

# Math 131BH - Notes

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**Definition:** A topological space  $(X, \tau)$  is a set  $X$  with a collection  $\tau \subseteq \mathcal{P}(X)$ , called a topology on  $X$ , such that:

- i)  $\emptyset, X \in \tau$
- ii)  $\tau$  is closed under finite intersections and arbitrary unions.

The elements in  $\tau$  are open sets, their complements are closed.

For example, for some set  $X$ ,  $\tau = \{\emptyset, X\}$  gives a trivial topology, and  $\tau = \mathcal{P}(X)$  gives the discrete topology.

**Definition:** A neighborhood  $V \subseteq X$  of  $x \in X$  is a set such that  $\exists U \in \tau$  such that  $x \in U \subseteq V$ .

**Definition:** We say that  $S \subseteq \mathcal{P}(X)$  generates a topology  $\tau$  on  $X$  if  $\tau$  is the smallest topology containing  $S$ .

**Definition:** Given  $Y \subseteq X$ , the subspace topology on  $Y$  has open sets of the form  $U \cap Y$ , with  $U$  being an open set in  $X$ .

Any metric on  $X$  generates a topology. The generating set is  $\{B_r(x) : x \in X, r > 0\}$ .

If given a norm  $\|\cdot\|$  in a vector space  $V$ , we can induce a metric (and thus also a topology) on  $V$  by  $d(v, w) = \|v - w\|$ . In particular,  $\mathbb{R}^n$  is a topological space with the Euclidean metric.

**Definition:** A subset  $S \subseteq X$  of a topological space is dense in  $X$  if every open set in  $X$  intersects  $S$  nontrivially.

**Remark:** If  $X$  is a metric space, this is equivalent to saying that every open ball intersects  $S$ .

**Definition:** A topological space is called separable if it contains a countable dense subset.

**Heuristic:** When trying to show a metric space is not separable, it may be useful to construct a collection of uncountable points s.t. every two points are a fixed distance apart. Then we can argue that no countable subset may intersect an uncountable number of mutually exclusive balls.

We can show that  $\mathbb{R}$  is separable, since  $\mathbb{Q}$  is a countable dense subset.

The same goes for  $\mathbb{R}^n$ , with  $\mathbb{Q}^n$  being a countable dense subset.

A discrete metric space is separable if and only if it is at most countably infinite, since the fact that every  $\{x\}$  is open shows that the only dense subset of  $X$  is itself.

Exercise: Let  $l^p(\mathbb{N})$  be a vector space with norm  $\|(a_n)_{n \geq 1}\|_{l^p} = (\sum_{n=1}^{\infty} |a_n|^p)^{\frac{1}{p}}$ .

Show that  $l^p(\mathbb{N})$  is separable for  $1 \leq p < \infty$ , but not  $p = \infty$ .

**Definition:** Suppose we have  $f : (X, \tau) \rightarrow (Y, \kappa)$ .

We say that  $f$  is continuous if the preimage of every open set is open. That is,  $f^{-1}(V) \in \tau$  for every  $V \in \kappa$ .

We say that  $f$  is continuous at  $x \in X$  if the preimage of any neighborhood of  $f(x)$  is a neighborhood of  $x$ .

**Definition:** A subset  $A \subseteq X$  of a topological space  $X$  is called connected if it is not disconnected. That is, there does not exist open, nonempty subsets  $U, V \subseteq X$  such that  $A \subseteq U \cup V$ ,  $A \cap U \neq \emptyset$ ,  $A \cap V \neq \emptyset$ ,  $U \cap V = \emptyset$ .

**Definition:** We say that  $A \subseteq X$  is path-connected if any two points  $x, y \in A$  are connected by a path in  $A$ . That is, a continuous function  $f : [0, 1] \rightarrow A$  such that  $f(0) = x$ ,  $f(1) = y$ .

**Proposition:** Let  $f : X \rightarrow Y$  be a continuous map of topological spaces. If  $X$  is connected, then  $f(X)$  is connected.

**Proof:** We proceed by contrapositive.

Suppose that  $f(X)$  is disconnected. Then we have open nonempty subsets  $U, V \subseteq Y$  such that  $f(X) \subseteq U \cup V$ ,  $f(X) \cap U \neq \emptyset$ ,  $f(X) \cap V \neq \emptyset$ ,  $U \cap V = \emptyset$ .

By the continuous nature of  $f$ , we have that  $f^{-1}(U), f^{-1}(V)$  are open.

We also have that  $f^{-1}(U) \cap f^{-1}(V) = \emptyset$  and  $f^{-1}(U) \cup f^{-1}(V) \subseteq f^{-1}(U \cup V) = f^{-1}(f(X)) = X$ .

Thus,  $X$  is disconnected.

**Proposition:** If  $f : X \rightarrow Y$  is continuous and  $X$  is path-connected, then  $f(X)$  is path-connected.

**Proof:** Exercise.

**Definition:** Let  $X$  be a topological space. An open cover of a set  $K \subseteq X$  is a collection  $\mathcal{F} = \{U_\alpha\}_{\alpha \in I}$  of open sets such that  $K \subseteq \bigcup_{\alpha \in I} U_\alpha$ .

**Definition:** The set  $K \subseteq X$  is called compact if every open cover of  $K$  has a finite subcover. That is, if  $K \subseteq \bigcup_{\alpha \in I} U_\alpha$ , there exists  $n_1, \dots, n_k$  such that  $K \subseteq \bigcup_{j=1}^k U_{n_j}$ .

We say that  $K$  is relatively compact or precompact if  $\bar{K}$  is compact.

**Definition:** Let  $X$  be a metric space. We say  $K \subseteq X$  is totally bounded if for all  $\epsilon > 0$ , there exists finitely many open balls  $B_1, \dots, B_n$  of radius  $\epsilon$  such that  $K \subseteq \bigcup_{j=1}^n B_j$ .

**Definition:** Let  $X$  be a metric space.  $K \subseteq X$  is sequentially compact if every sequence  $\{x_n\} \subseteq K$  has a convergent subsequence  $\{x_{n_k}\}$  with  $x_{n_k} \rightarrow x$  for some  $x \in K$ .

**Theorem:** The following are equivalent for a subset  $K \subseteq X$  of metric space  $X$ :

- (a)  $K$  is compact
- (b)  $K$  is sequentially compact
- (c)  $K$  is complete and totally bounded

We already know that in  $\mathbb{R}$ , every bounded sequence has a convergent subsequence.

**Theorem:** In  $\mathbb{R}^n$ ,  $A$  is sequentially compact  $\Leftrightarrow A$  is closed and bounded.

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**Proposition:** The following are equivalent for a subset  $K \subseteq X$  of a metric space  $X$ :

- (a)  $K$  is compact
- (b)  $K$  is sequentially compact
- (c)  $K$  is complete and totally bounded

**Proof:** We WTS (a) $\Rightarrow$ (c) $\Leftrightarrow$ (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c): If  $K$  is compact, then it is clearly totally bounded, since for any  $\epsilon > 0$  there exists a finite subcover for the open cover  $\{B(x, \epsilon) : x \in K\}$ .

Now we WTS completeness. Assume otherwise.

Let  $(x_n)_{n \in \mathbb{N}} \subseteq K$  be a Cauchy sequence that is not convergent. Note that this implies that there is no convergent subsequence (otherwise the Cauchy sequence would converge).

Define  $r_n = \inf_{m \neq n} d(x_n, x_m)$ ,  $B_n = B(x_n, r_n)$ . Notice that  $r_n > 0$  for all  $n$  (otherwise we can construct a subsequence converging to  $x_n$ ).

Note that by construction,  $B_n \cap (x_k)_{k \in \mathbb{N}} = \{x_n\}$ .

Now define  $V = X \setminus \overline{\bigcup_{n \geq 1} B(x_n, \frac{r_n}{2})}$ .

Claim:  $\{V\} \cup \{B_n\}_{n \geq 1}$  is an open cover of  $K$ . Assume otherwise.

Then  $\exists x \in X$  s.t.  $d(x, x_n) \geq r_n \forall n$  and  $\exists (y_{n_k})_{k \in \mathbb{N}}$  s.t.  $y_{n_k} \in B_{n_k}$  and  $y_{n_k} \rightarrow x$ .

Then  $d(x, x_{n_k}) = d(x, y_{n_k}) + d(y_{n_k}, x_{n_k}) < d(x, y_{n_k}) + r_{n_k} \rightarrow 0$ , we have a convergent subsequence, leading to a contradiction.

Therefore,  $\{V\} \cup \{B_n\}_{n \in \mathbb{N}}$  is an open cover of  $K$ . However, since each  $x_n$  is contained exactly in one  $B_n$ , and  $V$  contains none of the  $x_n$ , any finite subset of this open cover would fail to contain the entirety of  $\{x_n\}_{n \in \mathbb{N}} \subseteq K$ . This contradicts the compactness of  $K$ .

Therefore,  $K$  is complete.

(b) $\Rightarrow$ (c): Completeness is clear since any Cauchy sequence would have a convergent subsequence by sequential compactness.

Now we WTS totally boundedness. Assume otherwise.

Then for some  $\epsilon > 0$ , there is no finite cover by  $\epsilon$ -balls of  $K$ .

We can inductively select  $x_1, x_2, \dots$ :  $x_1 \in K$  is arbitrary.  $x_2 \in K \setminus B(x_1, \epsilon)$ ,  $x_3 \in K \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon))$ ,  $\dots$ .

But then we have that  $d(x_n, x_m) \geq \epsilon \forall n \neq m$ , it is impossible for any subsequence to be Cauchy. Therefore,  $(x_n)_{n \in \mathbb{N}} \subseteq K$  has no convergent subsequence, contradicting the sequential compactness of  $K$ .

(c) $\Rightarrow$ (b): We have some arbitrary sequence  $(x_n)_{n \in \mathbb{N}}$ . Let us inductively select a sequence of balls:  $(B_n)_{n \in \mathbb{N}}$ .

Define  $\epsilon_n = \frac{1}{2^n}$ . Since  $K$  is totally bounded, there exists a finite cover of  $K$  that consists of  $\epsilon_1$ -balls. Since the sequence has infinite elements, there exists  $B_1 = B(y_1, \epsilon_1)$  such that  $B_1 \cap \{x_n\}_{n \in \mathbb{N}}$  has infinite elements.

Now consider the finite cover that consists of  $\epsilon_2$ -balls. By the same logic, there exists  $B_2 = B(y_2, \epsilon_2)$  such that  $B_2 \cap B_1 \cap \{x_n\}_{n \in \mathbb{N}}$  has infinite elements.

Proceed by induction, we have a sequence of balls  $(B_n)_{n \in \mathbb{N}}$ .

Now we can select a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  where  $x_{n_k} \in B_k \forall k \in \mathbb{N}$ .

We have that  $\forall i, j \in \mathbb{N}$ ,  $d(x_{n_i}, x_{n_j}) < 2\epsilon_{\min(n_i, n_j)} \rightarrow 0$ , therefore  $(x_{n_k})_{k \in \mathbb{N}}$  is Cauchy.

Due to the completeness of  $K$ , we have that  $(x_n)_{n \in \mathbb{N}}$  has a convergent subsequence, therefore  $K$  is sequentially compact.

(b) $\Rightarrow$ (a): **Lemma:** If  $K$  is sequentially compact and  $\mathcal{F}$  is any open cover of  $K$ , then  $\exists \epsilon > 0$  s.t.  $\forall x \in K$ ,  $\exists U \in \mathcal{F}$  s.t.  $B(x, \epsilon) \subseteq U$ . The largest of such  $\epsilon$  is called the Lebesgue radius of  $\mathcal{F}$ . To prove this, assume otherwise. Then there exists a sequence of balls  $(B(x_n, r_n))_{n \in \mathbb{N}}$  with  $r_n \rightarrow 0$  that are not contained fully in any  $U \in \mathcal{F}$ .

By sequential compactness we have that  $x_n \rightarrow x \in K$ . Take  $U \in \mathcal{F}$  s.t.  $x \in U$ . Then  $\exists r > 0$  s.t.  $B(x, r) \subseteq U$ . But since  $x_{n_k} \rightarrow x$  and  $r_{n_k} \rightarrow 0$ , for some large enough  $k$  this contradicts  $B_{n_k}$ .

being not contained in  $U$ , thus the Lebesgue radius exists.

Now, let  $\mathcal{F}$  be an open cover of  $K$ , and  $\epsilon > 0$  be its Lebesgue radius. Since sequentially compact  $\Rightarrow$  totally bounded, let  $B_1, \dots, B_n$  be an  $\epsilon$ -cover of  $K$  and pick  $U_1, \dots, U_n \in \mathcal{F}$  s.t.  $B_k \subseteq U_k \forall 1 \leq k \leq n$ .

Then  $U_1, \dots, U_n$  is a finite subcover of  $K$  from  $\mathcal{F}$ .

**Proposition:** In a metric space, a closed subset of a compact set is compact.

**Proof:** We have compact  $\rightarrow$  sequentially bounded. Let  $K$  be compact,  $K' \subseteq K$  be closed.

Let  $(x_n)_{n \in \mathbb{N}} \subseteq K' \subseteq K$ , then  $\exists x \in K$  s.t.  $x_{n_k} \rightarrow x$ .

Since  $K'$  closed,  $x \in \bar{K}' = K'$ , therefore  $K$  is sequentially compact  $\Rightarrow$  compact.

Note that this is generally not true in arbitrary topological spaces.

**Definition:** A family of sets  $\mathcal{F} \subseteq \mathcal{P}(X)$  has the finite intersection property (FIP) if any finite intersection  $\bigcap_{k=1}^n A_k \neq \emptyset$  for any  $A_1, \dots, A_n \in \mathcal{F}$ .

**Proposition:** A subspace  $K \subseteq X$  of a topological space is compact  $\Leftrightarrow$  every family  $\mathcal{F}$  of closed subsets of  $K$  with FIP has nontrivial intersection.

**Proof:** If  $K$  is compact, and the intersection of the closed sets in  $\mathcal{F}$  is empty, then the union of their complements would contain the entire  $K$  and therefore form an open cover. However, since  $K$  is compact, that open cover would have a finite subcover, which, if we take another complement, would become a finite subset of  $\mathcal{F}$  with an empty intersection, contradicting FIP. If every family of closed subsets that has FIP has nontrivial intersection, and  $K$  is not compact, let  $\{U_i\}_{i \in I}$  be an open cover of  $K$  with no finite subcover, and let  $\mathcal{F} = \{U_i^c\}_{i \in I}$ . We have that  $\mathcal{F}$  has FIP, therefore the intersection is non-empty. However, if we take another complement, we would have that the union of  $\{U_i\}_{i \in I}$  is not the entire  $K$ , contradicting that this is an open cover.

**Proposition:** (Heine-Borel Theorem)  $K \subseteq \mathbb{R}^n$  is compact  $\Leftrightarrow$   $K$  is closed and bounded

**Proof:** By Bolzano-Weierstrass, for any sequence  $(x_k)_{k \in \mathbb{N}} \subseteq K$  where  $x_k = (x_k^{(1)}, \dots, x_k^{(n)})$ ,  $\exists x \in \mathbb{R}^n$  s.t.  $x_{k_j} \rightarrow x$ .

Since  $K$  is closed,  $x \in \bar{K} = K$ , so  $K$  is sequentially compact  $\Rightarrow$  compact.

If  $K$  is compact, then  $K$  is contained in a finite union of bounded balls (**this needs to be checked!!!**).

For example, the sphere in  $\mathbb{R}^n$ :  $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1\} \subseteq \mathbb{R}^n$  is compact.

**Remark:** In general, compact  $\Rightarrow$  closed and bounded, but the converse almost always fails.

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**Proposition:** Let  $f : X \rightarrow Y$  be a continuous function of topological spaces. If  $X$  is compact, then  $f(X)$  is compact.

**Proof:** Take any open cover  $\{U_i\}_{i \in I}$  of  $f(X)$  in  $Y$ . Then,  $\{f^{-1}(U_i)\}_{i \in I}$  is an open cover of  $X$ . This has a finite subcover  $\{f^{-1}(U_n)\}_{n \in \mathbb{N}}$  (Since  $X$  is compact).

Since  $X = \bigcup_{n \in \mathbb{N}} f^{-1}(U_n)$ , we have  $f(X) = \bigcup_{n \in \mathbb{N}} U_n$ .

Therefore  $f(X)$  is compact.

**Proposition:** (Extreme Value Theorem) For  $X$  compact topological space, any continuous function  $f : X \rightarrow \mathbb{R}$  attains a global maximum/minimum on  $X$ .

**Proof:** We have  $f(X)$  compact. By Heines-Borel and that  $f(X) \subseteq \mathbb{R}$ , we have  $f(X)$  is closed and bounded. So  $\sup f(X) < \infty$  and  $\inf f(X) > -\infty$ . And since  $f(X)$  is closed,  $\sup f(X)$  and  $\inf f(X)$  must be in  $f(X)$  (and therefore attained).

Since the  $n$ -sphere  $S^n$  is compact, any continuous  $f : S^n \rightarrow \mathbb{R}$  attains a global maximum/minimum.

**Proposition:** The following are equivalent for metric spaces  $X, Y$ , some fixed  $x \in X$ , and  $f : X \rightarrow Y$ :

- (a) If  $x_n \rightarrow x$  for some  $(x_n)_{n \in \mathbb{N}} \subseteq X$ , then  $(f(x_n))_{n \in \mathbb{N}} \subseteq Y$  is such that  $f(x_n) \rightarrow f(x)$ .
- (b)  $f$  is topologically continuous at  $x$ .
- (c) For all  $\epsilon > 0$ ,  $\exists \delta > 0$  depend on  $x$  s.t. for some  $y \in X$ , if  $d(x, y) < \delta$ , then  $d(f(x), f(y)) < \epsilon$ .

**Proof:** (a) $\Rightarrow$ (b): Suppose  $f$  is not topologically continuous at  $x_0$ . Then, there exists a neighborhood  $V$  of  $f(x_0)$  s.t.  $f^{-1}(V)$  not a neighborhood of  $x_0$ .

We know that  $B(f(x_0), \epsilon) \subseteq V$  for some  $\epsilon > 0$ . This means we can select a sequence  $x_n \in B(x_0, \frac{1}{n}) \setminus f^{-1}(V)$  (note that we can always select an element, otherwise  $V$  becomes a neighborhood of  $x_0$ ) so that  $x_n \rightarrow x_0$  but  $d(f(x_n), f(x_0)) \geq \epsilon \forall n \in \mathbb{N}$ . This contradicts (a).

(b) $\Rightarrow$ (c): Let  $V = B(f(x_0), \epsilon)$ . So  $\exists \delta > 0$  s.t.  $B(x_0, \delta) \subseteq f^{-1}(V)$  (since by (b)  $f^{-1}(V)$  is a neighborhood of  $x_0$ ). Thus,  $d(x, x_0) < \delta \Rightarrow x \in f^{-1}(V) \Rightarrow f(x) \in V \Rightarrow d(f(x_0), f(x)) < \epsilon$ .

(c) $\Rightarrow$ (a): Let  $\epsilon > 0$ . By (c) we have that  $\exists \delta > 0$  dependent on  $x$  s.t. if  $d(x_n, x) < \delta$  then  $d(f(x_n), f(x)) < \epsilon$ . Since  $x_n \rightarrow x$ ,  $\exists N \in \mathbb{N}$  s.t.  $d(x_n, x) < \delta \forall n \geq N$ . Thus  $d(f(x_n), f(x)) < \epsilon \forall n \geq N$ . Therefore,  $f(x_n) \rightarrow f(x)$ .

**Definition:** A function of metric spaces  $f : X \rightarrow Y$  is called uniformly continuous if for all  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t. for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

**Proposition:** The following are equivalent for  $X$  totally bounded and a map  $f : X \rightarrow Y$  between metric spaces:

- (a)  $f$  is uniformly continuous.
- (b) If  $(x_n)_{n \in \mathbb{N}}$  is Cauchy, then  $(f(x_n))_{n \in \mathbb{N}}$  is Cauchy.

**Proof:** (a) $\Rightarrow$ (b): Since  $f$  uniformly continuous, we have that  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon \forall x, y \in X$ .

Since  $(x_n)_{n \in \mathbb{N}}$  Cauchy, we have that  $d(x_n, x_m) < \delta$  for large enough  $n, m$ . Therefore,  $d(f(x_n), f(x_m)) < \epsilon$  for large enough  $n, m$ . Thus,  $(f(x_n))_{n \in \mathbb{N}}$  is Cauchy.

(b) $\Rightarrow$ (a): Suppose that  $f$  is not uniformly continuous, therefore  $\exists \epsilon > 0$  and  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  s.t.  $d(x_n, y_n) \rightarrow 0$  yet  $d(f(x_n), f(y_n)) \geq \epsilon \forall n \in \mathbb{N}$ .

Since  $X$  is totally bounded, we can find a Cauchy subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  and since  $d(x_{n_k}, y_{n_k}) \rightarrow 0$ ,  $(y_{n_k})_{k \in \mathbb{N}}$  is also Cauchy.

Therefore we construct a new sequence  $(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, \dots)$ . we note that this sequence is Cauchy since  $(x_{n_k})$  and  $(y_{n_k})$  both Cauchy and  $d(x_{n_k}, y_{n_k}) \rightarrow 0$ , but  $d(f(x_{n_k}), f(y_{n_k})) \geq \epsilon \forall k \in \mathbb{N}$ , which is a contradiction to (b).

**Definition:** Given a metric space  $X$ , its completion  $\bar{X}$  is the set of Cauchy sequences  $(x_n)_{n \in \mathbb{N}}$  in  $X$ , modulo the equivalence relation:  $(x_n)_n \sim (y_n)_n$  if and only if  $\forall \epsilon > 0, d(x_n, y_m) < \epsilon$  for large enough  $n, m$ . Then, the metric on  $\bar{X}$  can be given by  $d((x_n)_n, (y_n)_n) = \lim_{n \rightarrow \infty} d(x_n, y_n)$ . This limit exists because  $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$  for large enough  $n, m$  and that  $\mathbb{R}$  is complete.

Then  $X$  is a subspace of  $\bar{X}$ , with  $x \in X \Rightarrow \bar{x} \in \bar{X}$ , where  $\bar{x}$  is the equivalence class of sequences converging to  $x \in X$ .

**Proposition:** The completion  $\bar{X}$  of any metric space  $X$  is complete.

**Proof:** Take sequences  $X_1 = (x_n^{(1)})_{n \in \mathbb{N}}, X_2 = (x_n^{(2)})_{n \in \mathbb{N}}, \dots$  in  $X$ , and suppose they are Cauchy in  $\bar{X}$ . That is,  $\lim_{n \rightarrow \infty} d(x_n^{(m)}, x_n^{(m')}) < \epsilon$  for  $m, m'$  large enough. For each  $k \in \mathbb{N}$ , select  $N_k$  s.t.  $d(x_j^{(k)}, x_l^{(k)}) < \frac{1}{k}$  for  $j, l \geq N_k$ , select  $y_k = x_p^{(k)}$  for  $p \geq N_k$ .

Let  $Y = (y_k)_{k \in \mathbb{N}}$ . We will define  $Y_k = (y_k, y_k, \dots)$ . First, we note that  $d(X_k, Y_k) \leq \frac{1}{k} \forall k \in \mathbb{N}$  since  $d(X_k, Y_k) = \lim_{n \rightarrow \infty} d(x_n^{(k)}, y_n^{(k)})$ .

Next we claim that  $Y$  is Cauchy. By triangle inequality, we have  $d(y_j, y_l) \leq d(y_j, x_p^{(j)}) + d(x_p^{(j)}, x_p^{(l)}) + d(x_p^{(l)}, y_l)$ . We can select  $M$  s.t. for  $j, l \geq M$ , the first and third terms are  $< \epsilon$  for large enough  $p$ . Moreover, since  $(x_k)_{k \in \mathbb{N}}$  is Cauchy, the second term is  $< \epsilon$  for large enough  $p$ .

So  $(y_k)_{k \in \mathbb{N}}$  is Cauchy.

Now, since  $d(X_k, Y_k) \rightarrow 0$  and  $Y_k \rightarrow Y$ , we have that  $X_k \rightarrow Y \in \bar{X}$ .

Given  $X \subseteq \bar{X}$ , the embedding of  $X$  in  $\bar{X}$  is unique.

For example, the completion of  $\mathbb{Q}$  is identical to  $\mathbb{R}$  as a metric space.

**Theorem:** (Corollary) For some  $f : X \rightarrow Y$  with metric spaces  $X, Y$ , if  $X$  is totally bounded,  $Y$  is complete, and  $f$  is uniformly continuous, then  $f$  extends uniquely to  $f : \bar{X} \rightarrow Y$  where  $f((x_n)_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} f(x_n)$ .

**Proposition:** If  $f : X \rightarrow Y$  continuous and  $X$  is compact, then  $f$  is uniformly continuous.

**Proof:** If  $(x_n)_{n \in \mathbb{N}}$  is Cauchy in  $X$  then  $x_n \rightarrow x$  for some  $x \in X$ , then  $f(x_n) \rightarrow f(x)$ . That is,  $(f(x_n))_{n \rightarrow \infty}$  is Cauchy.

For example, any uniformly continuous map  $f : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$  extends continuously to a uniform continuous map  $f : [0, 1] \rightarrow \mathbb{R}$ .

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**Definition:** For  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}$ , the following are equal: (a) There exists  $L \in \mathbb{R}$  such that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \neq x_0$ ,  $|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$ .

(b) For all sequences  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \neq x_0 \forall n \in \mathbb{N}$ , if  $x_n \rightarrow x_0$ , then  $f(x_n) \rightarrow L$ .

If either holds, we say that the limit of  $f(x)$  equals  $L$  as  $x$  approaches  $x_0$ , and write  $\lim_{x \rightarrow x_0} f(x) = L$ .

**Proposition:** (a) and (b) are equivalent.

**Proof:** (a) $\Rightarrow$ (b): Suppose (b) is false, then there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \rightarrow x_0$ , but  $f(x_n) \geq L + \epsilon$  for infinitely many  $n$ . This implies that there is no  $\delta > 0$  for this  $\epsilon$  s.t.  $|f(x) - L| < \epsilon$  for all  $x$  near  $x_0$ .

(b) $\Rightarrow$ (a): If (a) false, there exists  $\epsilon > 0$  s.t.  $\forall \delta_n = \frac{1}{n}$ ,  $\exists x_n \in (x_0 - \delta_n, x_0 + \delta_n)$  s.t.  $|f(x_n) - L| \geq \epsilon$ . But then clearly  $f(x_n)$  does not  $\rightarrow L$ .

**Remark:** If one restrict to  $x < x_0$  or  $x > x_0$  in (a), we get the definition of left/right limits  $\lim_{x \rightarrow x_0^-} f(x)$  and  $\lim_{x \rightarrow x_0^+} f(x)$ .

**Remark:** If one takes  $\limsup_{n \rightarrow \infty} f(x_n)$  or  $\liminf_{n \rightarrow \infty} f(x_n)$  in (b) (these always exist), then we get the definition of  $\limsup_{x \rightarrow x_0} f(x)$  and  $\liminf_{x \rightarrow x_0} f(x)$ .

**Definition:** The oscillation of  $f : \mathbb{R} \rightarrow \mathbb{R}$  at  $x = c$  is  $\text{osc}(f)(c) = \lim_{\epsilon \rightarrow 0} \sup_{x, y \in (c - \epsilon, c + \epsilon)} |f(x) - f(y)|$ .

For example,  $\text{osc}(\sin(\frac{1}{x}))(0) = 2$ .

**Remark:** The actual value of the function at the point is included in the oscillation, but not in the limit definition.

**Proposition:**  $f$  is continuous at  $x \Leftrightarrow \text{osc}(f)(x) = 0$ .

**Proof:** Unravel definition of  $\text{osc}(f)$ .

We get that  $\text{osc}(f)(x) = 0 \Leftrightarrow$  for all  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.  $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \Leftrightarrow f$  is continuous at  $x$ .

**Definition:** We say that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in \mathbb{R}$  if  $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x \rightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0}$  exists.  
 $f'(x_0)$  is the derivative of  $f(x)$  at  $x_0$ .

**Proposition:** If  $f$  is differentiable at  $x_0$ , then  $f$  is continuous at  $x_0$ .

**Proof:** If  $f$  is differentiable at  $x_0$ , we have that for  $L = f'(x_0)$ ,  $|\frac{f(x_0+h)-f(x_0)}{h} - L| < \epsilon$  for  $|h| < \delta$ .

Multiply by  $|h|$  on both sides, we have  $||f(x_0+h)-f(x_0)|-|Lh|| \leq |f(x_0+h)-f(x_0)-Lh| < \epsilon|h|$ .  
Therefore,  $|f(x_0+h)-f(x_0)| \leq |L||h| + \epsilon|h| = (|L| + \epsilon)|h|$ . As  $h \rightarrow 0$ , the right hand side also  $\rightarrow 0$ , therefore  $\lim_{h \rightarrow 0} f(x_0+h) = f(x_0) \Leftrightarrow f$  is continuous at  $x_0$ .

**Proposition:** Suppose  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  differentiable. Then, (a)  $f + g$  differentiable,  $(f + g)' = f' + g'$ ,  $(cf)' = cf'$ .

(b)  $fg$  is differentiable,  $(fg)' = f'g + fg'$ .

(c)  $\frac{f}{g}$  is differentiable when  $g(x) \neq 0$  with  $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$ .

**Proof:** (a) Exercise

(b)  $\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h \rightarrow 0} \frac{(f(x+h)-f(x))g(x+h)+f(x)(g(x+h)-g(x))}{h}$ .

Which is equal to  $f'(x) \lim_{h \rightarrow 0} g(x+h) + g'(x)f(x) = f'(x)g(x) + f(x)g'(x)$ .

(c) Apply (b) to  $f$  and  $\frac{1}{g}$ .

**Proposition:** (Chain Rule) Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f$  differentiable at  $g(c) = d$  and  $g$  is differentiable at  $c$ .

Then the composite  $f \circ g$  would be differentiable at  $c$  and  $(f \circ g)'(c) = f'(g(c))g'(c)$ .

**Proof:**  $g(c) = d, g(c+h) = d + h'$ . Since  $g$  is continuous at  $c$ ,  $h \rightarrow 0 \Rightarrow h' \rightarrow 0$ .

$(f \circ g)'(c) = \lim_{h \rightarrow 0} \frac{f(g(c+h))-f(g(c))}{h} = \lim_{h \rightarrow 0} \frac{f(d+h')-f(d)}{h'} \cdot \frac{(d+h')-d}{h}$ .

Which is equal to  $\lim_{h \rightarrow 0} \frac{f(d+h')-f(d)}{h'} \lim_{h \rightarrow 0} \frac{g(c+h)-g(c)}{h} = f'(d)g'(c) = f'(g(c))g'(c)$ .

**Definition:** We say that  $f : [a, b] \rightarrow \mathbb{R}$  satisfies the intermediate value property (IVP) if for all  $c \in [f(a), f(b)]$ ,  $\exists x \in [a, b]$  s.t.  $f(x) = c$ .

**Proposition:** (Intermediate Value Theorem) If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  satisfies the intermediate value theorem.

**Proof:** Without loss of generality, assume  $f(a) < f(b)$ . take  $c \in (f(a), f(b))$ . Set  $x_0 = \sup\{x \in [a, b] : f(x) < c, y \in [a, x]\}$ .

Note that  $x_0 = b$  by continuity of  $f$  at  $b$  and  $f(b) > c$  ( $\epsilon = \frac{f(b)-c}{2}$ ).

We claim that  $f(x_0) = c$ . Indeed,  $\limsup_{x \rightarrow x_0^-} f(x) \leq c$  and  $\limsup_{x \rightarrow x_0^+} f(x) \geq c$ .

So  $\limsup_{x \rightarrow x_0} f(x) = c$ , so by continuity of  $f$ ,  $\limsup_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} f(x) = f(x_0) = c$ .

**Proposition:** (Rolle's Theorem) If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, differentiable on  $(a, b)$ , and  $f(a) = f(b)$ , then  $\exists x_0 \in (a, b)$  s.t.  $f'(x_0) = 0$ .

**Proof:** Unless  $f$  is constant, by Extreme Value Theorem  $f$  attains (without loss of generality) a global max at some  $x = x_0$  on  $(a, b)$ .

Then,  $f(x_0 + h) \leq f(x_0)$  for  $h > 0$ , and  $f(x_0 - h) \geq f(x_0)$  for  $h > 0$ .

So,  $\lim_{h \rightarrow 0^+} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$  and  $\lim_{h \rightarrow 0^+} \frac{f(x_0-h)-f(x_0)}{h} \geq 0$ .

So  $\lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = 0 = f'(x_0)$ .

**Proposition:** (Mean Value Theorem) If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then  $\exists c \in (a, b)$  s.t.  $\frac{f(b)-f(a)}{b-a} = f'(c)$ .

**Proof:** Apply Rolle's Theorem to  $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$ . We have that  $g(a) = g(b) = f(a)$ .

So,  $\exists c \in (a, b)$  s.t.  $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$ .

**Proposition:** (Darboux's Theorem) If  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable, then  $f'$  satisfies the Intermediate Value Theorem.

**Proof:** For  $[c, d] \subseteq (a, b)$ . Without loss of generality, assume  $f'(c) < f'(d)$  with  $r \in (f'(c), f'(d))$ .

Define  $g(x) = f(x) - rx$  which is differentiable.  $g'(x) = f'(x) - r$ , so  $g'(c) < 0$  and  $g'(d) > 0$ .

This implies that the global min of  $g$  is attained in the interior  $(c, d) \subseteq [c, d]$ .

Call  $x_0$  the global min of  $g$  in  $(c, d)$ . By the argument of Rolle's theorem,  $g'(x_0) = 0 \Rightarrow f'(x_0) = r \Rightarrow f'$  has IVP.

**Remark:** The derivative of a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  can be wild.

For example, there are functions whose derivative exists everywhere, but is discontinuous at uncountably many points.

**Proposition:** (1D Inverse Function Theorem) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable with  $f(c) = d$  and  $f'(c) \neq 0$ .

Then,  $\exists$  open intervals  $I, J$  s.t.  $c \in I, d \in J$  s.t.  $f : I \rightarrow J$  is a  $C^1$ -diffeomorphism, i.e.  $f : I \rightarrow J$  is bijective, and  $f^{-1} : J \rightarrow I$  is continuously differentiable.

Moreover,  $(f^{-1})'(d) = \frac{1}{f'(c)}$ .

**Proof:** First we WTS injectivity. If  $f$  is not injective on any interval around  $c$ , we can construct  $(x_n)_n, (y_n)_n$  inductively such that  $x_n \rightarrow c, y_n \rightarrow c$  and  $f(x_n) = f(y_n)$ .

By MVT,  $\frac{f(x_n) - f(y_n)}{x_n - y_n} = f'(z_n)$  for  $z_n \in (x_n, y_n)$ , so  $z_n \rightarrow c$ .

Therefore by continuity of  $f'$ ,  $f'(z_n) \rightarrow f'(c) = 0$ , leading to a contradiction.

So  $f : I \rightarrow f(I)$  bijective, where  $c \in I$ .

Let  $J = f(I)$ .  $J$  is an interval since  $I$  is interval and  $f$  continuous.

It remains to show  $g = f^{-1}$  is continuously differentiable.

$g(d + h') = c + h$ , where  $h \rightarrow 0 \Leftrightarrow h' \rightarrow 0$ .

$$g'(d) = (f^{-1})'(d) = \lim_{h' \rightarrow 0} \frac{g(d+h') - g(d)}{(d+h') - d} = \lim_{h' \rightarrow 0} \frac{c+h-c}{f+h-f(c)} = \frac{1}{\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}} = \frac{1}{f'(c)}.$$

We haven't shown that  $g$  is continuously differentiable. We will do this in the next lecture.

**Jan. 14**

**Proposition:** (1D Inverse Function Theorem, continued)

**Proof:** We still want to show that  $g$  is continuously differentiable.

By shrinking  $I$  if necessary and invoking the continuity of  $f$  (If  $f'(c) \neq 0$  then there exists some interval around  $c$  s.t.  $f(x) \neq 0$  on the interval), we may assume that  $f(x) \neq 0 \forall x \in I$ .

Now, let  $x$  be an arbitrary element in  $I$ , and let  $y = f(x) \in J$ . We have that  $f(x+h) = y+h'$ , with  $h \rightarrow 0 \Leftrightarrow h' \rightarrow 0$ .

$$\text{So, } g'(y) = \lim_{h' \rightarrow 0} \frac{g(y+h') - g(y)}{y+h'-y} = \lim_{h \rightarrow 0} \frac{x+h-x}{f(x+h)-f(x)} = \lim_{h \rightarrow 0} \frac{1}{\frac{f(x+h)-f(x)}{h}} = \frac{1}{f'(x)}.$$

Since  $g'$  is the reciprocal of a continuous, non-zero function  $f'$ , we have that  $g'$  is also continuous.

**Proposition:** (L'Hopital's Rule) Let  $f, g : (a, b) \rightarrow \mathbb{R}$  be two continuous functions.

Suppose that  $f, g$  are differentiable on  $(a, b)$ , except possibly at  $c$ .

Suppose that  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ , and that  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  exists.

Also suppose that  $g'(x) \neq 0$  on an open punctured interval  $I \setminus \{c\}$  around  $c$ .

Then,  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  exists and  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ .

**Proof:** First, notice that  $g$  must also be non-zero on some punctured interval  $I \setminus \{c\}$ , otherwise we can construct a sequence  $(x_n)_{n \in \mathbb{N}}$ , where without loss of generality  $x_n < c, x_n \rightarrow c$  and  $g(x_n) = 0$ . Then by MVT,  $\frac{g(x_{n+1}) - g(x_n)}{x_{n+1} - x_n} = g'(z_n) = 0$ , where  $z_n \in (x_n, x_{n+1})$ . So we have that  $z_n \rightarrow c$ , which contradicts the assumption that  $g'(x)$  is non-zero near  $c$ .

**Lemma** (Generalized Mean Value Theorem) Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous functions for  $a < b$ , that are differentiable on  $(a, b)$ . Assume that  $g'(x) \neq 0$  on  $(a, b)$  and  $g(a) \neq g(b)$ . Then,

there exists  $c \in (a, b)$  s.t.  $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ .

Proof Homework Exercise.

Using generalized MVT (and recall that  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ ), we have:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x \rightarrow c} \frac{f'(d)}{g'(d)} \text{ for some } d \in (c, x).$$

As  $x \rightarrow c$ , it is immediate that  $d \rightarrow c$ , therefore  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ .

**Remark:** The assumption  $g(x) \neq 0$  near  $c$  is crucial and often missed, as there is the counterexample  $f(x) = x + \sin x \cos x$ ,  $g(x) = (x + \sin x \cos x)e^{\sin x}$  as  $x \rightarrow \infty$ , where  $g'(x)$  vanishes periodically.

**Remark:** A similar method works for  $\frac{\infty}{\infty}$  by considering  $\frac{1}{f}, \frac{1}{g}$ . Can generalize to other indeterminate forms.

**Definition:** A partition  $P$  of  $[a, b]$  is a finite collection of points  $a = x_0 < x_1 < x_2 < \dots < x_{n+1} = b$ . We write  $\Delta x_k = x_{k+1} - x_k$ .

The mesh of a partition  $P$  is  $\text{mesh}P = \max_{0 \leq k \leq n} |\Delta x_k|$ .

A refinement  $P'$  of a partition  $P$  is a partition that includes all points of  $P$  and maybe other points.

We write  $P \subseteq P'$ .

The common refinement of partitions  $P, Q$  of  $[a, b]$  is  $P \cup Q$  as a set.

**Remark:** Clearly,  $\text{mesh}(P \cup Q) \leq \min\{\text{mesh}P, \text{mesh}Q\}$ .

**Definition:** Given  $f : [a, b] \rightarrow \mathbb{R}$  be bounded and let  $P$  be a partition of  $[a, b]$ .

We define the lower sum  $L(f, P)$  and upper sum  $U(f, P)$  as follows:

$$U(f, P) = \sum_{k=0}^n (\sup_{[x_k, x_{k+1}]} f(x)) \Delta x_k, \quad L(f, P) = \sum_{k=0}^n (\inf_{[x_k, x_{k+1}]} f(x)) \Delta x_k.$$

**Proposition:** If  $P \subseteq P'$ , then  $L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$ .

**Proof:** We label the inequalities as (1), (2) and (3) respectively.

(2) follows immediately from  $\inf_{[x_k, x_{k+1}]} f(x) \leq \sup_{[x_k, x_{k+1}]} f(x)$ .

(1) and (3) follow from that if  $[y_j, y_{j+1}] \subseteq [x_k, x_{k+1}]$ , then  $\inf_{[y_j, y_{j+1}]} f(x) \geq \inf_{[x_k, x_{k+1}]} f(x)$  and  $\sup_{[y_j, y_{j+1}]} f(x) \leq \sup_{[x_k, x_{k+1}]} f(x)$ .

Therefore,  $L(f, P) = \sum_{k=0}^n (\inf_{[x_k, x_{k+1}]} f(x)) \Delta x_k \leq \sum_{j=0}^m (\inf_{[y_j, y_{j+1}]} f(x)) \Delta y_j = L(f, P')$ .

The  $U(f, P) \geq U(f, P')$  case follows analogously.

**Definition:** The lower/upper Darboux integrals of a bounded  $f : [a, b] \rightarrow \mathbb{R}$  are:

$\bar{I}(f) = \inf_P U(f, P)$ ,  $\underline{I}(f) = \sup_P L(f, P)$ , where  $\inf, \sup$  are taken over all partitions of  $[a, b]$ .

We say that  $f : [a, b] \rightarrow \mathbb{R}$  is Darboux integrable if the upper and lower integrals of  $f$  are equal, in which case the Darboux integral is  $\int_a^b f(x)dx = \bar{I}(f) = \underline{I}(f) = I(f)$ .

**Proposition:** (Lebesgue Criterion for Darboux Integrability) A bounded  $f : [a, b] \rightarrow \mathbb{R}$  is Darboux Integrable if and only if  $\forall \epsilon > 0, \exists$  partition  $P$  of  $[a, b]$  s.t.  $U(f, P) - L(f, P) < \epsilon$ .

**Proof:** If  $f$  is Darboux integrable,  $\sup_P L(f, P) = \inf_P U(f, P)$ , then there exists partitions  $P, P'$  where  $|U(f, P) - L(f, P')| < \epsilon$ .

Take the common refinement  $P \cup P'$ , we have  $U(f, P \cup P') - L(f, P \cup P') \leq U(f, P) - L(f, P') < \epsilon$ .

Conversely, if  $U(f, P) - L(f, P) < \epsilon$  for some partition  $P$  depending on  $\epsilon$  for every  $\epsilon > 0$ , clearly  $\sup L(f, P) = \inf U(f, P) \Leftrightarrow \bar{I}(f) = \underline{I}(f)$ .

**Proposition:** A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Darboux integrable if and only if for all  $\epsilon > 0$ , there exists  $\delta > 0$  s.t. all partitions  $P$  of  $[a, b]$  s.t.  $\text{mesh}P < \delta$  have  $U(f, P) - L(f, P) < \epsilon$ .

**Proof:** If the mesh condition holds, clearly  $\sup_P L(f, P) = \inf_P U(f, P)$ , the above argument immediately follows.

Conversely, suppose  $f$  is Darboux integrable.

Lemma If  $P, Q$  are two partitions of  $[a, b]$  s.t.  $\text{mesh}Q$  is less than or equal to the length of any subinterval in  $P$ , then  $U(f, Q) - L(f, Q) \leq 3(U(f, P) - L(f, P))$ .

Proof By the assumption, the sum of lengths of any intervals in  $Q$  covering any fixed interval in  $P$  is at most  $I + 2I = 3I$ , that is,  $\sum_{j \in J_i} \Delta y_j \leq 3\Delta x_i$ .

Moreover, by the triangle inequality, the sum -  $\sum \sup f - \inf f$  over all intervals  $I_j$  in  $P$  covering some fixed interval  $J$  with index  $j$  in  $Q$  - bounds  $\sup_J f - \inf_J f$  from above, that is,  $\sup_j f - \inf_j f \leq \sum_{i \in I_j} \sup_i f - \inf_i f$ .

Since summing over intervals  $I_j$  in  $P$  (with index  $i$ ) covering each fixed interval  $J$  with index  $j$  for all  $J$  in  $Q$  is the same as summing over all intervals  $J_i$  in  $Q$  (with index  $j$ ) covering each fixed interval  $I$  with index  $i$  for all  $I \in P$ , on thus gets  $U(f, Q) - L(f, Q) = \sum_j (\sup_j f - \inf_j f) \Delta y_j \leq \sum_j \sum_{i \in I_j} (\sup_i f - \inf_i f) \Delta y_j = \sum_i \sum_{j \in J_i} (\sup_i f - \inf_i f) \Delta y_j = \sum_i (\sup_i f - \inf_i f) \sum_{j \in J_i} \Delta y_j \leq 3 \sum_i (\sup_i f - \inf_i f) \Delta x_i = 3(U(f, P) - L(f, P))$ .

Now, if  $f$  is Darboux integrable and  $U(f, Q) - L(f, Q) < \epsilon$ , for any partition  $Q$  with  $\text{mesh}Q$  less than the length of each subinterval in  $P$  (call the min of these lengths  $\delta > 0$ ), we get that  $\text{mesh}Q \Rightarrow U(f, Q) - L(f, Q) \leq 3(U(f, P) - L(f, P)) < 3\epsilon$ .

**Definition:** A tagged partition  $P^*$  is a partition with a choice of point  $x_k^*$  in each subinterval.

**Definition:** A bounded  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if  $\lim_{\text{mesh} P \rightarrow 0} \sum_{x_i \in P} f(x_i^*) \Delta x_i$  exists, in which case we say it is equal to  $\int_a^b f(x) dx$ , the Riemann integral of  $f$  on  $[a, b]$ .

**Proposition:**  $f$  is Riemann integrable  $\Leftrightarrow f$  is Darboux integrable, and the value of the two integrals coincide.

**Proof:** If  $f$  is Riemann integrable, for any  $\epsilon > 0$ , we can take  $\delta > 0$  s.t. for all partitions  $Q$  of  $[a, b]$  s.t.  $\text{mesh} Q < \delta$ , and  $Q^*, Q^{**}$  being any two tagged partition from  $Q$ :

$$|\sum_{x_i^* \in Q^*} f(x_i^*) \Delta x_i - \sum_{x_i^{**} \in Q^{**}} f(x_i^{**}) \Delta x_i| < \epsilon.$$

Since we are free to choose the tagging, we may approximate  $\sup f$  and  $\inf f$  on the subintervals by tagging. That is,  $f(x_i^*) \leq \inf f + \epsilon$  and  $f(x_i^{**}) \geq \sup f - \epsilon$ . This way we would have  $|U(f, Q) - L(f, Q) - \sum_{x_i \in Q} (f(x_i^{**}) - f(x_i^*)) \Delta x_i| = |(U(f, Q) - \sum_{x_i \in Q} f(x_i^{**}) \Delta x_i) + (\sum_{x_i \in Q} f(x_i^*) \Delta x_i - L(f, Q))|$ .

The two parts of this sum are, by definition of each individual  $x_i^*, x_i^{**}$ ,  $\leq \epsilon \sum_{x_i \in Q} \Delta x_i = \epsilon(b-a)$ . So  $|U(f, Q) - L(f, Q) - \sum_{x_i \in Q} (f(x_i^{**}) - f(x_i^*)) \Delta x_i| \leq 2\epsilon(b-a)$ . And since the third term in this sum is, as derived,  $< \epsilon$ , we have that  $U(f, Q) - L(f, Q) < \epsilon + 2\epsilon(b-a) = \epsilon(1 + 2(b-a))$ . This can be arbitrarily small, therefore  $f$  is Darboux integrable.

Conversely,  $f(x^*) - f(x^{**}) \leq \sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f$ , with  $x^*, x^{**} \in [x_i, x_{i+1}]$  being any two taggings.

Thus, we select a partition  $P$  of  $[a, b]$  s.t.  $\text{mesh} P$  is small enough so that  $U(f, P) - L(f, P) < \epsilon$  (this is possible due to the mesh condition of Darboux integrability).

Therefore for any tagging  $P^*, P^{**}$ , we have that  $|\sum_{x_i \in P} (f(x_i^*) - f(x_i^{**})) \Delta x_i| \leq U(f, P) - L(f, P) < \epsilon$ .

Since this can be arbitrarily small, we have that  $f$  is Riemann integrable.

We also have that given  $f$  is both Riemann and Darboux integrable, the Riemann integral is "squeezed" by the upper and lower sums to the Darboux integral, therefore the two values coincide.

## Jan. 16

**Remark:** Darboux integral can be thought of as approximating  $f : [a, b] \rightarrow \mathbb{R}$  by "simple functions" of the form  $\sum_{k=0}^n \chi_{I_k}(x)$ , where given  $A \subseteq \mathbb{R}$ , the characteristic function:

$$\chi_A(x) = \begin{cases} 1 : x \in A \\ 0 : x \notin A \end{cases}, \text{ and } I_k \text{ being the subintervals in our partition.}$$

Namely,  $f$  is Darboux integrable  $\Leftrightarrow \forall \epsilon > 0$ , there exists simple functions  $h_1, h_2$  s.t.  $h_1 \leq f \leq h_2$  and  $\int_a^b (h_2 - h_1) dx < \epsilon$ .

**Proposition:** (a) If  $f, g : [a, b] \rightarrow \mathbb{R}$  are integrable,  $c \in \mathbb{R}$ , then  $f + g$  and  $cf$  are integrable, with  $\int_a^b cf(x) dx = c \int_a^b f(x) dx$  and  $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$ .

- (b) If  $f, g : [a, b] \rightarrow \mathbb{R}$  integrable and  $f \leq g$ , then  $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ .
- (c) (Triangle Inequality) If  $f : [a, b] \rightarrow \mathbb{R}$  are integrable, then  $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$ .
- (d) If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then it is integrable.
- (e) If  $f : [a, c] \rightarrow \mathbb{R}$  is integrable and  $a < b < c$ , then  $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$ .
- (f) If  $f, g : [a, b] \rightarrow \mathbb{R}$  is integrable, then  $fg$  is integrable.

**Proof:** (a)  $\int_a^b cf(x)dx = c \int_a^b f(x)dx$  is left as exercise.

For  $f + g$ , use Darboux for integrability, and note that:

$$\sup(f + g) - \inf(f + g) \leq \sup f + \sup g - (\inf f + \inf g) = (\sup f - \inf f) + (\sup g - \inf g).$$

This easily implies that  $U(f + g, P) - L(f + g, P) \leq U(f, P) - L(f, P) + U(g, P) - L(g, P) < 2\epsilon$ , which can be arbitrarily small, implying that  $f + g$  is integrable.

Also, since  $\inf f + \inf g \leq \inf(f + g)$  and  $\sup(f + g) \leq \sup f + \sup g$ , we have that  $L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P)$ .

We have that  $L(f, P)$  and  $U(f, P)$ , as well as  $L(g, P)$  and  $U(g, P)$ , get arbitrarily close to each other, therefore by limiting behavior we have that  $\bar{I}(f + g) = \underline{I}(f + g) = I(f) + I(g)$ .

Which gives us  $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ .

(b) Use Riemann:  $\int_a^b f(x)dx = \lim_{\text{mesh}P \rightarrow 0} \sum_{x_i \in P} f(x_i^*)\Delta x_i \leq \lim_{\text{mesh}P \rightarrow 0} \sum_{x_i \in P} g(x_i^*)\Delta x_i = \int_a^b g(x)dx$ .

(c) Use Riemann and regular triangle inequality:

$$|\sum_{x_i \in P} f(x_i^*)\Delta x_i| \leq \sum_{x_i \in P} |f(x_i^*)|\Delta x_i.$$

Take the limit  $\text{mesh}P \rightarrow 0$ , we have  $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$ .

To show that  $|f|$  is integrable, use Darboux definition (left as exercise).

(d) Use Darboux:  $U(f, P) - L(f, P) = \sum_{x_i \in P} (\sup_{[x_i, x_{i+1}]} f(x) - \inf_{[x_i, x_{i+1}]} f(x))\Delta x_i$ .

Since  $f$  is continuous on a compact set, it is uniformly continuous, so  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ .

Therefore,  $\sup_I f - \inf_I f \leq \epsilon$  for any interval  $I$  with length less than  $\delta$ .

So if we take any partition  $P$  with  $\text{mesh}P < \delta$ , then  $U(f, P) - L(f, P) \leq \epsilon \sum \Delta x_i = \epsilon(b - a)$ .

This can be arbitrarily small, so  $f$  is integrable.

(e) This follows from the fact that a partition of  $[a, c]$  induces partitions of  $[a, b]$  and  $[b, c]$ .

The details are left as exercise.

(f) Left as exercise.

**Definition:** A function  $f : [a, \infty)$  is improper Riemann integrable if it is integrable on any closed, bounded subinterval of  $[a, \infty)$  and  $\lim_{b \rightarrow \infty} \int_a^b f(x)dx$  exists, in which case  $\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$ .

Similarly, we can define improper integrals for functions with asymptotes at real values.

For example,  $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}}dx$ , if this limit exists.

**Proposition:** (Integral MVT) If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $\exists c \in [a, b]$  s.t.  $\frac{1}{b-a} \int_a^b f(x)dx = f(c)$ .

**Proof:**  $\inf_{[a,b]} f \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \sup_{[a,b]} f$  since  $\inf_{[a,b]} f \leq f(x) \leq \sup_{[a,b]} f$ .

Since  $f$  is continuous, IVP holds for  $f(c) = \frac{1}{b-a} \int_a^b f(x)dx$  for  $c \in [a', b']$ , where  $f(a') = \inf_{[a,b]} f$  and  $f(b') = \sup_{[a,b]} f$ , therefore  $[a', b'] \subseteq [a, b]$ .

**Proposition:** (Integral Test) If  $f : [1, \infty) \rightarrow (0, \infty)$  is a monotonically decreasing function (that is, for  $x \leq y$ ,  $f(x) \geq f(y)$ ), then  $\sum_{n=1}^{\infty} f(n)$  converges if and only if  $\int_1^{\infty} f(x)dx$  converges.

**Proof:** Monotonically decreasing  $\Rightarrow$  integrable (left as exercise).

By lower and upper Riemann sums, we get  $\sum_{n=2}^N f(n) \leq \int_1^N f(x)dx \leq \sum_{n=1}^N f(n)$ .

Taking  $n \rightarrow \infty$  yields the desired result.

**Theorem:** (Corollary) For  $p > 0$ ,  $\int_1^{\infty} \frac{1}{x^p} dx < \infty \Leftrightarrow p > 1$ , so  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if and only if  $p > 1$ .

**Proposition:** (Lebesgue Criterion for Riemann Integrability) A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if  $\forall \epsilon > 0$ , there exists a countable collection  $(I_n)_{n \in \mathbb{N}}$  of open intervals that cover all the points of discontinuity of  $f$  and the sum of length of all the intervals is less than  $\epsilon$ .

**Proof:** ( $\Leftarrow$ ) Let us call the points of discontinuity of  $f$  "bad", and the points of continuity "good". We also call any interval that contains any "bad" points as "bad", otherwise it is a "good" interval.

Suppose that bad points can be covered by countably many open intervals of total length  $< \epsilon$ .

Recall that  $f$  is discontinuous at  $x \Leftrightarrow \text{osc}(f)(x) > 0$ .

Fix  $\alpha > 0$ , and note that  $\{x : \text{osc}(f)(x) < \alpha\}$  is open.

Thus, the complement,  $\{x : \text{osc}(f)(x) \geq \alpha\}$  is closed, and since it is also bounded, it is compact.

Notice that  $(I_k)_{k \in \mathbb{N}}$  is an open cover of this set. By compactness, there exists a finite subcover  $(I_{k_n})_n$  with total length  $< \epsilon$ .

On the good intervals, we get that  $f$  is continuous. By making these intervals closed, we get that  $f$  is a continuous function on a compact set (the finite union of closed intervals is closed) and therefore uniformly continuous.

Now, for any  $\epsilon > 0$ , pick the countable cover, and pick the finite subcover, and extend it to a partition by including the endpoints of the intervals. Then,

$$U(f, P) - L(f, P) = \sum (\sup f - \inf f) \Delta x_i = \sum_{\text{good}} (\sup f - \inf f) \Delta x_i + \sum_{\text{bad}} (\sup f - \inf f) \Delta x_i \leq \epsilon \sum_{\text{good}} \Delta x_i + M \sum_{\text{good}} \Delta x_i, \text{ where } M = \sup_{[a,b]} f - \inf_{[a,b]} f.$$

This is then  $< \epsilon(b-a) + M\epsilon = \epsilon(M+b-a)$ . Therefore  $f$  is integrable.

(Note) We really want an open cover of  $X = \{x : \text{osc}(f)(x) > 0\}$  but have covers of  $\{x :$

$\text{osc}(f)(x) > 0\}$ .

We can take the union of covers for  $\alpha_n = \frac{1}{n}$  for all  $n$ .

We also want  $X$  to be compact (?).

( $\Rightarrow$ ) Suppose, for the sake of contradiction, that the bad points cannot be covered by countably many intervals with total length arbitrarily small. We claim that for some  $\epsilon > 0$ , the set  $\{x : \text{osc}(f)(x) > \epsilon\}$  cannot be covered by countable union of intervals of arbitrarily small length.

If not, taking union of covers for  $\epsilon_n = \frac{1}{n}$  for all  $n$  with the total length of the  $n$ -th cover being  $\frac{\delta}{2^n}$  for some  $\delta > 0$ .

$\{x : \text{osc}(f)(x) \geq \frac{1}{n}\} \subseteq \bigcup I_j$ , of total length  $< \frac{\delta}{n^2}$ . Union of covers has length  $\leq \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta$ , leading to a contradiction.

Finally, this implies that for any partition  $P$  of  $[a, b]$ , the bad intervals have total length always at least  $\epsilon'$  for some  $\epsilon' > 0$ .

Thus,  $U(f, P) - L(f, P) \geq \sum_{\text{bad}} (\sup f - \inf f) \Delta x_i \geq \epsilon \sum_{\text{bad}} \Delta x_i \geq \epsilon \epsilon'$ , here we define "bad" as containing a point of  $\{x : \text{osc}(f)(x) > \epsilon\}$ .

This implies that  $f$  is not integrable, which is a contradiction.

**Jan. 21**

**Remark:** In the proof of Lebesgue criterion for Riemann integrability, we should instead denote "good" intervals as those with  $\text{osc}(f) \leq \alpha$ , and "bad" intervals as those with  $\text{osc}(f) > \alpha$ .

**Proposition:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable. Then  $F(x) = \int_a^x f(t)dt$  is (uniformly) continuous.

**Proof:**  $|F(x) - F(y)| = |\int_y^x f(t)dt| \leq \int_y^x |f(t)|dt \leq M|x - y| < \epsilon$  for  $|x - y| < \frac{\epsilon}{M} = \delta$ , where  $M = \sup_{[a, b]} |f|$ .

**Proposition:** (Fundamental Theorem of Calculus) If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $F(x) = \int_a^x f(t)dt$  is differentiable and  $F'(x) = f(x)$ . (Part I)

If  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable, and  $\exists F : [a, b] \rightarrow \mathbb{R}$  s.t.  $F' = f$ , then  $\int_a^b f(t)dt = F(b) - F(a)$ .

**Proof:** (Part I)  $\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt = f(c) \rightarrow f(x)$  as  $h \rightarrow 0$  for  $c \in [x, x+h] \Rightarrow c \rightarrow x$ . (Part II)  $F(b) - F(a) = F(b) - F(x_n) + F(x_n) - \dots + F(x_1) - F(a) = \sum_i F(x_{i+1}) - F(x_i) = \sum_i f(x_i^*) \Delta x_i$ . Push  $\text{mesh}(P) \rightarrow 0$ , this approaches  $\int_a^b f(x)dx$ . Therefore,  $F(b) - F(a) = \int_a^b f(x)dx$ .

**Proposition:** (Integration by Parts) If  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuously differentiable functions, then:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

**Proof:** Apply fundamental theorem of calculus to product rule.

**Definition:** Given a sequence of functions,  $(f_n)_n : X \rightarrow Y$  and  $f : X \rightarrow Y$ , we say that  $f_n \rightarrow f$  pointwise if  $f_n(x) \rightarrow f(x) \forall x \in X$ . We say that  $f_n \rightarrow f$  uniformly if for all  $\epsilon > 0$ , there exists  $N$  s.t.  $d(f_n(x), f(x)) < \epsilon \forall x \in X$  for  $n \geq N$ .

For example, we define  $f_n : [0, 1] \rightarrow \mathbb{R}$  as  $f_n(x) = x^n$ . It is apparent that each  $f_n$  is continuous.

However, we have that  $f_n$  pointwise converges to  $f(x) = \begin{cases} 0 : 0 \leq x < 1 \\ 1 : x = 1 \end{cases}$ , which is not continuous.

**Proposition:** Given  $(f_n) : X \rightarrow Y$  continuous,  $f : X \rightarrow Y$  s.t.  $f_n \rightarrow f$  uniformly, then  $f$  is continuous.

**Proof:** Fix  $x \in X$ , consider  $N$  large s.t.  $d(f_n(y), f(y)) < \frac{\epsilon}{3}$  for all  $y \in X$ .

Then  $d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$  for  $d(x, y) < \delta$ .

**Proposition:** (Dini's Theorem) Let  $X$  be compact, and let continuous  $f_n : X \rightarrow \mathbb{R}$  be a monotonically decreasing sequence. That is,  $f_1 \geq f_2 \geq f_3 \geq \dots$ , and suppose  $f_n \rightarrow f$  pointwise for some  $f : X \rightarrow \mathbb{R}$ .

Then, if  $f$  is continuous, the convergence is uniform.

**Proof:** Fix  $\epsilon > 0$ . Then, since  $X$  is compact,  $f$  is (uniformly) continuous.

Then  $\exists \delta > 0$  s.t.  $f(x) - \epsilon < f(y)$  for  $d(x, y) < \delta$ .

Moreover, for fixed  $N$ ,  $\exists \delta_N > 0$  (since  $f_N$  uniformly continuous) s.t.  $f_N(y) < f_N(x) + \epsilon$  for  $d(x, y) < \delta_N$ .

Then for each  $x \in X$ ,  $\exists N$  and  $\delta' = \min\{\delta, \delta_N\} > 0$  s.t. for  $n \geq N$  and  $d(x, y) < \delta'$ , one has:

$$f_n(y) - f(y) \leq f_N(y) - f(y) < f_N(x) + \epsilon - (f(x) - \epsilon) = f_N(x) - f(x) + 2\epsilon.$$

Taking the union of these balls for all  $x \in X$  gives an open cover of  $X$ . Since  $X$  compact,  $\exists$  finite subcover  $B(x_1, \delta_1), \dots, B(x_k, \delta_k)$ .

Then, for any  $y \in Y$ , if we take  $N = \max\{N_1, \dots, N_k\}$ ,  $f_n(y) - f(y) < 3\epsilon$  for  $n \geq N$ . Then, since  $\epsilon > 0$  is arbitrary and so is  $y$ , we converge uniformly.

**Proposition:** (Weierstrass M-Test) If  $f_n : X \rightarrow \mathbb{R}$  is a sequence of continuous functions s.t.  $|f_n(x)| \leq M_n$  for all  $x \in X$  and  $\sum_{n=1}^{\infty} M_n < \infty$ , then  $\sum_{n=1}^{\infty} f_n$  is uniformly convergent.

**Proof:** We want to show that  $\sum_{n=1}^{\infty} f_n$  is uniformly Cauchy, that is,  $|\sum_{n=N}^M f_n(x)| < \epsilon \forall x \in X$  for  $M, N$  large.

Indeed, we have this is  $\leq \sum_{n=N}^M M_n < \epsilon$  for  $M, N$  large.

**Theorem:** (Corollary) A summable series of continuous functions is continuous.

**Definition:** A collection  $\mathcal{F}$  of functions  $f_n : X \rightarrow \mathbb{R}$  is (uniformly) equicontinuous if for all  $\epsilon > 0$ , there exists  $\delta > 0$  s.t.  $\forall f \in \mathcal{F}$  and  $\forall x, y \in X$ ,  $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$ .

**Theorem:** (Arzela-Ascoli Theorem) Let  $X$  be a compact metric space, and let  $\mathcal{F}$  be a uniformly equicontinuous collection of functions  $f_n : X \rightarrow \mathbb{R}$ . Then, if  $\mathcal{F}$  is pointwise bounded, that is,  $\forall x \in X$ ,  $\sup_{f \in \mathcal{F}} |f(x)| < \infty$ , then  $\mathcal{F}$  is precompact in  $C(X)$ . That is, any sequence  $(f_n)_n \subseteq \mathcal{F}$  has a uniformly convergent subsequence.