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CORRECTION TO *A NOTE ON THE ENTSCHEIDUNGSPROBLEM*

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In *A note on the Entscheidungsproblem*[1] the author gave a proof of the unsolvability of the general case of the Entscheidungsproblem of the engere Funktionenkalkül. This proof, however, contains an error,[2] in order to correct which it is necessary to modify the "additional axioms" of the system L so that they contain no free variables (either free individual variables or free propositional function variables).[3]

The additional axioms of L other than $x=y \rightarrow [F(x) \rightarrow F(y)]$ contain no free propositional function variables, and hence it is sufficient to replace each one by an expression obtained from it by quantifying all the individual variables by means of universal quantifiers initially placed—thus, in particular, $x=x$ is replaced by $(x)[x=x]$. Moreover the axiom $x=y \rightarrow [F(x) \rightarrow F(y)]$ may be replaced by the following set of axioms:

$$(x)(y)(z)[x = y \rightarrow [x = z \rightarrow y = z]],$$

$$(x)(y)[x = y \rightarrow s(x) = s(y)],$$

$$(x)(y)(z)[x = y \rightarrow a(x, z) = a(y, z)],$$

$$(x)(y)(z)[x = y \rightarrow a(z, x) = a(z, y)],$$

and similar axioms for each of the functions $b_1, b_2, \cdots, b_k$.[4]

The "additional axioms" of L′ will then be modified correspondingly, so that they also will contain no free variables. In particular, since the additional axioms of L′ must include,

$$(y)(Ex)S(x, y),$$

$$(x)(y)(z)[S(x, z) \ \& \ S(y, z) \rightarrow x = y],$$

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[1] *The journal of symbolic logic*, vol. 1 (1936), pp. 40–41.

[2] The author is indebted to Paul Bernays for pointing out this error and suggesting the method of correcting it, as also for calling attention to the desirability of distinguishing in this connection (as is done below) between proofs which are constructive (finite) and those which are not.

[3] When a formula containing free variables is asserted, these free variables may be thought of as having been bound by suppressed universal quantifiers. And on combining several such formulas (or negating such a formula) it may be necessary to restore the suppressed quantifiers in order to avoid confusions of scope. Thus, if U contains free variables, the proposition meant when U→R is asserted is not an implication between the proposition meant when U is asserted and that meant when R is asserted (this observation, and the consequent technique of restoring suppressed quantifiers in such cases, are, of course, a familiar matter to users of the functional calculus). It is in this, or, more strictly, in formal matters which parallel it, that the error lies in the present instance.

[4] In the presence of the axioms and rules of the engere Funktionenkalkül, these axioms suffice for the derivation of any expression which may be obtained from $x=y \rightarrow [F(x) \rightarrow F(y)]$ by replacing F by a propositional function, which is expressible in the notation of L, and which does not involve free propositional function variables. Cf. Hilbert and Bernays, *Grundlagen der Mathematik*, vol. 1 (Berlin 1934), pp. 373–375.

and similar axioms for each of the propositional functions A, B_1, B_2, $\cdots$, B_k, the axiom $x=y\rightarrow[F(x)\rightarrow F(y)]$ of L′ may be replaced by the following set:

$$(x)(y)(z)[x = y \rightarrow [x = z \rightarrow y = z]],$$

$$(x)(y)(z)[x = y \rightarrow [S(x, z) \rightarrow S(y, z)]],$$

$$(x)(y)(z)[x = y \rightarrow [S(z, x) \rightarrow S(z, y)]],$$

$$(x)(y)(z)(t)[x = y \rightarrow [A(x, z, t) \rightarrow A(y, z, t)]],$$

$$(x)(y)(z)(t)[x = y \rightarrow [A(z, x, t) \rightarrow A(z, y, t)]],$$

$$(x)(y)(z)(t)[x = y \rightarrow [A(z, t, x) \rightarrow A(z, t, y)]],$$

and similar axioms for each of the propositional functions B_1, B_2, $\cdots$, B_k.

With these modifications in the axioms of L and L′, the proof given in the author's cited paper is thought to be correct by accepted mathematical standards.

It is desirable, however, to distinguish between constructive and non-constructive proof, and for this purpose, in order to avoid all question over the inference from "not all" to "some not," the property of L proved by Bernays should be restated as follows:[5] *If* P *contains no quantifiers and* (Ex)P *is provable in* L *then some one of* P_1, P_2, P_3, $\cdots$ *is provable in* L (*where* P_1, P_2, P_3, $\cdots$ *are respectively the results of substituting for x the symbols for* 1, 2, 3, $\cdots$ *throughout* P).[6] Then the argument given provides a constructive proof of the unsolvability of what we may call the *deducibility problem* of the engere Funktionenkalkül, that is the problem to find an effective procedure which is capable of determining, about any given expression in the notation of the engere Funktionenkalkül, whether it is deducible in that system. The inference, however, to the unsolvability of the other form of the Entscheidungsproblem, which concerns a procedure for determining universal validity, depends on the non-constructively proved theorem of Gödel that every universally valid expression is deducible in the engere Funktionenkalkül, as well as on the assumption of the converse of this, that every deducible expression is universally valid. The unsolvability of this second form of the Entscheidungsproblem of the engere Funktionenkalkül cannot, therefore, be regarded as established beyond question.

For the system L′, however, the unsolvability of both forms of the Entscheidungsproblem follows constructively.

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[5] Bernays's proof is adequate to establish the property constructively in this positive form. See the mimeographed notes of his lectures at The Institute for Advanced Study, p. 122.

[6] Similarly the condition of ω-consistency, on the last page of *An unsolvable problem of elementary number theory* (***American journal of mathematics***, vol. 58 (1936), pp. 345–363), should be replaced by the condition: *Where E is the existential quantifier over the class of positive integers, if* (Ex)P *is provable then of some one of the formulas* P_1, P_2, P_3, $\cdots$ *it is true that the negation is not provable*. This condition we may call *strong ω-consistency*.