# Surface Integrals

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## [ Summary ]

Let us consider a vector function $\mathbf{S}(x,y)$ which describes a parameterized surface in space. The surface integral of a vector function $\mathbf{F}$ over $\mathbf{S}$ can be evaluated as follows:

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathbf{F} \cdot \left[\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right] dx\,dy$$

The surface integral of a scalar function $f$ can be evaluated in a similar manner:

$$\int_S f \cdot d\mathbf{S} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f \cdot \left|\left|\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y}\right|\right| dx\,dy$$

## [ Discovery ]

Let us consider a vector function $\mathbf{S}(x,y)$ which describes a parameterized surface in space. We will shift each parameter by a differential amount and observe the offsets that this creates on the surface $\mathbf{S}$.

$$d_x\mathbf{S} = \mathbf{S}(x+dx,y) - \mathbf{S}(x,y)$$
$$d_y\mathbf{S} = \mathbf{S}(x,y+dy) - \mathbf{S}(x,y)$$

These offset vectors together define a differential parallelogram on the surface. The normal vector of this parallelogram can be obtained with a cross product

$$d\mathbf{S} = d_x\mathbf{S} \times d_y\mathbf{S}$$

The right side of this equation can be manipulated to create concrete partial derivatives

$$d\mathbf{S} = \left(\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y}\right) dx\,dy$$

The surface integral of a vector function $\mathbf{F}$ over $\mathbf{S}$ can therefore be evaluated as follows:

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathbf{F} \cdot \left[\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y}\right] dx\,dy$$

The surface integral of a scalar function $f$ over $\mathbf{S}$ can be evaluated in a similar manner:

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f \cdot \left|\left|\frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial y}\right|\right| dx\,dy$$

Recall that within these integrals the functions $\mathbf{F}$ and $f$ must be evaluated as $\mathbf{F}(\mathbf{S}(x,y,z))$ and $f(\mathbf{S}(x,y,z))$. We must be careful to evaluate them over the surface, and not over the surface's parameter space.