# Rotation About a Vector

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## [ Summary ]

The rotation $\mathbf{R}$ of the vector $\mathbf{r}$ through an angle $\psi$ about some unit vector $\hat{\boldsymbol{\omega}}$ can be computed with

$$\mathbf{R}(\mathbf{r}, \hat{\boldsymbol{\omega}}, \psi) = \sin\psi \cdot [\hat{\boldsymbol{\omega}} \times \mathbf{r}] + \cos\psi \cdot \mathbf{r} + \hat{\boldsymbol{\omega}}\,(1 - \cos\psi)\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r})$$

The matrix form of this result is

$$\mathbf{R}\,(\mathbf{r}, \hat{\boldsymbol{\omega}}, \psi) = \boldsymbol{\Psi}\mathbf{r}$$

where $\boldsymbol{\Psi}$ is the rotation matrix

$$\boldsymbol{\Psi} = \begin{bmatrix} c + \omega_x\omega_x\,(1-c) & -\omega_z s + \omega_x\omega_y\,(1-c) & +\omega_y s + \omega_x\omega_z\,(1-c) \\ +\omega_z s + \omega_y\omega_x\,(1-c) & c + \omega_y\omega_y\,(1-c) & -\omega_x s + \omega_y\omega_z\,(1-c) \\ -\omega_y s + \omega_z\omega_x\,(1-c) & +\omega_x s + \omega_z\omega_y\,(1-c) & c + \omega_z\omega_z\,(1-c) \end{bmatrix}$$

Note that we have used the shorthand $c = \cos\psi$ and $s = \sin\psi$.

## [ Discovery ]

We will consider the problem of taking any point in space $\mathbf{r}$ and rotating it through some angle $\psi$ about the unit vector $\hat{\boldsymbol{\omega}}$. To begin, let us use label $\theta$ to denote the angle between $\mathbf{r}$ and $\boldsymbol{\omega}$. We will split $\mathbf{r}$ into $\mathbf{r}_\parallel$ and $\mathbf{r}_\perp$. We will form $\mathbf{r}_\parallel$ by projecting $\mathbf{r}$ onto $\hat{\boldsymbol{\omega}}$ and we will form $\mathbf{r}_\perp$ by taking whatever is left over after the projection

$$\begin{aligned} \mathbf{r}_\parallel &= r\cos\theta \cdot \hat{\boldsymbol{\omega}} \\ &= \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}) \end{aligned}$$

$$\begin{aligned} \mathbf{r}_\perp &= \mathbf{r} - \mathbf{r}_\parallel \\ &= \mathbf{r} - \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}) \end{aligned}$$

We now consider the plane which is perpendicular $\mathbf{r}_\parallel$ and drawn at the tip of $\mathbf{r}_\parallel$. We wish to find basis vectors in that plane. The vector $\mathbf{r}_\perp$ is already one such basis vector. We know that our second basis vector $\boldsymbol{\rho}$ will be perpendicular to both $\mathbf{r}_\perp$ and $\hat{\boldsymbol{\omega}}$. We can therefore find $\boldsymbol{\rho}$ as follows

$$\boldsymbol{\rho} = \hat{\boldsymbol{\omega}} \times \mathbf{r}_\perp = \hat{\boldsymbol{\omega}} \times \left(\mathbf{r} - \mathbf{r}_\parallel\right) = \hat{\boldsymbol{\omega}} \times \mathbf{r}$$

Notice that $\boldsymbol{\rho}$ is conveniently of the same magnitude as $\mathbf{r}_\perp$. To find where $\mathbf{r}$ goes when rotated about $\hat{\boldsymbol{\omega}}$ we make a circular combination of $\mathbf{r}_\perp$ and $\boldsymbol{\rho}$, which we then add back to $\mathbf{r}_\parallel$, as follows:

$$\begin{aligned} \mathbf{R}(\mathbf{r}, \hat{\boldsymbol{\omega}}, \psi) &= \mathbf{r}_\parallel + \mathbf{r}_\perp\cos\psi + \boldsymbol{\rho}\sin\psi \\ &= \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}) + \cos\psi \cdot [\mathbf{r} - \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r})] + \sin\psi \cdot [\hat{\boldsymbol{\omega}} \times \mathbf{r}] \\ &= \sin\psi \cdot [\hat{\boldsymbol{\omega}} \times \mathbf{r}] + \cos\psi \cdot \mathbf{r} + \hat{\boldsymbol{\omega}}\,(1 - \cos\psi)\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}) \end{aligned}$$

We give the component expansion of this result below. For the sake of readability we will use $c$ and $s$ for $\cos\psi$ and $\sin\psi$.

$$\begin{aligned}
R_x &= s\left(\omega_y r_z - \omega_z r_y\right) + c r_x + \omega_x \left(1-c\right)\left(\omega_x r_x + \omega_y r_y + \omega_z r_z\right)\\
R_y &= s\left(\omega_z r_x - \omega_x r_z\right) + c r_y + \omega_y \left(1-c\right)\left(\omega_x r_x + \omega_y r_y + \omega_z r_z\right)\\
R_z &= s\left(\omega_x r_y - \omega_y r_x\right) + c r_z + \omega_z \left(1-c\right)\left(\omega_x r_x + \omega_y r_y + \omega_z r_z\right)
\end{aligned}$$

It is clear that each of these components can be written as a linear combination of $r_x$, $r_y$, and $r_z$. In order to see this explicitly we first expand out all multiplications

$$\begin{aligned}
R_x &= s\omega_y r_z - s\omega_z r_y + c r_x + \omega_x\omega_x r_x + \omega_x\omega_y r_y + \omega_x\omega_z r_z - c\omega_x\omega_x r_x - c\omega_x\omega_y r_y - c\omega_x\omega_z r_z\\
R_y &= s\omega_z r_x - s\omega_x r_z + c r_y + \omega_y\omega_x r_x + \omega_y\omega_y r_y + \omega_y\omega_z r_z - c\omega_y\omega_x r_x - c\omega_y\omega_y r_y - c\omega_y\omega_z r_z\\
R_z &= s\omega_x r_y - s\omega_y r_x + c r_z + \omega_z\omega_x r_x + \omega_z\omega_y r_y + \omega_z\omega_z r_z - c\omega_z\omega_x r_x - c\omega_z\omega_y r_y - c\omega_z\omega_z r_z
\end{aligned}$$

Then we collect the terms in order to make the linear combinations stand out

$$\begin{aligned}
R_x &= r_x\left(\omega_x\omega_x + c - c\omega_x\omega_x\right) + r_y\left(\omega_x\omega_y - c\omega_x\omega_y - s\omega_z\right) + r_z\left(\omega_x\omega_z - c\omega_x\omega_z + s\omega_y\right)\\
R_y &= r_x\left(\omega_x\omega_y - c\omega_x\omega_y + s\omega_z\right) + r_y\left(\omega_y\omega_y + c - c\omega_y\omega_y\right) + r_z\left(\omega_y\omega_z - c\omega_y\omega_z - s\omega_x\right)\\
R_z &= r_x\left(\omega_x\omega_z - c\omega_x\omega_z - s\omega_y\right) + r_y\left(\omega_y\omega_z - c\omega_y\omega_z + s\omega_x\right) + r_z\left(\omega_z\omega_z + c - c\omega_z\omega_z\right)
\end{aligned}$$

This result is best written in matrix form as

$$\mathbf{R}(\mathbf{r}, \hat{\boldsymbol{\omega}}, \psi) = \boldsymbol{\Psi}\mathbf{r}$$

where $\boldsymbol{\Psi}$ is the rotation matrix

$$\boldsymbol{\Psi} = \begin{bmatrix}
c + \omega_x\omega_x\left(1-c\right) & -\omega_z s + \omega_x\omega_y\left(1-c\right) & +\omega_y s + \omega_x\omega_z\left(1-c\right)\\
+\omega_z s + \omega_y\omega_x\left(1-c\right) & c + \omega_y\omega_y\left(1-c\right) & -\omega_x s + \omega_y\omega_z\left(1-c\right)\\
-\omega_y s + \omega_z\omega_x\left(1-c\right) & +\omega_x s + \omega_z\omega_y\left(1-c\right) & c + \omega_z\omega_z\left(1-c\right)
\end{bmatrix}$$