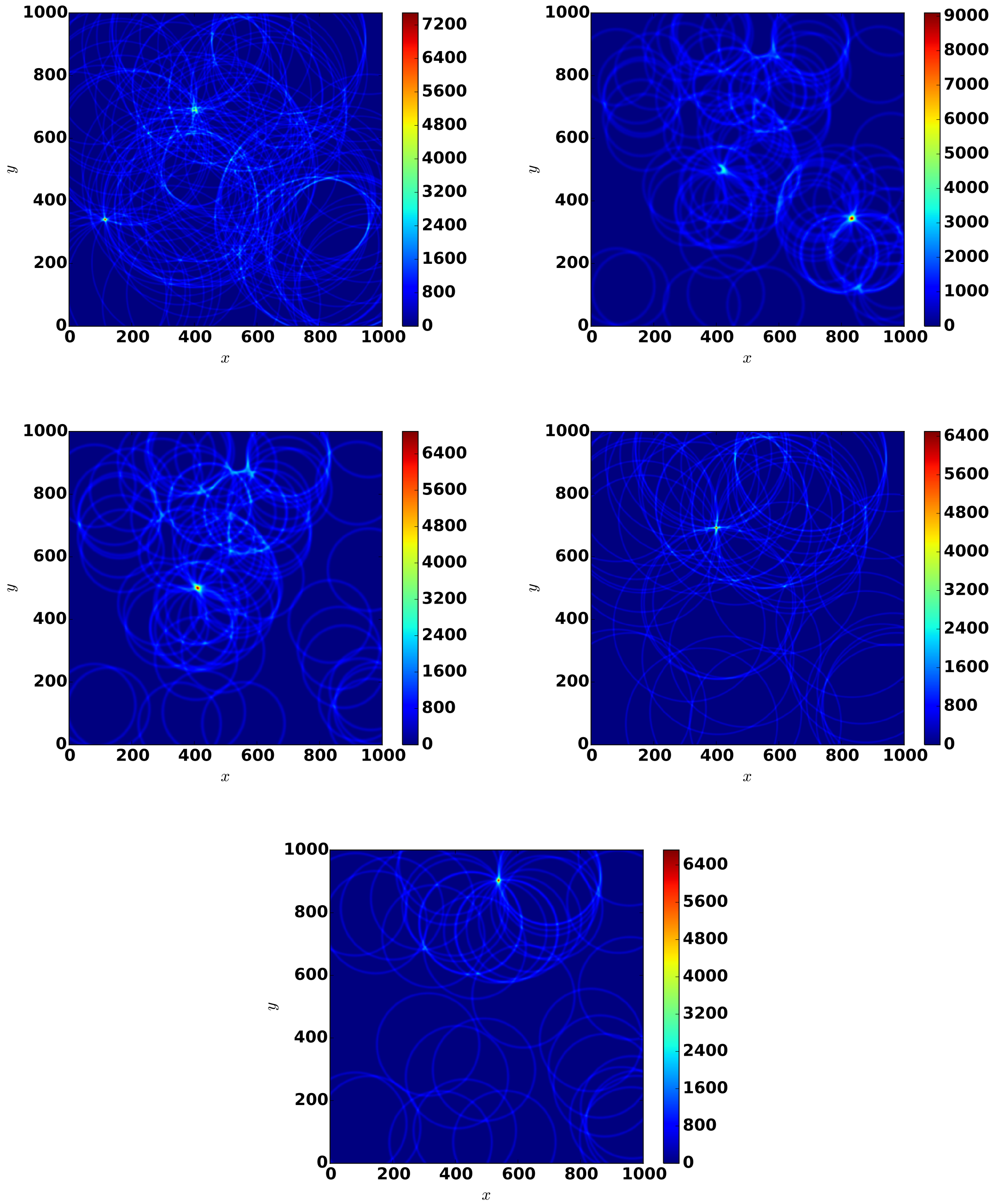


FIG. 4.10 – Center scores for all the centers for an event with 5 circles.

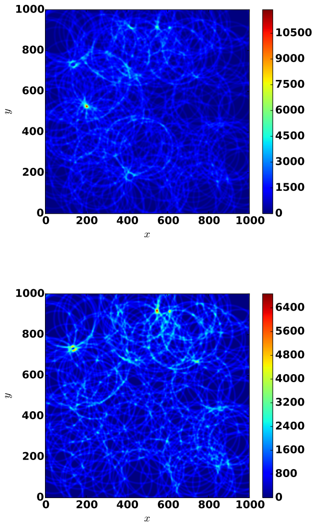


FIG. 4.11 – Center scores for 6 circles with 200 background hits. There is a lot more going on because of all the background hits that by accident contribute to a high score all over the grid.

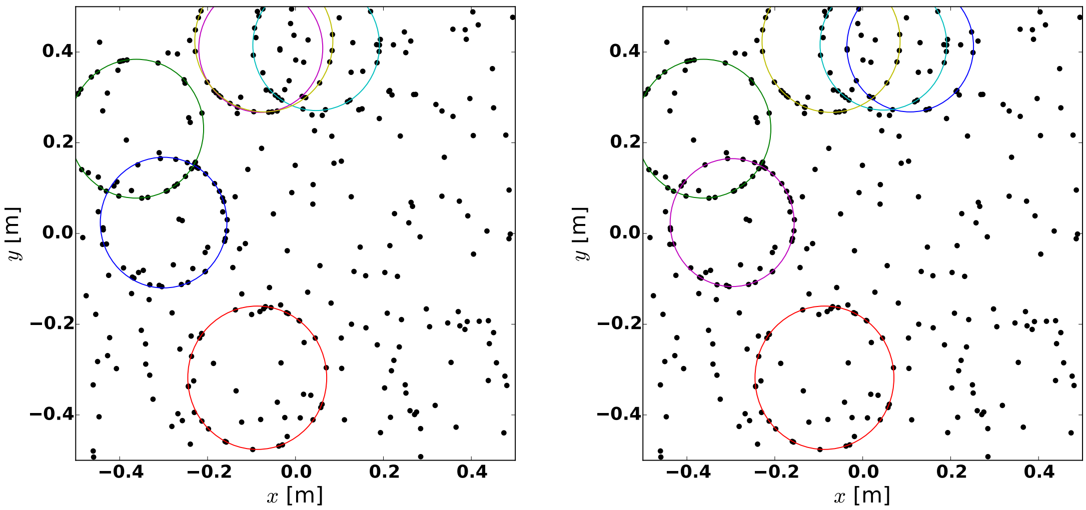


FIG. 4.12 – On the left side the wrong result obtained by the 2D Hough transform and the correct one on the right side

4.2.3 2D Hough transform, 6 circles, 200 background hits

As briefly discussed in the overview the algorithm does make mistakes as this event will show. 6 circles were generated with 200 background hits. The interesting part is not that amount of circles or the amount of background hits are the problem but the properties of the circles (center coordinate and radius). In this example there is a misidentification of a circle with points of another.

A possible way to fix this particular problem is to tune the parameters of the weight function namely reducing the standard deviation.

$$w(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\eta^2}{2\sigma^2}\right)$$

Having a higher σ means that a point that is a bit off of the circle still contributes a considerable value to the total score. The smaller the σ is the sharper the peak. However if the peak is too sharp then the algorithm might discard possible results because they are just a bit off the circle but since the peak is so narrow they don't contribute at all to the total score.

In the example before σ was equal to 0.001 while the space dimension was 1. So if the detector was 1 m per dimension a hit 1 mm off the perfect location contributes still more half of the score off the perfect location. If we set $\sigma = 0.0005$ a point 0.001 away from the perfect location it only contributes about 14% of the maximum score.

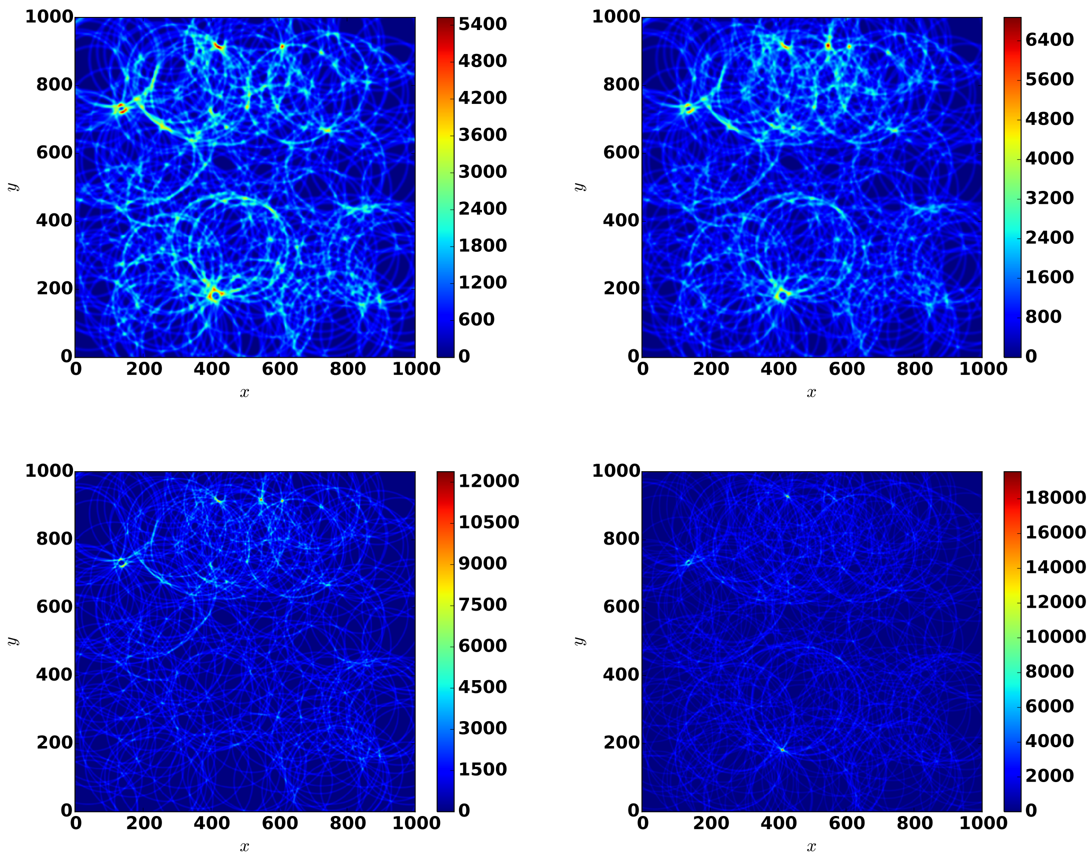


FIG. 4.13 – Old center score in the top row for circle 5 (magenta) and 6 (yellow). And below the same circles this time with the new $\sigma = 0.0005$. It is well visible how big the influence of sigma is for the center score. With a σ of 0.001 just by eye there seem to be many similar maxima whereas with a $\sigma = 0.0005$ the maximas get much more distinct and the maximum score is also higher. Where the high score in the top row is $\approx 5'400$ and $6'400$ in the bottom it is $\approx 12'000$ and $18'000$.

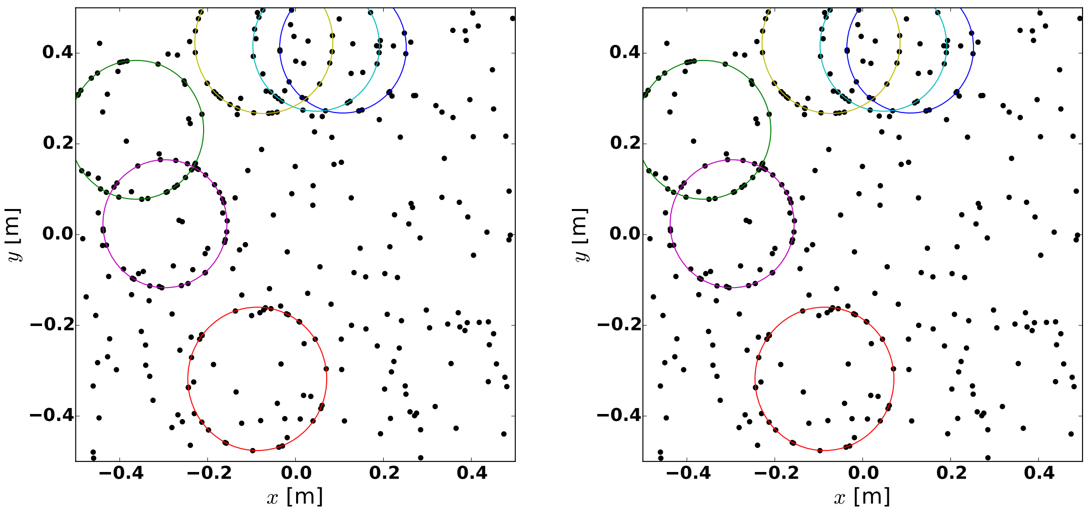


FIG. 4.14 – Again on the left the calculated result and the result taken from data on the right. With the new σ the algorithm is able to calculate all the circles correctly.

4.3 3D Hough transform Results

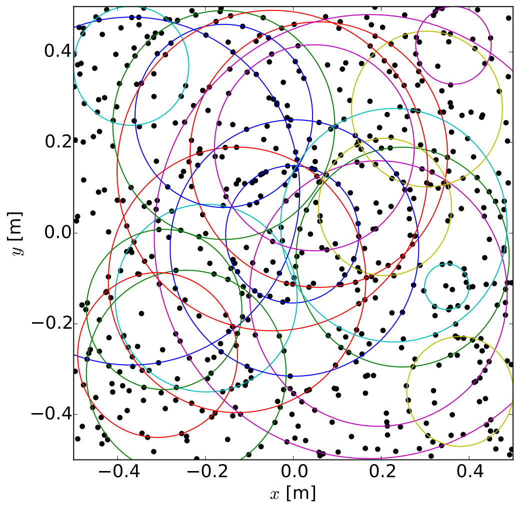
The 3D Hough transform has now to deal with 3 unknown parameters: x, y, r, which also means that the number of total circles is unknown. As explained in the code snippet 3.4 in Section 3.1.3 a threshold is needed to decide when the algorithm should stop finding a circle. This value depends heavily on the σ of the weight function. The σ was set to 0.001 and the threshold was set to 3500. After testing the algorithm for different circles and comparing their scores this threshold found all the circles. For comparison again the same events as in the 1D and 2D Hough transform are studied to compare the reliability of the algorithms and how another unknown dimension adds to the complexity of finding circles. The plots of the found circle for the 4 events are shown in Figure 4.15.

4.3.1 3D: Overview of the results

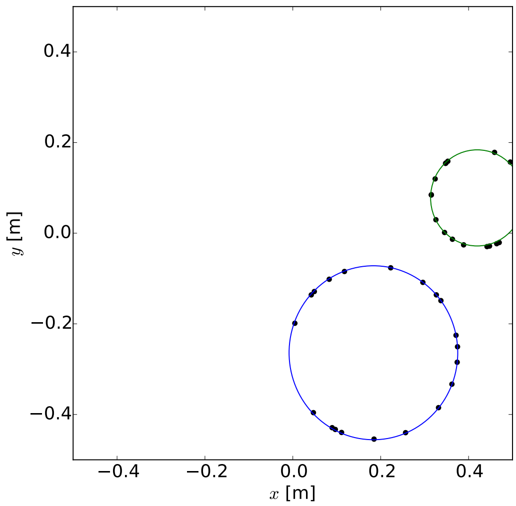
In general the 3D Hough transform works quite well. Even with the added complexity of an extra dimension it is capable to solve three of the four events correctly. One event that failed was the 1 circle and 600 background example where the signal to noise ratio was just too small for the algorithm to isolate the correct solution. The problem was that the threshold was set too low and thus it was unavoidable that the algorithm would fail. In Table 4.3 the scores for the 10 highest scores are shown. As a reminder, the threshold was set to 3500 initially. So the obvious thing seems to set the threshold to 6300 so the algorithm wouldn't pick up any fake rings. However, with a threshold that high legit rings from other events would get lost. For a fixed threshold there will always be a case where rings get lost or fake rings get picked up. On the other hand the 3D Hough transform managed to solve the 6 circles with 200 background hits despite having less information. This is probably due to the fact that the 3D Hough transform finds the circles sorted by score and doesn't depend on the order in which circles are searched (remember, in the 2D case there is the radius given and depending on which radius is given first we can solve it correctly or not).

TAB. 4.3 – The ten highest scores for the case of 1 circle with 600 background hits.

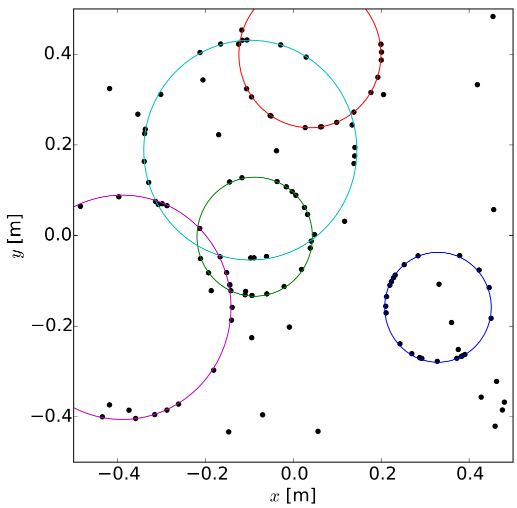
8122	6256	5672	5654
5373	5290	5047	5000
4727	4647		



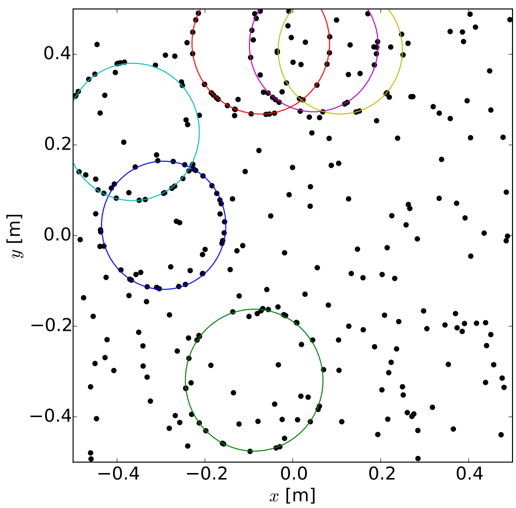
(A) 1 circle, 600 background hits. The circle is centered at $0.0/0.0$.



(B) 2 circles, 0 background hits



(C) 5 circles, 30 background hits



(D) 6 circles, 200 background hits

FIG. 4.15 – Circles found by the 3D Hough transform. Figure 4.15a shows that with a low enough signal/noise ratio even this seemingly simple event fails to be caluclated correctly. Even though the algorithm found the real circle it also found 22 others.

TAB. 4.4 – The 3D Hough transform works differently than the 1D and 2D since it has no information about how many rings in the data set are. If there is no background then there will be no fake ring because after finding all the rings there are no points left to form rings with (points used for a circle are removed from the data set). This is why the event with 2 circles and 0 background has no additional score since there were no points left to calculate a score.

	Highest score
1 circle 600 background hits	
	8122
Fake ring	6265
2 circles 0 background hits	
	7332
	5601
No fakes found	
5 circles 30 background hits	
	8170
	6979
	6477
	4416
	3536
Fake ring	2712
6 circles 200 background hits	
	10269
	9050
	8505
	8258
	5836
	3944
Fake ring	3485

4.4 Combinatorial triplet Hough transform results

4.4.1 Overview of the results

This method was tested against a set of 10'000 events generated in a Monte-Carlo simulation with parameters (number of rings, number of points per ring, number of noise points) tuned to reproduce distributions measured int LHC*b*. The whole set was tested several times, each time with a different threshold for the radius and center histograms. Each of these runs used the speed up method of splitting the data point list into two lists and building triplets from the sublists. The results show that this choice yielded an efficiency above 98%. Thethe performance was evaluated with an analysis script to measure efficiency and ghost rate (along with missed circles and fake circles).

The efficiency ϵ is defined as

$$\epsilon = 1 - \frac{m}{T} \tag{4.1}$$

where m is the number of circles that weren't matched by the algorithm and T the combined number of circles from all events. The ghost rate γ is defined as

$$\gamma = \frac{f}{T} \tag{4.2}$$

where f is the number of fake circles that were found by the algorithm but have no match in the test data and T again the total number of circles.

The the main parameters for this algorithm are

- Histogram thresholds (radius and center histogram)
- Distance between centers of pairs of rings
- Difference between radiuses of pairs of rings

The last two parameters are only parameters for the analysis to decide whether or not the algorithm has found a circle comparing radius and center coordinates with the test data and if a circle is within these bounds the circle will be considered as found. In a real application however, these parameters are useless because a priori there is no knowledge about the true circles.

- Distance between the center of a calculated ring and the corresponding true ring
- Difference between calculated radius and the true radius of a ring

4.4.2 Removing duplicates

Without cleaning up the algorithm finds a lot of duplicates. The reason for such duplicates can be background hits or a hits from another ring close to the actual ring. The triplets including one of these foreign points can shift the center or the radius of the ring they form just a little bit away from the pure ring. As an illustration the 1D radius histogram where two almost adjacent bins have a high score as seen in Figure 4.16. When the algorithm finds a maximum in the histogram it also considers the left and right neighbour. If their sum exceeds the threshold their values are set to 0 and the algorithm looks for more peaks. If the next maximum is just adjacent to one of the previous neighbours it means that there could be another ring with a similar radius than the one just found and it is possible that in fact they belong to the same ring. And even though they could belong to the same ring, they are treated now as independent because they were separated by more than 1 bin. The data extracted from the radius histogram also contains the coordinates of the ring center. Both candidates are handled by the center extraction (as shown below in the code snippet). Both centers have the maximum at the same (x, y) coordinate but the algorithm considers them as independent rings without cleaning. The result can be seen in Figure 4.17.

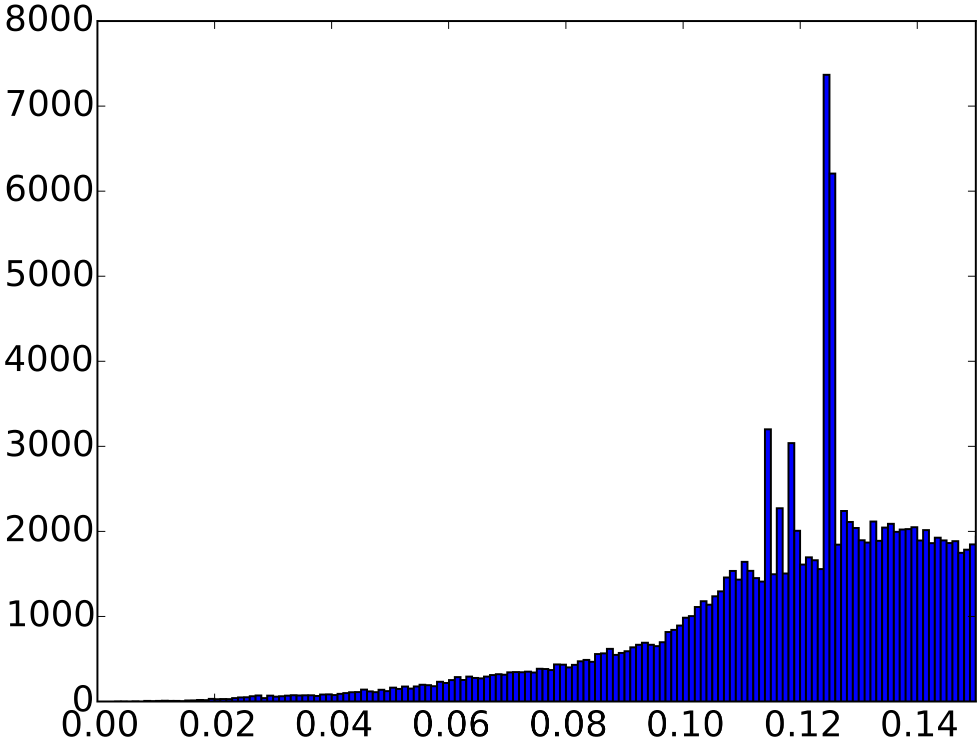


FIG. 4.16 – Two almost adjacent bins with a high score. Apart from the very clear peak of two bins next to each other at $r \approx 0.125$ with a score of 7000 and 6000 respectively, there are 3 smaller peaks separated around $r \approx 0.11 - -0.12$. The highest score is clearly visible. After the algorithm found this maximum and sets the bins to 0 the algorithm picks then the next maximas and its neighbours. But since there are several peaks separated by one bin means they are treated seperately and this introduces a duplicate.

The code for extracting the radiuses with their center data is shown in Code Snippet 4.1. Assuming bin index i contains the highest score of the histogram. The code then adds the score from bin $i - 1$ and $i + 1$ to the maximum score and checks if the sum is bigger than the threshold. If that's the case the next step adds the center information stored in center to a separate list. The function then returns a list of radiuses and for each radius in the list there is a list of center data which is used to extract the center of the ring.

```
edges #x edges of the histogram
center #center data
while True:
  i = max(H)
  n = NUMBER_OF_R_BINS
  n_entries = sum(H[i-1 if i>0 else i:i+2
                   if i<n-1 else i+1])
  if n_entries < RADIUS_THRESHOLD:
    # there are less than THRESHOLD
    # entries in 3 bins
    break
  radiuses.append(edges[i])
  index_list = range(i-1 if i>0 else i,i+2
                    if i<n-1 else i+1)
  for index in index_list:
    if center[index]:
      center_list += center[index]
    H[index] = 0
  center_data.append(center_list)
return radiuses, center_data
```

Code Snippet 4.1 – This code shows how the radius maximas are found and their respective center data is extracted. n_entries is the score of the maximum bin and its left and right neighbour. If the score exceeds the threshold the entries of the bins are set to 0 and the radius value together with the center data are returned.

As before once a maximum has been found and the sum of the maximum plus the adjacent neighbors is bigger than the threshold we have another candidate. The bin and the neighbors are then set to 0. This is repeated until the sum of a maximum plus the adjacent neighbors is smaller than the threshold, where the algorithm stops.

After this step the ring finding algorithm is done. The result is shown in Figure 4.17. A cleaning step is now performed to remove duplicates. The algorithm compares all found circles and if the center coordinates and radiuses of two circles are within a certain range they are considered to be duplicates and the one with a lower score is discarded. The range cuts were a parameter that had to be tuned. If the range was too small then some duplicates weren't detected and remained. If the cut was too large, true rings were removed because they happend to be close to another ring. In Tables 4.6 and 4.7 the effects of different cuts can be seen.

4.4.3 Ring finding threshold

The algorithm applies a very simple threshold to decide whether a candidate is a ring or not. Assume just one ring with 10 points. With these 10 points the algorithm creates

```
H #2d histogram with the center data
xedges
yedges
centers = []
n = NUMBER_OF_S_BINS
while True:
  i,j  = max(H)
  score =   sum(H[i-1 if i>0 else i:i+2
              if i<n else i+1,j]) +
            sum(H[i, j-1 if j>0 else
              j:j+2 if j+2 <= 3
              else j+1]) - H[i,j]
  if score < CENTER_THRESHOLD:
    break
  i_index = range(i-1 if i>0
            else i,i+2 if i<n else i+1)
  j_index = range(j-1 if j>0
            else j,j+2 if j<n else j+1)
  for ii in i_index:
    H[ii][j] = 0
  for jj in j_index:
    H[i][jj] = 0
  centers.append( {'center' : (xedges[i], yedges[j]),
                              'nEntries' : score } )

return centers
```

Code Snippet 4.2 – H is the center histogram. After finding the index for the maximum value the values of the adjacent neighbours are also added to the score. If the score exceeds the center threshold the algorithm stores them as a center for a circle candidate and sets the used bins to 0.

120 triplets $\left(\binom{10}{3}\right)$. For each of these triplets the radius and the ring are calculated. Since they all belong to the same ring the radius histogram has one peak at the ring radius and the center histogram has 1 peak at the ring center. If the threshold is now smaller than 120 then this ring will be found and otherwise ignored. So essentially we can decide how many points per ring are needed in order to be recognized as ring. This puts a requriement on the minimum number of points a ring should have and the threshold is

$$\binom{\text{Number of Points}}{3}$$

In Table 4.5 the different efficiencies for the algorithm for different values of the threshold can be seen. Intuitively one expects to have a larger number fake detections at low threshold since it is more likely for some random hits to pass the threshold. Starting from a certain threshold the efficiency will start to drop when legit rings will fall under the threshold. But analysing the test data set shows that approximately 98% of the rings

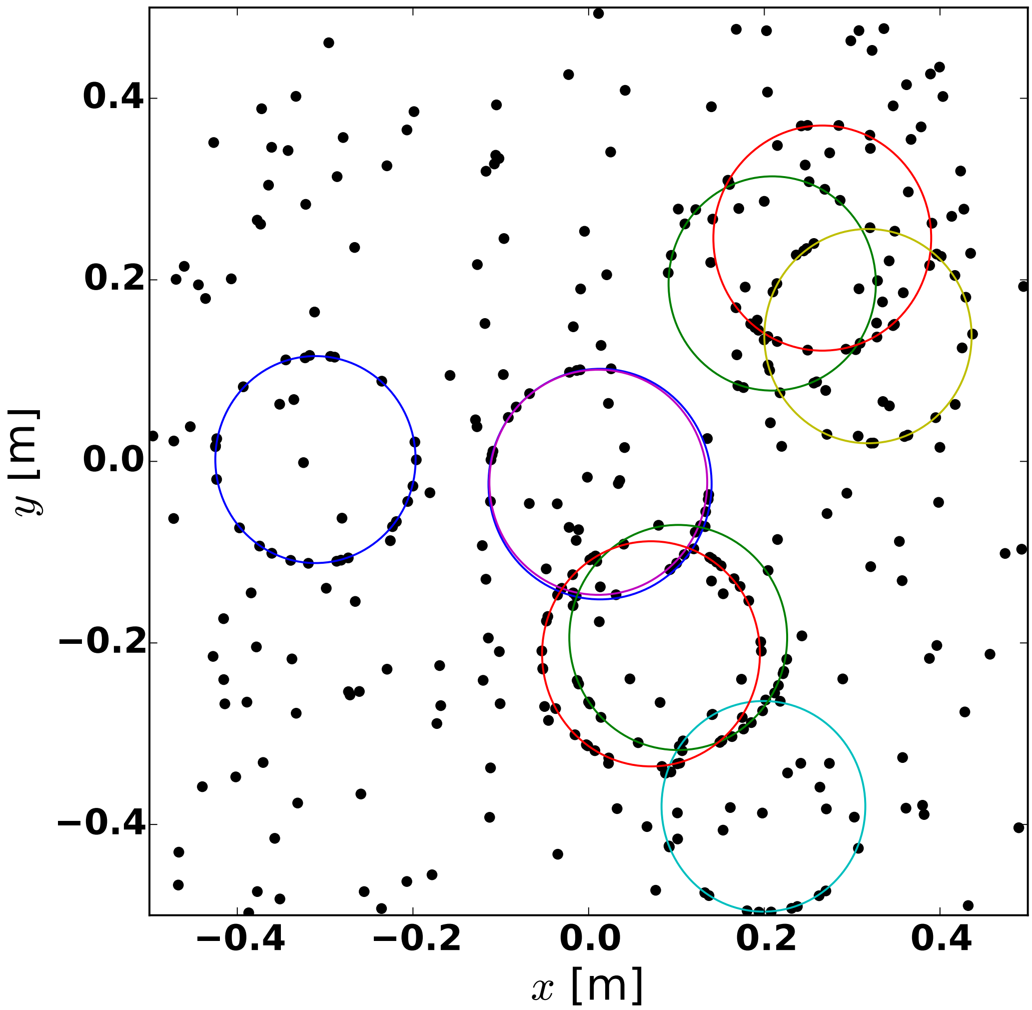


FIG. 4.17 – The result of the combinatorial triplet Hough transform before cleaning up the duplicates. Blue and purple are basically the same circle.

have more than 12 points (see Figure 4.18).

This gives an upper bound on the efficiency of the algorithm. No matter how accurate the algorithm is, if it doesn't consider circles below the threshold it will never find these. Thus the higher the needed number of points per circle is, the lower the efficiency will be.

4.4.4 Cuts for removing duplicates

Additional circles can get lost because they get identified as duplicate when they are in fact just another distinct circle in the vicinity of another circle. Duplicate circles that

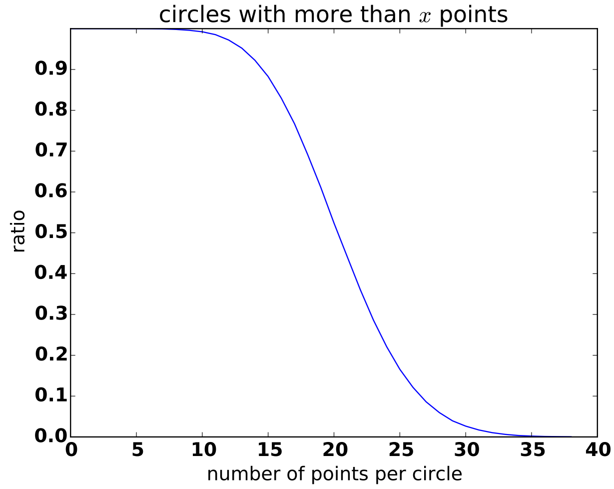


FIG. 4.18 – Reversed cumulative distribution of the points per circle. The plot shows the ratio of circles with more than x points over the total number of circles

TAB. 4.5 – Efficiencies for different thresholds (radius and center cuts are fixed at 0.006). The ghost efficiency explodes if the threshold is too low, that is when combinations from background/background, background/circle or circleA/circleB points classify as a circle

Threshold	ϵ	**Missed**	γ	**Fakes**
20	99.99%	7	183.22%	91573
35	99.98%	9	131.50%	65723
56	99.97%	16	9.07%	4533
84	99.92%	41	2.13%	1067
165	99.57%	214	0.13%	66
220	99.18%	410	0.05%	27
286	98.43%	784	0.01%	4

are typically created with a major fraction of points of a circle and for example a single background hit that is just far enough off that the radius and center histogram show it as a distinct circle. In Code Snippet 4.19 the duplicate removal is shown. First all ring candidates are sorted ascending by number of entries they had in the center histogram. The first circle is then popped from the list and is compared to all the remaining circles in the list. If the circle is within the threshold distance (see Equations in 4.3) the circle is flagged as not unique and the comparison for the rest of the circles stop.

$$\begin{aligned} |\mathbf{c_1} - \mathbf{c_2}| &< \texttt{DUPLICATE\_MAX\_CENTER\_DISTANCE} \\ |r_1 - r_2| &< \texttt{DUPLICATE\_MAX\_RADIUS\_DISTANCE} \end{aligned} \tag{4.3}$$

where c_1 and c_2 are the center coordinates and r_1 and r_2 the radiuses of the circles.

However if there is no other circle within the threshold distances the circle is unique and will be added to a result list which at the end will be returned with all unique circles.

If the cut is too loose, there is a chance that a true ring gets discarded as a duplicate if its center and radius happens to be very close to those of another ring.

```
res = []
sorted_results = sorted( results, key=lambda k:
                  k['nEntries'], reverse=True)
while len(sorted_results):
  circle = sorted_results.pop()
  unique = True
  for dic in sorted_results:
    if (np.linalg.norm(np.array(circle['center']) -
        np.array(dic['center']))) <
                DUPLICATE_MAX_CENTER_DISTANCE and\
       (abs(circle['radius'] - dic['radius']) <
                DUPLICATE_MAX_RADIUS_DISTANCE):
      unique = False
      break
  if unique:
    res.append(circle)
return res
```

FIG. 4.19 – Pseudo code for removing possible duplicates from the circles found by the algorithm. First results are sorted by their number of entries in the histogram so the least relevant comes first. If for a given circle another circle exists with the same center and radius (within the cuts) then the circle is considered a duplicate and will be removed. DUPLICATE_MAX_CENTER_DISTANCE and DUPLICATE_MAX_RADIUS_DISTANCE are parameters that can be tuned

Cut variation for a fixed threshold

In tables 4.6 and 4.7 different cut parameters were tested and compared for the efficiency and ghost rate.

TAB. 4.6 – Efficiency and ghost rate for a fixed radius cut (0.003) with varying center cut. The tighter the center cut is the more duplicate circles remain after

C cut	ϵ	Missed	γ	Fakes
0.003	99.19%	405	0.17%	86
0.006	99.19%	407	0.12%	62
0.009	99.18%	409	0.12%	60
0.012	99.17%	415	0.12%	60

Tables 4.8 show all combinations of cuts for both radius and center ranging from 0.003 to 0.0012 in steps of 0.003 for different ring finding thresholds.

Tab. 4.7 – Reducing ghost rate for a fixed center cut (0.003) with varying center cut. The tighter the center cut is the more duplicate circles fail being detected by the removeDuplicate code shown above

R **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	99.19%	405	0.17%	86
0.006	99.19%	405	0.15%	77
0.009	99.19%	405	0.15%	77
0.012	99.19%	405	0.15%	76

Tab. 4.8 – Results for all the cut combinations for radius and center cuts

Threshold = 35 (7 points for a circle)					
R **cut**	C **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	0.003	99.99%	4	178.96%	89444
0.003	0.006	99.99%	6	151.62%	75777
0.003	0.009	99.98%	8	144.59%	72264
0.003	0.012	99.97%	13	140.88%	70408
0.006	0.003	99.99%	4	174.01%	86968
0.006	0.006	99.98%	9	131.50%	65723
0.006	0.009	99.96%	19	118.91%	59432
0.006	0.012	99.94%	28	113.54%	56745
0.009	0.003	99.99%	4	173.53%	86729
0.009	0.006	99.98%	12	130.54%	65244
0.009	0.009	99.95%	25	112.21%	56079
0.009	0.012	99.93%	36	103.50%	51727
0.012	0.003	99.99%	4	173.30%	86612
0.012	0.006	99.97%	15	130.26%	65103
0.012	0.009	99.94%	32	111.64%	55795
0.012	0.012	99.91%	47	99.85%	49902
Threshold = 56 (8 points for a circle)					
R **cut**	C **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	0.003	99.98%	11	20.54%	10264
0.003	0.006	99.97%	13	14.52%	7255
0.003	0.009	99.97%	15	13.59%	6792
0.003	0.012	99.96%	21	13.24%	6616
0.006	0.003	99.98%	11	19.23%	9611
0.006	0.006	99.97%	16	9.07%	4533
0.006	0.009	99.95%	26	7.29%	3641
0.006	0.012	99.93%	35	6.82%	3409
0.009	0.003	99.98%	11	19.14%	9568
0.009	0.006	99.96%	19	8.91%	4454
0.009	0.009	99.94%	32	5.90%	2950
0.009	0.012	99.91%	43	4.94%	2467

0.012	0.003	99.98%	11	19.10%	9544
0.012	0.006	99.96%	22	8.87%	4432
0.012	0.009	99.92%	39	5.81%	2906
0.012	0.012	99.89%	54	4.29%	2143

Threshold = 84 (9 points for a circle)

R **cut**	C **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	0.003	99.93%	36	5.82%	2911
0.003	0.006	99.92%	38	4.00%	1999
0.003	0.009	99.92%	40	3.80%	1900
0.003	0.012	99.91%	46	3.72%	1861
0.006	0.003	99.93%	36	5.37%	2682
0.006	0.006	99.92%	41	2.13%	1067
0.006	0.009	99.90%	51	1.66%	830
0.006	0.012	99.88%	60	1.57%	787
0.009	0.003	99.93%	36	5.34%	2671
0.009	0.006	99.91%	44	2.09%	1044
0.009	0.009	99.89%	57	1.27%	636
0.009	0.012	99.86%	68	1.07%	534
0.012	0.003	99.93%	36	5.33%	2665
0.012	0.006	99.91%	47	2.08%	1040
0.012	0.009	99.87%	64	1.25%	626
0.012	0.012	99.84%	79	0.88%	439

Threshold = 165 (11 points for a circle)

R **cut**	C **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	0.003	99.58%	209	0.48%	238
0.003	0.006	99.58%	211	0.32%	162
0.003	0.009	99.57%	213	0.31%	156
0.003	0.012	99.56%	219	0.31%	155
0.006	0.003	99.58%	209	0.42%	209
0.006	0.006	99.57%	214	0.13%	66
0.006	0.009	99.55%	224	0.10%	50
0.006	0.012	99.53%	233	0.10%	49
0.009	0.003	99.58%	209	0.42%	209
0.009	0.006	99.57%	217	0.13%	64
0.009	0.009	99.54%	230	0.07%	34
0.009	0.012	99.52%	241	0.05%	26
0.012	0.003	99.58%	209	0.42%	208
0.012	0.006	99.56%	220	0.13%	64
0.012	0.009	99.53%	237	0.07%	34
0.012	0.012	99.50%	252	0.04%	18

Threshold = 220 (12 points for a circle)

R **cut**	C **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	0.003	99.19%	405	0.17%	86

0.003	0.006	99.19%	407	0.12%	62
0.003	0.009	99.18%	409	0.12%	60
0.003	0.012	99.17%	415	0.12%	60
0.006	0.003	99.19%	405	0.15%	77
0.006	0.006	99.18%	410	0.05%	27
0.006	0.009	99.16%	420	0.04%	21
0.006	0.012	99.14%	429	0.04%	21
0.009	0.003	99.19%	405	0.15%	77
0.009	0.006	99.17%	413	0.05%	25
0.009	0.009	99.15%	426	0.02%	11
0.009	0.012	99.13%	437	0.02%	9
0.012	0.003	99.19%	405	0.15%	76
0.012	0.006	99.17%	416	0.05%	25
0.012	0.009	99.13%	433	0.02%	11
0.012	0.012	99.10%	448	0.01%	6

Threshold = 286 (13 points for a circle)					
R **cut**	C **cut**	ϵ	**Missed**	γ	**Fakes**
0.003	0.003	98.44%	779	0.05%	25
0.003	0.006	98.44%	781	0.03%	17
0.003	0.009	98.43%	783	0.03%	16
0.003	0.012	98.42%	789	0.03%	16
0.006	0.003	98.44%	779	0.04%	21
0.006	0.006	98.43%	784	0.01%	4
0.006	0.009	98.41%	794	0.01%	3
0.006	0.012	98.39%	803	0.01%	3
0.009	0.003	98.44%	779	0.04%	21
0.009	0.006	98.43%	787	0.01%	4
0.009	0.009	98.40%	800	0.00%	2
0.009	0.012	98.38%	811	0.00%	2
0.012	0.003	98.44%	779	0.04%	21
0.012	0.006	98.42%	790	0.01%	4
0.012	0.009	98.39%	807	0.00%	2
0.012	0.012	98.36%	822	0.00%	2

These results show that the most important parameter for changing the ghost rate is the ring finding threshold. If this threshold is too low, too many fake rings are found from random point combination that happen to lie close to a circle.

The main reason for missed rings is again the ring finding threshold since it limits the number of rings being found. Since the algorithm uses two split lists, the number of missed circles is even slightly higher due to the fact that sometimes the points of a circle can be split just in a way that the two lists don't contain enough points individually to create an accepted circle (see subsection 4.4.3).

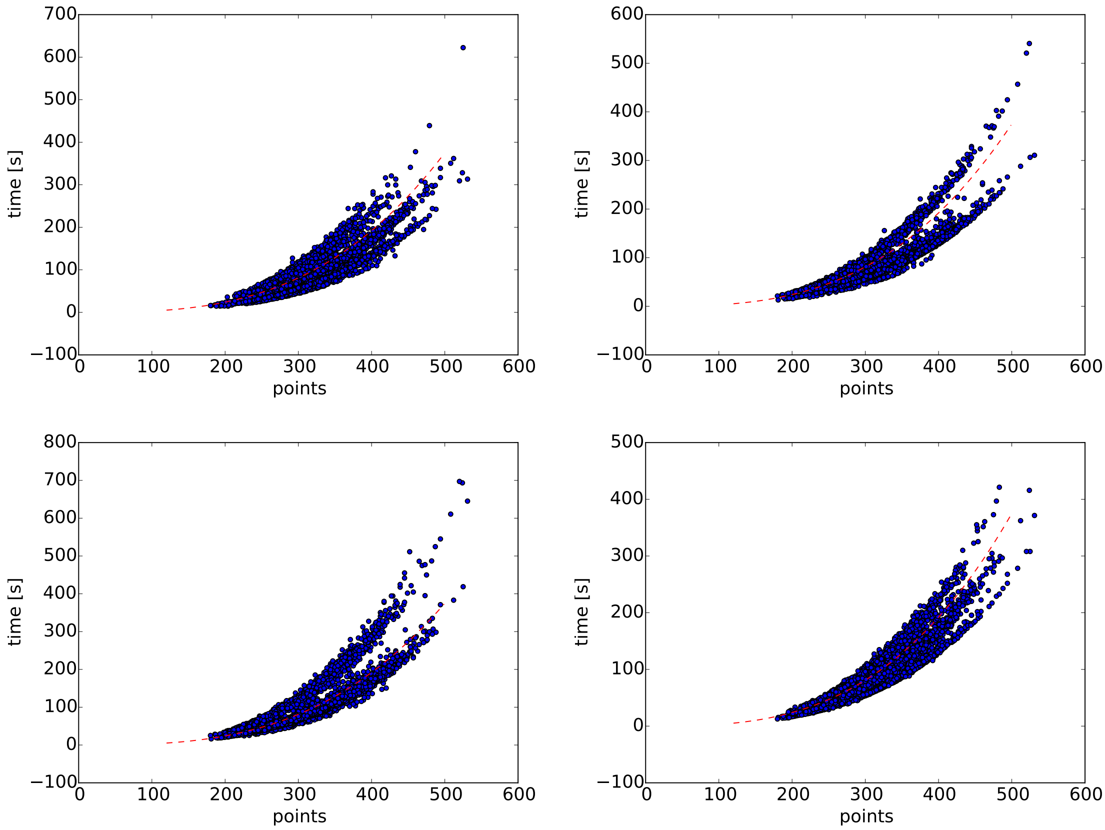


FIG. 4.20 – Runtimes of four different runs with differend ring finding thresholds. The algorithm was run on the farm-ui cluster of the University of Zurich. The worknodes are not identical. This could explain why there is branching in the runtime when some nodes are slightly quicker than the other. The fit in the plot is $1.8e{-}5\frac{N^3}{6}$ where N is the number of data points to show the dependency of the runtime on the points. The constant factor was determined by trial and error but is for all the plots the same.

4.4.5 Runtime

The runtime of the whole algorithm depends heavily on the number of data points. The runtime ranges from several seconds to several minutes as shown in Figure 4.20. The algorithm in its current state has a runtime from several seconds up to several minutes with the longest around 11 minutes. The runtime grows with the number of points

5 Conclusions

This thesis studied different Hough transforms for ring detection in the LHC*b* detector. In a first approach a 1D Hough transform was used and then extended to two and three dimensions. As a fourth approach a new method based on triplets of hits was developped for the ring detection.

First of all it should be noted that especially the one, two and three dimensional Hough transforms were only superficially studied. Their study served more of an introduction into the Hough transform by adding extra dimensions for each case. The statements regarding these methods should always be considered with this in mind.

The 1D Hough transform assumes the number of rings and ring centers are known and searches for maximas in the r distribution for each rings. It is a very stable method having no trouble to find different rings in different kinds of set ups. Even high background doesn't seem to impede the efficiency of the method. It is also a faster than all the other approaches since it only has to search one dimensional histogram.

The 2D Hough transform assumes number of rings and radiuses are known and searches for maxima in the (x, y) distribution for each ring. It shows a good performance as well but it suffers from lower speed due to the increased complexity namely searching 2 dimension for the parameters instead of just one. Another problem is also the misidentification of rings based on the order in which they are searched for and also the sensitivity of the weight function used to weigh the centers for a given data points and radius.

The 3D Hough transform does not make any assumptions on the number of rings or ring parameters. The main weakness of this method is the score threshold which tells the algorithm when to stop looking for a ring (the 1D and 2D transform didn't have this problem since due to the information given it was known, how many rings there were). If the threshold is too low, too many fake rings can be formed by random combinations of (background) hits. - if it's too high a valid ring might get lost. The difficulty in defining a score threshold is that it is based on the scoring function 3.2. Depending on the σ chosen the scoring function can behave quite differently. A problem related to that is that the 3D Hough transform strongly depends on the bin size of the parameter space. The bins shouldn't be too big or the algorithm becomes inaccurate because the scoring function uses inaccurate values. If the bins are too small, the algorithm takes forever due to the $\mathcal{O}(N^3)$ complexity. But the disadvantage can also have advantages because now the algorithm doesn't depend anymore on the order of which it looks for rings - it always picks the strongest candidates first and every subsequent candidate has a lower score until the score is so low that it's unlikely that it is a ring.

Finally the main part of this thesis discusses the combinatorial triplet Hough transform. It is based on the fact that 3 points define a ring. Given 3 points in a 2D space it is possible to calculate the center and radius of the ring on which these 3 points lie. Given a list of points a creating a list of all possible triplets can be used to find possible rings. The algorithm works very well and maintains a high efficiency with all tested parameters but suffers as the 3D Hough transform from a high runtime (up to several minutes). However given the high accuracy of the algorithm in determining the ring parameters makes this approach still very interesting and may offer potential for future investigations. The parameter with the largest impact is the ring finding threshold which defines how many points need to lie on a ring in order to qualify as a circle candidate. Choosing this threshold too low many fake circles appear and the ghost rate skyrockets. Setting this threshold too high means an inherent loss of efficiency from which the algorithm can never recover since circles with a point count lower than the threshold will never be considered as a candidate.

5.1 Outlook

As a next step the possibility of parallelising the algorithm should be considered. Once the two triplet lists are generated the radius and centers for these triplets can be calculated independently. Other options for optimisation would be improving the code. Some steps have already been done. For example instead of using the `np.linalg.norm` function of the numpy module (which has a high overhead see [15]) a simple norm function has been implemented. There was not enough time to test the speed improvement systematically but a profiling of one event showed an improvement of the speed from 188 s to 95 s. In general, the method where radius and center of triplets get calculated would be a good start to optimise the code since that's the place where the algorithm most its time (shown by profiling the python script but not shown in this thesis).

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