A SET OF POSTULATES FOR THE FOUNDATION OF LOGIC.[1]

(SECOND PAPER.)

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1. **Revision of the list of formal postulates.** In a recent paper[2] the author proposed a set of postulates which, it was believed, would lead to a system of mathematical logic free of some of the complications entailed by Bertrand Russell's theory of types, and would at the same time avoid the well known paradoxes, in particular the Russell paradox, by weakening the classical principle of reductio ad absurdum. But, in the course of working with these postulates, it has since become clear that the set as originally given requires some modification in order to render it free from contradiction.

Postulate 19 makes it possible to prove $\sim\Sigma(\{\mathbf{F}\}(\mathbf{U}))$ if there can be found some consequence of $\Sigma(\{\mathbf{F}\}(y))$ which is false of $\mathbf{U}$, subject only to the restriction that $\{\mathbf{F}\}(\mathbf{U}, x)$ shall be known to be false for some x. This is not far from assuming the principle of reductio ad absurdum in its full strength as applied to propositions of the form $\Sigma(\mathbf{A})$. But, given a proposition which is not of the form $\Sigma(\mathbf{A})$, it is ordinarily possible in various ways, to write another proposition of much the same import, which is of the form $\Sigma(\mathbf{A})$. And in this way we find that Postulate 19 contains enough of the principle of reductio ad absurdum to lead to a modified form of the Russell paradox, in outline as follows.

We take $\mathfrak{Y}$ to be $\lambda u \lambda v E(u) E(v)$ and prove, in a straightforward manner, $\sim\sim\{\mathfrak{Y}\}\Big(\varLambda\varphi . \sim\varphi(\{\mathfrak{Y}\}(\varphi)), \{\mathfrak{Y}\}\big(\{\mathfrak{Y}\}(\varLambda\varphi . \sim\varphi(\{\mathfrak{Y}\}(\varphi)))\big)\Big)$, where $\varLambda$ is defined as in § 5 below. Hence, by Rule III, $\sim\{\lambda\varphi . \sim\varphi(\{\mathfrak{Y}\}(\varphi))\}\big(\{\mathfrak{Y}\}(\varLambda\varphi . \sim\varphi(\{\mathfrak{Y}\}(\varphi)))\big)$. Hence, by Theorem 8 below, $\sim\{\varLambda\varphi . \sim\varphi(\{\mathfrak{Y}\}(\varphi))\}\big(\{\mathfrak{Y}\}(\varLambda\varphi . \sim\varphi(\{\mathfrak{Y}\}(\varphi)))\big)$. And hence we prove $\Sigma y \Sigma x . \sim x(y(x)) . x = \varLambda\varphi . \sim\varphi(y(\varphi))$. But, assuming $\Sigma x . \sim x(y(x)) . x = \varLambda\varphi . \sim\varphi(y(\varphi))$, we can prove $\Sigma r . r(r) . \sim r(y(r))$, because $\sim\{\varLambda\varphi . \sim\varphi(y(\varphi))\}\big(y(\varLambda\varphi . \sim\varphi(y(\varphi)))\big)$ is convertible into $\{\lambda\varphi . \sim\varphi(y(\varphi))\}(\varLambda\varphi . \sim\varphi(y(\varphi)))$, which, by Theorem 6 below, yields in turn $\{\varLambda\varphi . \sim\varphi(y(\varphi))\}(\varLambda\varphi . \sim\varphi(y(\varphi)))$. Therefore, by Theorem I, $\Sigma(\{\mathfrak{F}\}(y)) \supset_y \{\mathfrak{G}\}(y)$, where $\{\mathfrak{F}\}(y)$ stands for $\lambda x . \sim x(y(x)) . x = \varLambda\varphi . \sim\varphi(y(\varphi))$ and $\{\mathfrak{G}\}(y)$ stands for $\Sigma r . r(r) . \sim r(y(r))$. Now we

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[2] These Annals, vol. 33 (1932), pp. 346–366.

take $\mathfrak{U}$ to be λzz and, with the aid of Postulate 19, we prove $\sim\{\mathfrak{G}\}(\mathfrak{U})$. Therefore, by Postulate 19, $\sim\Sigma(\{\mathfrak{F}\}(\mathfrak{U}))$, that is, $\sim\Sigma x.\sim x(x).x = \mathit{\Delta}\varphi.\sim\varphi(\varphi)$. Then, by means of Theorem I, we prove $\sim\varphi(\varphi)\supset_\varphi\Sigma x.\sim x(x).x=\varphi$. Therefore, by Theorem 9 below, $\sim\{\mathit{\Delta}\varphi.\sim\varphi(\varphi)\}(\mathit{\Delta}\varphi.\sim\varphi(\varphi))$. Therefore, by Rule III, $\{\lambda\varphi.\sim\varphi(\varphi)\}(\mathit{\Delta}\varphi.\sim\varphi(\varphi))$. Therefore, by Theorem 6 below, $\{\mathit{\Delta}\varphi.\sim\varphi(\varphi)\}(\mathit{\Delta}\varphi.\sim\varphi(\varphi))$, a contradiction.

For this reason we shall omit Postulate 19 from our list. And with it we shall omit Postulates 20, 28, and 29, because these three postulates entirely lose their point after the omission of Postulate 19.

The chief effect of the omission of Postulate 19 is that we are no longer able to prove formulas of the form $\sim\Sigma(\mathbf{A})$,[3] We believe, however, that this restriction will not render our system inadequate, because the formula $\sim\{\mathit{\Delta}y\Sigma(\{\mathbf{F}\}(y))\}(\mathbf{U})$ can often be proved and used instead of $\sim\Sigma(\{\mathbf{F}\}(\mathbf{U}))$.

Let us call a formula *vacuous* when it has, or is convertible into, the form $\Pi(\mathbf{F},\mathbf{G})$, and there is no formula $\mathbf{M}$ such that $\{\mathbf{F}\}(\mathbf{M})$ is provable. The omission of Postulates 28 and 29 does away with our chief means of proving vacuous formulas. It remains possible to prove vacuous formulas of certain particular kinds by means of Postulate 25 or of Postulate 34, but this property of these two postulates is largely accidental, and we propose to eliminate it by means of appropriate slight changes in them. It then becomes impossible, we think, to prove any vacuous formula. And we recognize this property of our system by adding to it (tentatively) the postulate, $\Sigma\psi\Pi(\varphi,\psi)\supset_\varphi\Sigma(\varphi)$. This postulate is, probably, not indispensable, but it is sometimes convenient. Its addition to the list renders Postulate 32 non-independent, and the latter postulate may, therefore, be omitted.

And it is now also necessary to modify Postulates 17 and 18 to read, respectively, $\Sigma x[\varphi(x).\sim\psi(x)]\supset_{\varphi\psi}\sim\Pi(\varphi,\psi)$ and $\sim\Pi(\varphi,\psi)\supset_{\varphi\psi}\Sigma x.\varphi(x).\sim\psi(x)$. For otherwise it would be impossible to prove any formula $\sim\Pi(\mathbf{F},\mathbf{G})$ unless $\Sigma(\mathbf{G})$ were provable.

And finally, Postulate 2 fails to be independent, and may, therefore, be omitted from the list. For the proofs of cases 9 and 12 under Theorem I can be modified so that Postulate 7 takes the place of Postulate 2, and Theorem I can therefore be proved from Postulates 1, 3–11.[4]

[3] It is to be observed, however, that another symbol, meaning, "There exists", can be defined as follows:

$$\mathfrak{E}\rightarrow\lambda\varphi\lambda\psi\sim.\varphi(x)\supset_x\sim\psi(x).$$

It will then be found that $\mathfrak{E}(\mathbf{A},\mathbf{B})$ is provable whenever $\Sigma x.\{\mathbf{A}\}(x)\{\mathbf{B}\}(x)$ is provable and conversely. And, for particular formulas $\mathbf{A}$ and $\mathbf{B}$, it frequently is possible to prove $\sim\mathfrak{E}(\mathbf{A},\mathbf{B})$.

The expression $\mathfrak{E}(\mathbf{A},\mathbf{B})$ is to be read, "Some $\mathbf{A}$ is $\mathbf{B}$".

[4] This observation is due to Mr. J. B. Rosser.

The question whether our other postulates are all independent remains at present open to doubt, but there appears to be no reason why this question cannot be investigated by the usual method of constructing independence examples.

Our revised list of formal postulates is now as follows:

1. $\Sigma(\varphi) \supset_{\varphi} \Pi(\varphi, \varphi)$.
3. $\Sigma(\sigma) \supset_{\sigma} . [\sigma(x) \supset_{x} \varphi(x)] \supset_{\varphi} . \Pi(\varphi, \psi) \supset_{\psi} . \sigma(x) \supset_{x} \psi(x)$.
4. $\Sigma(\varrho) \supset_{\varrho} . \Sigma y [\varrho(x) \supset_{x} \varphi(x, y)] \supset_{\varphi} . [\varrho(x) \supset_{x} \Pi(\varphi(x), \psi(x))] \supset_{\psi} .$ $[\varrho(x) \supset_{x} \varphi(x, y)] \supset_{y} . \varrho(x) \supset_{x} \psi(x, y)$.
5. $\Sigma(\varphi) \supset_{\varphi} . \Pi(\varphi, \psi) \supset_{\psi} . \varphi(f(x)) \supset_{fx} \psi(f(x))$.
6. $\text{‘}x . \varphi(x) \supset_{\varphi} . \Pi(\varphi, \psi(x)) \supset_{\psi} \psi(x, x)$.
7. $\varphi(x, f(x)) \supset_{\varphi f x} . \Pi(\varphi(x), \psi(x)) \supset_{\psi} \psi(x, f(x))$.
8. $\Sigma(\varrho) \supset_{\varrho} . \Sigma y [\varrho(x) \supset_{x} \varphi(x, y)] \supset_{\varphi} . [\varrho(x) \supset_{x} \Pi(\varphi(x), \psi)] \supset_{\psi} .$ $[\varrho(x) \supset_{x} \varphi(x, y)] \supset_{y} \psi(y)$.
9. $\text{‘}x . \varphi(x) \supset_{\varphi} \Sigma(\varphi)$.
10. $\Sigma x \varphi(f(x)) \supset_{f\varphi} \Sigma(\varphi)$.
11. $\varphi(x, x) \supset_{\varphi x} \Sigma(\varphi(x))$.
12. $\Sigma(\varphi) \supset_{\varphi} \Sigma x \varphi(x)$.
13. $\Sigma(\varphi) \supset_{\varphi} . [\varphi(x) \supset_{x} \psi(x)] \supset_{\psi} \Pi(\varphi, \psi)$.
14. $p \supset_{p} . q \supset_{q} pq$.
15. $pq \supset_{qp} p$.
16. $pq \supset_{pq} q$.
17. $\Sigma x [\varphi(x) . \sim \psi(x)] \supset_{\varphi\psi} \sim \Pi(\varphi, \psi)$.
18. $\sim \Pi(\varphi, \psi) \supset_{\varphi\psi} \Sigma x . \varphi(x) . \sim \psi(x)$.
21. $p \supset_{p} . \sim q \supset_{q} \sim . pq$.
22. $\sim p \supset_{p} . q \supset_{q} \sim . pq$.
23. $\sim p \supset_{p} . \sim q \supset_{q} \sim . pq$.
24. $p \supset_{p} . [\sim . pq] \supset_{q} \sim q$.
25. $\Big[\sim [\varphi(u) \psi(u)] . [[\varphi(x) . \sim \psi(x)] \supset_{x} \varrho(x)] . [[\sim \varphi(x) . \psi(x)] \supset_{x} \varrho(x)] .$ $[\sim \varphi(x) . \sim \psi(x)] \supset_{x} \varrho(x)\Big] \supset_{\varphi\psi\varrho u} \varrho(u)$.
26. $p \supset_{p} \sim \sim p$.
27. $\sim \sim p \supset_{p} p$.
30. $\text{‘}x . \varphi(x) \supset_{\varphi} . [\theta(x) . \psi(x) \supset_{\psi} \Pi(\theta, \psi)] \supset_{\theta} \varphi(\iota(\theta))$.
31. $\varphi(\iota(\theta)) \supset_{\theta\varphi} \Pi(\theta, \varphi)$.
33. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_{z} \varphi(x, z)] [\varphi(x, y) \supset_{xy} \varphi(y, x)] \supset_{\varphi} . \varphi(u, v)$ $\supset_{uv} . \psi(A(\varphi, u)) \supset_{\psi} \psi(A(\varphi, v))$.

34. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z)] [\varphi(x, y) \supset_{xy} \varphi(y, x)] [\varphi(x, y) \supset_{xy} \theta(x, y)]$
$[\sim \theta(u, v)] \supset_{\varphi\theta uv} \sim . \psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v))$.

35. $[\psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v))] \supset_{\varphi uv} \varphi(u, v)$.

36. $E(A(\varphi)) \supset_\varphi . \varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z)$.

37. $E(A(\varphi)) \supset_\varphi . \varphi(x, y) \supset_{xy} \varphi(y, x)$.

38. $\Sigma \psi \Pi(\varphi, \psi) \supset_\varphi \Sigma(\varphi)$.

2. **Possibility of a proof of freedom from contradiction.** Our present project is to develop the consequences of the foregoing set of postulates, until a contradiction is obtained from them, or until the development has been carried so far consistently as to make it empirically probably that no contradiction can be obtained from them. And in this connection it is to be remembered that just such empirical evidence, although admittedly inconclusive, is the only existing evidence of the freedom from contradiction of any system of mathematical logic which has a claim to adequacy.

It is worth observing, however, that there may be a possibility of proving that there is no formula **A** such that both **A** and $\sim$.**A** are consequences of our postulates, or, failing this, of proving the same theorem about some related set of postulates. This is conceivable on account of the entirely formal character of the system which makes it possible to abstract from the meaning of the symbols and to regard the proving of theorems (of formal logic) as a game played with marks on paper according to a certain arbitrary set of rules. It is then proposed to use our intuitive logic (some portion of which we have already had to presuppose) to prove about this game, regarded objectively, that if it be played according to the rules it cannot lead to combinations of marks of certain particular kinds. Indeed the whole set of undefined terms of the formal logic, including enumerably many symbols available for use as variables, is enumerable, and hence the whole set of formulas and of possible proofs is also enumerable. And it is therefore even to be hoped that the portion of intuitive logic necessary to the proof of freedom from contradiction should not go beyond the logic of enumerable classes or employ any number system more elaborate than the system of positive integers.[5]

The impossibility of such a proof of freedom from contradiction for the system of *Principia Mathematica* has recently been established by Kurt Gödel.[6]

[5] The conception of a consistency proof of this kind is due to David Hilbert. See *Die logischen Grundlagen der Mathematik,* Math. Annalen, vol. 88 (1923), pp. 151–165, and *Die Grundlagen der Mathematik,* Abh. a. d. Math. Sem. d. Hamb. Univ., vol. 6 (1928), pp. 65–92.

[6] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I,* Monatshefte für Mathematik und Physik, vol. 38 (1931), pp. 173–198.

His argument, however, makes use of the relation of implication U between propositions in a way which would not be permissible under the system of this paper, and there is no obvious way of modifying the argument so as to make it apply to the system of this paper. It therefore remains, at least for the present, conceivable that there should be found a proof of freedom from contradiction for our system.

3. **Abbreviations of the statement of proofs.** In the proofs of theorems which follow, we indicate after each formula in the proof the means by which it is inferred, that is, we give the number of the rule of procedure and a reference to the premise or premises. Formal postulates are referred to by the number written alone, and theorems are referred to by the abbreviation "Thm" followed by the number.

We use Theorems I, II, and III, and the corollary of Theorem I as if they were rules of procedure. This is justified because our proof of these theorems was constructive in character, so that their use in this way can be regarded as a mere device for abbreviating the statement of a proof. For instance, whenever we have two particular formulas $\mathbf{M}$ and $\mathbf{N}$, each of which contains $\mathbf{x}$ as a free variable, and a particular proof of $\Sigma\mathbf{x} . \mathbf{M}$, and a particular scheme for proving $\mathbf{N}$ as a consequence of $\mathbf{M}$, then, by the method used in the proof of Theorem I, we can obtain a particular proof of $[\mathbf{M}] \supset_{\mathbf{x}} . \mathbf{N}$.

Moreover we allow uses of Theorems I, II, and III, and the corollary of Theorem I, in the role of rules of procedure, even in the course of proving $\mathbf{N}$ as a consequence of $\mathbf{M}$ preparatory to another application of one of the Theorems I, II, III, or Theorem I Corollary. For our proof of these theorems remains valid even after the formula $\mathbf{M}$ is added to our list as an additional formal postulate, provided only that the variable $\mathbf{x}$, which occurs in $\mathbf{M}$ as a free variable, is not used elsewhere as a bound variable.

In the proof of a theorem we give before each formula a symbol which will be used in referring to it, namely a letter $\mathbf{A}$, $\mathbf{C}$, or $\mathbf{D}$, with a subscript. The formulas marked $\mathbf{A}_i$ have been proved outright. Those marked $\mathbf{D}_i$ are assumed, in order to derive other formulas from them, preparatory to an application of Theorem I, or II, or III, or Theorem I Corollary. And those marked $\mathbf{C}_i$ are proved on the basis of the assumption of certain of the formulas $\mathbf{D}_i$, as indicated in each case.

We abbreviate further by omitting steps which involve only an application of one of the rules of procedure I, II, III, or by condensing several such steps into one, using "conv" to mean, "By conversion", or, "By applications of Rules I, II, and III".

We use IV^n to stand for n successive applications of Rule IV. And we use V^n to stand for n successive applications of Rule V, the result of each application of Rule V being used in turn as the major premise[7] for the next application of Rule V. After V^n, so used, we give a reference, first to the major premise of the first application of Rule V, and then to the successive minor premises[7] in order. But in passing from $\{\mathbf{F}\}(\mathbf{x}, \mathbf{y}) \supset_{\mathbf{xy}} \{\mathbf{G}\}(\mathbf{x}, \mathbf{y})$ and $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ to $\{\mathbf{G}\}(\mathbf{A}, \mathbf{B})$ or in passing from $\{\mathbf{F}\}(\mathbf{x}, \mathbf{y}) \supset_{\mathbf{xy}} \{\mathbf{G}\}(\mathbf{y})$ and $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ to $\{\mathbf{G}\}(\mathbf{B})$, we refer to $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ as the only minor premise, omitting reference to the other minor premise $\Sigma\, \mathbf{y}\, \{\mathbf{F}\}(\mathbf{A}, \mathbf{y})$. Similarly we shall sometimes refer to a formula $\mathbf{M}$ as minor premise when the true minor premise is not $\mathbf{M}$ but some formula obtained from $\mathbf{M}$ by use of Rule IV or conversion or both. And we shall sometimes refer to two minor premises, $\mathbf{P}$ and $\mathbf{Q}$, when the true minor premise is the logical product $[\mathbf{P}]\,[\mathbf{Q}]$, obtained from the $\mathbf{P}$ and $\mathbf{Q}$ by use of Postulate 14.

4. Identity. We adopt the following definitions of identity, or equality,[8] and of distinctness, or inequality:

$$= \longrightarrow \lambda\mu\, \lambda\nu \,.\, \psi(\mu) \supset_{\psi} \psi(\nu).$$
$$\neq \longrightarrow \lambda\mu\, \lambda\nu \sim .\, \psi(\mu) \supset_{\psi} \psi(\nu).$$

We shall abbreviate $\{=\}(\mathbf{A}, \mathbf{B})$ as $[\mathbf{A}] = [\mathbf{B}]$, and $\{\neq\}(\mathbf{A}, \mathbf{B})$ as $[\mathbf{A}] \neq [\mathbf{B}]$.

In terms of these definitions, Postulates 33, 34, 35 can be rewritten as follows:

33′. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z)]\; [\varphi(x, y) \supset_{xy} \varphi(y, x)] \supset_{\varphi} . \varphi(u, v) \supset_{uv} . A(\varphi, u) = A(\varphi, v).$

34′. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z)]\; [\varphi(x, y) \supset_{xy} \varphi(y, x)]\; [\varphi(x, y) \supset_{xy} \theta(x, y)]\; [\sim \theta(u, v)] \supset_{\varphi\theta uv} . A(\varphi, u) \neq A(\varphi, v).$

35′. $[A(\varphi, u) = A(\varphi, v)] \supset_{\varphi uv} \varphi(u, v).$

It is seen that the propositions 33′, 34′, 35′ are derivable by conversion from Postulates 33, 34, 35, respectively. And Postulate 30 is seen to be essentially the same as Theorem 4 below.

THEOREM 1. $\text{‘}x \,.\, x = x.$

$\mathbf{D}_1$: $E(x)$.

$\mathbf{C}_1$: $\Sigma\psi \,.\, \psi(x)$ —conv, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\psi(x) \supset_{\psi} \psi(x)$ —V, 1, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $x = x$ —conv, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma\psi \,.\, \psi(\lambda\varphi\, \Pi(\varphi, \varphi))$ —IV, 1.

[7] In an application of Rule V, we call the formula $\Pi(\mathbf{F}, \mathbf{G})$ the *major premise* and the formula $\{\mathbf{F}\}(\mathbf{A})$ the *minor premise*.

[8] This definition is similar to the one used in *Principia Mathematica*. It is a translation into symbolic notation of the definition originally given by Leibniz. Cf. C. I. Lewis, *A Survey of Symbolic Logic*, 1918, p. 373.

A$_2$: $E\varphi\Pi(\varphi, \varphi)$ —conv, **A**$_1$.

A$_3$: $\Sigma x E(x)$ —IV, **A**$_2$.

A$_4$: $'x . x = x$ —Thm I, **C**$_3$, **A**$_3$.

THEOREM 2. $x = y \supset_{xy} . y = x$

D$_1$: $x = y$.

C$_1$: $\Sigma\varphi . \varphi(x)$ —IV, **D**$_1$.

C$_2$: $E(x)$ —III, **C**$_1$; **D**$_1$.

C$_3$: $x = x$ —V, Thm 1, **C**$_2$; **D**$_1$.

C$_4$: $\{\lambda z . z = x\}(x)$ —III, **C**$_3$; **D**$_1$.

C$_5$: $\{\lambda z . z = x\}(y)$ —V, **D**$_1$, **C**$_4$; **D**$_1$.

C$_6$: $y = x$ —II, **C**$_5$; **D**$_1$.

A$_1$: $Ex . x = x$ —IV, Thm 1.

A$_2$: $[\lambda x . x = x] = . \lambda x . x = x$ —V, Thm 1, **A**$_1$.

A$_3$: $\Sigma x \Sigma y . x = y$ —IV^2, **A**$_2$.

A$_4$: $x = y \supset_{xy} . y = x$ —Thm I Cor, **C**$_6$, **A**$_3$.

THEOREM 3. $x = y \supset_{xy} . y = z \supset_z . x = z$.

D$_1$: $x = y$.

D$_2$: $y = z$.

C$_1$: $x = z$ —V, **D**$_2$, **D**$_1$.

C$_2$: $E(y)$ —IV, **D**$_1$.

C$_3$: $y = y$ —V, Thm 1, **C**$_2$; **D**$_1$.

C$_4$: $\Sigma z . y = z$ —IV, **C**$_3$; **D**$_1$.

C$_5$: $y = z \supset_z . x = z$ —Thm I, **C**$_1$, **C**$_4$; **D**$_1$.

A$_1$: $x = y \supset_{xy} . y = z \supset_z . x = z$ —Thm I Cor, **C**$_5$, **A**$_3$ under Thm 2.

THEOREM 4. $[\theta(x) . \theta(y) \supset_y . x = y] \supset_{\theta x} . x = \iota(\theta)$.

D$_1$: $\theta(x) . \theta(y) \supset_y . x = y$.

C$_1$: $\theta(x)$ —V^2, 15, **D**$_1$.

C$_2$: $E(x)$ —IV, **C**$_1$; **D**$_1$.

C$_3$: $x = x$ —V, Thm 1, **C**$_2$; **D**$_1$.

C$_4$: $\theta(y) \supset_y . x = y$ —V^2, 16, **D**$_1$.

D$_2$: $\theta(y)$.

C$_5$: $x = y$ —V, **C**$_4$, **D**$_2$; **D**$_1$, **D**$_2$.

D$_3$: $\psi(x)$.

C$_6$: $\psi(y)$ —V, **C**$_5$, **D**$_3$; **D**$_1$, **D**$_2$, **D**$_3$.

$\mathbf{C}_7$: $\Sigma(\theta)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_8$: $\pi(\theta, \psi)$ —Thm II, $\mathbf{C}_6$, $\mathbf{C}_7$; $\mathbf{D}_1$, $\mathbf{D}_8$.

$\mathbf{C}_9$: $\Sigma\psi \,.\, \psi(x)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_{10}$: $\psi(x) \supset_\psi \pi(\theta, \psi)$ —Thm I, $\mathbf{C}_8$, $\mathbf{C}_9$; $\mathbf{D}_1$.

$\mathbf{C}_{11}$: $x = \iota(\theta)$ —V^8, 30, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_1$, $\mathbf{C}_{10}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma y \,.\, [\lambda x \,.\, x = x] = y$ —IV, $\mathbf{A}_2$ under Thm 2.

$\mathbf{A}_2$: $[[\lambda x \,.\, x = x] = y] \supset_y . [\lambda x \,.\, x = x] = y$ —V, 1, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[[\lambda x \,.\, x = x] = . \lambda x \,.\, x = x] \,.\, [[\lambda x \,.\, x = x] = y] \supset_y . [\lambda x \,.\, x = x] = y$
—V^2, 14, $\mathbf{A}_2$ under Thm 2, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Sigma\theta\, \Sigma x \,.\, \theta(x) \,.\, \theta(y) \supset_y . x = y$ —IV^2, $\mathbf{A}_3$.

$\mathbf{A}_5$: $[\theta(x) \,.\, \theta(y) \supset_y . x = y] \supset_{\theta x} . x = \iota(\theta)$ —Thm I Cor, $\mathbf{C}_{11}$, $\mathbf{A}_4$.

THEOREM 5. ${}^{\iota}x \,.\, x = . \iota y \,.\, x = y$.

$\mathbf{D}_1$: $E(x)$.

$\mathbf{C}_1$: $x = x$ —V, Thm 1, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Sigma y \,.\, x = y$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $[x = y] \supset_y . x = y$ —V, 1, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $x = . \iota y \,.\, x = y$ —V, Thm 4, $\mathbf{C}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{A}_1$: ${}^{\iota}x \,.\, x = . \iota y \,.\, x = y$ —Thm I, $\mathbf{C}_4$, $\mathbf{A}_3$ under Thm 1.

5. **The completion of a propositional function.** We call a propositional function **F** *significant* for the value **A** of the independent variable if {**F**} (**A**) is either a true proposition or a false proposition.

We define the symbol Δ as follows:

$$\Delta \longrightarrow \lambda\mu\, \lambda\pi \,.\, \Pi(\mu, \gamma) \supset_\gamma \gamma(\pi).$$

If **F** is any propositional function, then Δ(**F**) is a propositional function which we call the *completion* of **F**. We shall prove (in Theorems 6 and 7) that if **F** is at least once true then Δ(**F**) is equivalent to **F**, that is, that for every value of the independent variable for which either is true the other is also true. We shall prove (in Theorem 9 and its corollary) that, of all propositional functions which are equivalent to **F**, Δ(**F**) is significant for the greatest range of values of the independent variable. And we shall prove (in Theorems 14 and 15) that there exist functions **F** for which the range of significance of Δ(**F**) is greater than that of **F**.

We define the symbol G as follows:

$$G \longrightarrow \lambda\nu \,.\, \sim \Delta(\nu, x) \supset_x \sim \nu(x).$$

And G(**F**) is to be read, "**F** is complete". In Theorem 16 we prove that if **F** is at least once true and at least once false, then Δ(**F**) is complete

And Theorem 20 can be regarded as asserting a certain form of the principle of reductio ad absurdum, as applied to complete propositional functions.

THEOREM 6. $\Sigma(\varphi) \supset_{\varphi} \Pi(\varphi, \Delta(\varphi))$.

D$_1$: $\Sigma(\varphi)$.

C$_1$: $\Pi(\varphi, \varphi)$ —V, 1, **D**$_1$.

C$_2$: $\Sigma \gamma \Pi(\varphi, \gamma)$ —IV, **C**$_1$; **D**$_1$.

D$_2$: $\varphi(x)$.

D$_3$: $\Pi(\varphi, \gamma)$.

C$_3$: $\gamma(x)$ —V, **D**$_3$, **D**$_2$.

C$_4$: $\Pi(\varphi, \gamma) \supset_{\gamma} \gamma(x)$ —Thm I, **C**$_3$, **C**$_2$; **D**$_1$, **D**$_2$.

C$_5$: $\Delta(\varphi, x)$ —conv, **C**$_4$; **D**$_1$, **D**$_2$.

C$_6$: $\Pi(\varphi, \Delta(\varphi))$ —Thm II, **C**$_5$, **D**$_1$; **D**$_1$.

A$_1$: $\Sigma(\Pi(\lambda \varphi \Sigma(\varphi)))$ —IV, 1.

A$_2$: $\Sigma \varphi \Sigma(\varphi)$ —IV, **A**$_1$.

A$_3$: $\Sigma(\varphi) \supset_{\varphi} \Pi(\varphi, \Delta(\varphi))$ —Thm I, **C**$_6$, **A**$_2$.

COROLLARY. $\varphi(x) \supset_{\varphi x} \Delta(\varphi, x)$.

THEOREM 7. $\Sigma(\varphi) \supset_{\varphi} \Pi(\Delta(\varphi), \varphi)$.

D$_1$: $\Sigma(\varphi)$.

C$_1$: $\Pi(\varphi, \varphi)$ —V, 1, **D**$_1$.

D$_2$: $\Delta(\varphi, x)$.

C$_2$: $\Pi(\varphi, \gamma) \supset_{\gamma} \gamma(x)$ —conv, **D**$_2$.

C$_3$: $\varphi(x)$ —V, **C**$_2$, **C**$_1$; **D**$_1$, **D**$_2$.

C$_4$: $\Pi(\varphi, \Delta(\varphi))$ —V, Thm 6, **D**$_1$.

D$_3$: $\varphi(y)$.

C$_5$: $\Delta(\varphi, y)$ —V, **C**$_4$, **D**$_3$; **D**$_1$, **D**$_3$.

C$_6$: $\Sigma(\Delta(\varphi))$ —IV, **C**$_5$; **D**$_1$, **D**$_3$.

C$_7$: $\Sigma(\Delta(\varphi))$ —Thm III, **C**$_6$, **D**$_1$; **D**$_1$.

C$_8$: $\Pi(\Delta(\varphi), \varphi)$ —Thm II, **C**$_3$, **C**$_7$; **D**$_1$.

A$_1$: $\Sigma(\varphi) \supset_{\varphi} \Pi(\Delta(\varphi), \varphi)$ —Thm I, **C**$_8$, **A**$_2$ under Thm 6.

COROLLARY. $\Delta(\varphi, x) \supset_{\varphi x} \varphi(x)$.

Postulate 38 is necessary to the proof of the corollary.

THEOREM 8. $[\Sigma(\varphi) \,.\sim \varphi(x)] \supset_{\varphi x} \sim \Delta(\varphi, x)$.

D$_1$: $\Sigma(\varphi) \,.\sim \varphi(x)$.

C$_1$: $\Sigma(\varphi)$ —V^2, 15, **D**$_1$.

C$_2$: $\Pi(\varphi, \varphi)$ —V, 1, **C**$_1$; **D**$_1$.

C$_3$: $\sim \varphi(x)$ —V^2, 16, **D**$_1$.

C$_4$: $\Pi(\varphi, \varphi) . \sim \varphi(x)$ —V^2, 14, **C**$_2$, **C**$_3$; **D**$_1$.

C$_5$: $\Sigma \gamma . \Pi(\varphi, \gamma) . \sim \gamma(x)$ —IV, **C**$_4$; **D**$_1$.

C$_6$: $\sim \Pi(\varphi, \gamma) \supset_\gamma \gamma(x)$ —V^2, 17, **C**$_5$; **D**$_1$.

C$_7$: $\sim \Delta(\varphi, x)$ —conv, **C**$_6$.

A$_1$: $\sim \sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V, 26, 1.

A$_2$: $\Sigma(\sim)$ —IV, **A**$_1$.

A$_3$: $\Sigma(\sim) . \sim \sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V, 14, **A**$_2$, **A**$_1$.

A$_4$: $\Sigma \varphi \Sigma x . \Sigma(\varphi) . \sim \varphi(x)$ —IV^2, **A**$_3$.

A$_5$: $[\Sigma(\varphi) . \sim \varphi(x)] \supset_{\varphi x} \sim \Delta(\varphi, x)$ —Thm I Cor, **C**$_7$, **A**$_4$.

THEOREM 9. $[\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\varphi \psi x} \sim \Delta(\varphi, x)$.

D$_1$: $\Pi(\varphi, \psi) . \sim \psi(x)$.

C$_1$: $\sim \cdot \Pi(\varphi, \gamma) \supset_\gamma \gamma(x)$ —V^2, 17, **D**$_1$.

C$_2$: $\sim \Delta(\varphi, x)$ —conv, **C**$_1$; **D**$_1$.

A$_1$: $\sim \sim \cdot \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V, 26, 1.

A$_2$: $\Sigma(\sim)$ —IV, **A**$_1$.

A$_3$: $\Pi(\sim, \sim)$ —V, 1, **A**$_2$.

A$_4$: $\Pi(\sim, \sim) . \sim \sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V^2, 14, **A**$_3$, **A**$_1$.

A$_5$: $\Sigma \varphi \Sigma \psi \Sigma x . \Pi(\varphi, \psi) . \sim \psi(x)$ —IV^3, **A**$_4$.

A$_6$: $[\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\varphi \psi x} \sim \Delta(\varphi, x)$ —Thm I Cor, **C**$_2$, **A**$_5$.

COROLLARY. $[Q(\varphi, \psi) . \sim \psi(x)] \supset_{\varphi \psi x} \sim \Delta(\varphi, x)$.

THEOREM 10. $\sim \Pi(E, \sim)$.

A$_1$: $\sim \sim . p \supset_p \sim \sim p$ —V, 26, 26.

A$_2$: $E(p \supset_p \sim \sim p)$ —IV, **A**$_1$.

A$_3$: $E(p \supset_p \sim \sim p) . \sim \sim . p \supset_p \sim \sim p$ —V^2, 14, **A**$_2$, **A**$_1$.

A$_4$: $\sim \Pi(E, \sim)$ —V^2, 17, **A**$_3$.

THEOREM 11. $\sim \Pi(E, \lambda z z)$.

A$_1$: $\sim \sim . p \supset_p \sim \sim p$ —V, 26, 26.

A$_2$: $E(\sim . p \supset_p \sim \sim p)$ —IV, **A**$_1$.

A$_3$: $E(\sim . p \supset_p \sim \sim p) . \sim \{\lambda z z\} (\sim . p \supset_p \sim \sim p)$ —V^2, 14, **A**$_2$, **A**$_1$.

A$_4$: $\sim \Pi(E, \lambda z z)$ —V^2, 17, **A**$_3$.

THEOREM 12. $\Sigma(\iota)$.

A$_1$: $\text{ʿ}x . x = . \iota y . x = y$ —Thm 5.

A$_2$: $E(\text{ʿ}x . x = . \iota y . x = y)$ —IV, **A**$_1$.

A$_3$: $[{}^{\iota}x \cdot x = . \iota y . x = y] = . \iota y . [{}^{\iota}x \cdot x = . \iota y . x = y] = y$ —V, **A**$_1$, **A**$_2$.
A$_4$: $\iota y . [{}^{\iota}x \cdot x = . \iota y . x = y] = y$ —V, **A**$_3$, **A**$_1$.
A$_5$: $\Sigma(\iota)$ —IV, **A**$_4$.

THEOREM 13. $\Sigma x \sim \iota(x)$.

A$_1$: $\sim\sim . p \supset_p \sim\sim p$ —V, 26, 26.
A$_2$: $E(\sim . p \supset_p \sim\sim p)$ —IV, **A**$_1$.
A$_3$: $[\sim . p \supset_p \sim\sim p] = . \iota y . [\sim . p \supset_p \sim\sim p] = y$ —V, Thm 5, **A**$_2$.
A$_4$: $\sim \iota y . [\sim . p \supset_p \sim\sim p] = y$ —V, **A**$_3$, **A**$_1$.
A$_5$: $\Sigma x \sim \iota(x)$ —IV, **A**$_4$.

THEOREM 14. $\sim \Delta(\iota, E)$.

D$_1$: $\iota(x)$.
C$_1$: $\Pi(x, \lambda z z)$ —V^2, 31, **D**$_1$.
A$_1$: $\Pi(\iota, \lambda x \Pi(x, \lambda z z))$ —Thm II, **C**$_1$, Thm 12.
A$_2$: $\Pi(\iota, \lambda x \Pi(x, \lambda z z)) . \sim \Pi(E, \lambda z z)$ —V^2, 14, **A**$_1$, Thm 11.
A$_3$: $\sim . \Pi(\iota, \gamma) \supset_\gamma \gamma(E)$ —V^2, 17, **A**$_2$.
A$_4$: $\sim \Delta(\iota, E)$ —conv, **A**$_3$.

THEOREM 15. $\sim \{\Delta x \sim \iota(x)\}(E)$.

D$_1$: $\sim \iota(x)$.
C$_1$: $\Pi(x, \sim)$ —V^2, 31, **D**$_1$.
A$_1$: $\sim \iota(x) \supset_x \Pi(x, \sim)$ —Thm I, **C**$_1$, Thm 13.
A$_2$: $[\sim \iota(x) \supset_x \Pi(x, \sim)] \sim \Pi(E, \sim)$ —V^2, 14, **A**$_1$, Thm 10.
A$_3$: $\sim . \Pi(\lambda x \sim \iota x, \gamma) \supset_\gamma \gamma(E)$ —V^2, 17, **A**$_2$.
A$_4$: $\sim \{\Delta x \sim \iota(x)\}(E)$ — conv, **A**$_3$.

THEOREM 16. $\Sigma x \Sigma y [\varphi(x) . \sim \varphi(y)] \supset_\varphi G(\Delta(\varphi))$.

D$_1$: $\varphi(z)$.
C$_1$: $\Delta(\varphi, z)$ —V^2, Thm 6 Cor, **D**$_1$.
C$_2$: $\Delta(\Delta(\varphi), z)$ —V^2, Thm 6 Cor, **C**$_1$; **D**$_1$.
D$_2$: $\varphi(x) . \sim \varphi(y)$.
C$_3$: $\varphi(x)$ —V^2, 15, **D**$_2$.
C$_4$: $\Sigma(\varphi)$ —IV, **C**$_3$; **D**$_2$.
C$_5$: $\Pi(\varphi, \Delta(\Delta(\varphi)))$ —Thm II, **C**$_2$, **C**$_4$; **D**$_2$.
D$_3$: $\sim \Delta(\Delta(\varphi), t)$.
C$_6$: $\sim \Delta(\varphi, t)$ —V^3, Thm 9, **C**$_5$, **D**$_3$; **D**$_2$, **D**$_3$.
C$_7$: $\Delta(\varphi, x)$ —V^2, Thm 6 Cor, **C**$_3$; **D**$_2$.

C$_8$: $\Sigma(\Delta(\varphi))$ —IV, **C**$_7$; **D**$_2$.
C$_9$: $\sim\varphi(y)$ —V^2, 16, **D**$_2$.
C$_{10}$: $\sim\Delta(\varphi, y)$ —V^2, Thm 8, **C**$_4$, **C**$_9$; **D**$_2$.
C$_{11}$: $\sim\Delta(\Delta(\varphi), y)$ —V^2, Thm 8, **C**$_8$, **C**$_{10}$; **D**$_2$.
C$_{12}$: $\Sigma t\sim\Delta(\Delta(\varphi), t)$ —IV, **C**$_{11}$; **D**$_2$.
C$_{13}$: $\sim\Delta(\Delta(\varphi), t)\supset_t\sim\Delta(\varphi, t)$ —Thm I, **C**$_6$, **C**$_{12}$; **D**$_2$.
C$_{14}$: $G(\Delta(\varphi))$ —conv, **C**$_{13}$; **D**$_2$.
D$_4$: $\Sigma x\,\Sigma y\,.\,\varphi(x)\,.\sim\varphi(y)$.
C$_{15}$: $G(\Delta(\varphi))$ —Thm I Cor, **C**$_{14}$, **D**$_4$; **D**$_4$.
A$_1$: $E(\Pi(E, \sim))$ —IV, Thm 10.
A$_2$: $\Sigma(E)$ —IV, **A**$_1$.
A$_3$: $\Pi(E, E)$ —V, 1, **A**$_2$.
A$_4$: $\Pi(E, E)\,.\sim\Pi(E, \sim)$ —V^2, 14, **A**$_3$, Thm 10.
A$_5$: $\Sigma\varphi\,\Sigma x\,\Sigma y\,.\,\varphi(x)\,.\sim\varphi(y)$ —IV^3, **A**$_4$.
A$_6$: $\Sigma x\,\Sigma y[\varphi(x)\,.\sim\varphi(y)]\supset_\varphi G(\Delta(\varphi))$ —Thm I, **C**$_{15}$, **A**$_5$.

Corollary. $\Sigma x\,\Sigma y[\varphi(x)\,.\sim\Delta(\varphi, y)]\supset_\varphi G(\Delta(\varphi))$.

Theorem 17. $\Sigma(G)$.

A$_1$: $\Sigma x\,\Sigma y\,.\,\Pi(E, x)\,.\sim\Pi(E, y)$ —IV^2, **A**$_4$ under Thm 16.
A$_2$: $G(\Delta(\Pi(E)))$ —V, Thm 16, **A**$_1$.
A$_3$: $\Sigma(G)$ —IV, **A**$_2$.

Corollary. $\Sigma\varphi\,G(\varphi)$.

Theorem 18. $G(\varphi)\supset_\varphi\Sigma(\varphi)$.

D$_1$: $G(\varphi)$.
C$_1$: $\sim\Delta(\varphi, x)\supset_x\sim\varphi(x)$ —II, **D**$_1$.
C$_2$: $\Sigma x\sim\Delta(\varphi, x)$ —V, 38, **C**$_1$; **D**$_1$.
D$_2$: $\sim\Delta(\varphi, x)$.
C$_3$: $\sim.\,\Pi(\varphi, \gamma)\supset_\gamma\gamma(x)$ —conv, **D**$_2$.
C$_4$: $\Sigma\gamma\,.\,\Pi(\varphi, \gamma)\,.\sim\gamma(x)$ —V^2, 18, **C**$_3$; **D**$_2$.
D$_3$: $\Pi(\varphi, \gamma)\,.\sim\gamma(x)$.
C$_5$: $\Pi(\varphi, \gamma)$ —V^2, 15, **D**$_3$.
C$_6$: $\Sigma(\varphi)$ —V, 38, **C**$_5$; **D**$_3$.
C$_7$: $\Sigma(\varphi)$ —Thm I, **C**$_6$, **C**$_4$; **D**$_2$.
C$_8$: $\Sigma(\varphi)$ —Thm I, **C**$_7$, **C**$_2$; **D**$_1$.
A$_1$: $G(\varphi)\supset_\varphi\Sigma(\varphi)$ —Thm I, **C**$_8$, Thm 17 Cor.

THEOREM 19. $G(\varphi) \supset_\varphi \Sigma x \sim \varphi(x)$.

$\mathbf{D}_1$: $G(\varphi)$.

$\mathbf{C}_1$: $\sim \Delta(\varphi, x) \supset_x \sim \varphi(x)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Sigma x \sim \Delta(\varphi, x)$ —V, 38, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\sim \Delta(\varphi, x)$.

$\mathbf{C}_3$: $\sim \varphi(x)$ —V, $\mathbf{C}_1$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma x \sim \varphi(x)$ —IV, $\mathbf{C}_3$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Sigma x \sim \varphi(x)$ —Thm I, $\mathbf{C}_4$, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $G(\varphi) \supset_\varphi \Sigma x \sim \varphi(x)$ —Thm I, $\mathbf{C}_5$, Thm 17 Cor.

THEOREM 20. $G(\varphi) \supset_\varphi . [\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\psi x} \sim \varphi(x)$.

$\mathbf{D}_1$: $\Pi(\varphi, \psi) . \sim \psi(x)$.

$\mathbf{C}_1$: $\sim \Delta(\varphi, x)$ —V, Thm 9, $\mathbf{D}_1$.

$\mathbf{D}_2$: $G(\varphi)$.

$\mathbf{C}_2$: $\sim \Delta(\varphi, x) \supset_x \sim \varphi(x)$ —II, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\sim \varphi(x)$ —V, $\mathbf{C}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma(\varphi)$ —V, Thm 18, $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Pi(\varphi, \varphi)$ —V, 1, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{C}_6$: $\Sigma y \sim \varphi(y)$ —V, Thm 19, $\mathbf{D}_2$.

$\mathbf{D}_3$: $\sim \varphi(y)$.

$\mathbf{C}_7$: $\Pi(\varphi, \varphi) . \sim \varphi(y)$ —V^2, 14, $\mathbf{C}_5$, $\mathbf{D}_3$; $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_8$: $\Sigma \psi \Sigma x . \Pi(\varphi, \psi) . \sim \psi(x)$ —IV^2, $\mathbf{C}_7$; $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_9$: $\Sigma \psi \Sigma x . \Pi(\varphi, \psi) . \sim \psi(x)$ —Thm I, $\mathbf{C}_8$, $\mathbf{C}_6$; $\mathbf{D}_2$.

$\mathbf{C}_{10}$: $[\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\psi x} \sim \varphi(x)$ —Thm I Cor, $\mathbf{C}_3$, $\mathbf{C}_9$; $\mathbf{D}_2$.

$\mathbf{A}_1$: $G(\varphi) \supset_\varphi . [\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\psi x} \sim \varphi(x)$ —Thm I, $\mathbf{C}_{10}$, Thm 17 Cor.

THEOREM 21. $G(\varphi) \supset_\varphi Gx \sim \{\Delta y \sim \varphi(y)\}(x)$.

$\mathbf{D}_1$: $G(\varphi)$.

$\mathbf{C}_1$: $\Sigma y \sim \varphi(y)$ —V, Thm 19, $\mathbf{D}_1$.

$\mathbf{D}_2$: $\varphi(u)$.

$\mathbf{C}_2$: $\sim \sim \varphi(u)$ —V, 26, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\sim \{\Delta y \sim \varphi(y)\}(u)$ —V^2, Thm 8, $\mathbf{C}_1$, $\mathbf{C}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{D}_3$: $\Pi(\lambda x \sim \{\Delta y \sim \varphi(y)\}(x), \gamma) . \sim \gamma(z)$.

$\mathbf{C}_4$: $\Pi(\lambda x \sim \{\Delta y \sim \varphi(y)\}(x), \gamma)$ —V^2, 15, $\mathbf{D}_3$.

$\mathbf{C}_5$: $\gamma(u)$ —V, $\mathbf{C}_4$, $\mathbf{C}_3$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_6$: $\Sigma(\varphi)$ —V, Thm 18, $\mathbf{D}_1$.

C$_7$: $\Pi(\varphi, \gamma)$ —Thm II, **C**$_5$, **C**$_6$; **D**$_1$, **D**$_3$.

C$_8$: $\sim \gamma(z)$ —V, 16, **D**$_3$.

C$_9$: $\Pi(\varphi, \gamma) . \sim \gamma(z)$ —V^2, 14, **C**$_7$, **C**$_8$; **D**$_1$, **D**$_3$.

C$_{10}$: $\Sigma\gamma . \Pi(\varphi, \gamma) . \sim \gamma(z)$ —IV, **C**$_9$; **D**$_1$, **D**$_3$.

C$_{11}$: $\sim A(\varphi, z)$ —V^2, 17, **C**$_{10}$; **D**$_1$, **D**$_3$.

D$_4$: $\sim \{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z)$.

C$_{12}$: $\Sigma\gamma . \Pi(\lambda x \sim \{Ay \sim \varphi(y)\}(x), \gamma) . \sim \gamma(z)$ —V^2, 18, **D**$_4$.

C$_{13}$: $\sim A(\varphi, z)$ —Thm I, **C**$_{11}$, **C**$_{12}$; **D**$_1$, **D**$_4$.

C$_{14}$: $\sim A(\varphi, x) \supset_x \sim \varphi(x)$ —II, **D**$_1$.

C$_{15}$: $\sim \varphi(z)$ —V, **C**$_{14}$, **C**$_{13}$; **D**$_1$, **D**$_4$.

C$_{16}$: $\{Ay \sim \varphi(y)\}(z)$ —V^2, Thm 6 Cor, **C**$_{15}$; **D**$_1$, **D**$_4$.

C$_{17}$: $\sim\sim \{Ay \sim \varphi(y)\}(z)$ —V, 26, **C**$_{16}$; **D**$_1$, **D**$_4$.

D$_5$: $\sim \varphi(v)$.

C$_{18}$: $\{Ay \sim \varphi(y)\}(v)$ —V^2, Thm 6 Cor, **D**$_5$.

C$_{19}$: $\sim\sim \{Ay \sim \varphi(y)\}(v)$ —V, 26, **C**$_{18}$; **D**$_5$.

C$_{20}$: $\Sigma x \sim \{Ay \sim \varphi(y)\}(x)$ —IV, **C**$_3$; **D**$_1$, **D**$_2$.

C$_{21}$: $\Sigma x \sim \{Ay \sim \varphi(y)\}(x)$ —Thm III, **C**$_{20}$, **C**$_6$; **D**$_1$.

C$_{22}$: $\sim \{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(v)$ —V^2, Thm 8, **C**$_{21}$, **C**$_{19}$; **D**$_1$, **D**$_5$.

C$_{23}$: $\Sigma z \sim \{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z)$ —IV, **C**$_{22}$; **D**$_1$, **D**$_5$.

C$_{24}$: $\Sigma z \sim \{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z)$ —Thm I, **C**$_{23}$, **C**$_1$; **D**$_1$.

C$_{25}$: $\sim\{Ax\sim\{Ay\sim\varphi(y)\}(x)\}(z) \supset_z \sim\sim\{Ay\sim\varphi(y)\}(z)$ —Thm I, **C**$_{17}$, **C**$_{24}$; **D**$_1$.

C$_{26}$: $Gx \sim \{Ay \sim \varphi(y)\}(x)$ —conv, **C**$_{25}$; **D**$_1$.

A$_1$: $G(\varphi) \supset_\varphi Gx \sim \{Ay \sim \varphi(y)\}(x)$ —Thm I, **C**$_{26}$, Thm 17 Cor.

COROLLARY. $Gx \sim \varphi(x) \supset_\varphi Gx \sim A(\varphi, x)$.

THEOREM 22. $[\Sigma z . x \neq z] \supset_x Gy . x = y$.

D$_1$: $\Sigma z . x \neq z$.

C$_1$: $E(x)$ —IV, **D**$_1$.

C$_2$: $x = x$ —V, Thm 1, **C**$_1$; **D**$_1$.

C$_3$: $\{\lambda u . x = u\}(x)$ —III, **C**$_2$; **D**$_1$.

C$_4$: $\{Au . x = u\}(x)$ —V^2, Thm 6 Cor, **C**$_3$; **D**$_1$.

D$_2$: $\sim \{Au . x = u\}(y)$.

C$_5$: $\{Au . x = u\}(x) . \sim \{Au . x = u\}(y)$ —V^2, 14, **C**$_4$, **D**$_2$; **D**$_1$, **D**$_2$.

C$_6$: $\Sigma\varrho . \varrho(x) . \sim \varrho(y)$ —IV, **C**$_5$; **D**$_1$, **D**$_2$.

C$_7$: $x \neq y$ —V^2, 17, **C**$_6$; **D**$_1$, **D**$_2$.

C$_8$: $\Sigma u . x = u$ —IV, **C**$_2$; **D**$_1$.

D$_3$: $x \neq z$.
C$_9$: $\sim \{\lambda u \,.\, x = u\}(z)$ —III, **D**$_3$.
C$_{10}$: $\sim \{\Lambda u \,.\, x = u\}(z)$ —V^2, Thm 8, **C**$_8$, **C**$_9$; **D**$_1$, **D**$_3$.
C$_{11}$: $\Sigma y \sim \{\Lambda u \,.\, x = u\}(y)$ —IV, **C**$_{10}$; **D**$_1$, **D**$_3$.
C$_{12}$: $\Sigma y \sim \{\Lambda u \,.\, x = u\}(y)$ —Thm I, **C**$_{11}$, **D**$_1$; **D**$_1$.
C$_{13}$: $\sim \{\Lambda u \,.\, x = u\}(y) \supset_y x \neq y$ —Thm I, **C**$_7$, **C**$_{12}$; **D**$_1$.
C$_{14}$: $Gy \,.\, x = y$ —conv, **C**$_{13}$; **D**$_1$.
A$_1$: $\Sigma \varrho \,.\, \varrho(E) \,.\, \sim \varrho(\sim)$ —IV, **A**$_4$ under Thm 16.
A$_2$: $E \neq \sim$ —V^2, 17, **A**$_1$.
A$_3$: $\Sigma x \Sigma z \,.\, x \neq z$ —IV^2, **A**$_2$.
A$_4$: $[\Sigma z \,.\, x \neq z] \supset_x Gy \,.\, x = y$ —Thm I, **C**$_{14}$, **A**$_3$.

6. **Classes.** We recall the following definitions:

$$Q \longrightarrow \lambda \mu \, \lambda \nu \,.\, \Pi(\mu, \nu) \,.\, \Pi(\nu, \mu).$$
$$K \longrightarrow A(Q).$$

And to these we add two new definitions:

$$\varepsilon \longrightarrow \lambda \alpha \, \lambda \beta \,.\, [\beta = K(\varphi)] \supset_\varphi \varphi(\alpha).$$
$$O \longrightarrow \lambda \gamma \, K \pi \sim \{\varepsilon\} \, (\pi, \gamma).$$

We shall abbreviate $\{\varepsilon\}$ (**A**, **B**) as [**A**] ε [**B**], and hence we may rewrite the definition of O as follows:

$$O \longrightarrow \lambda \gamma \, K \pi \sim . \, \pi \, \varepsilon \, \gamma.$$

If **F** and **G** are propositional functions, Q(**F**, **G**) is to be read, "**F** and **G** are equivalent", and K(**F**) is to be read, "The class of x's such that {**F**} (x) is true". And [**A**] ε [**B**] is to be read, "**A** is a member of the class **B**". And O(**B**) is to be read, "The class of things which are not members of the class **B**", or, "The complement of the class **B**".

THEOREM 23. $Q(x, y) \supset_{xy} . \, Q(y, z) \supset_z Q(x, z)$.

D$_1$: $Q(x, y)$.
C$_1$: $\Pi(x, y)$ —V^2, 15, **D**$_1$.
D$_2$: $x(r)$.
C$_2$: $y(r)$ —V, **C**$_1$, **D**$_2$; **D**$_1$, **D**$_2$.
D$_3$: $Q(y, z)$.
C$_3$: $\Pi(y, z)$ —V^2, 15, **D**$_3$.
C$_4$: $z(r)$ —V, **C**$_3$, **C**$_2$; **D**$_1$, **D**$_2$, **D**$_3$.

$\mathbf{C}_5$: $\Sigma(x)$ —V, 38, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $\Pi(x, z)$ —Thm II, $\mathbf{C}_4$, $\mathbf{C}_5$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_7$: $\Pi(z, y)$ —V, 16, $\mathbf{D}_3$.

$\mathbf{D}_4$: $z(s)$.

$\mathbf{C}_8$: $y(s)$ —V, $\mathbf{C}_7$, $\mathbf{D}_4$; $\mathbf{D}_3$, $\mathbf{D}_4$.

$\mathbf{C}_9$: $\Pi(y, x)$ —V, 16, $\mathbf{D}_1$.

$\mathbf{C}_{10}$: $x(s)$ —V, $\mathbf{C}_9$, $\mathbf{C}_8$; $\mathbf{D}_1$, $\mathbf{D}_3$, $\mathbf{D}_4$.

$\mathbf{C}_{11}$: $\Sigma(z)$ —V, 38, $\mathbf{C}_7$; $\mathbf{D}_3$.

$\mathbf{C}_{12}$: $\Pi(z, x)$ —Thm II, $\mathbf{C}_{10}$, $\mathbf{C}_{11}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{13}$: $Q(x, z)$ —V^2, 14, $\mathbf{C}_6$, $\mathbf{C}_{12}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{14}$: $\Sigma(y)$ —V, 38, $\mathbf{C}_9$; $\mathbf{D}_1$.

$\mathbf{C}_{15}$: $\Pi(y, y)$ —V, 1, $\mathbf{C}_{14}$; $\mathbf{D}_1$.

$\mathbf{C}_{16}$: $Q(y, y)$ —V^2, 14, $\mathbf{C}_{15}$, $\mathbf{C}_{15}$; $\mathbf{D}_1$.

$\mathbf{C}_{17}$: $\Sigma z\, Q(y, z)$ —IV, $\mathbf{C}_{16}$; $\mathbf{D}_1$.

$\mathbf{C}_{18}$: $Q(y, z) \supset_z Q(x, z)$ —Thm I, $\mathbf{C}_{13}$, $\mathbf{C}_{17}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $Q(E, E)$ —V^2, 14, $\mathbf{A}_3$ under Thm 16, $\mathbf{A}_3$ under Thm 16.

$\mathbf{A}_2$: $\Sigma x \Sigma y\, Q(x, y)$ —IV^2, $\mathbf{A}_1$.

$\mathbf{A}_3$: $Q(x, y) \supset_{xy} . Q(y, z) \supset_z Q(x, z)$ —Thm I Cor, $\mathbf{C}_{18}$, $\mathbf{A}_2$.

THEOREM 24. $Q(x, y) \supset_{xy} Q(y, x)$.

$\mathbf{D}_1$: $Q(x, y)$.

$\mathbf{C}_1$: $\Pi(x, y)$ —V^2, 15, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Pi(y, x)$ —V^2, 16, $\mathbf{D}_1$.

$\mathbf{C}_3$: $Q(y, x)$ —V^2, 14, $\mathbf{C}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $Q(x, y) \supset_{xy} Q(y, x)$ —Thm I Cor, $\mathbf{C}_3$, $\mathbf{A}_2$ under Thm 23.

THEOREM 25. $Q(\varphi, \psi) \supset_{\varphi\psi} . K(\varphi) = K(\psi)$ — V, 33, Thm 23, Thm 24.

THEOREM 26. $[K(\varphi) = K(\psi)] \supset_{\varphi\psi} Q(\varphi, \psi)$.

$\mathbf{A}_1$: $K(E) = K(E)$ —V^2, Thm 25, $\mathbf{A}_1$ under Thm 23.

$\mathbf{A}_2$: $\Sigma u \Sigma v . K(u) = K(v)$ —IV^2, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[K(u) = K(v)] \supset_{uv} Q(u, v)$ —V, 35, $\mathbf{A}_2$.

$\mathbf{A}_4$: $[K(\varphi) = K(\psi)] \supset_{\varphi\psi} Q(\varphi, \psi)$ —conv, $\mathbf{A}_3$.

THEOREM 27. $\sim Q(\varphi, \psi) \supset_{\varphi\psi} . K(\varphi) \neq K(\psi)$.

$\mathbf{D}_1$: $\Sigma y\, Q(x, y)$.

$\mathbf{C}_1$: $Q(x, y) \supset_y Q(x, y)$ —V, 1, $\mathbf{D}_1$.

A$_1$: $Q(x, y) \supset_{xy} Q(x, y)$ —Thm I, **C**$_1$, **A**$_2$ under Thm 23.
D$_2$: $\sim Q(\varphi, \psi)$.
C$_2$: $K(\varphi) \neq K(\psi)$ —V^4, 34, Thm 23, Thm 24, **A**$_1$, **D**$_2$.
D$_3$: $\sim p$.
C$_3$: $E(p)$ —IV, **D**$_3$.
A$_2$: $\Pi(\sim, E)$ —Thm II, **C**$_3$, **A**$_2$ under Thm 9.
A$_3$: $\sim Q(E, \sim)$ —V^2, 22, Thm 10, **A**$_2$.
A$_4$: $\Sigma\varphi\Sigma\psi \sim Q(\varphi, \psi)$ —IV^2, **A**$_3$.
A$_5$: $\sim Q(\varphi, \psi) \supset_{\varphi\psi} . K(\varphi) \neq K(\psi)$ —Thm I Cor, **C**$_2$, **A**$_4$.

THEOREM 28. $\Sigma x \Sigma y [\varphi(x) . \sim \varphi(y)] \supset_\varphi G\psi . K(\varphi) = K(\psi)$.

D$_1$: $Q(x, y)$.
C$_1$: $K(x) = K(y)$ —V^2, Thm 25, **D**$_1$.
C$_2$: $\{\Lambda\psi . K(x) = K(\psi)\}(y)$ —V^2, Thm 6 Cor, **C**$_1$; **D**$_1$.
A$_1$: $Q(x,y) \supset_{xy} \{\Lambda\psi . K(x) = K(\psi)\}(y)$ —Thm I Cor, **C**$_2$, **A**$_2$ under Thm 23.
D$_2$: $\sim \{\Lambda\psi . K(u) = K(\psi)\}(v)$.
C$_3$: $K(u) \neq K(v)$ —V^4, 34, Thm 23, Thm 24, **A**$_1$, **D**$_2$.
D$_3$: $u(x) . \sim u(y)$.
C$_4$: $u(x)$ —V^2, 15, **D**$_3$.
C$_5$: $\sim\sim u(x)$ —V, 26, **C**$_4$; **D**$_3$.
C$_6$: $\sim \Pi(u, \lambda x \sim u(x))$ —V^2, 17, **C**$_4$, **C**$_5$; **D**$_3$.
C$_7$: $\sim u(y)$ —V^2, 16, **D**$_3$.
C$_8$: $\sim \Pi(\lambda x \sim u(x), u)$ —V^2, 17, **C**$_7$, **C**$_7$; **D**$_3$.
C$_9$: $\sim Q(u, \lambda x \sim u(x))$ —V^2, 23, **C**$_6$, **C**$_8$; **D**$_3$.
C$_{10}$: $K(u) \neq K x \sim u(x)$ —V^2, Thm 27, **C**$_9$; **D**$_3$.
C$_{11}$: $\Sigma(u)$ —IV, **C**$_4$; **D**$_3$.
C$_{12}$: $\Pi(u, u)$ —V, 1, **C**$_{11}$; **D**$_3$.
C$_{13}$: $Q(u, u)$ —V^2, 14, **C**$_{12}$, **C**$_{12}$; **D**$_3$.
C$_{14}$: $K(u) = K(u)$ —V^2, Thm 25, **C**$_{13}$; **D**$_3$.
C$_{15}$: $\Sigma\psi . K(u) = K(\psi)$ —IV, **C**$_{14}$; **D**$_3$.
C$_{16}$: $\sim \{\Lambda\psi . K(u) = K(\psi)\}(\lambda x \sim u(x))$ —V^2, Thm 8, **C**$_{15}$, **C**$_{10}$; **D**$_3$.
C$_{17}$: $\Sigma v \sim \{\Lambda\psi . K(u) = K(\psi)\}(v)$ —IV, **C**$_{16}$; **D**$_3$.
C$_{18}$: $\sim \{\Lambda\psi . K(u) = K(\psi)\}(v) \supset_v . K(u) \neq K(v)$ —Thm I, **C**$_3$, **C**$_{17}$; **D**$_3$.
C$_{19}$: $G\psi . K(u) = K(\psi)$ —conv, **C**$_{18}$; **D**$_3$.
D$_4$: $\Sigma x \Sigma y . u(x) . \sim u(y)$.
C$_{20}$: $G\psi . K(u) = K(\psi)$ —Thm I Cor, **C**$_{19}$, **D**$_4$; **D**$_4$.

A$_2$: $\Sigma u \Sigma x \Sigma y \,.\, u(x) \,.\, \sim u(y)$ —I, **A**$_5$ under Thm 16.

A$_3$: $\Sigma x \Sigma y\, [u(x) \,.\, \sim u(y)] \supset_u G\psi \,.\, K(u) = K(\psi)$ —Thm I, **C**$_{20}$, **A**$_2$.

A$_4$: $\Sigma x \Sigma y\, [\varphi(x) \,.\, \sim \varphi(y)] \supset_\varphi G\psi \,.\, K(\varphi) = K(\psi)$ —I, **A**$_3$.

THEOREM 29. $\varphi(x) \supset_{\varphi x} . \, x \,\epsilon\, K(\varphi)$.

D$_1$: $K(\psi) = K(\varphi)$.

C$_1$: $Q(\psi, \varphi)$ —V^2, Thm 26, **D**$_1$.

C$_2$: $\Pi(\psi, \varphi)$ —V^2, 15, **C**$_1$; **D**$_1$.

D$_2$: $\psi(x)$.

C$_3$: $\varphi(x)$ —V, **C**$_2$, **D**$_2$; **D**$_1$, **D**$_2$.

C$_4$: $\Sigma(\psi)$ —IV, **D**$_2$.

C$_5$: $\Pi(\psi, \psi)$ —V, 1, **C**$_4$; **D**$_2$.

C$_6$: $Q(\psi, \psi)$ —V^2, 14, **C**$_3$, **C**$_5$; **D**$_2$.

C$_7$: $K(\psi) = K(\psi)$ —V^2, Thm 25, **C**$_6$; **D**$_2$.

C$_8$: $\Sigma\varphi \,.\, K(\psi) = K(\varphi)$ —IV, **C**$_7$; **D**$_2$.

C$_9$: $[K(\psi) = K(\varphi)] \supset_\varphi \varphi(x)$ —Thm I, **C**$_3$, **C**$_8$; **D**$_2$.

C$_{10}$: $x \,\epsilon\, K(\psi)$ —conv, **C**$_9$; **D**$_2$.

A$_1$: $\Sigma\psi \Sigma x \,.\, \psi(x)$ —IV^2, 1.

A$_2$: $\psi(x) \supset_{\psi x} . \, x \,\epsilon\, K(\psi)$ —Thm I Cor, **C**$_{10}$, **A**$_1$.

A$_3$: $\varphi(x) \supset_{\varphi x} . \, x \,\epsilon\, K(\varphi)$ —I, **A**$_2$.

THEOREM 30. $[x \,\epsilon\, K(\varphi)] \supset_{\varphi x} \varphi(x)$.

D$_1$: $x \,\epsilon\, K(\psi)$.

C$_1$: $E(K(\psi))$ —IV, **D**$_1$.

C$_2$: $K(\psi) = K(\psi)$ —V, Thm 1, **C**$_1$; **D**$_1$.

C$_3$: $[K(\psi) = K(\varphi)] \supset_\varphi \varphi(x)$ —conv, **D**$_1$.

C$_4$: $\psi(x)$ —V, **C**$_3$, **C**$_2$; **D**$_1$.

A$_1$: $[\lambda\varphi\, \Pi(\varphi, \varphi)] \,\epsilon\, K(\Pi(\lambda\varphi\, \Sigma(\varphi)))$ —V^2, Thm 29, 1.

A$_2$: $\Sigma\psi \Sigma x \,.\, x \,\epsilon\, K(\psi)$ —IV^2, **A**$_1$.

A$_3$: $[x \,\epsilon\, K(\psi)] \supset_{\psi x} \psi(x)$ —Thm I Cor, **C**$_4$, **A**$_2$.

A$_4$: $[x \,\epsilon\, K(\varphi)] \supset_{\varphi x} \varphi(x)$ —I, **A**$_3$.

COROLLARY 1. $\Sigma(\varphi) \supset_\varphi Q(\varphi, \lambda x \,.\, x \,\epsilon\, K(\varphi))$.

COROLLARY 2. $\Sigma(\varphi) \supset_\varphi . \, K(\varphi) = Kx \,.\, x \,\epsilon\, K(\varphi)$.

THEOREM 31. $E(K(\varphi)) \supset_\varphi \Sigma(\varphi)$.

D$_1$: $E(K(\varphi))$.

C$_1$: $K(\varphi) = K(\varphi)$ —V, Thm 1, **D**$_1$.

$\mathbf{C}_2$: $Q(\varphi, \varphi)$ —V^2, Thm 26, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $\Pi(\varphi, \varphi)$ —V^2, 15, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\Sigma(\varphi)$ —V, 38, $\mathbf{C}_3$; $\mathbf{D}_1$.
$\mathbf{A}_1$: $E\Big(K(\Pi(\lambda\varphi\,\Sigma(\varphi)))\Big)$ —IV, $\mathbf{A}_1$ under Thm 30.
$\mathbf{A}_2$: $\Sigma\varphi\,E(K(\varphi))$ —IV, $\mathbf{A}_1$.
$\mathbf{A}_3$: $E(K(\varphi)) \supset_\varphi \Sigma(\varphi)$ —Thm I, $\mathbf{C}_4$, $\mathbf{A}_2$.

COROLLARY 1. $E(K(\varphi)) \supset_\varphi \Sigma x \,.\, x \,\varepsilon\, K(\varphi)$.

COROLLARY 2. $\Sigma x[\sim . \, x \,\varepsilon\, \alpha] \supset_\alpha \Sigma x \,.\, x \,\varepsilon\, \alpha$.

THEOREM 32. $\Sigma x[x \,\varepsilon\, \alpha] \supset_\alpha . \, \alpha = Kx \,.\, x \,\varepsilon\, \alpha$.

$\mathbf{D}_1$: $\alpha = K(\varphi)$.
$\mathbf{C}_1$: $E(K(\varphi))$ —IV, $\mathbf{D}_1$.
$\mathbf{C}_2$: $\Sigma(\varphi)$ —V, Thm 31, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $K(\varphi) = Kx \,.\, x \,\varepsilon\, K(\varphi)$ —V, Thm 30 Cor 2, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\alpha = Kx \,.\, x \,\varepsilon\, K(\varphi)$ —V^3, Thm 3, $\mathbf{D}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.
$\mathbf{C}_5$: $K(\varphi) = \alpha$ —V^2, Thm 2, $\mathbf{D}_1$.
$\mathbf{C}_6$: $\alpha = Kx \,.\, x \,\varepsilon\, \alpha$ —V, $\mathbf{C}_5$, $\mathbf{C}_4$; $\mathbf{D}_1$.
$\mathbf{D}_2$: $x \,\varepsilon\, \alpha$.
$\mathbf{C}_7$: $[\alpha = K(\varphi)] \supset_\varphi \varphi(x)$ —conv, $\mathbf{D}_2$.
$\mathbf{C}_8$: $\Sigma\varphi \,.\, \alpha = K(\varphi)$ —V, 38, $\mathbf{C}_7$; $\mathbf{D}_2$.
$\mathbf{C}_9$: $\alpha = Kx \,.\, x \,\varepsilon\, \alpha$ —Thm I, $\mathbf{C}_6$, $\mathbf{C}_8$; $\mathbf{D}_2$.
$\mathbf{D}_3$: $\Sigma x \,.\, x \,\varepsilon\, \alpha$.
$\mathbf{C}_{10}$: $\alpha = Kx \,.\, x \,\varepsilon\, \alpha$ – Thm I, $\mathbf{C}_9$, $\mathbf{D}_3$; $\mathbf{D}_3$.
$\mathbf{A}_1$: $\Sigma\alpha\,\Sigma x \,.\, x \,\varepsilon\, \alpha$ —IV^2, $\mathbf{A}_1$ under Thm 30.
$\mathbf{A}_2$: $\Sigma x[x \,\varepsilon\, \alpha] \supset_\alpha . \, \alpha = Kx \,.\, x \,\varepsilon\, \alpha$ —Thm I, $\mathbf{C}_{10}$, $\mathbf{A}_1$.

COROLLARY. $[[x \,\varepsilon\, \alpha] \equiv_x . \, x \,\varepsilon\, \beta] \supset_{\alpha\beta} . \, \alpha = \beta$.

THEOREM 33. $[\Sigma(\varphi) \,.\, \sim \varphi(x)] \supset_{\varphi x} \sim . \, x \,\varepsilon\, K(\varphi)$.

$\mathbf{D}_1$: $\psi(y)$.
$\mathbf{C}_1$: $y \,\varepsilon\, K(\psi)$ —V^2, Thm 29, $\mathbf{D}_1$.
$\mathbf{C}_2$: $E(K(\psi))$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $K(\psi) = K(\psi)$ —V, Thm 1, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{D}_2$: $\Sigma(\psi) \,.\, \sim \psi(x)$.
$\mathbf{C}_4$: $\sim \psi(x)$ —V^2, 16, $\mathbf{D}_2$.
$\mathbf{C}_5$: $K(\psi) = K(\psi) \,.\, \sim \psi(x)$ —V^2, 14, $\mathbf{C}_3$, $\mathbf{C}_4$; $\mathbf{D}_1$, $\mathbf{D}_2$.
$\mathbf{C}_6$: $\Sigma\varphi \,.\, K(\psi) = K(\varphi) \,.\, \sim \varphi(x)$ —IV, $\mathbf{C}_5$; $\mathbf{D}_1$, $\mathbf{D}_2$.

C$_7$: $\sim . [K(\psi) = K(\varphi)] \supset_\varphi \varphi(x)$ —V^2, 17, **C**$_6$; **D**$_1$, **D**$_2$.

C$_8$: $\sim . x \, \varepsilon \, K(\psi)$ —conv, **C**$_7$; **D**$_1$, **D**$_2$.

C$_9$: $\Sigma(\psi)$ —V^2, 15, **D**$_2$.

C$_{10}$: $\sim . x \, \varepsilon \, K(\psi)$ —Thm III, **C**$_8$, **C**$_9$; **D**$_2$.

A$_1$: $\Sigma \psi \Sigma x . \Sigma(\psi) . \sim \psi(x)$ —I, **A**$_4$ under Thm 8.

A$_2$: $[\Sigma(\psi) . \sim \psi(x)] \supset_{\psi x} \sim . x \, \varepsilon \, K(\psi)$ —Thm I Cor, **C**$_{10}$, **A**$_1$.

A$_3$: $[\Sigma(\varphi) . \sim \varphi(x)] \supset_{\varphi x} \sim . x \, \varepsilon \, K(\varphi)$ —I, **A**$_2$.

THEOREM 34. $\Sigma y [\sim . y \, \varepsilon \, \alpha] \supset_\alpha G x . x \, \varepsilon \, \alpha$.

D$_1$: $x \, \varepsilon \, \alpha$.

C$_1$: $\Sigma x . x \, \varepsilon \, \alpha$ —IV, **D**$_1$.

C$_2$: $\Pi(\Delta x . x \, \varepsilon \, \alpha, \lambda x . x \, \varepsilon \, \alpha)$ —V, Thm 7, **C**$_1$; **D**$_1$.

C$_3$: $\Pi(\lambda x . x \, \varepsilon \, \alpha, \Delta x . x \, \varepsilon \, \alpha)$ —V, Thm 6, **C**$_1$; **D**$_1$.

C$_4$: $Q(\Delta x . x \, \varepsilon \, \alpha, \lambda x . x \, \varepsilon \, \alpha)$ —V^2, 14, **C**$_2$, **C**$_3$; **D**$_1$.

C$_5$: $K(\Delta x . x \, \varepsilon \, \alpha) = K x . x \, \varepsilon \, \alpha$ —V^2, Thm 25, **C**$_4$; **D**$_1$.

C$_6$: $\alpha = K x . x \, \varepsilon \, \alpha$ —V, Thm 32, **C**$_1$; **D**$_1$.

C$_7$: $K x [x \, \varepsilon \, \alpha] = \alpha$ —V^2, Thm 2, **C**$_6$; **D**$_1$.

C$_8$: $K(\Delta x . x \, \varepsilon \, \alpha) = \alpha$ —V^3, Thm 3, **C**$_5$, **C**$_7$; **D**$_1$.

C$_9$: $\{\Delta x . x \, \varepsilon \, \alpha\}(x)$ —V, **C**$_3$, **D**$_1$; **D**$_1$.

C$_{10}$: $\Sigma(\Delta x . x \, \varepsilon \, \alpha)$ —IV, **C**$_9$; **D**$_1$.

D$_2$: $\sim \{\Delta x . x \, \varepsilon \, \alpha\}(z)$.

C$_{11}$: $\sim . z \, \varepsilon \, K(\Delta x . x \, \varepsilon \, \alpha)$ —V^2, Thm 33, **C**$_{10}$, **D**$_2$; **D**$_1$, **D**$_2$.

C$_{12}$: $\sim . z \, \varepsilon \, \alpha$ —V, **C**$_8$, **C**$_{11}$; **D**$_1$, **D**$_2$.

D$_3$: $\sim . y \, \varepsilon \, \alpha$.

C$_{13}$: $\sim \{\Delta x . x \, \varepsilon \, \alpha\}(y)$ —V^2, Thm 8, **C**$_1$, **D**$_3$; **D**$_1$, **D**$_3$.

C$_{14}$: $\Sigma z \sim \{\Delta x . x \, \varepsilon \, \alpha\}(z)$ —IV, **C**$_{13}$; **D**$_1$, **D**$_3$.

C$_{15}$: $\sim \{\Delta x . x \varepsilon \alpha\}(z) \supset_z \sim . z \, \varepsilon \, \alpha$ —Thm I, **C**$_{12}$, **C**$_{14}$; **D**$_1$, **D**$_3$.

C$_{16}$: $G x . x \, \varepsilon \, \alpha$ —conv, **C**$_{15}$; **D**$_1$, **D**$_3$.

D$_4$: $\Sigma y \sim . y \, \varepsilon \, \alpha$.

C$_{17}$: $\Sigma x . x \, \varepsilon \, \alpha$ —V, Thm 31 Cor 2, **D**$_4$.

C$_{18}$: $G x . x \, \varepsilon \, \alpha$ —Thm I, **C**$_{16}$, **C**$_{17}$; **D**$_3$, **D**$_4$.

C$_{19}$: $G x . x \, \varepsilon \, \alpha$ —Thm I, **C**$_{18}$, **D**$_4$; **D**$_4$.

A$_1$: $\Sigma(\Pi(E))$ —IV, **A**$_3$ under Thm 16.

A$_2$: $\sim . [\sim] \, \varepsilon \, K(\Pi(E))$ —V^2, Thm 33, **A**$_1$, Thm 10.

A$_3$: $\Sigma \alpha \Sigma y \sim . y \, \varepsilon \, \alpha$ —IV^2, **A**$_2$.

A$_4$: $\Sigma y [\sim . y \, \varepsilon \, \alpha] \supset_\alpha G x . x \, \varepsilon \, \alpha$ —Thm I, **C**$_{19}$, **A**$_3$.

THEOREM 35. $[\sim . x \varepsilon \alpha] \supset_{x\alpha} . x \varepsilon O(\alpha)$.

$\mathbf{D}_1$: $\sim . x \varepsilon \alpha$.

$\mathbf{C}_1$: $x \varepsilon K\pi \sim . \pi \varepsilon \alpha$ —V², Thm 29, $\mathbf{D}_1$.

$\mathbf{C}_2$: $x \varepsilon O(\alpha)$ —III, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma x \Sigma \alpha \sim . x \varepsilon \alpha$ —IV², $\mathbf{A}_2$ under Thm 34.

$\mathbf{A}_2$: $[\sim . x \varepsilon \alpha] \supset_{x\alpha} . x \varepsilon O(\alpha)$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_1$.

THEOREM 36. $[x \varepsilon O(\alpha)] \supset_{x\alpha} \sim . x \varepsilon \alpha$.

$\mathbf{D}_1$: $x \varepsilon O(\alpha)$.

$\mathbf{C}_1$: $x \varepsilon K\pi \sim . \pi \varepsilon \alpha$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim . x \varepsilon \alpha$ —V², Thm 30, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $[\sim] \varepsilon O(K(\Pi(E)))$ —V², Thm 35, $\mathbf{A}_2$ under Thm 34.

$\mathbf{A}_2$: $\Sigma x \Sigma \alpha . x \varepsilon O(\alpha)$ —IV², $\mathbf{A}_1$.

$\mathbf{A}_3$: $[x \varepsilon O(\alpha)] \supset_{x\alpha} \sim . x \varepsilon \alpha$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_2$.

THEOREM 37. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [x \varepsilon \alpha] \supset_x \sim . x \varepsilon O(\alpha)$.

$\mathbf{D}_1$: $x \varepsilon \alpha$.

$\mathbf{C}_1$: $\sim \sim . x \varepsilon \alpha$ —V, 26, $\mathbf{D}_1$.

$\mathbf{D}_2$: $\Sigma y \sim . y \varepsilon \alpha$.

$\mathbf{C}_2$: $\sim . x \varepsilon K y \sim . y \varepsilon \alpha$ —V², Thm 33, $\mathbf{D}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\sim . x \varepsilon O(\alpha)$ —conv, $\mathbf{C}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma x . x \varepsilon \alpha$ —V, Thm 31 Cor 2, $\mathbf{D}_2$.

$\mathbf{C}_5$: $[x \varepsilon \alpha] \supset_x \sim . x \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_3$, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{A}_1$: $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [x \varepsilon \alpha] \supset_x \sim . x \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_5$, $\mathbf{A}_3$ under Thm 34.

COROLLARY 1. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [x \varepsilon \alpha] \supset_x x \varepsilon O(O(\alpha))$.

COROLLARY 2. $[\sim . x \varepsilon \alpha] \supset_{x\alpha} \sim . x \varepsilon O(O(\alpha))$.

COROLLARY 3. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [\sim . x \varepsilon O(O(\alpha))] \supset_x \sim . x \varepsilon \alpha$.

COROLLARY 4. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [\sim . x \varepsilon O(O(\alpha))] \supset_x . x \varepsilon O(\alpha)$.

THEOREM 38. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha G x \sim . x \varepsilon O(\alpha)$.

$\mathbf{D}_1$: $\Sigma y \sim . y \varepsilon \alpha$.

$\mathbf{D}_2$: $x \varepsilon \alpha$.

$\mathbf{C}_1$: $\sim . x \varepsilon O(\alpha)$ —V², Thm 37, $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_2$: $\{\Lambda y \sim . y \varepsilon O(\alpha)\}(x)$ —V², Thm 6 Cor, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\Sigma y . y \varepsilon \alpha$ —V, Thm 31 Cor 2, $\mathbf{D}_1$.

$\mathbf{C}_4$: $\Pi(\lambda y . y \varepsilon \alpha, \Lambda y \sim . y \varepsilon O(\alpha))$ —Thm II, $\mathbf{C}_2$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $G y . y \varepsilon \alpha$ —V, Thm 34, $\mathbf{D}_1$.

$\mathbf{D}_3$: $\sim \{A y \sim . y \varepsilon O(\alpha)\} (z)$.

$\mathbf{C}_6$: $\sim . z \varepsilon \alpha$ —V^3, Thm 20, $\mathbf{C}_5$, $\mathbf{C}_4$, $\mathbf{D}_3$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_7$: $z \varepsilon O(\alpha)$ —V^2, Thm 35, $\mathbf{C}_6$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_8$: $\sim \sim . z \varepsilon O(\alpha)$ —V, 26, $\mathbf{C}_7$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{D}_4$: $\sim . u \varepsilon \alpha$.

$\mathbf{C}_9$: $u \varepsilon O(\alpha)$ —V^2, Thm 35, $\mathbf{D}_4$.

$\mathbf{C}_{10}$: $\sim \sim . u \varepsilon O(\alpha)$ —V, 26, $\mathbf{C}_9$; $\mathbf{D}_4$.

$\mathbf{C}_{11}$: $\Sigma y \sim . y \varepsilon O(\alpha)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_{12}$: $\sim \{A y \sim . y \varepsilon O(\alpha)\} (u)$ —V^2, Thm 8, $\mathbf{C}_{11}$, $\mathbf{C}_{10}$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_4$.

$\mathbf{C}_{13}$: $\Sigma z \sim \{A y \sim . y \varepsilon O(\alpha)\} (z)$ —IV, $\mathbf{C}_{12}$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_4$.

$\mathbf{C}_{14}$: $\Sigma z \sim \{A y \sim . y \varepsilon O(\alpha)\} (z)$ —Thm I, $\mathbf{C}_{13}$, $\mathbf{D}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_{15}$: $\Sigma z \sim \{A y \sim . y \varepsilon O(\alpha)\} (z)$ —Thm I, $\mathbf{C}_{14}$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_{16}$: $\sim \{A y \sim . y \varepsilon O(\alpha)\} (z) \supset_z \sim \sim . z \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_8$, $\mathbf{C}_{15}$; $\mathbf{D}_1$.

$\mathbf{C}_{17}$: $G x \sim . x \varepsilon O(\alpha)$ —conv, $\mathbf{C}_{16}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha G x \sim . x \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_{17}$, $\mathbf{A}_3$ under Thm 34.

7. **The Russell paradox.** The existence of the propositional function $\lambda \varphi . \sim \varphi(\varphi)$ follows readily from our assumptions. In fact, it can be shown that the value of this function is a true proposition for the value $\lambda x . \sim E(x)$ of the independent variable (among others), and that the value of the function is a false proposition for each of the values E and Σ of the independent variable (among others). If, however, we inquire whether the value of the function $\lambda \varphi . \sim \varphi(\varphi)$ is a true proposition or a false proposition for the value $\lambda \varphi . \sim \varphi(\varphi)$ of the independent variable, we are led at once into difficulty. For let $\mathfrak{P}$ stand for the formula $\{\lambda \varphi . \sim \varphi(\varphi)\} (\lambda \varphi . \sim \varphi(\varphi))$. Then $\mathfrak{P}$ is convertible into $\sim . \mathfrak{P}$, and $\sim . \mathfrak{P}$ is convertible into $\mathfrak{P}$. And hence if $\mathfrak{P}$ be a true proposition it must also be a false proposition, and if $\mathfrak{P}$ be a false proposition it must also be a true proposition.

We are not, however, forced to the conclusion that our system is self-contradictory. For no way appears by which we can prove either of the formulas $\mathfrak{P}$ or $\sim . \mathfrak{P}$, and in the absence of such a proof there is no obvious method of obtaining a contradiction out of the situation just described. There is, in fact, no reason to suppose that the propositional function $\lambda \varphi . \sim \varphi(\varphi)$ is significant for all values of the independent variable, or that the formula $\mathfrak{P}$ is anything but a meaningless aggregate of symbols.

Of course, if there were any way of constructing a propositional function **F**, equivalent to $\lambda \varphi . \sim \varphi(\varphi)$ and significant for all values of the independent variable, then a contradiction could be obtained by considering the formula $\{\mathbf{F}\}(\mathbf{F})$. But there is no reason to suppose that such a function $\mathbf{F}$ can be constructed. In particular, it does not appear that $\Lambda\varphi\,.\sim\varphi(\varphi)$ could be taken as $\mathbf{F}$.

The property of the formula $\mathfrak{P}$, that its falsehood is provable as a consequence of its truth, and its truth as a consequence of its falsehood, is shared by many other formulas. We cite as examples the formulas $\{\lambda\varphi\,\lambda x\,.\sim\varphi(x,\varphi)\}\,(\lambda\varphi\,\lambda x\,.\sim\varphi(x,\varphi),\,\lambda\varphi\,\lambda x\,.\sim\varphi(x,\varphi))$, and $\{\Lambda\varphi\,.\sim\varphi(\varphi)\}$ $(\Lambda\varphi\,.\sim\varphi(\varphi))$, and $Kx[\sim.\,x\,\varepsilon\,x]\,\varepsilon\,Kx\sim.\,x\,\varepsilon\,x$.

Let $\mathfrak{Q}$ stand for the formula $Kx[\sim.x\varepsilon x]\,\varepsilon\,Kx\sim.x\varepsilon x$. Then if $\mathfrak{Q}$ were true, Theorem 30would enable us to prove $\sim.\mathfrak{Q}$. And if $\sim.\mathfrak{Q}$ were true, Theorem 29 would enable us to prove $\mathfrak{Q}$. This means that if we assume $\mathfrak{Q}$ we can deduce the contradiction $\mathfrak{Q}\,.\sim.\,\mathfrak{Q}$, and hence, under the classical principle of reductio ad absurdum, we should be justified in inferring that $\mathfrak{Q}$ is false, that is, $\sim.\mathfrak{Q}$. And once we had $\sim.\mathfrak{Q}$, we could infer $\mathfrak{Q}$ by Theorem 29.

In this way we see that the classical principle of reductio ad absurdum, and the assumption that for every propositional function there is an equivalent function significant for all values of the independent variable, are from a certain point of view the same.

And our omission of the principle of reductio ad absurdum from our list of assumptions is justified by reference to cases, like that just discussed, to which the principle appears to be inapplicable.

8. **Properties of the expression $\lambda\varphi.\sim\varphi(x(\varphi))$.** We now proceed to the proof of a number of theorems which bear a close relation to the Russell paradox.

THEOREM 39. $\Sigma x\{\lambda\varphi\,.\sim\varphi(x(\varphi))\}\Big(\lambda\varphi\,.\sim\varphi(x(\varphi))\Big)$.

$\mathbf{A}_1$: $Ep\,.\,q\supset_q pq$ —IV, 14.

$\mathbf{A}_2$: $E(E)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $E(E)\,E(E)$ —V^2, 14, $\mathbf{A}_2$, $\mathbf{A}_2$.

$\mathbf{A}_4$: $Ev\,.\,E(E)\,E(v)$ —IV, $\mathbf{A}_3$.

$\mathbf{A}_5$: $\sim\sim Ev\,.\,E(E)\,E(v)$ —V, 26, $\mathbf{A}_4$.

$\mathbf{A}_6$: $E\varphi\,.\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))$ —IV, $\mathbf{A}_5$.

$\mathbf{A}_7$: $E\varphi[\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))]\,E(E)$ —V^2, 14, $\mathbf{A}_6$, $\mathbf{A}_2$.

$\mathbf{A}_8$: $Ew\,.\,E\varphi[\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))]\,E(w)$ —IV, $\mathbf{A}_7$.

$\mathbf{A}_9$: $Ew[E\varphi[\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))]\,E(w)]\,E(E)$ —V^2, 14, $\mathbf{A}_8$, $\mathbf{A}_2$.

$\mathbf{A}_{10}$: $Ey\,.\,Ew[E\varphi[\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))]\,E(w)]\,E(y)$ —IV, $\mathbf{A}_9$.

$\mathbf{A}_{11}$: $E\varphi[\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))]\,.\,Ey\,.\,Ew[E\varphi[\sim\varphi(\lambda v\,.\,E(\varphi)\,E(v))]\,E(w)]\,E(y)$ —V^2, 14, $\mathbf{A}_6$, $\mathbf{A}_{10}$.

$\mathbf{A}_{12}$: $\sim\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) E(v))] . Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi)\, E(v))]\, E(w)]$
$E(y)$ —V, 26, $\mathbf{A}_{11}$.

$\mathbf{A}_{13}$: $\{\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi)\, E(v))\}\,(\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi)\, E(v)))$ —conv, $\mathbf{A}_{12}$.

$\mathbf{A}_{14}$: $\Sigma x\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$ —IV, $\mathbf{A}_{13}$.

THEOREM 40. $\Sigma x \sim . \{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$.

$\mathbf{A}_1$: $Ep . \sim q \supset_q \sim . pq$ —IV, 21.

$\mathbf{A}_2$: $E(E)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\sim\sim E(E)$ —IV, 26, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\sim . E(E) . \sim E(E)$ —V^2, 21, $\mathbf{A}_2$, $\mathbf{A}_3$.

$\mathbf{A}_5$: $Ev . E(E) . \sim E(v)$ —IV, $\mathbf{A}_4$.

$\mathbf{A}_6$: $\sim\sim Ev . E(E) . \sim E(v)$ —V, 26, $\mathbf{A}_5$.

$\mathbf{A}_7$: $E\varphi . \sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))$ —IV, $\mathbf{A}_6$.

$\mathbf{A}_8$: $\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(E)$ —V^2, 21, $\mathbf{A}_7$, $\mathbf{A}_3$.

$\mathbf{A}_9$: $Ew . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)$ —IV, $\mathbf{A}_8$.

$\mathbf{A}_{10}$: $\sim . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(E)$
—V^2, 21, $\mathbf{A}_9$, $\mathbf{A}_3$.

$\mathbf{A}_{11}$: $Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(y)$ —IV, $\mathbf{A}_9$.

$\mathbf{A}_{12}$: $\sim\sim Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(y)$
—V, 26, $\mathbf{A}_{11}$.

$\mathbf{A}_{13}$: $\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi)$
$. \sim E(v))] . \sim E(w)] . \sim E(y)$ —V^2, 21, $\mathbf{A}_7$, $\mathbf{A}_{12}$.

$\mathbf{A}_{14}$: $\sim\sim\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi)$
$. \sim E(v))] . \sim E(w)] . \sim E(y)$ —V, 26, $\mathbf{A}_{13}$.

$\mathbf{A}_{15}$: $\sim . \{\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))\}\,(\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi) . \sim E(v)))$
—conv, $\mathbf{A}_{14}$.

$\mathbf{A}_{16}$: $\Sigma x \sim . \{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$ —IV, $\mathbf{A}_{15}$.

THEOREM 41. $\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi))) \supset_x . x \neq \lambda zz$.

$\mathbf{D}_1$: $\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$.

$\mathbf{C}_1$: $\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(x(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim(x(\varphi)))$ —V, 26, $\mathbf{D}_1$.

$\mathbf{C}_3$: $\sim\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(\{\lambda zz\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big)$ —III, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(x(\lambda\varphi . \sim\varphi(x(\varphi)))\big) . \sim\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}$
$\big(\{\lambda zz\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big)$ —V^2, 14, $\mathbf{C}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $\Sigma y . y(x) . \sim y(\lambda zz)$ —IV, $\mathbf{C}_4$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $x \ddagger \lambda zz$ —V², 17, $\mathbf{C}_5$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi))) \supset_x . \, x \ddagger \lambda zz$ —Thm I, $\mathbf{C}_6$, Thm 39.

COROLLARY. $\sim . \{\varDelta x\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))\}\,(\lambda zz)$.

THEOREM 42. $\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi))) \supset_x . \, x \ddagger \lambda zz$.

$\mathbf{D}_1$: $\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))$.

$\mathbf{C}_1$: $\sim\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,\big(x(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))\big)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim\sim\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))$ —V, 26, $\mathbf{D}_1$.

$\mathbf{C}_3$: $\sim\sim\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,\big(\{\lambda zz\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))\big)$ —III, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $\sim\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,\big(x(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))\big) \,.\, \sim\sim\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}$
$\big(\{\lambda zz\}\,(\lambda\varphi \,.\, \sim\varphi(x(x)))\big)$ —V², 14, $\mathbf{C}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $\Sigma y \,.\, y(x) \,.\, \sim y(\lambda zz)$ —IV, $\mathbf{C}_4$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $x \ddagger \lambda zz$ —V², 17, $\mathbf{C}_5$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi))) \supset_x . \, x \ddagger \lambda zz$ —Thm I, $\mathbf{C}_6$, Thm 40.

COROLLARY. $\sim . \{\varDelta x \sim\{\lambda\varphi \,.\, \sim\varphi(x(\varphi))\}\,(\lambda\varphi \,.\, \sim\varphi(x(\varphi)))\}\,(\lambda zz)$.

9. **Positive integers.** We define:

$$1 \longrightarrow \lambda f \lambda x \,.\, f(x).$$
$$S \longrightarrow \lambda \varrho \lambda f \lambda x \,.\, f(\varrho(f, x)).$$
$$N \longrightarrow \lambda y \,.\, [\varphi(1) \,.\, \varphi(x) \supset_x \varphi(S(x))] \supset_\varphi \varphi(y).$$

The symbol 1 is to be read, "One", and $S(\mathbf{A})$ is to be read, "The successor of $\mathbf{A}$", and $N(\mathbf{A})$ is to be read, "$\mathbf{A}$ is a positive integer".

If we define $2 \longrightarrow S(1)$, and $3 \longrightarrow S(2)$, and so on, we find that 2 is convertible into $\lambda f \lambda x \,.\, f(f(x))$, and 3 is convertible into $\lambda f \lambda x \,.\, f(f(f(x)))$, and so on. The form of these expressions provides a convenient method of writing definitions by induction, as we may illustrate in the case of the definition of the sum, $m + n$, of two positive integers m and n. The equivalent of the recursion formulas, $m + 1 = S(m)$, and $m + (k + 1) = S(m + k)$, is, in fact, obtained by defining:

$$+ \longrightarrow \lambda m \lambda n \,.\, n(S, m).$$

And the operations of subtraction and multiplication may then be defined as follows:

$$- \longrightarrow \lambda r \lambda s \iota x \,.\, \{+\}\,(x, s) = r.$$
$$\times \longrightarrow \lambda m \lambda n \,.\, \{-\}\,(m(n(S), 1), 1).$$

Peano's axioms for the positive integers[9] may be expressed in our notation as follows:

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[9] Rivista di Mathematica, vol. 1 (1891), pp. 87–102.

1. $N(1)$.
2. $N(x) \supset_x N(S(x))$.
3. $[N(x) \,.\, N(y) \,.\, S(x) = S(y)] \supset_{xy} . \, x = y$.
4. $N(x) \supset_x . \, S(x) \neq 1$.
5. $[\varphi(1) \,.\, \varphi(x) \supset_x \varphi(S(x))] \supset_\varphi . \, N(y) \supset_y \varphi(y)$.

It is believed that each of these five propositions will turn out to be a theorem which can be proved as a consequence of our postulates. And indeed proofs of 1, 2, and 5 are immediately evident.

Our program is to develop the theory of positive integers on the basis which we have just been describing, and then, by known methods or appropriate modifications of them, to proceed to a theory of rational numbers and a theory of real numbers.

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