The Hyperbolic Functions

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[Summary]

The hyperbolic functions take hyperbolic areas as inputs, and they return the locations on the hyperbola which produce those areas. They are

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

These functions obey the identity

$$\cosh^2 - \sinh^2 = 1$$

Additionally, they are each others derivatives:

$$\cosh' = \sinh$$
$$\sinh' = \cosh$$

It is not difficult to show by manually expanding out the exponentials that

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$
$$\sinh(x+y) = \cosh(x)\sinh(y) + \cosh(y)\sinh(x)$$

The inverses of these functions are

$$\operatorname{arccosh}(x) = \log\left(x \pm \sqrt{x^2-1}\right)$$
$$\operatorname{arcsinh}(x) = \log\left(x - \sqrt{1+x^2}\right)$$

Note that either $+$ or $-$ is allowed inside of the log for arccosh, but not for arcsinh.

[Discovery]

Consider a function $f(x)$ which gives the unit hyperbola:

$$f(x) = \sqrt{x^2-1}$$

Now, let us consider the area

$$A(x) = xf - 2\int_1^x f(x)\,dx = \left(x\sqrt{x^2-1}\right) + 2\int_1^x \sqrt{x^2-1}\,dx$$

The derivative of this is

$$A'(x) = \left[\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right] - 2\sqrt{x^2-1} = \frac{x^2-1+x^2-2\left(x^2-1\right)}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

Now, let us call the inverse of A by the name $f(x)$. We know that $f(x)$ must obey

$$\frac{d}{dx}\Big[A(f(x))\Big] = A'(f(x)) \cdot f'(x) = \frac{1}{\sqrt{f(x)^2-1}} \cdot f'(x) = 1$$

or in other words

$$f' = \sqrt{f^2-1}$$

Let us square this:

$$f'^2 = f^2 - 1$$

and then take another derivative:

$$2f'f'' = 2ff'$$

We now have

$$f'' = f$$

We know from any basic study of differential equations that the solution to this is

$$f(x) = \alpha e^x + \beta e^{-x}$$

for some values α and β. Now, we know that the area function will have $A(1) = 0$, and so we have $f(0) = 1$. Moreover, we know from $f' = \sqrt{f^2-1}$ that $f'(0) = 0$, since $f(0) = 1$. Based on this we know that

$$1 = \alpha + \beta$$
$$0 = \alpha - \beta$$

and so α and β are both $1/2$. The inverse function of A is therefore

$$f(x) = \frac{e^x + e^{-x}}{2}$$

This function f is usually called cosh:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

The cosh function takes a hyperbolic area as an input, and it identifies which x-axis location on the parabola will create the given area. Since cosh is a horizontal location on the parabola we can expect that there be some other function sinh which obeys

$$\cosh^2 - \sinh^2 = 1$$

From this we can derive an expression for sinh:

$$\sinh = \sqrt{\cosh^2 - 1} = \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4} - 1} = \sqrt{\frac{e^{2x} - 2 + e^{-2x}}{4}} = \frac{e^x - e^{-x}}{2}$$

[Derivatives]

It is immediately apparent that cosh and sinh are derivatives of each other:

$$\cosh' = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh$$
$$\sinh' = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh$$

[Inverses]

The inverse of cosh is not difficult to find

$$\begin{aligned}
\cosh &= \frac{e^x + e^{-x}}{2} \\
2\cosh &= e^x + e^{-x} \\
e^x 2\cosh &= e^{x2} + 1 \\
0 &= e^{x2} - e^x 2\cosh + 1 \\
e^x &= \frac{2\cosh \pm \sqrt{4\cosh^2 - 4}}{2} \\
e^x &= \cosh \pm \sqrt{\cosh^2 - 1} \\
x &= \log\left(\cosh \pm \sqrt{\cosh^2 - 1}\right)
\end{aligned}$$

The inverse of cosh is therefore

$$\operatorname{arccosh}(x) = \log\left(x \pm \sqrt{x^2 - 1}\right)$$

Typically the $\pm$ is resolved to $+$, but $-$ is allowable as well. The inverse of sinh is easy to find as well

$$\begin{aligned}
\sinh &= \frac{e^x - e^{-x}}{2} \\
2\sinh &= e^x - e^{-x} \\
e^x 2\sinh &= e^{x2} - 1 \\
0 &= e^{x2} - e^x 2\sinh - 1 \\
e^x &= \frac{2\sinh \pm \sqrt{4\sinh^2 + 4}}{2} \\
e^x &= \sinh \pm \sqrt{\sinh^2 + 1} \\
x &= \log\left(\sinh \pm \sqrt{\sinh^2 + 1}\right)
\end{aligned}$$

The inverse of sinh is therefore

$$\operatorname{arcsinh}(x) = \log\left(x + \sqrt{1 + x^2}\right)$$

Note that we must resolve the $\pm$ to $+$, unlike with arccosh where $-$ is allowed as well. Resolving $\pm$ to $-$ for arcsinh would mean taking the log of a negative value.

[An Oddity of arccosh]

Recall that arccosh is

$$\operatorname{arccosh}(x) = \log\left(x \pm \sqrt{x^2 - 1}\right)$$

and that its derivative is

$$\operatorname{arccosh}'(x) = \frac{1}{\sqrt{x^2 - 1}}$$

Now, as x goes to ∞ we see that arccosh approaches

$$\operatorname{arccosh}(x) \approx \log(2x)$$

and that its derivative approches

$$\operatorname{arccosh}'(x) \approx \frac{1}{x}$$

How can this be? Should it not be the case that the derivative approaches $1/2x$, since the function itself becomes $\log(2x)$? So far, this remains a mystery to me.