The Transport Theorem

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[Summary]

Given a fluid motion described by

$$\mathbf{u}(\boldsymbol{\phi}(\mathbf{r},t),t) = \frac{\partial \boldsymbol{\phi}}{\partial t}$$

the transport theorem tell us that

$$\frac{\partial}{\partial t} \iiint_{V_\phi} f(x,y,z,t)\, d^3V = \iiint_{V_\phi} \frac{Df}{Dt} + f\left[\nabla \cdot \mathbf{u}\right]\, d^3V$$

[Discovery]

We will observe the motion of a fluid particle which starts at $\mathbf{r}$, the future position of which we shall denote as $\boldsymbol{\phi}(\mathbf{r},t)$. We will then use this function to define a vector field $\mathbf{u}$ representing the fluid's velocity. This velocity field will obey the identity

$$\mathbf{u}(\boldsymbol{\phi}(\mathbf{r},t),t) = \frac{\partial \boldsymbol{\phi}}{\partial t}$$

We can observe a cluster of such fluid particles all at their initial positions. We will say that this cluster makes a volume V_0. After some time t every fluid particle moves as per $\boldsymbol{\phi}(\mathbf{r},t)$ and the volume V_0 is transformed into V_ϕ. Now, we wish to find the time derivative of some function $f(x,y,z,t)$ over the volume V_ϕ as the volume moves:

$$\frac{\partial}{\partial t} \iiint_{V_\phi} f(x,y,z,t)\, d^3V$$

In order to do this we will make use of the Jacobian determinant J of $\boldsymbol{\phi}$, for some fixed time t:

$$J(x,y,z,t) = \det \begin{bmatrix} \dfrac{\partial \phi_x}{\partial x} & \dfrac{\partial \phi_x}{\partial y} & \dfrac{\partial \phi_x}{\partial z} \\ \dfrac{\partial \phi_y}{\partial x} & \dfrac{\partial \phi_y}{\partial y} & \dfrac{\partial \phi_y}{\partial z} \\ \dfrac{\partial \phi_z}{\partial x} & \dfrac{\partial \phi_z}{\partial y} & \dfrac{\partial \phi_z}{\partial z} \end{bmatrix}$$

This allows us to change our integral to

$$
\begin{aligned}
\cdots &= \frac{\partial}{\partial t} \iiint_{V_\phi} f(x,y,z,t)\, d^3V \\
&= \frac{\partial}{\partial t} \iiint_{V_0} f(\boldsymbol{\phi},t) \cdot J\, d^3V \\
&= \iiint_{V_0} \frac{\partial}{\partial t}\Big[f(\boldsymbol{\phi},t)\Big]\, J + f(\boldsymbol{\phi},t)\cdot \frac{\partial J}{\partial t}\, d^3V \\
&= \iiint_{V_0} \left[\nabla f(\boldsymbol{\phi},t)\cdot \frac{\partial \boldsymbol{\phi}}{\partial t} + \frac{\partial f}{\partial t}\circ(\boldsymbol{\phi},t)\right] J + f(\boldsymbol{\phi},t)\cdot \frac{\partial J}{\partial t}\, d^3V \\
&= \iiint_{V_0} \left[\nabla f\cdot \mathbf{u} + \frac{\partial f}{\partial t}\right]\circ(\boldsymbol{\phi},t)\cdot J + f(\boldsymbol{\phi},t)\cdot \frac{\partial J}{\partial t}\, d^3V
\end{aligned}
$$

In order to proceed we must now find the time-derivative of J. First, let us expand J out in full:

$$
\begin{aligned}
J = &+ \frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial z} - \frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial y} \\
&+ \frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial x} - \frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial z} \\
&+ \frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial y} - \frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial x}
\end{aligned}
$$

Now, observe that taking a time derivative of any of the sub-terms in the above will yield a result like

$$
\frac{\partial}{\partial t}\frac{\partial \boldsymbol{\phi}_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial t} = \frac{\partial}{\partial y}\Big[u_y(\phi(\mathbf{r},t),t)\Big] = \frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial y} + \frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial y} + \frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial y}
$$

We can now take the time-derivative of J

$$
\begin{aligned}
\frac{\partial J}{\partial t} = &+ \left[\frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial x} + \frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial x} + \frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial x}\right]\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial z} \\
&+ \frac{\partial \boldsymbol{\phi}_x}{\partial x}\left[\frac{\partial u_y}{\partial x}\frac{\partial \phi_x}{\partial y} + \frac{\partial u_y}{\partial y}\frac{\partial \phi_y}{\partial y} + \frac{\partial u_y}{\partial z}\frac{\partial \phi_z}{\partial y}\right]\frac{\partial \boldsymbol{\phi}_z}{\partial z} \\
&+ \frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\left[\frac{\partial u_z}{\partial x}\frac{\partial \phi_x}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial \phi_y}{\partial z} + \frac{\partial u_z}{\partial z}\frac{\partial \phi_z}{\partial z}\right] \\
&- \left[\frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial x} + \frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial x} + \frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial x}\right]\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial y} \\
&- \frac{\partial \boldsymbol{\phi}_x}{\partial x}\left[\frac{\partial u_y}{\partial x}\frac{\partial \phi_x}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial \phi_y}{\partial z} + \frac{\partial u_y}{\partial z}\frac{\partial \phi_z}{\partial z}\right]\frac{\partial \boldsymbol{\phi}_z}{\partial y} \\
&- \frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\left[\frac{\partial u_z}{\partial x}\frac{\partial \phi_x}{\partial y} + \frac{\partial u_z}{\partial y}\frac{\partial \phi_y}{\partial y} + \frac{\partial u_z}{\partial z}\frac{\partial \phi_z}{\partial y}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial y}+\frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial y}+\frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial y}\right]\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial x}\\
&+\frac{\partial \boldsymbol{\phi}_x}{\partial y}\left[\frac{\partial u_y}{\partial x}\frac{\partial \phi_x}{\partial z}+\frac{\partial u_y}{\partial y}\frac{\partial \phi_y}{\partial z}+\frac{\partial u_y}{\partial z}\frac{\partial \phi_z}{\partial z}\right]\frac{\partial \boldsymbol{\phi}_z}{\partial x}\\
&+\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\left[\frac{\partial u_z}{\partial x}\frac{\partial \phi_x}{\partial x}+\frac{\partial u_z}{\partial y}\frac{\partial \phi_y}{\partial x}+\frac{\partial u_z}{\partial z}\frac{\partial \phi_z}{\partial x}\right]\\
&-\left[\frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial y}+\frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial y}+\frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial y}\right]\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial z}\\
&-\frac{\partial \boldsymbol{\phi}_x}{\partial y}\left[\frac{\partial u_y}{\partial x}\frac{\partial \phi_x}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \phi_y}{\partial x}+\frac{\partial u_y}{\partial z}\frac{\partial \phi_z}{\partial x}\right]\frac{\partial \boldsymbol{\phi}_z}{\partial z}\\
&-\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\left[\frac{\partial u_z}{\partial x}\frac{\partial \phi_x}{\partial z}+\frac{\partial u_z}{\partial y}\frac{\partial \phi_y}{\partial z}+\frac{\partial u_z}{\partial z}\frac{\partial \phi_z}{\partial z}\right]\\
&+\left[\frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial z}+\frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial z}+\frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial z}\right]\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial y}\\
&+\frac{\partial \boldsymbol{\phi}_x}{\partial z}\left[\frac{\partial u_y}{\partial x}\frac{\partial \phi_x}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \phi_y}{\partial x}+\frac{\partial u_y}{\partial z}\frac{\partial \phi_z}{\partial x}\right]\frac{\partial \boldsymbol{\phi}_z}{\partial y}\\
&+\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\left[\frac{\partial u_z}{\partial x}\frac{\partial \phi_x}{\partial y}+\frac{\partial u_z}{\partial y}\frac{\partial \phi_y}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \phi_z}{\partial y}\right]\\
&-\left[\frac{\partial u_x}{\partial x}\frac{\partial \phi_x}{\partial z}+\frac{\partial u_x}{\partial y}\frac{\partial \phi_y}{\partial z}+\frac{\partial u_x}{\partial z}\frac{\partial \phi_z}{\partial z}\right]\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial x}\\
&-\frac{\partial \boldsymbol{\phi}_x}{\partial z}\left[\frac{\partial u_y}{\partial x}\frac{\partial \phi_x}{\partial y}+\frac{\partial u_y}{\partial y}\frac{\partial \phi_y}{\partial y}+\frac{\partial u_y}{\partial z}\frac{\partial \phi_z}{\partial y}\right]\frac{\partial \boldsymbol{\phi}_z}{\partial x}\\
&-\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\left[\frac{\partial u_z}{\partial x}\frac{\partial \phi_x}{\partial x}+\frac{\partial u_z}{\partial y}\frac{\partial \phi_y}{\partial x}+\frac{\partial u_z}{\partial z}\frac{\partial \phi_z}{\partial x}\right]
\end{aligned}
$$

After 12 cancellations we are left with

$$
\begin{aligned}
\frac{\partial J}{\partial t}=&+\frac{\partial u_x}{\partial x}\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial z}+\frac{\partial u_y}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial z}+\frac{\partial u_z}{\partial z}\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial z}\\
&-\frac{\partial u_x}{\partial x}\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial y}-\frac{\partial u_y}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial y}-\frac{\partial u_z}{\partial z}\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial y}\\
&+\frac{\partial u_x}{\partial x}\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial x}+\frac{\partial u_z}{\partial z}\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial x}\\
&-\frac{\partial u_x}{\partial x}\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial z}-\frac{\partial u_y}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial z}-\frac{\partial u_z}{\partial z}\frac{\partial \boldsymbol{\phi}_x}{\partial y}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial z}\\
&+\frac{\partial u_x}{\partial x}\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial y}+\frac{\partial u_y}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial x}\frac{\partial \boldsymbol{\phi}_z}{\partial y}\\
&-\frac{\partial u_x}{\partial x}\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial x}-\frac{\partial u_y}{\partial y}\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial x}-\frac{\partial u_z}{\partial z}\frac{\partial \boldsymbol{\phi}_x}{\partial z}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial x}
\end{aligned}
$$

We can then group these terms on derivatives of $\mathbf{u}$

$$
\frac{\partial J}{\partial t}=+\frac{\partial u_x}{\partial x}\left[+\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial y}\frac{\partial \boldsymbol{\phi}_z}{\partial z}-\frac{\partial \boldsymbol{\phi}_x}{\partial x}\frac{\partial \boldsymbol{\phi}_y}{\partial z}\frac{\partial \boldsymbol{\phi}_z}{\partial y}\right.
$$

$$
\begin{aligned}
&\qquad + \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial z}\frac{\partial \phi_z}{\partial x} - \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial x}\frac{\partial \phi_z}{\partial z} \\
&\qquad + \frac{\partial \phi_x}{\partial z}\frac{\partial \phi_y}{\partial x}\frac{\partial \phi_z}{\partial y} - \frac{\partial \phi_x}{\partial z}\frac{\partial \phi_y}{\partial y}\frac{\partial \phi_z}{\partial x} \Bigg] \\
&+ \frac{\partial u_y}{\partial y}\Bigg[+ \frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y}\frac{\partial \phi_z}{\partial z} - \frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial z}\frac{\partial \phi_z}{\partial y} \\
&\qquad + \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial z}\frac{\partial \phi_z}{\partial x} - \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial x}\frac{\partial \phi_z}{\partial z} \\
&\qquad + \frac{\partial \phi_x}{\partial z}\frac{\partial \phi_y}{\partial x}\frac{\partial \phi_z}{\partial y} - \frac{\partial \phi_x}{\partial z}\frac{\partial \phi_y}{\partial y}\frac{\partial \phi_z}{\partial x} \Bigg] \\
&+ \frac{\partial u_z}{\partial z}\Bigg[+ \frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y}\frac{\partial \phi_z}{\partial z} - \frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial z}\frac{\partial \phi_z}{\partial y} \\
&\qquad + \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial z}\frac{\partial \phi_z}{\partial x} - \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial x}\frac{\partial \phi_z}{\partial z} \\
&\qquad + \frac{\partial \phi_x}{\partial z}\frac{\partial \phi_y}{\partial x}\frac{\partial \phi_z}{\partial y} - \frac{\partial \phi_x}{\partial z}\frac{\partial \phi_y}{\partial y}\frac{\partial \phi_z}{\partial x} \Bigg]
\end{aligned}
$$

We appear to have been left with Jacobian determinants, and so we can now write

$$
\begin{aligned}
\frac{\partial J}{\partial t} &= \frac{\partial u_x}{\partial x}J + \frac{\partial u_y}{\partial y}J + \frac{\partial u_z}{\partial z}J \\
&= \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right] J \\
&= \Big[(\nabla \cdot \mathbf{u}) \circ (\boldsymbol{\phi}, t)\Big] \cdot J
\end{aligned}
$$

We can then plug this into our integral from before

$$
\begin{aligned}
\cdots &= \iiint_{V_0} \left[\nabla f \cdot \mathbf{u} + \frac{\partial f}{\partial t}\right] \circ (\boldsymbol{\phi}, t) \cdot J + f(\boldsymbol{\phi}, t) \cdot \frac{\partial J}{\partial t}\, d^3V \\
&= \iiint_{V_0} \left[\nabla f \cdot \mathbf{u} + \frac{\partial f}{\partial t}\right] \circ (\boldsymbol{\phi}, t) \cdot J + f(\boldsymbol{\phi}, t) \cdot \big([\nabla \cdot \mathbf{u}] \circ (\boldsymbol{\phi}, t)\big) \cdot J\, d^3V \\
&= \iiint_{V_0} \left[\nabla f \cdot \mathbf{u} + \frac{\partial f}{\partial t} + f\,[\nabla \cdot \mathbf{u}]\right] \circ (\boldsymbol{\phi}, t) \cdot J\, d^3V \\
&= \iiint_{V_\phi} \nabla f \cdot \mathbf{u} + \frac{\partial f}{\partial t} + f\,[\nabla \cdot \mathbf{u}]\ d^3V \\
&= \iiint_{V_\phi} \frac{Df}{Dt} + f\,[\nabla \cdot \mathbf{u}]\ d^3V
\end{aligned}
$$

And so we have our final result:

$$
\frac{\partial}{\partial t}\iiint_{V_\phi} f(x, y, z, t)\, d^3V = \iiint_{V_\phi} \frac{Df}{Dt} + f\,[\nabla \cdot \mathbf{u}]\ d^3V
$$

[Incompressible Fluids]

If we set $f(x, y, z, t) = 1$ then the integral in the transport theorem will compute a fluid region's volume as it moves. If we know the fluid to be incompressible then the time-derivative of this volume integral should be 0, because an incompressible region will be able to deform, but never to change grow or shrink in volume. Under this circumstance the transport theorem will tell us that

$$0 = \iiint_{V_\phi} \nabla \cdot \mathbf{u} \; d^3V$$

Thus we know that incompressible fluids have no velocity divergence

$$\nabla \cdot \mathbf{u} = 0$$

[Mass Flow]

We know that volumes in all fluids will maintain a constant mass as they move, regardless of the type of fluid. The divergence theorem can thus give us some insight into how a fluid's mass density ρ evolves over time

$$0 = \frac{\partial}{\partial t} \iiint_{V_\phi} \rho \; d^3V = \iiint_{V_\phi} \nabla\rho \cdot \mathbf{u} + \frac{\partial \rho}{\partial t} + \rho \left[\nabla \cdot \mathbf{u}\right] \; d^3V$$

We thus know that

$$\frac{\partial \rho}{\partial t} = (-\nabla\rho) \cdot \mathbf{u} - \rho \left[\nabla \cdot \mathbf{u}\right]$$