The Angular Acceleration Vector

by Patrick Rutkowski, Columbia University *www.patrick-rutkowski.com*

[Discovery]

Our goal is to find the time derivative of the angular velocity vector $\boldsymbol{\omega}$ in terms of the torque on a body. We will frequently be crossing vectors with $\boldsymbol{\omega}$, so let us establish some matrix notation for this operation:

$$\boldsymbol{\omega} \times \mathbf{r} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \Omega r$$

Now posit some frame which rotates according to $\boldsymbol{\omega}$. There will be a unitary matrix R^T which transforms coordinates from inertial space into rotating space. Taking a cross product in rotating space must yield the same result as taking it in inertial space and then transforming the resulting coordinates:

$$\Omega' r' = R^T \Omega r$$

Since $r' = R^T r$ we can conclude that

$$\Omega = R\Omega' R^T \qquad \Omega' = R^T \Omega R$$

These relations allow us to take a cross product with $\boldsymbol{\omega}$ in one frame given the coordinates of $\boldsymbol{\omega}$ in the other.

We need to establish two more lemmas before we can tackle the problem at hand. Recall that the basis vectors of the rotating frame are the columns of the matrix R. The time derivatives of these vectors can be found by crossing them with $\boldsymbol{\omega}$, and so we have

$$\dot{R} = \Omega R$$

Finally, consider the following

$$L = I\omega \quad L' = I'\omega' \quad L' = R^T L \quad \omega' = R^T \omega$$

We can combine these to get

$$I = RI'R^T$$

Now we can proceed very rapidly:

$$\tau = \frac{d}{dt}\Big(I\omega\Big) = \frac{d}{dt}\Big(RI'R^T R\,\omega'\Big) = \frac{d}{dt}\Big(RI'\omega'\Big) = \dot{R}I'\omega' + RI'\dot{\omega}' = \Omega RI'\omega' + RI'\dot{\omega}'$$

Next we multiply on the left by R^T:

$$R^T\tau = R^T \Omega R I'\omega' + I'\dot{\omega}'$$

Finally we apply $\tau' = R^T\tau$ and $\Omega' = R^T\Omega R$ to get

$$\tau' = \Omega' I' \omega' + I' \dot{\omega}'$$

When we expand this out in component form we get Euler's equations:

$$\tau_x = \omega_y \omega_z \left(\lambda_3 - \lambda_2\right) + \lambda_1 \dot{\omega}_x$$

$$\tau_y = \omega_z \omega_x \left(\lambda_1 - \lambda_3\right) + \lambda_2 \dot{\omega}_y$$

$$\tau_z = \omega_x \omega_y \left(\lambda_2 - \lambda_1\right) + \lambda_3 \dot{\omega}_z$$

Note that we have omitted writing the primes for brevity. Reference books typically omit the primes as well, and they assume that the reader is aware that the equations must be evaluated in the rotating frame. We can rearrange these to find the coordinates of the angular acceleration vector in the rotating frame

$$\dot{\omega}_x = \frac{\tau_x - \omega_y \omega_z (\lambda_3 - \lambda_2)}{\lambda_1} \qquad \dot{\omega}_y = \frac{\tau_y - \omega_z \omega_x (\lambda_1 - \lambda_3)}{\lambda_2} \qquad \dot{\omega}_z = \frac{\tau_z - \omega_x \omega_y (\lambda_2 - \lambda_1)}{\lambda_3}$$

We will often want to translate the angular acceleration back into inertial coordinates. Translating a vector derivative out of a rotating frame typically takes great care, but in this case the process simplifies nicely. Observe that

$$R^T\omega = \omega' \quad \rightarrow \quad \omega = R\omega' \quad \rightarrow \quad \dot{\omega} = \dot{R}\omega' + R\dot{\omega}' \quad \rightarrow \quad \dot{\omega} = \Omega R\omega' + R\dot{\omega}'$$

This then reduces to

$$\dot{\omega} = \Omega\omega + R\dot{\omega}'$$

But $\Omega\omega = 0$, and so we have

$$\dot{\omega} = R\dot{\omega}'$$

In summary, if we want to find the angular acceleration vector in inertial coordinates then we have to transform the torque and angular velocity into rotating coordinates, apply Euler's equations, and then translate the result back into inertial coordinates.