The Derivatives of sin, cos, arcsin, and arccos

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[Summary]

The derivatives of sin and cos are

$$\begin{aligned}\sin' &= \cos\\ \cos' &= -\sin\end{aligned}$$

The derivatives of arcsin and arccos are

$$\begin{aligned}\arcsin' x &= \frac{1}{\sqrt{1-x^2}}\\ \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

[Derivation]

Consider the function that draws the upper half of the unit circle:

$$f(x) = \sqrt{1-x^2}$$

To every x we can assign some angle $\theta = \arccos(x)$. To that θ we can assign some pie-shaped slice of the unit circle, and this slice will have area $A = \theta/2$. It also possible to define a function $A(x)$ which gives the area of that very same slice, but only in terms of x:

$$\begin{aligned}A(x) &= \frac{1}{2}xf(x) + \int_x^1 f(x)\,dx\\ &= \frac{1}{2}x\sqrt{1-x^2} + \int_x^1 \sqrt{1-x^2}\,dx\end{aligned}$$

Let us explore this function $A(x)$ by finding its derivative

$$\begin{aligned}A'(x) &= \frac{1}{2}\left[\sqrt{1-x^2} + x\cdot\frac{1}{2}\left(1-x^2\right)^{-1/2}\cdot -2x\right] - \sqrt{1-x^2}\\ &= \frac{1}{2}\left[\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} - 2\sqrt{1-x^2}\right]\\ &= -\frac{1}{2}\left[\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\right]\\ &= -\frac{1}{2}\left[\frac{1-x^2+x^2}{\sqrt{1-x^2}}\right]\\ &= -\frac{1}{2}\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Now, consider the expression

$$A(\cos\theta)$$

On the one hand we can evaluate this as

$$A(\cos(\theta)) = \frac{1}{2}\cdot\cos\theta\cdot f(\cos\theta) + \int_{\cos\theta}^{1} f(\cos\theta)\,dx$$

Yet on the other hand when we look at $A(\cos\theta)$ we see that an angle is going in and an area is coming out, and so we know immediately that

$$A(\cos\theta) = \theta/2$$

We thus have

$$\frac{1}{2}\theta = \frac{1}{2}\cdot\cos\theta\cdot f(\cos\theta) + \int_{\cos\theta}^{1} f(x)\,dx$$

Differentiating this will yield a novel result, and most of the work has already been done:

$$\begin{aligned}
\frac{1}{2} &= A'(\cos\theta)\cdot\cos'\theta \\
\frac{1}{2} &= -\frac{1}{2}\frac{1}{\sqrt{1-\cos^2\theta}}\cos'\theta \\
1 &= -\frac{1}{\sqrt{1-\cos^2\theta}}\cos'\theta \\
\cos'\theta &= -\sin\theta
\end{aligned}$$

This completes the proof. Finding $\sin'$ is then easy:

$$\begin{aligned}
1 &= \sin^2+\cos^2 \\
0 &= 2\sin\sin' + 2\cos\cos' \\
-2\sin\sin' &= 2\cos\cos' \\
-\sin\sin' &= -\cos\sin' \\
\sin' &= \cos
\end{aligned}$$

[**The derivative of arccos and arcsin**]

Notice that in the previous section we had

$$A(\cos\theta) = \frac{1}{2}\theta$$

If we simply set $\theta = \arccos(x)$ then we have

$$A(x) = \frac{1}{2}\arccos(x)$$

We can then differentiate this using the tools we have already developed:

$$
\begin{aligned}
A'(x) &= \frac{1}{2}\arccos'(x) \\
-\frac{1}{2}\frac{1}{\sqrt{1-x^2}} &= \frac{1}{2}\arccos'(x)
\end{aligned}
$$

And so we see that

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$$

The derivative $\arcsin'$ is then easy to find:

$$
\begin{aligned}
\sin(\arcsin x) &= x \\
\sin'(\arcsin x) \cdot \arcsin' x &= 1 \\
\arcsin' x &= \frac{1}{\sin'(\arcsin x)} \\
\arcsin' x &= \frac{1}{\cos(\arcsin x)} \\
\arcsin' x &= \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} \\
\arcsin' x &= \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$