Polar Vector Derivatives

by Patrick Rutkowski, Columbia University *www.patrick-rutkowski.com*

[Summary]

The first order derivative of a vector in polar coordinates is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}}$$

The second order derivative of a vector in polar coordinates is

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\omega^2\right)\hat{\mathbf{r}} + \left(r\alpha + 2\dot{r}\omega\right)\hat{\boldsymbol{\theta}}$$

This can also be written as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\omega^2\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left[r^2\omega\right]\hat{\boldsymbol{\theta}}$$

[Discovery of the First Derivative]

Let us consider a vector function $\mathbf{r}(t)$ at times t_0 and t. We will use the abbreviation $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{r}_0 = \mathbf{r}(t_0)$. We split $\mathbf{r}$ into parts $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ that are respectively parallel and perpendicular to $\mathbf{r}_0$

$$\boldsymbol{\sigma} = r\cos\theta\,\hat{\mathbf{r}}_{\mathbf{0}}$$
$$\boldsymbol{\gamma} = r\sin\theta\,\hat{\boldsymbol{\theta}}_{\mathbf{0}}$$

The vector $\hat{\boldsymbol{\theta}}$ here is a unit vector defined as

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \times \hat{\boldsymbol{k}}$$

The derivative of $\mathbf{r}$ is then

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{d}{dt}\left[\boldsymbol{\sigma} + \boldsymbol{\gamma}\right] \\ &= \dot{r}\cos\theta\,\hat{\mathbf{r}}_{\mathbf{0}} - r\omega\sin\theta\,\hat{\mathbf{r}}_{\mathbf{0}} + \dot{r}\sin\theta\,\hat{\boldsymbol{\theta}}_{\mathbf{0}} + r\omega\cos\theta\,\hat{\boldsymbol{\theta}}_{\mathbf{0}}\end{aligned}$$

We are most interested in the derivative at time t_0 because at this time we will have $\theta = 0$, which simplifies things greatly to

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\omega\,\hat{\boldsymbol{\theta}}$$

Note that we have written $\hat{\mathbf{r}}$ in place of $\hat{\mathbf{r}}_{\mathbf{0}}$ and $\hat{\boldsymbol{\theta}}$ in place of $\hat{\boldsymbol{\theta}}_{\mathbf{0}}$ since at time $t = t_0$ these are the same.

[Discovery of the Second Derivative]

One approach to finding the second derivative of $\mathbf{r}$ at time t_0 would be to start from the point in the previous section before we set $\theta = 0$. We could simply forgo setting $\theta = 0$ and instead take another derivative with repsect to time. Afterward we could then eventually set $\theta = 0$, and this would grant us our result. However, we will not do this. We will instead opt for a more novel approach. Consider the following

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d}{dt}\left[\dot{r}\,\hat{\mathbf{r}} + r\omega\,\hat{\boldsymbol{\theta}}\right] \\ &= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\omega\,\hat{\boldsymbol{\theta}} + r\dot{\omega}\,\hat{\boldsymbol{\theta}} + r\omega\,\frac{d\hat{\boldsymbol{\theta}}}{dt}\end{aligned}$$

In order to proceed further we must find $d\hat{\mathbf{r}}/dt$ and $d\hat{\boldsymbol{\theta}}/dt$. These are both first order vector derivatives themselves, and so we can make use the result from the previous section.

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \omega\hat{\boldsymbol{\theta}} \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= \frac{d}{dt}\left[\hat{\mathbf{r}} \times \hat{\boldsymbol{k}}\right] = \frac{d\hat{\mathbf{r}}}{dt} \times \hat{\boldsymbol{k}} = \omega\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{k}} = -\omega\hat{\mathbf{r}}\end{aligned}$$

The second derivative of $\mathbf{r}$ is then

$$\begin{aligned}\cdots &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\omega\hat{\boldsymbol{\theta}} + \dot{r}\omega\hat{\boldsymbol{\theta}} + r\dot{\omega}\hat{\boldsymbol{\theta}} - r\omega^2\,\hat{\mathbf{r}} \\ &= \left(\ddot{r} - r\omega^2\right)\hat{\mathbf{r}} + \left(r\alpha + 2\dot{r}\omega\right)\hat{\boldsymbol{\theta}}\end{aligned}$$

This result can also be written as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\omega^2\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left[r^2\omega\right]\hat{\boldsymbol{\theta}}$$