Harmonic Oscillation in Energy Wells

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[Summary]

An object sitting at the bottom of an energy well, when slightly perturbed, will experience harmonic oscillation about the bottom of the well. If the bottom of the well is at x_0 then the frequency of the oscillation will be

$$\omega = \sqrt{\frac{1}{m}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}}$$

[Discovery]

Consider some potential energy function $U(x)$. Let us take a Taylor expansion of this function about some point x_0:

$$U(x) = U(x_0) + \frac{\partial U}{\partial x}\bigg|_{x_0}(x - x_0) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}(x - x_0)^2 + \mathcal{O}\Big((x - x_0)^3\Big)$$

If we take x_0 to be a local minimum of $U(x)$ then we must have

$$\frac{\partial U}{\partial x}\bigg|_{x_0} = 0 \qquad \frac{\partial^2 U}{\partial x^2}\bigg|_{x_0} \geq 0$$

This reduces the Taylor expansion to

$$U(x) = U(x_0) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}(x - x_0)^2 + \mathcal{O}\Big((x - x_0)^3\Big)$$

If we also stipulate that x never strays too far from x_0 then we can throw away higher order terms in $x - x_0$. The potential is now just

$$U(x) = U(x_0) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}(x - x_0)^2$$

We can get the related force by taking the negated gradient:

$$F(x) = -\frac{\partial}{\partial x}\Big[U(x)\Big] = -\frac{\partial}{\partial x}\left[U(x_0) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}(x - x_0)^2\right] = -\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}(x - x_0)$$

In summary we have

$$F(x) = -\frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}(x - x_0)$$

It might feel odd that this force seems to depend on the negated *second* derivative of U, when forces typically depend on negated *first* derivatives of U. The key point here is that we are using $U''(x_0)$ as a constant, so this is nothing like the usual relationship $\mathbf{F} = -\nabla U$. Since $U''(x_0) \geq 0$ this equation of motion is clearly a harmonic oscillation with angular frequency

$$\omega = \sqrt{\frac{1}{m} \frac{\partial^2 U}{\partial x^2}\bigg|_{x_0}}$$

We have thus shown that an object sitting at the bottom of an energy well, when slightly perturbed, will experience harmonic motion about the bottom of the well.