Volume Integrals

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[**Summary**]

Let us consider a vector function $\mathbf{V}(x, y, z)$ which describes a parameterized volume. The volume integral of a function f over $\mathbf{V}$ can be evaluated as follows

$$\int_V f\,dV = \int_V f \cdot \left|\left|\frac{\partial \mathbf{V}}{\partial x} \cdot \left[\frac{\partial \mathbf{V}}{\partial y} \times \frac{\partial \mathbf{V}}{\partial z}\right]\right|\right| \,dx\,dy\,dz$$

When this triple product is expanded out it makes what is called a Jacobian determinant, which takes the form below

$$\frac{\partial \mathbf{V}}{\partial x} \cdot \left(\frac{\partial \mathbf{V}}{\partial y} \times \frac{\partial \mathbf{V}}{\partial z}\right) = \begin{vmatrix} \frac{\partial S_x}{\partial x} & \frac{\partial S_y}{\partial x} & \frac{\partial S_z}{\partial x} \\ \frac{\partial S_x}{\partial y} & \frac{\partial S_y}{\partial y} & \frac{\partial S_z}{\partial y} \\ \frac{\partial S_x}{\partial z} & \frac{\partial S_y}{\partial z} & \frac{\partial S_z}{\partial z} \end{vmatrix}$$

[**Discovery**]

Let us consider a vector function $\mathbf{V}(x, y, z)$ which describes a parameterized volume. We will shift each parameter by a differential amount and observe the total offset that this creates within the volume.

$$\begin{aligned} d_x\mathbf{V} &= \mathbf{V}(x + dx, y, z) - \mathbf{V}(x, y, z) \\ d_y\mathbf{V} &= \mathbf{V}(x, y + dy, z) - \mathbf{V}(x, y, z) \\ d_z\mathbf{V} &= \mathbf{V}(x, y, z + dz) - \mathbf{V}(x, y, z) \end{aligned}$$

These offset vectors taken together define a differential parallelepiped within $\mathbf{V}$. The volume of this parallelepiped can be obtained with a triple product

$$dV = \left|\left|d_x\mathbf{V} \cdot (d_y\mathbf{V} \times d_z\mathbf{V})\right|\right|$$

The right side of this equation can be manipulated to create concrete partial derivatives

$$dV = \left|\left|\frac{\partial \mathbf{V}}{\partial x} \cdot \left[\frac{\partial \mathbf{V}}{\partial y} \times \frac{\partial \mathbf{V}}{\partial z}\right]\right|\right| \,dx\,dy\,dz$$

When this this triple product of partial derivatives is expanded out it makes what is called a Jacobian determinant, and this produces the form below

$$dV = \text{abs}\left(\det \begin{bmatrix} \frac{\partial S_x}{\partial x} & \frac{\partial S_y}{\partial x} & \frac{\partial S_z}{\partial x} \\ \frac{\partial S_x}{\partial y} & \frac{\partial S_y}{\partial y} & \frac{\partial S_z}{\partial y} \\ \frac{\partial S_x}{\partial z} & \frac{\partial S_y}{\partial z} & \frac{\partial S_z}{\partial z} \end{bmatrix} \right) dx\, dy\, dz$$

The volume integral of a function f over $\mathbf{V}$ can then be evaluated as follows

$$\int_V f\, dV = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f \cdot \text{abs}\left(\det \begin{bmatrix} \frac{\partial S_x}{\partial x} & \frac{\partial S_y}{\partial x} & \frac{\partial S_z}{\partial x} \\ \frac{\partial S_x}{\partial y} & \frac{\partial S_y}{\partial y} & \frac{\partial S_z}{\partial y} \\ \frac{\partial S_x}{\partial z} & \frac{\partial S_y}{\partial z} & \frac{\partial S_z}{\partial z} \end{bmatrix} \right) dx\, dy\, dz$$

Recall that the functions $\mathbf{F}$ and f are actually $\mathbf{F}(\mathbf{V}(x, y, z))$ and $f(\mathbf{V}(x, y, z))$ when taken in the context of these integrals. We must be careful to evaluate them over the volume itself, and not over the volume's parameter space.