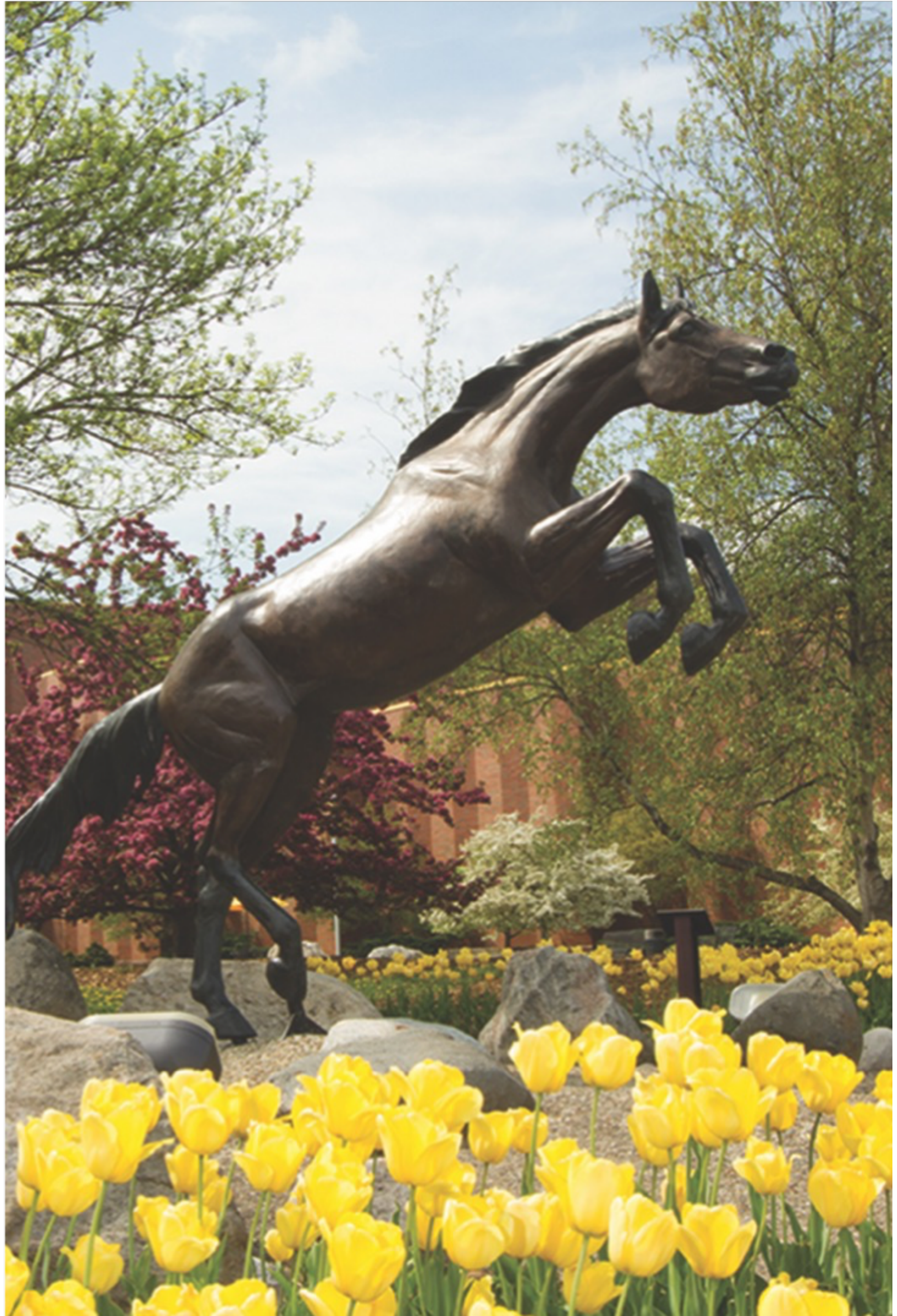
Western Michigan University

CS 5541 – Computer Systems

"Based on lecture notes developed by Randal E. Bryant and David R. O'Hallaron in conjunction with their textbook "Computer Systems: A Programmer's Perspective"



Module 1

Representing Numbers

Part 5 – More Floating Point

From: Computer Systems, Chapter 2

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Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Floating Point Operations: Basic Idea

- $\mathbf{x} +_f \mathbf{y} = \mathrm{Round}(\mathbf{x} + \mathbf{y})$

- $\mathbf{x} \times_f \mathbf{y} = \mathrm{Round}(\mathbf{x} \times \mathbf{y})$

- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down ($-\infty$)	$1	$1	$1	$2	–$2
Round up ($+\infty$)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

Closer Look at Round-To-Even

- Default Rounding Mode
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

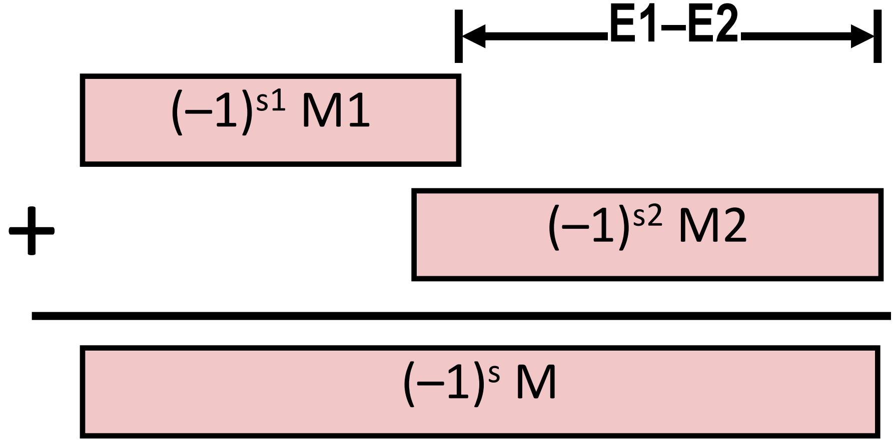
FP Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2
- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit **`frac`** precision
- Implementation
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume E1 > E2

Get binary points lined up



- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1

- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit **`frac`** precision

Mathematical Properties of FP Add

- Compare to those of Abelian Group
  - Closed under addition? Yes
    - But may generate infinity or NaN
  - Commutative? Yes
  - Associative? No
    - Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0 is additive identity? Yes
  - Every element has additive inverse? Almost
    - Yes, except for infinities & NaNs
- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$? Almost
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- Compare to Commutative Ring
  - Closed under multiplication? Yes
    - But may generate infinity or NaN
  - Multiplication Commutative? Yes
  - Multiplication is Associative? No
    - Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`
  - 1 is multiplicative identity? Yes
  - Multiplication distributes over addition? No
    - Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)=0.0, 1e20*1e20 – 1e20*1e20 =NaN`
- Monotonicity
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? Almost
    - Except for infinities & NaNs

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Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (double)(float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0      ⇒  ((d*2) < 0.0)`
- `d > f        ⇒  -f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

Creating Floating Point Number

- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

s	exp	frac
1	4-bits	3-bits

- Case Study
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Normalize

s	exp	frac
1	4-bits	3-bits

- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	Fraction	GRS	Incr?	Rounded
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

Interesting Numbers

`{single,double}`

Description	*exp*	*frac*	*Numeric Value*
• Zero	00…00	00…00	0.0
• Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
• Single ≈ 1.4×10^{-45}			
• Double ≈ 4.9×10^{-324}			
• Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
• Single ≈ 1.18×10^{-38}			
• Double ≈ 2.2×10^{-308}			
• Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
• Just larger than largest denormalized			
• One	01…11	00…00	1.0
• Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
• Single ≈ 3.4×10^{38}			
• Double ≈ 1.8×10^{308}			

Module 1 (Part 5)

Summary

- Implement floating point addition and multiplication
- Implement floating point Encoding (Normilization, Rounding, etc.)