Linear Algebra Cheat Sheet

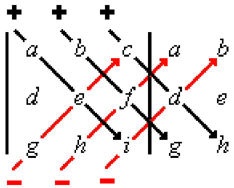
Matrices

basic operations

transpose: $[A^{\mathrm{T}}]_{ij} = [A]_{ji}$: "mirror over main diagonal"
conjungate transpose / adjugate: $A^* = (\overline{A})^{\mathrm{T}} = \overline{A^{\mathrm{T}}}$
"transpose and complex conjugate all entries"
(same as transpose for real matrices)
multiply: $A_{N\times K} * B_{K\times M} = M_{N\times M}$
invert: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

determinants

$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma_i}$
For 3×3 matrices (Sarrus rule):



arithmetic rules:
$\det(A\cdot B) = \det(A)\cdot\det(B)$
$\det(A^{-1}) = \det(A)^{-1}$
$\det(rA) = r^n \det A$, for all $A^{n\times n}$ and scalars r

rank

Let A be a matrix.
$\operatorname{rank}(A) = \operatorname{columnSpace}(A) = \operatorname{rowSpace}(A)$
= number of linearly independent column vectors of A
= number of non-zero rows in A after applying Gauss

row space

The row space of a matrix is the set of all possible linear combinations of its row vectors.
Let A be a matrix and R a row-echelon form of A.
Then the set of nonzero rows in R is a basis for the row space of A.

column space

Let A be a matrix and R a row-echelon form of A.
A basis for the column space of A can be obtained by taking the columns of A that correspond to the columns with leading entries in R.

kernel == nullspace

$\operatorname{kern}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$ (the set of vectors mapping to 0)

rank and nullity
$rank(A) + nullity(A) = n$

trace

defined on n×n square matrices: $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$
(sum of the elements on the main diagonal)

span

Let $v_1, \dots, v_r$ be the column vectors of A. Then:
The span of A may be defined as the set of all finite linear combinations of elements of A.
$\operatorname{span}(A) = \{\lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_1, \dots, \lambda_r \in \mathbb{R}\}$

properties

square: $N \times N$
symmetric: $A = A^T$
diagonal: 0 except a_{kk}

orthogonal
$A^T = A^{-1} \Rightarrow$ normal and diagonalizable

nonsingular
$A^{n\times n}$ is nonsingular = invertible iff:

- There is a matrix $B := A^{-1}$ such that $AB = I = BA$
- $det(A) \neq 0$
- $Ax = b$ has exactly one solution for each b, $b = 0$ included
- The reduced row-echelon form of A is an identity matrix
- A can be expressed as a product of elementary matrices.
- The column vectors of A are linearly independent
- The rows of A form a basis for $\mathbb{R}^n$
- The columns of A form a basis for $\mathbb{R}^n$
- $\operatorname{rank}(A) = n$

$\Rightarrow det(A^{-1}) = \frac{1}{det(A)}$
$\Rightarrow (A^{-1})^{-1} = A$
$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

block matrices
Let B, C be submatrices, and A, D square submatrices. Then:
$\det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D)$

permutation matrix
Permutation matrix $P = R_k \dots R_1$.
Row swap matrices R_i are symmetric and that they are their own inverses.
$P^{-1} = R_1 \dots R_k = R_1^T \dots R_k^T$.
Thus $P^{-1} = P^T$.

transpose properties
$(A^T)^T = A$
$(AB)^T = A^T B^T$
$det(A^T) = det(A)$
$(A^T)^{-1} = (A^{-1})^T$

compute powers

$A = BDB^{-1}$. D is a diagonal matrix.
$A^n = BD^nB^{-1}$.
$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = B\begin{bmatrix} \phi_+ & 0 \\ 0 & \phi_{-1} \end{bmatrix} B^{-1}$
$\phi_+ = \frac{1+\sqrt{5}}{2}$; $\phi_- = \frac{1-\sqrt{5}}{2}$; $\phi_+\phi_- = -1$
$B = \begin{bmatrix} 1 & 1 \\ \phi_+ & \phi_- \end{bmatrix}$
$B^{-1} = \frac{1}{\phi_+ - \phi_-}\begin{bmatrix} -\phi_- & 1 \\ \phi_+ & -1 \end{bmatrix}$
$fib[n] = \frac{\phi_+^n - \phi_-^n}{\phi_+ - \phi_-}$
$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \frac{1}{\phi_- - \phi_+}\begin{bmatrix} \phi_+^{n-1} - \phi_-^{n-1} & \phi_-^n - \phi_+^n \\ \phi_-^n - \phi_+^n & -\phi_+^{n+1} + \phi_-^{n+1} \end{bmatrix}$

Cramers Rule

$Ax = b$
$x_1 = \frac{det(A_{1\leftarrow b})}{det(A)}$ $x_2 = \frac{det(A_{2\leftarrow b})}{det(A)}$ $x_3 = \frac{det(A_{3\leftarrow b})}{det(A)}$

Cofactor

Let M_{ij} be the matrix A with the i^{th} row and j^{th} column removed.
$C_{ij} = (-1)^{i+j} det(M_{ij})$
$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(M_{ij})$
$A^{-1} = \frac{C^T}{det(A)} \Rightarrow AC^T = det(A) I_n$

Orthogonality

Two vectors are orthogonal if and only if
$u^T v = 0$

subset vs subspace

A subset is just a set of elements from the vector space.
A subspace of a vector space is a subset that follow the 3 rules.

subspace
The $\cap$ of two subspaces of $\mathbb{R}^n$ is still a subspace of $\mathbb{R}^n$.
The $\cup$ of two subspaces of $\mathbb{R}^n$ may not be a subspace of $\mathbb{R}^n$.

dimension

The dimension of a vector space V , denoted by $dim(V)$, is defined to be the number of vectors in a basis for V.
In addition, we define the dimension of the zero space to be zero.

solving $[A|b]$
Do Gaussian elimination on the augmented matrix $[A|b]$.
If $rank([A|b]) > rank(A) \Rightarrow Ax = b$ does not have a solution $\Rightarrow$
b is not in the column space of A

dimension general case

Vector space $M(m,n)$ of all m-by-n matrices.
The dimension of this space is $m \times n$
Let E_{ij} be the m-by-n matrix that is all zero except for a 1 in the (i,j) entry.
The all the E matrices are a basis for $M(m,n)$

Reasoning about dimension

Let $S \subseteq \mathbb{R}^n$ be a subspace:
if vectors $v_1, \dots, v_k \in S$ are linearly independent, then
$dim(S) \geq k$
if $span(v_1, \dots, v_k) = S$ then
$dim(S) \leq k$

General solution for $Ax = b$

$x =$ (the general solution of $Ax = 0$)
+ (one particular solution of $Ax = b$).
e.g
$x = s * v_1 + t * v_2 + a$
v_i spans nullspace of A
a is a particular solution.