



Deduction in Matching Logic

Master's Thesis

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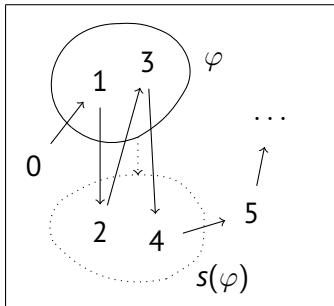
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Matching logic

Intuition



$$\mathfrak{M} = (\mathbb{N}, \{\underline{0}, s^{\mathfrak{M}}\})$$

Figure: Matching the pattern $\varphi \equiv (\underline{1} \vee \underline{2} \vee \underline{3}) \wedge \neg x$ with $x := 2$.

Matching logic

Motivation

Matching logic (ML) is designed for reasoning about operational semantics.

1. Operational semantics can be defined as a ML theory Γ .
2. Then we can prove various program properties, e.g.,

$$\Gamma \vdash \langle \varphi_{state}, x := 1 \rangle \rightarrow \langle \varphi_{state}, y := 1 \rangle.$$

(PT)	φ if φ is a propositional tautology over patterns
(MP)	$\frac{\varphi_1 \quad \varphi_1 \rightarrow \varphi_2}{\varphi_2}$
(\forall)	$(\forall x. \varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \forall x. \varphi_2)$ if $x \notin \text{FV}(\varphi_1)$
(Sub)	$(\forall x. \varphi) \rightarrow \varphi[y/x]$
(Gen)	$\frac{\varphi}{\forall x. \varphi}$
(Propagation $_{\perp}$)	$C_{\sigma}[\perp] \rightarrow \perp$
(Propagation $_{\vee}$)	$C_{\sigma}[\varphi_1 \vee \varphi_2] \rightarrow (C_{\sigma}[\varphi_1] \vee C_{\sigma}[\varphi_2])$
(Propagation $_{\exists}$)	$C_{\sigma}[\exists x. \varphi] \rightarrow \exists x. C_{\sigma}[\varphi]$ if $x \notin \text{FV}(C_{\sigma}[\exists x. \varphi])$
(Framing)	$\frac{\varphi_1 \rightarrow \varphi_2}{C_{\sigma}[\varphi_1] \rightarrow C_{\sigma}[\varphi_2]}$ where C_{σ} is a single symbol context.
(Ex)	$\exists x. x$
(Singleton)	$\neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg \varphi])$ where C_1, C_2 are nested symbol contexts.

Figure: System \mathcal{H}

Properties of System \mathcal{H}

Soundness

Theorem (Chen and Roşu, 2019b)

System \mathcal{H} is sound, i.e., $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$.

Is System \mathcal{H} complete?

Properties of System \mathcal{H}

Completeness

Let $\lceil \cdot \rceil$ be some unary symbol. We introduce the axiom

$$\text{(Definedness)} \qquad \forall x. \lceil x \rceil$$

Theorem (Chen and Roşu, 2019b)

Let Γ be a theory containing (Definedness). If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

Problem

Is System \mathcal{H} complete w.r.t. *all* theories?

1. Not all theories contain (Definedness).
2. We want to know why (Definedness) is so important.
3. ML has deep connections with other logics.

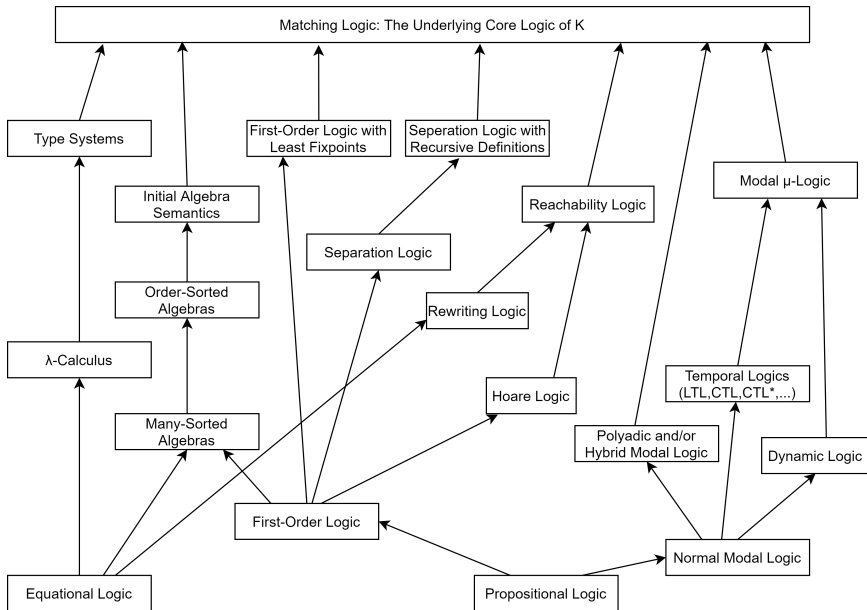


Figure: Logics successfully embedded into ML (Chen and Roşu, 2019a).

New results

We identified a new characterization of completeness for System \mathcal{H} :

Theorem

Let $\lceil \cdot \rceil$ be some *fresh* symbol not contained in Γ and φ . The following two statements are equivalent.

1. $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.
2. $\Gamma \cup \{\forall x. \lceil x \rceil\} \vdash \varphi$ implies $\Gamma \vdash \varphi$.

New results

Our characterization led to new results:

Theorem

System \mathcal{H} is complete iff System \mathcal{H} is complete w.r.t. *finite theories*.

Theorem

System \mathcal{H} is complete w.r.t. the fragment of ML without symbols.

Theorem (Compactness)

Let Γ be any theory. Then Γ is satisfiable iff every finite subset of Γ is satisfiable.

New results

We showed new connections with FOL:

Theorem (Full FOL embedding)

Let Φ be a FOL theory. Then there exists a ML theory Γ such that $\Phi \models_{\text{FOL}} \varphi$ iff $\Phi \cup \Gamma \models_{\text{ML}} \varphi$ for every FOL formula φ .

Theorem (Henkin's characterization in ML)

Under certain technical assumptions, the following two are equivalent.

1. System \mathcal{H} is complete.
2. Every consistent theory has a model.

New results

We devised a new¹ *canonical model* construction with the following property:

Theorem (Canonical model)

Let Γ be any theory containing (Definedness). If $\Gamma \not\models \varphi$, then we can construct a model \mathfrak{M} such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \varphi$.

¹Inspired by (Blackburn and Tzakova, 1998), (Chen and Roşu, 2019b).

Future work

- Are there any other techniques for *conservative extensions*?

Recall that for a fresh $\lceil \cdot \rceil$, it suffices to prove

$$\Gamma \cup \{\forall x. \lceil x \rceil\} \vdash \varphi \text{ implies } \Gamma \vdash \varphi.$$

- Does the connection with FOL still carry some answers?
- Are we able to extend our canonical models to all theories?

Conclusion

We have

1. identified a new characterization of completeness,
2. proved that completeness can be reduced to *finite theories*,
3. proved the *compactness theorem*,
4. showed some new classes of theories for which \mathcal{H} is complete,
5. discovered new connections with FOL,
6. devised a new technique for constructing canonical models.

Question 1

Is there a practical motivation?

- In practice: we can simply add (Definedness) to our theories.
- In theory: why is (Definedness) so important? Does it play a role in (semi-)decidability?

Question 2

Is $\langle \cdot, \cdot \rangle$ a function symbol?

$$\varphi_{\text{SUM}} \equiv \exists x \exists y \exists z. \langle x, \langle y, z \rangle \rangle \wedge x + y = z.$$

Claim: “ $\langle \cdot, \cdot \rangle$ is neither a predicate symbol nor a function symbol.”

Meaning:

- $\langle \cdot, \cdot \rangle$ cannot be a FOL symbol; it is both *top-level* and *nested*.
- For example,

$$\forall x. P(P(x)), \forall x. f(f(x))$$

are not FOL formulas.

Question 3

Equality in FOL vs. in ML

Claim:

- There is no “authentic” theory of term equality $t_1 = t_2$ in FOL.
- Equality $\varphi_1 = \varphi_2$ can be defined as a ML theory.

Questions: Is this different from FOL? Is this interesting?

Answer: It is just stating a difference (e.g., finite models).

Question 4

Motivation behind equality extensions?

Definition (Equality extension)

Let Γ be a theory over some signature Σ . We call $\Gamma_{=} \equiv \Gamma \cup \{[x]\}$ an *equality extension* of Γ if $[\cdot] \notin \Sigma$. We assume some fixed $[\cdot]$ is determined for each Σ .

Motivations:

1. Theories with equality vs. equality extensions.
2. (Definedness) vs. a definedness axiom.
3. Clear characterization of completeness.

Question 5

How did I decide what proofs to include?

1. No proof: a trivial result I have not found in the literature.
2. No proof + citation: a known result.
3. Proof: a new result I have not found in the literature.
4. Proof + citation: an old result with proof included for motivation.

Bibliography

- Blackburn, Patrick and Miroslava Tzakova (July 1998). “Hybrid completeness”. In: *Logic Journal of the IGPL* 6.4, pp. 625–650. issn: 1367-0751. doi: 10.1093/jigpal/6.4.625. url: <https://academic.oup.com/jigpal/article-pdf/6/4/625/1878635/060625.pdf>.
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