## [ Summary ]

The total angular momentum of a body is

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \mathbf{I}\boldsymbol{\omega}$$

where $\mathbf{R}$ is the location of the body's center of mass, $\mathbf{V}$ is the velocity of the body's center of mass, $\boldsymbol{\omega}$ is the body's angular velocity, and $\mathbf{I}$ is the body's tensor of inertia:

$$\mathbf{I} = \sum m_i \begin{bmatrix} \rho_y^2 + \rho_z^2 & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_y\rho_x & \rho_x^2 + \rho_z^2 & -\rho_y\rho_z \\ -\rho_z\rho_x & -\rho_z\rho_y & \rho_x^2 + \rho_y^2 \end{bmatrix}$$

where $\boldsymbol{\rho}_i = (\rho_x, \rho_y, \rho_z)$ denotes the vector to the particle $m_i$ from the center of mass. We have left out the $i$ subscripts on the $\rho$'s here, just for brevity. Keep in mind that $\rho_x$ needs to be read as $\rho_{ix}$, or the $x$-location of the $i$-th particle (and likewise for $\rho_y$ and $\rho_z$).

## [ Derivation ]

Let us consider the angular momentum of a rigid body which is allowed to both translate and rotate. We will denote the body's center of mass by $\mathbf{R}$. We will say that a given particle $i$ within the body has an offset from the center of mass of $\boldsymbol{\rho}_i = \mathbf{r}_i - \mathbf{R}$, where $\mathbf{r}_i$ is the absolute location of the particle relative to a fixed origin. We will also say that $\mathbf{V} = \dot{\mathbf{R}}$ is the velocity of the center of mass, and that $M = \sum m_i$ is the body's mass. The body's total angular momentum is then

$$\begin{aligned}
\mathbf{L} &= \sum \mathbf{r}_i \times \mathbf{p}_i \\
&= \sum \mathbf{r}_i \times m_i\dot{\mathbf{r}}_i \\
&= \sum (\boldsymbol{\rho}_i + \mathbf{R}) \times m_i\,(\dot{\boldsymbol{\rho}}_i + \mathbf{V}) \\
&= \sum (\boldsymbol{\rho}_i \times m_i\dot{\boldsymbol{\rho}}_i) + (\boldsymbol{\rho}_i \times m_i\mathbf{V}) + (\mathbf{R} \times m_i\dot{\boldsymbol{\rho}}_i) + (\mathbf{R} \times m_i\mathbf{V}) \\
&= \sum (\boldsymbol{\rho}_i \times m_i\dot{\boldsymbol{\rho}}_i) + (m_i\boldsymbol{\rho}_i \times \mathbf{V}) + (\mathbf{R} \times m_i\dot{\boldsymbol{\rho}}_i) + m_i\,(\mathbf{R} \times \mathbf{V}) \\
&= \sum \Big(\boldsymbol{\rho}_i \times m_i\dot{\boldsymbol{\rho}}_i\Big) + \Big(\sum m_i\boldsymbol{\rho}_i\Big) \times \mathbf{V} + \Big(\mathbf{R} \times \sum m_i\dot{\boldsymbol{\rho}}_i\Big) + \Big(\mathbf{R} \times \mathbf{V}\Big) \sum m_i
\end{aligned}$$

Now, let us define a notation for the center of mass in a coordinate system where the center of mass is itself the origin:

$$\mathbf{R}' = \frac{\sum m_i\boldsymbol{\rho}_i}{\sum m_i} = \frac{\sum m_i\boldsymbol{\rho}_i}{M}$$

We then have

$$M\mathbf{R}' = \sum m_i\boldsymbol{\rho}_i$$

and also

$$M\mathbf{V}' = \sum m_i\dot{\boldsymbol{\rho}}_i$$

Our expression for the angular momentum is thus

$$\mathbf{L} = \sum \Big( \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i \Big) + \Big( M\mathbf{R}' \times \mathbf{V} \Big) + \Big( \mathbf{R} \times M\mathbf{V}' \Big) + \Big( \mathbf{R} \times M\mathbf{V} \Big)$$

But $\mathbf{R}' = 0$ and $\mathbf{V}' = 0$, since the location of the center of mass is 0 in a coordinate system where the center of mass is itself at the origin. Two terms thus drop out of our expression, and this leave us with

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \sum \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$$

Now, a particle's distance from the center of mass $\rho_i = ||\boldsymbol{\rho}||$ must remain constant. The velocity of the particle can thus be described with an angular velocity vector:

$$\dot{\boldsymbol{\rho}}_{\mathbf{i}} = \boldsymbol{\omega} \times \boldsymbol{\rho}_i$$

We thus have

$$\begin{aligned} \mathbf{L} &= \mathbf{R} \times M\mathbf{V} + \sum m_i \left[ \boldsymbol{\rho}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \right] \\ &= \mathbf{R} \times M\mathbf{V} + \sum m_i \left[ \boldsymbol{\omega} \rho_i^2 - \boldsymbol{\rho}_i \left( \boldsymbol{\rho}_i \cdot \boldsymbol{\omega} \right) \right] \end{aligned}$$

Let us consider the interesting sum $\boldsymbol{\omega} \rho_i^2 - \boldsymbol{\rho}_i \left( \boldsymbol{\rho}_i \cdot \boldsymbol{\omega} \right)$. We can expand this vector expression into component form (for now we will drop the $i$ subscripts, just for brevity):

$$\begin{bmatrix} \omega_x \left( \rho_x^2 + \rho_y^2 + \rho_z^2 \right) - \rho_x \left( \rho_x \omega_x + \rho_y \omega_y + \rho_z \omega_z \right) \\ \omega_y \left( \rho_x^2 + \rho_y^2 + \rho_z^2 \right) - \rho_y \left( \rho_x \omega_x + \rho_y \omega_y + \rho_z \omega_z \right) \\ \omega_z \left( \rho_x^2 + \rho_y^2 + \rho_z^2 \right) - \rho_z \left( \rho_x \omega_x + \rho_y \omega_y + \rho_z \omega_z \right) \end{bmatrix}$$

When can now expand a little bit further

$$\begin{bmatrix} \omega_x \rho_x^2 + \omega_x \rho_y^2 + \omega_x \rho_z^2 - \rho_x^2 \omega_x - \rho_x \rho_y \omega_y - \rho_x \rho_z \omega_z \\ \omega_y \rho_x^2 + \omega_y \rho_y^2 + \omega_y \rho_z^2 - \rho_y \rho_x \omega_x - \rho_y^2 \omega_y - \rho_y \rho_z \omega_z \\ \omega_z \rho_x^2 + \omega_z \rho_y^2 + \omega_z \rho_z^2 - \rho_z \rho_x \omega_x - \rho_z \rho_y \omega_y - \rho_z^2 \omega_z \end{bmatrix}$$

Some cancellations are possible here, and carrying them out leaves us with

$$\begin{bmatrix} \omega_x \rho_y^2 + \omega_x \rho_z^2 - \rho_x \rho_y \omega_y - \rho_x \rho_z \omega_z \\ \omega_y \rho_x^2 + \omega_y \rho_z^2 - \rho_y \rho_x \omega_x - \rho_y \rho_z \omega_z \\ \omega_z \rho_x^2 + \omega_z \rho_y^2 - \rho_z \rho_x \omega_x - \rho_z \rho_y \omega_y \end{bmatrix}$$

We can now arrange this as a linear combination of $\omega_x$, $\omega_y$, and $\omega_z$. Note that all we are doing here is grouping terms with parenthesis:

$$\begin{bmatrix} \omega_x \left( \rho_y^2 + \rho_z^2 \right) + \omega_y \left( -\rho_x \rho_y \right) + \omega_z \left( -\rho_x \rho_z \right) \\ \omega_y \left( \rho_x^2 + \rho_z^2 \right) + \omega_x \left( -\rho_y \rho_x \right) + \omega_z \left( -\rho_y \rho_z \right) \\ \omega_z \left( \rho_x^2 + \rho_y^2 \right) + \omega_x \left( -\rho_z \rho_x \right) + \omega_y \left( -\rho_z \rho_y \right) \end{bmatrix}$$

This result now lends itself to being written in matrix form:

$$\begin{bmatrix} \rho_y^2 + \rho_z^2 & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_y\rho_x & \rho_x^2 + \rho_z^2 & -\rho_y\rho_z \\ -\rho_z\rho_x & -\rho_z\rho_y & \rho_x^2 + \rho_y^2 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

We therefore have the following (recall that you should still be visualizing $i$ subscripts on every element of $\boldsymbol{\rho}$):

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \sum m_i \begin{bmatrix} \rho_y^2 + \rho_z^2 & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_y\rho_x & \rho_x^2 + \rho_z^2 & -\rho_y\rho_z \\ -\rho_z\rho_x & -\rho_z\rho_y & \rho_x^2 + \rho_y^2 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Every particle in the body moves (relative to the body) with the same angular velocity vector $\boldsymbol{\omega}$. We can therefore factor the vector $\boldsymbol{\omega}$ out of the sum:

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \left( \sum m_i \begin{bmatrix} \rho_y^2 + \rho_z^2 & -\rho_x\rho_y & -\rho_x\rho_z \\ -\rho_y\rho_x & \rho_x^2 + \rho_z^2 & -\rho_y\rho_z \\ -\rho_z\rho_x & -\rho_z\rho_y & \rho_x^2 + \rho_y^2 \end{bmatrix} \right) \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

The matrix sum in this equation is usually denoted as $\mathbf{I}$, and is called the *tensor of inertia*. The terms on the diagonal are called *moments of inertia* and those off of the diagonal are called *products of inertia*. The whole result is more simply written as

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \mathbf{I}\boldsymbol{\omega}$$