Euler Angles

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[Summary]

The orientation of a body can be specified by three angles θ, ϕ, and ψ. The first angle θ specifies a rotation about a $\hat{\boldsymbol{k}}$ axis which is fixed to the body. The second angle ϕ specifies a rotation about an $\hat{\boldsymbol{\imath}}$ axis which is likewise fixed to the body. The third and final angle ψ specifies another rotation about the body's $\hat{\boldsymbol{k}}$ (which has by this point also been rotated by ϕ). In order to translate a vector's coordinates into this body-attached system we can use the matrix below:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} +c\theta c\psi - s\theta c\phi s\psi & +s\theta c\psi + c\theta c\phi s\psi & s\phi s\psi \\ -c\theta s\psi - s\theta c\phi c\psi & -s\theta s\psi + c\theta c\phi c\psi & s\phi c\psi \\ s\theta s\phi & -c\theta s\phi & c\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

[Discovery]

We need a method for translating coordinates from a set of external axes to a set of axes attached to a body. In order to do this we need to find the coordinates of the body axes in the external system. First let us rotate the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ axes about the $\hat{\boldsymbol{k}}$ axis. Recall that the general rotation of some vector $\mathbf{r}$ about a fixed axis $\hat{\boldsymbol{\omega}}$ is given by

$$\mathbf{R} = (\hat{\boldsymbol{\omega}} \times \mathbf{r}) \sin\theta + \mathbf{r} \cos\theta + \hat{\boldsymbol{\omega}} \, (1 - \cos\theta) \, (\hat{\boldsymbol{\omega}} \cdot \mathbf{r})$$

The rotations of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ about the $\hat{\boldsymbol{k}}$ axis are thus

$$\hat{\boldsymbol{\imath}}_1 = \left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}\right) \sin\theta + \hat{\boldsymbol{\imath}} \cos\theta = \hat{\boldsymbol{\jmath}} \sin\theta + \hat{\boldsymbol{\imath}} \cos\theta$$

$$\hat{\boldsymbol{\jmath}}_1 = \left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}\right) \sin\theta + \hat{\boldsymbol{\jmath}} \cos\theta = \hat{\boldsymbol{\jmath}} \cos\theta - \hat{\boldsymbol{\imath}} \sin\theta$$

$$\hat{\boldsymbol{k}}_1 = \hat{\boldsymbol{k}}$$

The rotations of these about $\hat{\boldsymbol{\imath}}_1$ are then

$$\hat{\boldsymbol{\imath}}_2 = \hat{\boldsymbol{\imath}}_1 = \left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}\right) \sin\theta + \hat{\boldsymbol{\imath}} \cos\theta$$

$$\begin{aligned} \hat{\boldsymbol{\jmath}}_2 &= \left(\hat{\boldsymbol{\imath}}_1 \times \hat{\boldsymbol{\jmath}}_1\right) \sin\phi + \hat{\boldsymbol{\jmath}}_1 \cos\phi = \hat{\boldsymbol{k}}_1 \sin\phi + \hat{\boldsymbol{\jmath}}_1 \cos\phi \\ &= \hat{\boldsymbol{k}} \sin\phi + \left(\hat{\boldsymbol{\jmath}} \cos\theta - \hat{\boldsymbol{\imath}} \sin\theta\right) \cos\phi \end{aligned}$$

$$\begin{aligned} \hat{\boldsymbol{k}}_2 &= \left(\hat{\boldsymbol{\imath}}_1 \times \hat{\boldsymbol{k}}_1\right) \sin\phi + \hat{\boldsymbol{k}}_1 \cos\phi = \hat{\boldsymbol{k}}_1 \cos\phi - \hat{\boldsymbol{\jmath}}_1 \sin\phi \\ &= \hat{\boldsymbol{k}} \cos\phi - \left(\hat{\boldsymbol{\jmath}} \cos\theta - \hat{\boldsymbol{\imath}} \sin\theta\right) \sin\phi \end{aligned}$$

The rotations of these about $\hat{\boldsymbol{k}}_2$ are then

$$\begin{aligned}
\hat{\boldsymbol{\imath}}_3 &= \left(\hat{\boldsymbol{k}}_2 \times \hat{\boldsymbol{\imath}}_2\right) \sin\psi + \hat{\boldsymbol{\imath}}_2 \cos\psi = \hat{\boldsymbol{\jmath}}_2 \sin\psi + \hat{\boldsymbol{\imath}}_2 \cos\psi \\
&= \left(\hat{\boldsymbol{k}} \sin\phi + \left(\hat{\boldsymbol{\jmath}} \cos\theta - \hat{\boldsymbol{\imath}} \sin\theta\right) \cos\phi\right) \sin\psi + \left(\hat{\boldsymbol{\jmath}} \sin\theta + \hat{\boldsymbol{\imath}} \cos\theta\right) \cos\psi
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\jmath}}_3 &= \left(\hat{\boldsymbol{k}}_2 \times \hat{\boldsymbol{\jmath}}_2\right) \sin\psi + \hat{\boldsymbol{\jmath}}_2 \cos\psi = \hat{\boldsymbol{\jmath}}_2 \cos\psi - \hat{\boldsymbol{\imath}}_2 \sin\psi \\
&= \left(\hat{\boldsymbol{k}} \sin\phi + \left(\hat{\boldsymbol{\jmath}} \cos\theta - \hat{\boldsymbol{\imath}} \sin\theta\right) \cos\phi\right) \cos\psi - \left(\hat{\boldsymbol{\jmath}} \sin\theta + \hat{\boldsymbol{\imath}} \cos\theta\right) \sin\psi
\end{aligned}$$

$$\hat{\boldsymbol{k}}_3 = \hat{\boldsymbol{k}}_2 = \hat{\boldsymbol{k}} \cos\phi - \left(\hat{\boldsymbol{\jmath}} \cos\theta - \hat{\boldsymbol{\imath}} \sin\theta\right) \sin\phi$$

The body axes are thus

$$\hat{\boldsymbol{\imath}}' = \begin{bmatrix} \cos\theta\cos\psi - \sin\theta\cos\phi\sin\psi \\ \sin\theta\cos\psi + \cos\theta\cos\phi\sin\psi \\ \sin\phi\sin\psi \end{bmatrix}$$

$$\hat{\boldsymbol{\jmath}}' = \begin{bmatrix} -\cos\theta\sin\psi - \sin\theta\cos\phi\cos\psi \\ -\sin\theta\sin\psi + \cos\theta\cos\phi\cos\psi \\ \sin\phi\cos\psi \end{bmatrix}$$

$$\hat{\boldsymbol{k}}' = \begin{bmatrix} \sin\theta\sin\phi \\ -\cos\theta\sin\phi \\ \cos\phi \end{bmatrix}$$

In order to project a vector (x, y, z) onto these body axes we simply need to take three dot products, one with each body axis. These three dot products can be accomplished with the following matrix multiplication, in which the coordinates of the body axes make up the rows:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} +c\theta c\psi - s\theta\, c\phi s\psi & +s\theta c\psi + c\theta\, c\phi s\psi & s\phi s\psi \\ -c\theta s\psi - s\theta\, c\phi c\psi & -s\theta s\psi + c\theta\, c\phi c\psi & s\phi c\psi \\ s\theta\, s\phi & -c\theta\, s\phi & c\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$