**Heidelberg University**
**Institute of Computer Science**
**Database Systems Research Group**

**Lecture: Complex Network Analysis**

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# Assignment 7

## Degree Assortativity and Robustness

`https://github.com/nilskre/CNA_assignments`

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# 1 Problem 7-2 Molloy-Reed Criterion

Consider a configuration model network that has nodes of degree 1, 2, and 3 only, with probabilities $p_1$, $p_2$, and $p_3$, respectively. The degree distribution is given by:

$$p_k = \delta_{k,1}p_1 + \delta_{k,2}p_2 + \delta_{k,3}p_3, \begin{cases} \delta_{k,1} = 3 & \text{if } k = 1 \\ \delta_{k,2} = 2 & \text{if } k = 2 \\ \delta_{k,3} = 2 & \text{if } k = 3 \end{cases}$$

1. Compute the first moment $\langle k \rangle$ and the second moment $\langle k^2 \rangle$ of the degree distribution.

   We assume $\delta_{k,k'}$ to be the dirac-delta-function. For the first and second moment is follows:

$$\langle k \rangle = \sum_{k=1}^{3} kp_k = 3p_1 + 4p_2 + 6p_3$$

$$\langle k^2 \rangle = \sum_{k=1}^{3} k^2 p_k = 3p_1 + 8p_2 + 18p_3$$

2. Using the Molloy-Reed criterion, show that there is a giant component if and only if $p_1 < 3p_3$.

   The Molloy-Reed criteria propagates a giant component exists in case $\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} > 2$. $\kappa$ can be calculated as:

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{3p_1 + 8p_2 + 18p_3}{3p_1 + 4p_2 + 6p_3}$$

   One way for $\kappa$ being greater than 2 is when all components of the sum in the numerator are at least twice as big as the matching component of the sum in the denominator (with one component being slightly bigger than twice as big for $\kappa$ being strictly bigger than 2). This means $6p_3 > 3p_1$ and therefore $2p_3 > p_1$. We therefore assume the thesis in the task desciption is wrong.

3. In terms of the structure of the network, discuss the meaning of the condition $p_3 < 3p_1$. Why does the result not depend on $p_2$?

The fact that in the calculation of $\kappa$ the component of $p_2$ already goes in twice as big in the numerator ($8p_2 > 4p_2$), makes the goal of $\kappa > 2$ only dependent on the scale of $p_1$ compared to $p_3$. If $p_3 < 3p_1$ holds, a giant component only exists in case $\frac{1}{3}p_3 < p_1 < 2p_3$. Other than that, we cannot really see what $p_3 < 3p_1$ should lead to.

# 2 Problem 7-4 Random Failures in Uncorrelated Networks

Compute the critical threshold $f_c$ for each of the following degree distributions, under the assumption that the networks do not exhibit any degree correlation.

1. Poisson distribution, i.e.,

$$p_k = e^{-\mu}\frac{\mu^k}{k!}$$

2. Discrete exponential distribution, i.e.,

$$p_k = (1 - e^{-\lambda})e^{-\lambda k}$$

3. Dirac delta distribution, i.e.,

$$p_k = \delta_{k,k_0} = \begin{cases} 1 & \text{if } k = k_0, \\ 0 & \text{otherwise.} \end{cases}$$

Discuss the consequences of your results for network robustness.
*Hint:* You may use the first and second moment from the lecture or other literature without a proof.

We know $f_c$ can be calculated by:

$$f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$$

1. For the poisson distribution, we receive:

$$f_c = 1 - \frac{1}{\frac{\mu^2+\mu}{\mu} - 1} = 1 - \frac{1}{\mu}$$

This means the higher $\mu$ the more robust the network is towards random failures. If $\mu \to \infty$ we receive maximum robustness as all nodes would theoretically need to fail for the network to be considered fragmented (even if this does not make sense since there would not be a network present anymore).

2. For the discrete exponential distribution, we receive (using Wolfram Alpha):

$$\langle k \rangle = \frac{(1 - e^{-\lambda})}{\lambda^2}$$

$$\langle k^2 \rangle = \frac{2(1 - e^{-\lambda})}{\lambda^3}$$

$$f_c = 1 - \frac{1}{\frac{2}{\lambda} - 1}$$

   This means the lower $\lambda$ the more robust the network is towards random failures. If $\lambda \to 0$ we receive maximum robustness.

3. For the dirac delta distribution, we receive:

$$f_c = 1 - \frac{1}{\frac{k_0^2}{k_0} - 1} = 1 - \frac{1}{k_0 - 1}$$

   Similar to the poisson distribution, the dirac delta distribution becomes more robust towards random failures the higher $k_0$ is. This becomes maximum for clique like structures.