

**Heidelberg University**  
**Institute of Computer Science**  
**Database Systems Research Group**

**Lecture: Complex Network Analysis**

Prof. Dr. Michael Gertz

**Assignment 1**  
**Graph Theory and Networks in Python**

[https://github.com/nilskre/CNA\\_assignments](https://github.com/nilskre/CNA_assignments)

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## 1 Problem 3-1 Clustering Coefficients

1. Given an undirected complete graph with  $N$  nodes, calculate the total number of triangles. Note that the order of nodes matters, e.g.,  $A \rightarrow B \rightarrow C$  is not the same as  $C \rightarrow B \rightarrow A$ .

First one needs to find the number of unique triangles in the complete graph, which is calculated by  $\binom{N}{3}$ . For the calculation of the global clustering coefficient, this is the amount of triangles to be used and matches the post on Moodle from Shideh Almasian:

*I delete the post and the comments because my answer was wrong. So the correct answer is that "the order of nodes does \*\*not\*\* matter",  $ABC$  is a single triangle and that counts towards the global clustering coefficient.*

In case one additionally wants to define the total number of triangles by the order of nodes, one has multiply with  $3!$ . So the total number of triangles in a complete graph asked by the task description is  $3! \cdot \binom{N}{3}$ .

2. Can the number of triangles in a graph be larger than the number of edges? Are you able to find a graph with more triangles than edges? If so, draw such a graph.

We use the stricter definition of the number of triangles in a graph that does not consider the order of nodes. We again use the example of a complete graph. The number of edges in such a graph is defined by  $L = \binom{N}{2} = \frac{N(N-1)}{2}$ .  $\binom{N}{3} > \binom{N}{2}$  holds for  $N = 6$ . See the following graph in Figure 1:

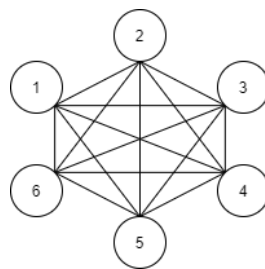


Figure 1: A complete graph with  $N = 6$ .

This means the number of triangles in a graph can be larger than the number of edges.

3. Prove that the number of connected triples in an undirected graph is equal to half the **sum** of non-zero off-diagonal entries of the adjacency matrix  $A^2$ , where  $A$  denotes the adjacency matrix.

Let  $(n_i, n_j, n_k)$  be a connected triple in a graph with  $N \geq 3$ . This means that there is a path from  $n_i$  to  $n_k$  of length 2. We know that for such a path is must hold  $A_{ij}A_{jk} = 1$ . Therefore,  $A^2$  holds paths of length 2, i.e. connected triples in its non-zero off-diagonal entries. As an illustration, consider the following simple graph in Figure 2 and the following Equation 3:

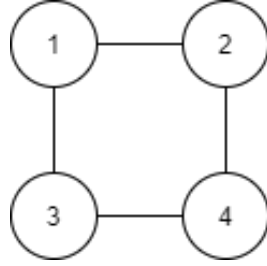


Figure 2: A simple graph with 4 connected triples.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Half the sum of all non-zero off-diagonal entries of the adjacency matrix  $A^2$  equals exactly the number of connected triples, namely 4.

4. Given a graph with its adjacency matrix  $A$ , find an expression for the global clustering coefficient in terms of the elements of the adjacency matrix. Notice that the number of triangles in a graph is **related** to the trace of the matrix  $A^3$ .

We know

$$C = \frac{3 \cdot \#triangles}{\#connected\ triples}.$$

The also know that the relation between the number of triangles  $n_{triangles}$  and the trace of  $A^3$  is exactly  $6 \cdot n_{triangles} = tr(A^3)$ . Therefore it follows:

$$C = \frac{tr(A^3)}{\sum_{i \neq j} A_{ij}^2}$$

## 2 Problem 3-2 Parametrized Random Networks

1. Determine the average degree. We use the equation  $\langle k \rangle = p(N - 1)$ :

- $a = 0.5, z = 1$

$$\langle k \rangle = \lim_{N \rightarrow \infty} \frac{0.5}{N} (N - 1) = \lim_{N \rightarrow \infty} 0.5 - \frac{0.5}{N} = 0.5 \quad (1)$$

$\langle k \rangle = 0.5$  and  $\langle k \rangle < 1$ , which means there is **no GC**.

- $a = 2, z = 1$

$$\langle k \rangle = \lim_{N \rightarrow \infty} \frac{2}{N} (N - 1) = \lim_{N \rightarrow \infty} 2 - \frac{2}{N} = 2 \quad (2)$$

$\langle k \rangle = 2$  and  $\langle k \rangle > 1$ , which means there is a **GC**.

- $a > 0, z = 2$

$$\langle k \rangle = \lim_{N \rightarrow \infty} \frac{a}{N^2} (N - 1) = \lim_{N \rightarrow \infty} \frac{a}{N} - \frac{a}{N^2} = 0 \quad (3)$$

$\langle k \rangle = 0$  and  $\langle k \rangle < 1$ , which means there is **no GC**.

- $a > 0, z = 0.5$

$$\langle k \rangle = \lim_{N \rightarrow \infty} \frac{a}{\sqrt{N}} (N - 1) = \lim_{N \rightarrow \infty} a\sqrt{N} - \frac{a}{\sqrt{N}} = \infty \quad (4)$$

$\langle k \rangle = \infty$  and  $\langle k \rangle > 1$ , which means there is a **GC**.

2. Determine the average degree. Again with  $\langle k \rangle = p(N - 1)$ :

$$\langle k \rangle = \lim_{N \rightarrow \infty} \frac{a}{N^z} (N - 1) = \lim_{N \rightarrow \infty} aN^{1-z} - \frac{a}{N^z} \quad (5)$$

This means:

$$\langle k \rangle = \begin{cases} \infty, & \text{if } z < 1 \\ 0, & \text{if } z \geq 1 \end{cases}$$

3. Determine the conditions on  $a$  and  $z$  for which these random networks are critical, again in the limit  $N \rightarrow \infty$ .

Networks are critical, if  $\langle k \rangle = 1$

We have this equation from the previous subtask:

$$\langle k \rangle = \lim_{N \rightarrow \infty} aN^{1-z} - \frac{a}{N^z} \quad (6)$$

Since here  $1 - z$  must be 0 and  $aN^0$  must be 1, the conditions are:  $z$  must be 1 and  $a$  must be 1.