

Math 131BH - Notes

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Definition: A topological space (X, τ) is a set X with a collection $\tau \subseteq \mathcal{P}(X)$, called a topology on X , such that:

- i) $\emptyset, X \in \tau$
- ii) τ is closed under finite intersections and arbitrary unions.

The elements in τ are open sets, their complements are closed.

For example, for some set X , $\tau = \{\emptyset, X\}$ gives a trivial topology, and $\tau = \mathcal{P}(X)$ gives the discrete topology.

Definition: A neighborhood $V \subseteq X$ of $x \in X$ is a set such that $\exists U \in \tau$ such that $x \in U \subseteq V$.

Definition: We say that $S \subseteq \mathcal{P}(X)$ generates a topology τ on X if τ is the smallest topology containing S .

Definition: Given $Y \subseteq X$, the subspace topology on Y has open sets of the form $U \cap Y$, with U being an open set in X .

Any metric on X generates a topology. The generating set is $\{B_r(x) : x \in X, r > 0\}$.

If given a norm $\|\cdot\|$ in a vector space V , we can induce a metric (and thus also a topology) on V by $d(v, w) = \|v - w\|$. In particular, \mathbb{R}^n is a topological space with the Euclidean metric.

Definition: A subset $S \subseteq X$ of a topological space is dense in X if every open set in X intersects S nontrivially.

Remark: If X is a metric space, this is equivalent to saying that every open ball intersects S .

Definition: A topological space is called separable if it contains a countable dense subset.

Heuristic: When trying to show a metric space is not separable, it may be useful to construct a collection of uncountable points s.t. every two points are a fixed distance apart. Then we can argue that no countable subset may intersect an uncountable number of mutually exclusive balls.

We can show that \mathbb{R} is separable, since \mathbb{Q} is a countable dense subset.

The same goes for \mathbb{R}^n , with \mathbb{Q}^n being a countable dense subset.

A discrete metric space is separable if and only if it is at most countably infinite, since the fact that every $\{x\}$ is open shows that the only dense subset of X is itself.

Exercise: Let $l^p(\mathbb{N})$ be a vector space with norm $\|(a_n)_{n \geq 1}\|_{l^p} = (\sum_{n=1}^{\infty} |a_n|^p)^{\frac{1}{p}}$.

Show that $l^p(\mathbb{N})$ is separable for $1 \leq p < \infty$, but not $p = \infty$.

Definition: Suppose we have $f : (X, \tau) \rightarrow (Y, \kappa)$.

We say that f is continuous if the preimage of every open set is open. That is, $f^{-1}(V) \in \tau$ for every $V \in \kappa$.

We say that f is continuous at $x \in X$ if the preimage of any neighborhood of $f(x)$ is a neighborhood of x .

Definition: A subset $A \subseteq X$ of a topological space X is called connected if it is not disconnected. That is, there does not exist open, nonempty subsets $U, V \subseteq X$ such that $A \subseteq U \cup V$, $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$, $U \cap V = \emptyset$.

Definition: We say that $A \subseteq X$ is path-connected if any two points $x, y \in A$ are connected by a path in A . That is, a continuous function $f : [0, 1] \rightarrow A$ such that $f(0) = x$, $f(1) = y$.

Proposition: Let $f : X \rightarrow Y$ be a continuous map of topological spaces. If X is connected, then $f(X)$ is connected.

Proof: We proceed by contrapositive.

Suppose that $f(X)$ is disconnected. Then we have open nonempty subsets $U, V \subseteq Y$ such that $f(X) \subseteq U \cup V$, $f(X) \cap U \neq \emptyset$, $f(X) \cap V \neq \emptyset$, $U \cap V = \emptyset$.

By the continuous nature of f , we have that $f^{-1}(U), f^{-1}(V)$ are open.

We also have that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) \subseteq f^{-1}(U \cup V) = f^{-1}(f(X)) = X$. Thus, X is disconnected.

Proposition: If $f : X \rightarrow Y$ is continuous and X is path-connected, then $f(X)$ is path-connected.

Proof: Exercise.

Definition: Let X be a topological space. An open cover of a set $K \subseteq X$ is a collection $\mathcal{F} = \{U_\alpha\}_{\alpha \in I}$ of open sets such that $K \subseteq \bigcup_{\alpha \in I} U_\alpha$.

Definition: The set $K \subseteq X$ is called compact if every open cover of K has a finite subcover. That is, if $K \subseteq \bigcup_{\alpha \in I} U_\alpha$, there exists n_1, \dots, n_k such that $K \subseteq \bigcup_{j=1}^k U_{n_j}$.

We say that K is relatively compact or precompact if \bar{K} is compact.

Definition: Let X be a metric space. We say $K \subseteq X$ is totally bounded if for all $\epsilon > 0$, there exists finitely many open balls B_1, \dots, B_n of radius ϵ such that $K \subseteq \bigcup_{j=1}^n B_j$.

Definition: Let X be a metric space. $K \subseteq X$ is sequentially compact if every sequence $\{x_n\} \subseteq K$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \rightarrow x$ for some $x \in K$.

Theorem: The following are equivalent for a subset $K \subseteq X$ of metric space X :

- (a) K is compact
- (b) K is sequentially compact
- (c) K is complete and totally bounded

We already know that in \mathbb{R} , every bounded sequence has a convergent subsequence.

Theorem: In \mathbb{R}^n , A is sequentially compact $\Leftrightarrow A$ is closed and bounded.

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Proposition: The following are equivalent for a subset $K \subseteq X$ of a metric space X :

- (a) K is compact
- (b) K is sequentially compact
- (c) K is complete and totally bounded

Proof: We WTS (a) \Rightarrow (c) \Leftrightarrow (b) \Rightarrow (a).

(a) \Rightarrow (c): If K is compact, then it is clearly totally bounded, since for any $\epsilon > 0$ there exists a finite subcover for the open cover $\{B(x, \epsilon) : x \in K\}$.

Now we WTS completeness. Assume otherwise.

Let $(x_n)_{n \in \mathbb{N}} \subseteq K$ be a Cauchy sequence that is not convergent. Note that this implies that there is no convergent subsequence (otherwise the Cauchy sequence would converge).

Define $r_n = \inf_{m \neq n} d(x_n, x_m)$, $B_n = B(x_n, r_n)$. Notice that $r_n > 0$ for all n (otherwise we can construct a subsequence converging to x_n).

Note that by construction, $B_n \cap (x_k)_{k \in \mathbb{N}} = \{x_n\}$.

Now define $V = X \setminus \overline{\bigcup_{n \geq 1} B(x_n, \frac{r_n}{2})}$.

Claim: $\{V\} \cup \{B_n\}_{n \geq 1}$ is an open cover of K . Assume otherwise.

Then $\exists x \in X$ s.t. $d(x, x_n) \geq r_n \forall n$ and $\exists (y_{n_k})_{k \in \mathbb{N}}$ s.t. $y_{n_k} \in B_{n_k}$ and $y_{n_k} \rightarrow x$.

Then $d(x, x_{n_k}) = d(x, y_{n_k}) + d(y_{n_k}, x_{n_k}) < d(x, y_{n_k}) + r_n \rightarrow 0$, we have a convergent subsequence, leading to a contradiction.

Therefore, $\{V\} \cup \{B_n\}_{n \in \mathbb{N}}$ is an open cover of K . However, since each x_n is contained exactly in one B_n , and V contains none of the x_n , any finite subset of this open cover would fail to contain the entirety of $\{x_n\}_{n \in \mathbb{N}} \subseteq K$. This contradicts the compactness of K .

Therefore, K is complete.

(b) \Rightarrow (c): Completeness is clear since any Cauchy sequence would have a convergent subsequence by sequential compactness.

Now we WTS totally boundedness. Assume otherwise.

Then for some $\epsilon > 0$, there is no finite cover by ϵ -balls of K .

We can inductively select x_1, x_2, \dots : $x_1 \in K$ is arbitrary. $x_2 \in K \setminus B(x_1, \epsilon)$, $x_3 \in K \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon))$, \dots

But then we have that $d(x_n, x_m) \geq \epsilon \forall n \neq m$, it is impossible for any subsequence to be Cauchy. Therefore, $(x_n)_{n \in \mathbb{N}} \subseteq K$ has no convergent subsequence, contradicting the sequential compactness of K .

(c) \Rightarrow (b): We have some arbitrary sequence $(x_n)_{n \in \mathbb{N}}$. Let us inductively select a sequence of balls: $(B_n)_{n \in \mathbb{N}}$.

Define $\epsilon_n = \frac{1}{2^n}$. Since K is totally bounded, there exists a finite cover of K that consists of ϵ_1 -balls. Since the sequence has infinite elements, there exists $B_1 = B(y_1, \epsilon_1)$ such that $B_1 \cap \{x_n\}_{n \in \mathbb{N}}$ has infinite elements.

Now consider the finite cover that consists of ϵ_2 -balls. By the same logic, there exists $B_2 = B(y_2, \epsilon_2)$ such that $B_2 \cap B_1 \cap \{x_n\}_{n \in \mathbb{N}}$ has infinite elements.

Proceed by induction, we have a sequence of balls $(B_n)_{n \in \mathbb{N}}$.

Now we can select a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ where $x_{n_k} \in B_k \forall k \in \mathbb{N}$.

We have that $\forall i, j \in \mathbb{N}$, $d(x_{n_i}, x_{n_j}) < 2\epsilon_{\min(n_i, n_j)} \rightarrow 0$, therefore $(x_{n_k})_{k \in \mathbb{N}}$ is Cauchy.

Due to the completeness of K , we have that $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence, therefore K is sequentially compact.

(b) \Rightarrow (a): **Lemma:** If K is sequentially compact and \mathcal{F} is any open cover of K , then $\exists \epsilon > 0$ s.t. $\forall x \in K$, $\exists U \in \mathcal{F}$ s.t. $B(x, \epsilon) \subseteq U$. The largest of such ϵ is called the Lebesgue radius of \mathcal{F} . To prove this, assume otherwise. Then there exists a sequence of balls $(B(x_n, r_n))_{n \in \mathbb{N}}$ with $r_n \rightarrow 0$ that are not contained fully in any $U \in \mathcal{F}$.

By sequential compactness we have that $x_n \rightarrow x \in K$. Take $U \in \mathcal{F}$ s.t. $x \in U$. Then $\exists r > 0$ s.t. $B(x, r) \subseteq U$. But since $x_{n_k} \rightarrow x$ and $r_{n_k} \rightarrow 0$, for some large enough k this contradicts B_{n_k} .

being not contained in U , thus the Lebesgue radius exists.

Now, let \mathcal{F} be an open cover of K , and $\epsilon > 0$ be its Lebesgue radius. Since sequentially compact \Rightarrow totally bounded, let B_1, \dots, B_n be an ϵ -cover of K and pick $U_1, \dots, U_n \in \mathcal{F}$ s.t. $B_k \subseteq U_k \forall 1 \leq k \leq n$.

Then U_1, \dots, U_n is a finite subcover of K from \mathcal{F} .

Proposition: In a metric space, a closed subset of a compact set is compact.

Proof: We have compact \rightarrow sequentially bounded. Let K be compact, $K' \subseteq K$ be closed.

Let $(x_n)_{n \in \mathbb{N}} \subseteq K' \subseteq K$, then $\exists x \in K$ s.t. $x_{n_k} \rightarrow x$.

Since K' closed, $x \in \bar{K}' = K'$, therefore K is sequentially compact \Rightarrow compact.

Note that this is generally not true in arbitrary topological spaces.

Definition: A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has the finite intersection property (FIP) if any finite intersection $\bigcap_{k=1}^n A_k \neq \emptyset$ for any $A_1, \dots, A_n \in \mathcal{F}$.

Proposition: A subspace $K \subseteq X$ of a topological space is compact \Leftrightarrow every family \mathcal{F} of closed subsets of K with FIP has nontrivial intersection.

Proof: If K is compact, and the intersection of the closed sets in \mathcal{F} is empty, then the union of their complements would contain the entire K and therefore form an open cover. However, since K is compact, that open cover would have a finite subcover, which, if we take another complement, would become a finite subset of \mathcal{F} with an empty intersection, contradicting FIP. If every family of closed subsets that has FIP has nontrivial intersection, and K is not compact, let $\{U_i\}_{i \in I}$ be an open cover of K with no finite subcover, and let $\mathcal{F} = \{U_i^c\}_{i \in I}$. We have that \mathcal{F} has FIP, therefore the intersection is non-empty. However, if we take another complement, we would have that the union of $\{U_i\}_{i \in I}$ is not the entire K , contradicting that this is an open cover.

Proposition: (Heine-Borel Theorem) $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow K$ is closed and bounded

Proof: By Bolzano-Weierstrass, for any sequence $(x_k)_{k \in \mathbb{N}} \subseteq K$ where $x_k = (x_k^{(1)}, \dots, x_k^{(n)})$, $\exists x \in \mathbb{R}^n$ s.t. $x_{k_j} \rightarrow x$.

Since K is closed, $x \in \bar{K} = K$, so K is sequentially compact \Rightarrow compact.

If K is compact, then K is contained in a finite union of bounded balls (**this needs to be checked!!!**).

For example, the sphere in \mathbb{R}^n : $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1\} \subseteq \mathbb{R}^n$ is compact.

Remark: In general, compact \Rightarrow closed and bounded, but the converse almost always fails.

Proposition: Let $f : X \rightarrow Y$ be a continuous function of topological spaces. If X is compact, then $f(X)$ is compact.

Proof: Take any open cover $\{U_i\}_{i \in I}$ of $f(X)$ in Y . Then, $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of X . This has a finite subcover $\{f^{-1}(U_n)\}_{n \in \mathbb{N}}$ (Since X is compact).

Since $X = \bigcup_{n \in \mathbb{N}} f^{-1}(U_n)$, we have $f(X) = \bigcup_{n \in \mathbb{N}} U_n$.

Therefore $f(X)$ is compact.

Proposition: (Extreme Value Theorem) For X compact topological space, any continuous function $f : X \rightarrow \mathbb{R}$ attains a global maximum/minimum on X .

Proof: We have $f(X)$ compact. By Heines-Borel and that $f(X) \subseteq \mathbb{R}$, we have $f(X)$ is closed and bounded. So $\sup f(X) < \infty$ and $\inf f(X) > -\infty$. And since $f(X)$ is closed, $\sup f(X)$ and $\inf f(X)$ must be in $f(X)$ (and therefore attained).

Since the n -sphere S^n is compact, any continuous $f : S^n \rightarrow \mathbb{R}$ attains a global maximum/minimum.

Proposition: The following are equivalent for metric spaces X, Y , some fixed $x \in X$, and $f : X \rightarrow Y$:

- (a) If $x_n \rightarrow x$ for some $(x_n)_{n \in \mathbb{N}} \subseteq X$, then $(f(x_n))_{n \in \mathbb{N}} \subseteq Y$ is such that $f(x_n) \rightarrow f(x)$.
- (b) f is topologically continuous at x .
- (c) For all $\epsilon > 0$, $\exists \delta > 0$ depend on x s.t. for some $y \in X$, if $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

Proof: (a) \Rightarrow (b): Suppose f is not topologically continuous at x_0 . Then, there exists a neighborhood V of $f(x_0)$ s.t. $f^{-1}(V)$ not a neighborhood of x_0 .

We know that $B(f(x_0), \epsilon) \subseteq V$ for some $\epsilon > 0$. This means we can select a sequence $x_n \in B(x_0, \frac{1}{n}) \setminus f^{-1}(V)$ (note that we can always select an element, otherwise V becomes a neighborhood of x_0) so that $x_n \rightarrow x_0$ but $d(f(x_n), f(x_0)) \geq \epsilon \forall n \in \mathbb{N}$. This contradicts (a).

(b) \Rightarrow (c): Let $V = B(f(x_0), \epsilon)$. So $\exists \delta > 0$ s.t. $B(x_0, \delta) \subseteq f^{-1}(V)$ (since by (b) $f^{-1}(V)$ is a neighborhood of x_0). Thus, $d(x, x_0) < \delta \Rightarrow x \in f^{-1}(V) \Rightarrow f(x) \in V \Rightarrow d(f(x_0), f(x)) < \epsilon$.

(c) \Rightarrow (a): Let $\epsilon > 0$. By (c) we have that $\exists \delta > 0$ dependent on x s.t. if $d(x_n, x) < \delta$ then $d(f(x_n), f(x)) < \epsilon$. Since $x_n \rightarrow x$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < \delta \forall n \geq N$. Thus $d(f(x_n), f(x)) < \epsilon \forall n \geq N$. Therefore, $f(x_n) \rightarrow f(x)$.

Definition: A function of metric spaces $f : X \rightarrow Y$ is called uniformly continuous if for all $\epsilon > 0$, $\exists \delta > 0$ s.t. for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Proposition: The following are equivalent for X totally bounded and a map $f : X \rightarrow Y$ between metric spaces:

- (a) f is uniformly continuous.
- (b) If $(x_n)_{n \in \mathbb{N}}$ is Cauchy, then $(f(x_n))_{n \in \mathbb{N}}$ is Cauchy.

Proof: (a) \Rightarrow (b): Since f uniformly continuous, we have that $\forall\epsilon > 0$, $\exists\delta > 0$ s.t. $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon \forall x, y \in X$.
 Since $(x_n)_{n \in \mathbb{N}}$ Cauchy, we have that $d(x_n, x_m) < \delta$ for large enough n, m . Therefore, $d(f(x_n), f(x_m)) < \epsilon$ for large enough n, m . Thus, $(f(x_n))_{n \in \mathbb{N}}$ is Cauchy.
 (b) \Rightarrow (a): Suppose that f is not uniformly continuous, therefore $\exists\epsilon > 0$ and $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ s.t. $d(x_n, y_n) \rightarrow 0$ yet $d(f(x_n), f(y_n)) \geq \epsilon \forall n \in \mathbb{N}$.
 Since X is totally bounded, we can find a Cauchy subsequence $(x_{n_k})_{k \in \mathbb{N}}$ and since $d(x_{n_k}, y_{n_k}) \rightarrow 0$, $(y_{n_k})_{k \in \mathbb{N}}$ is also Cauchy.
 Therefore we construct a new sequence $(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, \dots)$. we note that this sequence is Cauchy since (x_{n_k}) and (y_{n_k}) both Cauchy and $d(x_{n_k}, y_{n_k}) \rightarrow 0$, but $d(f(x_{n_k}), f(y_{n_k})) \geq \epsilon \forall k \in \mathbb{N}$, which is a contradiction to (b).

Definition: Given a metric space X , its completion \bar{X} is the set of Cauchy sequences $(x_n)_{n \in \mathbb{N}}$ in X , modulo the equivalence relation: $(x_n)_n \sim (y_n)_n$ if and only if $\forall\epsilon > 0$, $d(x_n, y_n) < \epsilon$ for large enough n, m . Then, the metric on \bar{X} can be given by $d((x_n)_n, (y_n)_n) = \lim_{n \rightarrow \infty} d(x_n, y_n)$. This limit exists because $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$ for large enough n, m and that \mathbb{R} is complete.

Then X is a subspace of \bar{X} , with $x \in X \Rightarrow \bar{x} \in \bar{X}$, where \bar{x} is the equivalence class of sequences converging to $x \in X$.

Proposition: The completion \bar{X} of any metric space X is complete.

Proof: Take sequences $X_1 = (x_n^{(1)})_{n \in \mathbb{N}}, X_2 = (x_n^{(2)})_{n \in \mathbb{N}}, \dots$ in X , and suppose they are Cauchy in \bar{X} . That is, $\lim_{n \rightarrow \infty} d(x_n^{(m)}, x_n^{(m')}) < \epsilon$ for m, m' large enough. For each $k \in \mathbb{N}$, select N_k s.t. $d(x_j^{(k)}, x_l^{(k)}) < \frac{1}{k}$ for $j, l \geq N_k$, select $y_k = x_p^k$ for $p \geq N_k$.

Let $Y = (y_k)_{k \in \mathbb{N}}$. We will define $Y_k = (y_k, y_k, \dots)$. First, we note that $d(X_k, Y_k) \leq \frac{1}{k} \forall k \in \mathbb{N}$ since $d(X_k, Y_k) = \lim_{n \rightarrow \infty} d(x_n^{(k)}, y_n^{(k)})$.

Next we claim that Y is Cauchy. By triangle inequality, we have $d(y_j, y_l) \leq d(y_j, x_p^{(j)}) + d(x_p^{(j)}, x_p^{(l)}) + d(x_p^{(l)}, y_l)$. We can select M s.t. for $j, l \geq M$, the first and third terms are $< \epsilon$ for large enough p . Moreover, since $(x_k)_{k \in \mathbb{N}}$ is Cauchy, the second term is $< \epsilon$ for large enough p . So $(y_k)_{k \in \mathbb{N}}$ is Cauchy.

Now, since $d(X_k, Y_k) \rightarrow 0$ and $Y_k \rightarrow Y$, we have that $X_k \rightarrow Y \in \bar{X}$.

Given $X \subseteq \bar{X}$, the embedding of X in \bar{X} is unique.

For example, the completion of \mathbb{Q} is identical to \mathbb{R} as a metric space.

Theorem: (Corollary) For some $f : X \rightarrow Y$ with metric spaces X, Y , if X is totally bounded, Y is complete, and f is uniformly continuous, then f extends uniquely to $f : \bar{X} \rightarrow Y$ where $f((x_n)_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} f(x_n)$.

Proposition: If $f : X \rightarrow Y$ continuous and X is compact, then f is uniformly continuous.

Proof: If $(x_n)_{n \in \mathbb{N}}$ is Cauchy in X then $x_n \rightarrow x$ for some $x \in X$, then $f(x_n) \rightarrow f(x)$. That is, $(f(x_n))_{n \rightarrow \infty}$ is Cauchy.

For example, any uniformly continuous map $f : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$ extends continuously to a uniform continuous map $f : [0, 1] \rightarrow \mathbb{R}$.

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Definition: For $f : \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$, the following are equal: (a) There exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \neq x_0$, $|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

(b) For all sequences $(x_n)_{n \in \mathbb{N}}$, $x_n \neq x_0 \forall n \in \mathbb{N}$, if $x_n \rightarrow x_0$, then $f(x_n) \rightarrow L$.

If either holds, we say that the limit of $f(x)$ equals L as x approaches x_0 , and write $\lim_{x \rightarrow x_0} f(x) = L$.

Proposition: (a) and (b) are equivalent.

Proof: (a) \Rightarrow (b): Suppose (b) is false, then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \rightarrow x_0$, but $f(x_n) \geq L + \epsilon$ for infinitely many n . This implies that there is no $\delta > 0$ for this ϵ s.t. $|f(x) - L| < \epsilon$ for all x near x_0 .

(b) \Rightarrow (a): If (a) false, there exists $\epsilon > 0$ s.t. $\forall \delta_n = \frac{1}{n}$, $\exists x_n \in (x_0 - \delta_n, x_0 + \delta_n)$ s.t. $|f(x_n) - L| \geq \epsilon$. But then clearly $f(x_n)$ does not $\rightarrow L$.

Remark: If one restrict to $x < x_0$ or $x > x_0$ in (a), we get the definition of left/right limits $\lim_{x \rightarrow x_0^-} f(x)$ and $\lim_{x \rightarrow x_0^+} f(x)$.

Remark: If one takes $\limsup_{n \rightarrow \infty} f(x_n)$ or $\liminf_{n \rightarrow \infty} f(x_n)$ in (b) (these always exist), then we get the definition of $\limsup_{x \rightarrow x_0} f(x)$ and $\liminf_{x \rightarrow x_0} f(x)$

Definition: The oscillation of $f : \mathbb{R} \rightarrow \mathbb{R}$ at $x = c$ is $\text{osc}(f)(c) = \lim_{\epsilon \rightarrow 0} \sup_{x, y \in (c - \epsilon, c + \epsilon)} |f(x) - f(y)|$.

For example, $\text{osc}(\sin(\frac{1}{x}))(0) = 2$.

Remark: The actual value of the function at the point is included in the oscillation, but not in the limit definition.

Proposition: f is continuous at $x \Leftrightarrow \text{osc}(f)(x) = 0$.

Proof: Unravel definition of $\text{osc}(f)$.

We get that $\text{osc}(f)(x) = 0 \Leftrightarrow$ for all $\epsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \Leftrightarrow f$ is continuous at x .

Definition: We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}$ if $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x \rightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists.
 $f'(x_0)$ is the derivative of $f(x)$ at x_0 .

Proposition: If f is differentiable at x_0 , then f is continuous at x_0 .

Proof: If f is differentiable at x_0 , we have that for $L = f'(x_0)$, $|\frac{f(x_0+h)-f(x_0)}{h} - L| < \epsilon$ for $|h| < \delta$.

Multiply by $|h|$ on both sides, we have $|f(x_0+h)-f(x_0)| - |Lh| \leq |f(x_0+h)-f(x_0)-Lh| < \epsilon|h|$. Therefore, $|f(x_0+h)-f(x_0)| \leq |L||h| + \epsilon|h| = (|L| + \epsilon)|h|$. As $h \rightarrow 0$, the right hand side also $\rightarrow 0$, therefore $\lim_{h \rightarrow 0} f(x_0+h) = f(x_0) \Leftrightarrow f$ is continuous at x_0 .

Proposition: Suppose $f, g : \mathbb{R} \rightarrow \mathbb{R}$ differentiable. Then, (a) $f + g$ differentiable, $(f + g)' = f' + g'$, $(cf)' = cf'$.

(b) fg is differentiable, $(fg)' = f'g + fg'$.

(c) $\frac{f}{g}$ is differentiable when $g(x) \neq 0$ with $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$.

Proof: (a) Exercise

$$(b) \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h \rightarrow 0} \frac{(f(x+h)-f(x))g(x+h)+(g(x+h)-g(x))f(x)}{h}.$$

Which is equal to $f'(x) \lim_{h \rightarrow 0} g(x+h) + g'(x)f(x) = f'(x)g(x) + f(x)g'(x)$.

(c) Apply (b) to f and $\frac{1}{g}$.

Proposition: (Chain Rule) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$, f differentiable at $g(c) = d$ and g is differentiable at c .

Then the composite $f \circ g$ would be differentiable at c and $(f \circ g)'(c) = f'(g(c))g'(c)$.

Proof: $g(c) = d$, $g(c+h) = d + h'$. Since g is continuous at c , $h \rightarrow 0 \Rightarrow h' \rightarrow 0$.

$$(f \circ g)'(c) = \lim_{h \rightarrow 0} \frac{f(g(c+h))-f(g(c))}{h} = \lim_{h \rightarrow 0} \frac{f(d+h')-f(d)}{h'} \cdot \frac{(d+h')-d}{h}.$$

Which is equal to $\lim_{h \rightarrow 0} \frac{f(d+h')-f(d)}{h'} \lim_{h \rightarrow 0} \frac{g(c+h)-g(c)}{h} = f'(d)g'(c) = f'(g(c))g'(c)$.

Definition: We say that $f : [a, b] \rightarrow \mathbb{R}$ satisfies the intermediate value property (IVP) if for all $c \in [f(a), f(b)]$, $\exists x \in [a, b]$ s.t. $f(x) = c$.

Proposition: (Intermediate Value Theorem) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f satisfies the intermediate value theorem.

Proof: Without loss of generality, assume $f(a) < f(b)$. take $c \in (f(a), f(b))$. Set $x_0 = \sup\{x \in [a, b] : f(x) < c, y \in [a, x]\}$.

Note that $x_0 \neq b$ by continuity of f at b and $f(b) > c$ ($\epsilon = \frac{f(b)-c}{2}$).

We claim that $f(x_0) = c$. Indeed, $\limsup_{x \rightarrow x_0^-} f(x) \leq c$ and $\limsup_{x \rightarrow x_0^+} f(x) \geq c$.

So $\limsup_{x \rightarrow x_0} f(x) = c$, so by continuity of f , $\limsup_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} f(x) = f(x_0) = c$.

Proposition: (Rolle's Theorem) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, differentiable on (a, b) , and $f(a) = f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f'(x_0) = 0$.

Proof: Unless f is constant, by Extreme Value Theorem f attains (without loss of generality) a global max at some $x = x_0$ on (a, b) .

Then, $f(x_0 + h) \leq f(x_0)$ for $h > 0$, and $f(x_0 - h) \geq f(x_0)$ for $h > 0$.

So, $\lim_{h \rightarrow 0^+} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$ and $\lim_{h \rightarrow 0^+} \frac{f(x_0-h)-f(x_0)}{h} \geq 0$.

So $\lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = 0 = f'(x_0)$.

Proposition: (Mean Value Theorem) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then $\exists c \in (a, b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Proof: Apply Rolle's Theorem to $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. We have that $g(a) = g(b) = f(a)$.

So, $\exists c \in (a, b)$ s.t. $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}(a-a) = 0 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proposition: (Darboux's Theorem) If $f : (a, b) \rightarrow \mathbb{R}$ is differentiable, then f' satisfies the Intermediate Value Theorem.

Proof: For $[c, d] \subseteq (a, b)$. Without loss of generality, assume $f'(c) < f'(d)$ with $r \in (f'(c), f'(d))$.

Define $g(x) = f(x) - rx$ which is differentiable. $g'(x) = f'(x) - r$, so $g'(c) < 0$ and $g'(d) > 0$.

This implies that the global min of g is attained in the interior $(c, d) \subseteq [c, d]$.

Call x_0 the global min of g in (c, d) . By the argument of Rolle's theorem, $g'(x_0) = 0 \Rightarrow f'(x_0) = r \Rightarrow f'$ has IVP.

Remark: The derivative of a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be wild.

For example, there are functions whose derivative exists everywhere, but is discontinuous at uncountably many points.

Proposition: (1D Inverse Function Theorem) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable with $f(c) = d$ and $f'(c) \neq 0$.

Then, \exists open intervals I, J s.t. $c \in I, d \in J$ s.t. $f : I \rightarrow J$ is a C^1 -diffeomorphism, i.e. $f : I \rightarrow J$ is bijective, and $f^{-1} : J \rightarrow I$ is continuously differentiable.

Moreover, $(f^{-1})'(d) = \frac{1}{f'(c)}$.

Proof: First we WTS injectivity. If f is not injective on any interval around c , we can construct $(x_n)_n, (y_n)_n$ inductively such that $x_n \rightarrow c, y_n \rightarrow c$ and $f(x_n) = f(y_n)$.

By MVT, $\frac{f(x_n) - f(y_n)}{x_n - y_n} = f'(z_n)$ for $z_n \in (x_n, y_n)$, so $z_n \rightarrow c$.

Therefore by continuity of f' , $f'(z_n) \rightarrow f'(c) = 0$, leading to a contradiction.

So $f : I \rightarrow f(I)$ bijective, where $c \in I$.

Let $J = f(I)$. J is an interval since I is interval and f continuous.

It remains to show $g = f^{-1}$ is continuously differentiable.

$g(d + h') = c + h$, where $h \rightarrow 0 \Leftrightarrow h' \rightarrow 0$.

$$g'(d) = (f^{-1})'(d) = \lim_{h' \rightarrow 0} \frac{g(d + h') - g(d)}{(d + h') - d} = \lim_{h' \rightarrow 0} \frac{c + h - c}{f(d + h) - f(d)} = \frac{1}{\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h}} = \frac{1}{f'(c)}.$$

We haven't shown that g is continuously differentiable. We will do this in the next lecture.

Jan. 14

Proposition: (1D Inverse Function Theorem, continued)

Proof: We still want to show that g is continuously differentiable.

By shrinking I if necessary and invoking the continuity of f (If $f'(c) \neq 0$ then there exists some interval around c s.t. $f(x) \neq 0$ on the interval), we may assume that $f(x) \neq 0 \forall x \in I$.

Now, let x be an arbitrary element in I , and let $y = f(x) \in J$. We have that $f(x + h) = y + h'$, with $h \rightarrow 0 \Leftrightarrow h' \rightarrow 0$.

$$\text{So, } g'(y) = \lim_{h' \rightarrow 0} \frac{g(y + h') - g(y)}{y + h' - y} = \lim_{h \rightarrow 0} \frac{x + h - x}{f(x + h) - f(x)} = \lim_{h \rightarrow 0} \frac{\frac{1}{f(x + h) - f(x)}}{\frac{h}{h}} = \frac{1}{f'(x)}.$$

Since g' is the reciprocal of a continuous, non-zero function f' , we have that g' is also continuous.

Proposition: (L'Hopital's Rule) Let $f, g : (a, b) \rightarrow \mathbb{R}$ be two continuous functions.

Suppose that f, g are differentiable on (a, b) , except possibly at c .

Suppose that $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$, and that $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ exists.

Also suppose that $g'(x) \neq 0$ on an open punctured interval $I \setminus \{c\}$ around c .

Then, $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ exists and $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$.

Proof: First, notice that g must also be non-zero on some punctured interval $I \setminus \{c\}$, otherwise we can construct a sequence $(x_n)_{n \in \mathbb{N}}$, where without loss of generality $x_n < c, x_n \rightarrow c$ and $g(x_n) = 0$. Then by MVT, $\frac{g(x_{n+1}) - g(x_n)}{x_{n+1} - x_n} = g'(z_n) = 0$, where $z_n \in (x_n, x_{n+1})$. So we have that $z_n \rightarrow c$, which contradicts the assumption that $g'(x)$ is non-zero near c .

Lemma (Generalized Mean Value Theorem) Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions for $a < b$, that are differentiable on (a, b) . Assume that $g'(x) \neq 0$ on (a, b) and $g(a) \neq g(b)$. Then,

there exists $c \in (a, b)$ s.t. $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof Homework Exercise.

Using generalized MVT (and recall that $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$), we have:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x \rightarrow c} \frac{f'(d)}{g'(d)} \text{ for some } d \in (c, x).$$

$$\text{As } x \rightarrow c, \text{ it is immediate that } d \rightarrow c, \text{ therefore } \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

Remark: The assumption $g(x) \neq 0$ near c is crucial and often missed, as there is the counterexample $f(x) = x + \sin x \cos x$, $g(x) = (x + \sin x \cos x)e^{\sin x}$ as $x \rightarrow \infty$, where $g'(x)$ vanishes periodically.

Remark: A similar method works for $\frac{\infty}{\infty}$ by considering $\frac{1}{f}, \frac{1}{g}$. Can generalize to other indeterminate forms.

Definition: A partition P of $[a, b]$ is a finite collection of points $a = x_0 < x_1 < x_2 < \dots < x_{n+1} = b$. We write $\Delta x_k = x_{k+1} - x_k$.

The mesh of a partition P is $\text{mesh}P = \max_{0 \leq k \leq n} |\Delta x_k|$.

A refinement P' of a partition P is a partition that includes all points of P and maybe other points.

We write $P \subseteq P'$.

The common refinement of partitions P, Q of $[a, b]$ is $P \cup Q$ as a set.

Remark: Clearly, $\text{mesh}(P \cup Q) \leq \min\{\text{mesh}P, \text{mesh}Q\}$.

Definition: Given $f : [a, b] \rightarrow \mathbb{R}$ be bounded and let P be a partition of $[a, b]$.

We define the lower sum $L(f, P)$ and upper sum $U(f, P)$ as follows:

$$U(f, P) = \sum_{k=0}^n (\sup_{[x_k, x_{k+1}]} f(x)) \Delta x_k, \quad L(f, P) = \sum_{k=0}^n (\inf_{[x_k, x_{k+1}]} f(x)) \Delta x_k.$$

Proposition: If $P \subseteq P'$, then $L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$.

Proof: We label the inequalities as (1), (2) and (3) respectively.

(2) follows immediately from $\inf_{[x_k, x_{k+1}]} f(x) \leq \sup_{[x_k, x_{k+1}]} f(x)$.

(1) and (3) follow from that if $[y_j, y_{j+1}] \subseteq [x_k, x_{k+1}]$, then $\inf_{[y_j, y_{j+1}]} f(x) \geq \inf_{[x_k, x_{k+1}]} f(x)$ and $\sup_{[y_j, y_{j+1}]} f(x) \leq \sup_{[x_k, x_{k+1}]} f(x)$.

Therefore, $L(f, P) = \sum_{k=0}^n (\inf_{[x_k, x_{k+1}]} f(x)) \Delta x_k \leq \sum_{j=0}^m (\inf_{[y_j, y_{j+1}]} f(x)) \Delta y_j = L(f, P')$.

The $U(f, P) \geq U(f, P')$ case follows analogously.

Definition: The lower/upper Darboux integrals of a bounded $f : [a, b] \rightarrow \mathbb{R}$ are:

$$\bar{I}(f) = \inf_P U(f, P), \underline{I}(f) = \sup_P L(f, P), \text{ where inf, sup are taken over all partitions of } [a, b].$$

We say that $f : [a, b] \rightarrow \mathbb{R}$ is Darboux integrable if the upper and lower integrals of f are equal, in which case the Darboux integral is $\int_a^b f(x)dx = \bar{I}(f) = \underline{I}(f) = I(f)$.

Proposition: (Lebesgue Criterion for Darboux Integrability) A bounded $f : [a, b] \rightarrow \mathbb{R}$ is Darboux Integrable if and only if $\forall \epsilon > 0, \exists$ partition P of $[a, b]$ s.t. $U(f, P) - L(f, P) < \epsilon$.

Proof: If f is Darboux integrable, $\sup_P L(f, P) = \inf_P U(f, P)$, then there exists partitions P, P' where $|U(f, P) - L(f, P')| < \epsilon$.

Take the common refinement $P \cup P'$, we have $U(f, P \cup P') - L(f, P \cup P') \leq U(f, P) - L(f, P') < \epsilon$.

Conversely, if $U(f, P) - L(f, P) < \epsilon$ for some partition P depending on ϵ for every $\epsilon > 0$, clearly $\sup L(f, P) = \inf U(f, P) \Leftrightarrow \bar{I}(f) = \underline{I}(f)$.

Proposition: A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is Darboux integrable if and only if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. all partitions P of $[a, b]$ s.t. $\text{mesh}P < \delta$ have $U(f, P) - L(f, P) < \epsilon$.

Proof: If the mesh condition holds, clearly $\sup_P L(f, P) = \inf_P U(f, P)$, the above argument immediately follows.

Conversely, suppose f is Darboux integrable.

Lemma If P, Q are two partitions of $[a, b]$ s.t. $\text{mesh}Q$ is less than or equal to the length of any subinterval in P , then $U(f, Q) - L(f, Q) \leq 3(U(f, P) - L(f, P))$.

Proof By the assumption, the sum of lengths of any intervals in Q covering any fixed interval in P is at most $I + 2I = 3I$, that is, $\sum_{j \in J_i} \Delta y_j \leq 3\Delta x_i$.

Moreover, by the triangle inequality, the sum - $\sum \sup f - \inf f$ over all intervals I_j in P covering some fixed interval J with index j in Q - bounds $\sup_J f - \inf_J f$ from above, that is, $\sup_j f - \inf_j f \leq \sum_{i \in I_j} \sup_i f - \inf_i f$.

Since summing over intervals I_j in P (with index i) covering each fixed interval J with index j for all J in Q is the same as summing over all intervals J_i in Q (with index j) covering each fixed interval I with index i for all $I \in P$, on thus gets $U(f, Q) - L(f, Q) = \sum_j (\sup_j f - \inf_j f) \Delta y_j \leq \sum_j \sum_{i \in I_j} (\sup_i f - \inf_i f) \Delta y_j = \sum_i \sum_{j \in J_i} (\sup_i f - \inf_i f) \Delta y_j = \sum_i (\sup_i f - \inf_i f) \sum_{j \in J_i} \Delta y_j \leq 3 \sum_i (\sup_i f - \inf_i f) \Delta x_i = 3(U(f, P) - L(f, P))$.

Now, if f is Darboux integrable and $U(f, Q) - L(f, Q) < \epsilon$, for any partition Q with $\text{mesh}Q$ less than the length of each subinterval in P (call the min of these lengths $\delta > 0$), we get that $\text{mesh}Q \Rightarrow U(f, Q) - L(f, Q) \leq 3(U(f, P) - L(f, P)) < 3\epsilon$.

Definition: A tagged partition P^* is a partition with a choice of point x_k^* in each subinterval.

Definition: A bounded $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if $\lim_{\text{mesh}P \rightarrow 0} \sum_{x_i \in P} f(x_i^*) \Delta x_i$ exists, in which case we say it is equal to $\int_a^b f(x) dx$, the Riemann integral of f on $[a, b]$.

Proposition: f is Riemann integrable $\Leftrightarrow f$ is Darboux integrable, and the value of the two integrals coincide.

Proof: If f is Riemann integrable, for any $\epsilon > 0$, we can take $\delta > 0$ s.t. for all partitions Q of $[a, b]$ s.t. $\text{mesh}Q < \delta$, and Q^*, Q^{**} being any two tagged partition from Q :

$$|\sum_{x_i^* \in Q^*} f(x_i^*) \Delta x_i - \sum_{x_i^{**} \in Q^{**}} f(x_i^{**}) \Delta x_i| < \epsilon.$$

Since we are free to choose the tagging, we may approximate $\sup f$ and $\inf f$ on the subintervals by tagging. That is, $f(x_i^*) \leq \inf f + \epsilon$ and $f(x_i^{**}) \geq \sup f - \epsilon$. This way we would have $|U(f, Q) - L(f, Q) - \sum_{x_i \in Q} (f(x_i^{**}) - f(x_i^*)) \Delta x_i| = |(U(f, Q) - \sum_{x_i \in Q} f(x_i^{**}) \Delta x_i) + (\sum_{x_i \in Q} f(x_i^*) \Delta x_i - L(f, Q))|$.

The two parts of this sum are, by definition of each individual $x_i^*, x_i^{**}, \leq \epsilon \sum_{x_i \in Q} \Delta x_i = \epsilon(b-a)$. So $|U(f, Q) - L(f, Q) - \sum_{x_i \in Q} (f(x_i^{**}) - f(x_i^*)) \Delta x_i| \leq 2\epsilon(b-a)$. And since the third term in this sum is, as derived, $< \epsilon$, we have that $U(f, Q) - L(f, Q) < \epsilon + 2\epsilon(b-a) = \epsilon(1 + 2(b-a))$. This can be arbitrarily small, therefore f is Darboux integrable.

Conversely, $f(x^*) - f(x^{**}) \leq \sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f$, with $x^*, x^{**} \in [x_i, x_{i+1}]$ being any two taggings.

Thus, we select a partition P of $[a, b]$ s.t. $\text{mesh}P$ is small enough so that $U(f, P) - L(f, P) < \epsilon$ (this is possible due to the mesh condition of Darboux integrability).

Therefore for any tagging P^*, P^{**} , we have that $|\sum_{x_i \in P} (f(x_i^*) - f(x_i^{**})) \Delta x_i| \leq U(f, P) - L(f, P) < \epsilon$.

Since this can be arbitrarily small, we have that f is Riemann integrable.

We also have that given f is both Riemann and Darboux integrable, the Riemann integral is "squeezed" by the upper and lower sums to the Darboux integral, therefore the two values coincide.

Jan. 16

Remark: Darboux integral can be thought of as approximating $f : [a, b] \rightarrow \mathbb{R}$ by "simple functions" of the form $\sum_{k=0}^n \chi_{I_k}(x)$, where given $A \subseteq \mathbb{R}$, the characteristic function:

$$\chi_A(x) = \begin{cases} 1 : x \in A \\ 0 : x \notin A \end{cases}, \text{ and } I_k \text{ being the subintervals in our partition.}$$

Namely, f is Darboux integrable $\Leftrightarrow \forall \epsilon > 0$, there exists simple functions h_1, h_2 s.t. $h_1 \leq f \leq h_2$ and $\int_a^b (h_2 - h_1) dx < \epsilon$.

Proposition: (a) If $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable, $c \in \mathbb{R}$, then $f + g$ and cf are integrable, with $\int_a^b cf(x) dx = c \int_a^b f(x) dx$ and $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$.

- (b) If $f, g : [a, b] \rightarrow \mathbb{R}$ integrable and $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
- (c) (Triangle Inequality) If $f : [a, b] \rightarrow \mathbb{R}$ are integrable, then $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.
- (d) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then it is integrable.
- (e) If $f : [a, c] \rightarrow \mathbb{R}$ is integrable and $a < b < c$, then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.
- (f) If $f, g : [a, b] \rightarrow \mathbb{R}$ is integrable, then fg is integrable.

Proof: (a) $\int_a^b cf(x)dx = c \int_a^b f(x)dx$ is left as exercise.

For $f + g$, use Darboux for integrability, and note that:

$$\sup(f + g) - \inf(f + g) \leq \sup f + \sup g - (\inf f + \inf g) = (\sup f - \inf f) + (\sup g - \inf g).$$

This easily implies that $U(f + g, P) - L(f + g, P) \leq U(f, P) - L(f, P) + U(g, P) - L(g, P) < 2\epsilon$, which can be arbitrarily small, implying that $f + g$ is integrable.

Also, since $\inf f + \inf g \leq \inf(f + g)$ and $\sup(f + g) \leq \sup f + \sup g$, we have that $L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P)$.

We have that $L(f, P)$ and $U(f, P)$, as well as $L(g, P)$ and $U(g, P)$, get arbitrarily close to each other, therefore by limiting behavior we have that $\bar{I}(f + g) = \underline{I}(f + g) = I(f) + I(g)$.

Which gives us $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

(b) Use Riemann: $\int_a^b f(x)dx = \lim_{\text{mesh}P \rightarrow 0} \sum_{x_i \in P} f(x_i^*) \Delta x_i \leq \lim_{\text{mesh}P \rightarrow 0} \sum_{x_i \in P} g(x_i^*) \Delta x_i = \int_a^b g(x)dx$.

(c) Use Riemann and regular triangle inequality:

$$|\sum_{x_i \in P} f(x_i^*) \Delta x_i| \leq \sum_{x_i \in P} |f(x_i^*)| \Delta x_i.$$

Take the limit $\text{mesh}P \rightarrow 0$, we have $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

To show that $|f|$ is integrable, use Darboux definition (left as exercise).

(d) Use Darboux: $U(f, P) - L(f, P) = \sum_{x_i \in P} (\sup_{[x_i, x_{i+1}]} f(x) - \inf_{[x_i, x_{i+1}]} f(x)) \Delta x_i$.

Since f is continuous on a compact set, it is uniformly continuous, so $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Therefore, $\sup_I f - \inf_I f \leq \epsilon$ for any interval I with length less than δ .

So if we take any partition P with $\text{mesh}P < \delta$, then $U(f, P) - L(f, P) \leq \epsilon \sum \Delta x_i = \epsilon(b - a)$.

This can be arbitrarily small, so f is integrable.

(e) This follows from the fact that a partition of $[a, c]$ induces partitions of $[a, b]$ and $[b, c]$.

The details are left as exercise.

(f) Left as exercise.

Definition: A function $f : [a, \infty)$ is improper Riemann integrable if it is integrable on any closed, bounded subinterval of $[a, \infty)$ and $\lim_{b \rightarrow \infty} \int_a^b f(x)dx$ exists, in which case $\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$.

Similarly, we can define improper integrals for functions with asymptotes at real values.

For example, $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx$, if this limit exists.

Proposition: (Integral MVT) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $\exists c \in [a, b]$ s.t. $\frac{1}{b-a} \int_a^b f(x)dx = f(c)$.

Proof: $\inf_{[a,b]} f \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \sup_{[a,b]} f$ since $\inf_{[a,b]} f \leq f(x) \leq \sup_{[a,b]} f$.

Since f is continuous, IVP holds for $f(c) = \frac{1}{b-a} \int_a^b f(x)dx$ for $c \in [a', b']$, where $f(a') = \inf_{[a,b]} f$ and $f(b') = \sup_{[a,b]} f$, therefore $[a', b'] \subseteq [a, b]$.

Proposition: (Integral Test) If $f : [1, \infty) \rightarrow (0, \infty)$ is a monotonically decreasing function (that is, for $x \leq y$, $f(x) \geq f(y)$), then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

Proof: Monotonically decreasing \Rightarrow integrable (left as exercise).

By lower and upper Riemann sums, we get $\sum_{n=2}^N f(n) \leq \int_1^N f(x)dx \leq \sum_{n=1}^N f(n)$.

Taking $n \rightarrow \infty$ yields the desired result.

Theorem: (Corollary) For $p > 0$, $\int_1^{\infty} \frac{1}{x^p} dx < \infty \Leftrightarrow p > 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proposition: (Lebesgue Criterion for Riemann Integrability) A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is integrable if and only if $\forall \epsilon > 0$, there exists a countable collection $(I_n)_{n \in \mathbb{N}}$ of open intervals that cover all the points of discontinuity of f and the sum of length of all the integrals is less than ϵ .

Proof: (\Leftarrow) Let us call the points of discontinuity of f "bad", and the points of continuity "good". We also call any interval that contains any "bad" points as "bad", otherwise it is a "good" interval.

Suppose that bad points can be covered by countably many open intervals of total length $< \epsilon$.

Recall that f is discontinuous at $x \Leftrightarrow \text{osc}(f)(x) > 0$.

Fix $\alpha > 0$, and note that $\{x : \text{osc}(f)(x) < \alpha\}$ is open.

Thus, the complement, $\{x : \text{osc}(f)(x) \geq \alpha\}$ is closed, and since it is also bounded, it is compact.

Notice that $(I_k)_{k \in \mathbb{N}}$ is an open cover of this set. By compactness, there exists a finite subcover $(I_{k_n})_n$ with total length $< \epsilon$.

On the good intervals, we get that f is continuous. By making these intervals closed, we get that f is a continuous function on a compact set (the finite union of closed intervals is closed) and therefore uniformly continuous.

Now, for any $\epsilon > 0$, pick the countable cover, and pick the finite subcover, and extend it to a partition by including the endpoints of the intervals. Then,

$$U(f, P) - L(f, P) = \sum (\sup f - \inf f) \Delta x_i = \sum_{\text{good}} (\sup f - \inf f) \Delta x_i + \sum_{\text{bad}} (\sup f - \inf f) \Delta x_i \leq \epsilon \sum_{\text{good}} \Delta x_i + M \sum_{\text{good}} \Delta x_i, \text{ where } M = \sup_{[a,b]} f - \inf_{[a,b]} f.$$

This is then $< \epsilon(b-a) + M\epsilon = \epsilon(M+b-a)$. Therefore f is integrable.

(Note) We really want an open cover of $X = \{x : \text{osc}(f)(x) > 0\}$ but have covers of $\{x :$

$\text{osc}(f)(x) > 0\}.$

We can take the union of covers for $\alpha_n = \frac{1}{n}$ for all n .

We also want X to be compact (?).

(\Rightarrow) Suppose, for the sake of contradiction, that the bad points cannot be covered by countably many integrals with total length arbitrarily small. We claim that for some $\epsilon > 0$, the set $\{x : \text{osc}(f)(x) > \epsilon\}$ cannot be covered by countable union of intervals of arbitrarily small length.

If not, taking union of covers for $\epsilon_n = \frac{1}{n}$ for all n with the total length of the n -th cover being $\frac{\delta}{2^n}$ for some $\delta > 0$.

$\{x : \text{osc}(f)(x) \geq \frac{1}{n}\} \subseteq \bigcup I_j$, of total length $< \frac{\delta}{n^2}$. Union of covers has length $\leq \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta$, leading to a contradiction.

Finally, this implies that for any partition P of $[a, b]$, the bad intervals have total length always at least ϵ' for some $\epsilon' > 0$.

Thus, $U(f, P) - L(f, P) \geq \sum_{\text{bad}} (\sup f - \inf f) \Delta x_i \geq \epsilon \sum_{\text{bad}} \Delta x_i \geq \epsilon \epsilon'$, here we define "bad" as containing a point of $\{x : \text{osc}(f)(x) > \epsilon\}$.

This implies that f is not integrable, which is a contradiction.

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Remark: In the proof of Lebesgue criterion for Riemann integrability, we should instead denote "good" intervals as those with $\text{osc}(f) \leq \alpha$, and "bad" intervals as those with $\text{osc}(f) > \alpha$.

Proposition: Let $f : [a, b] \rightarrow \mathbb{R}$ be integrable. Then $F(x) = \int_a^x f(t)dt$ is (uniformly) continuous.

Proof: $|F(x) - F(y)| = |\int_y^x f(t)dt| \leq \int_y^x |f(t)|dt \leq M|x - y| < \epsilon$ for $|x - y| < \frac{\epsilon}{M} = \delta$, where $M = \sup_{[a,b]} |f|$.

Proposition: (Fundamental Theorem of Calculus) If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)$. (Part I)

If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable, and $\exists F : [a, b] \rightarrow \mathbb{R}$ s.t. $F' = f$, then $\int_a^b f(t)dt = F(b) - F(a)$.

Proof: (Part I) $\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt = f(c) \rightarrow f(x)$ as $h \rightarrow 0$ for $c \in [x, x+h] \Rightarrow c \rightarrow 0$.

(Part II) $F(b) - F(a) = F(b) - F(x_n) + F(x_n) - \dots + F(x_1) - F(a) = \sum_i F(x_{i+1}) - F(x_i) = \sum_i f(x_i^*) \Delta x_i$. Push mesh(P) $\rightarrow 0$, this approaches $\int_a^b f(x)dx$. Therefore, $F(b) - F(a) = \int_a^b f(x)dx$.

Proposition: (Integration by Parts) If $f, g : [a, b] \rightarrow \mathbb{R}$ are continuously differentiable functions, then:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

Proof: Apply fundamental theorem of calculus to product rule.

Definition: Given a sequence of functions, $(f_n)_n : X \rightarrow Y$ and $f : X \rightarrow Y$, we say that $f_n \rightarrow f$ pointwise if $f_n(x) \rightarrow f(x) \forall x \in X$. We say that $f_n \rightarrow f$ uniformly if for all $\epsilon > 0$, there exists N s.t. $d(f_n(x), f(x)) < \epsilon \forall x \in X$ for $n \geq N$.

For example, we define $f_n : [0, 1] \rightarrow \mathbb{R}$ as $f_n(x) = x^n$. It is apparent that each f_n is continuous.

However, we have that f_n pointwise converges to $f(x) = \begin{cases} 0 : 0 \leq x < 1 \\ 1 : x = 1 \end{cases}$, which is not continuous.

Proposition: Given $(f_n) : X \rightarrow Y$ continuous, $f : X \rightarrow Y$ s.t. $f_n \rightarrow f$ uniformly, then f is continuous.

Proof: Fix $x \in X$, consider N large s.t. $d(f_n(y), f(y)) < \frac{\epsilon}{3}$ for all $y \in X$.

Then $d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ for $d(x, y) < \delta$.

Proposition: (Dini's Theorem) Let X be compact, and let continuous $f_n : X \rightarrow \mathbb{R}$ be a monotonically decreasing sequence. That is, $f_1 \geq f_2 \geq f_3 \geq \dots$, and suppose $f_n \rightarrow f$ pointwise for some $f : X \rightarrow \mathbb{R}$.

Then, if f is continuous, the convergence is uniform.

Proof: Fix $\epsilon > 0$. Then, since X is compact, f is (uniformly) continuous.

Then $\exists \delta > 0$ s.t. $f(x) - \epsilon < f(y)$ for $d(x, y) < \delta$.

Moreover, for fixed N , $\exists \delta_N > 0$ (since f_N uniformly continuous) s.t. $f_N(y) < f_N(x) + \epsilon$ for $d(x, y) < \delta_N$.

Then for each $x \in X$, $\exists N$ and $\delta' = \min\{\delta, \delta_N\} > 0$ s.t. for $n \geq N$ and $d(x, y) < \delta'$, one has:

$$f_n(y) - f(y) \leq f_N(y) - f(y) < f_N(x) + \epsilon - (f(x) - \epsilon) = f_N(x) - f(x) + 2\epsilon.$$

Taking the union of these balls for all $x \in X$ gives an open cover of X . Since X compact, \exists finite subcover $B(x_1, \delta_1), \dots, B(x_k, \delta_k)$.

Then, for any $y \in Y$, if we take $N = \max\{N_1, \dots, N_k\}$, $f_n(y) - f(y) < 3\epsilon$ for $n \geq N$. Then, since $\epsilon > 0$ is arbitrary and so is y , we converge uniformly.

Proposition: (Weierstrass M-Test) If $f_n : X \rightarrow \mathbb{R}$ is a sequence of continuous functions s.t. $|f_n(x)| \leq M_n$ for all $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

Proof: We want to show that $\sum_{n=1}^{\infty}$ is uniformly Cauchy, that is, $|\sum_{n=N}^M f_n(x)| < \epsilon \forall x \in X$ for M, N large.

Indeed, we have this is $\leq \sum_N^M M_n < \epsilon$ for M, N large.

Theorem: (Corollary) A summable series of continuous functions is continuous.

Definition: A collection \mathcal{F} of functions $f_n : X \rightarrow \mathbb{R}$ is (uniformly) equicontinuous if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $\forall f \in \mathcal{F}$ and $\forall x, y \in X$, $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.

Theorem: (Arzela-Ascoli Theorem) Let X be a compact metric space, and let \mathcal{F} be a uniformly equicontinuous collection of functions $f_n : X \rightarrow \mathbb{R}$. Then, if \mathcal{F} is pointwise bounded, that is, $\forall x \in X$, $\sup_{f \in \mathcal{F}} |f(x)| < \infty$, then \mathcal{F} is precompact in $C(X)$. That is, any sequence $(f_n)_n \subseteq \mathcal{F}$ has a uniformly convergent subsequence.