Spherical Coordinates

by Patrick Rutkowski, Columbia University www.patrick-rutkowski.com

[The Basis Vectors]

In the calculus of non-Cartesian coordinate systems we typically define the basis vectors in terms of the gradients of the coordinates themselves. In spherical coordinates this means that

$$\begin{aligned} \hat{\mathbf{r}} &= \operatorname{norm} \nabla r \\ \hat{\boldsymbol{\theta}} &= \operatorname{norm} \nabla \theta \\ \hat{\boldsymbol{\varphi}} &= \operatorname{norm} \nabla \phi \end{aligned}$$

The gradients themselves might not be of unit length. If we want basis vectors of unit length we must explicitly perform a normalization, which is why we have written norm ∇r instead of just ∇r (though it just so happens that ∇r comes out normalized by default).

The gradient $\nabla\theta$ will point in the direction of increasing θ. Defining $\hat{\boldsymbol{\theta}}$ in this way ensures that it points in the direction of increasing θ. This goes likewise for r and $\hat{\mathbf{r}}$, φ and $\hat{\boldsymbol{\varphi}}$, and even x and $\hat{\mathbf{x}}$ in Cartesian coordinates. It should make immediate sense that $\hat{\mathbf{r}}$ points in the direction of increasing r, but take a moment and try to convince yourself of this for $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ as well (for their respective coordinates). This gradient-based definition of the basis vectors works in other systems too, like for example cylindrical and polar. The basis vectors used for curved space-times in general relativity are defined in a similar manner, though the situation there is more complicated, and the basis vectors are usually not normalized.

Let us try to find the basis vectors in the spherical system by taking the gradients of r, θ, and φ. The mapping from spherical coordinates to Cartesian coordinates is

$$\begin{aligned} x &= r\cos\varphi\sin\theta \\ y &= r\sin\varphi\sin\theta \\ z &= r\cos\theta \end{aligned}$$

The inverse mapping is

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad \varphi = \arctan\left(\frac{y}{x}\right)$$

For brevity let us borrow some notation from the cylindrical system and write

$$\rho = \sqrt{x^2+y^2} = r\sin\theta$$

The inverse mapping can then be written more compactly in terms of ρ as

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\left(\frac{\rho}{z}\right) \qquad \varphi = \arctan\left(\frac{y}{x}\right)$$

Now there is noting left to do but to manually turn the crank and produce the gradients. We start by finding the gradient of r:

$$\begin{aligned}
\nabla r &= \frac{\partial r}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial r}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial r}{\partial z}\,\hat{\boldsymbol{k}} \\
&= \frac{x}{r}\,\hat{\boldsymbol{\imath}} + \frac{y}{r}\,\hat{\boldsymbol{\jmath}} + \frac{z}{r}\,\hat{\boldsymbol{k}} \\
&= \frac{r\cos\varphi\sin\theta}{r}\,\hat{\boldsymbol{\imath}} + \frac{r\sin\varphi\sin\theta}{r}\,\hat{\boldsymbol{\jmath}} + \frac{r\cos\theta}{r}\,\hat{\boldsymbol{k}} \\
&= \cos\varphi\sin\theta\,\hat{\boldsymbol{\imath}} + \sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}}
\end{aligned}$$

Next we find gradient of θ:

$$\begin{aligned}
\nabla \theta &= \frac{\partial \theta}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial \theta}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial \theta}{\partial z}\,\hat{\boldsymbol{k}} \\
&= \frac{\partial}{\partial x}\arctan\left(\frac{\rho}{z}\right)\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\arctan\left(\frac{\rho}{z}\right)\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\arctan\left(\frac{\rho}{z}\right)\hat{\boldsymbol{k}} \\
&= \frac{1}{1+\rho^2/z^2}\frac{\partial}{\partial x}\left(\frac{\rho}{z}\right)\hat{\boldsymbol{\imath}} + \frac{1}{1+\rho^2/z^2}\frac{\partial}{\partial y}\left(\frac{\rho}{z}\right)\hat{\boldsymbol{\jmath}} + \frac{1}{1+\rho^2/z^2}\frac{\partial}{\partial z}\left(\frac{\rho}{z}\right)\hat{\boldsymbol{k}} \\
&= \frac{1}{1+\rho^2/z^2}\left(\frac{x}{z\rho}\right)\hat{\boldsymbol{\imath}} + \frac{1}{1+\rho^2/z^2}\left(\frac{y}{z\rho}\right)\hat{\boldsymbol{\jmath}} - \frac{1}{1+\rho^2/z^2}\left(\frac{\rho}{z^2}\right)\hat{\boldsymbol{k}} \\
&= \frac{xz}{\rho r^2}\,\hat{\boldsymbol{\imath}} + \frac{yz}{\rho r^2}\,\hat{\boldsymbol{\jmath}} - \frac{\rho}{r^2}\,\hat{\boldsymbol{k}} \\
&= \frac{r\cos\varphi\sin\theta\cdot r\cos\theta}{r\sin\theta\cdot r^2}\,\hat{\boldsymbol{\imath}} + \frac{r\sin\varphi\sin\theta\cdot r\cos\theta}{r\sin\theta\cdot r^2}\,\hat{\boldsymbol{\jmath}} - \frac{r\sin\theta}{r^2}\,\hat{\boldsymbol{k}} \\
&= \frac{\cos\varphi\cos\theta}{r}\,\hat{\boldsymbol{\imath}} + \frac{\sin\varphi\cos\theta}{r}\,\hat{\boldsymbol{\jmath}} - \frac{\sin\theta}{r}\,\hat{\boldsymbol{k}}
\end{aligned}$$

Finally we find the gradient of φ:

$$\begin{aligned}
\nabla \varphi &= \frac{\partial \varphi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial \varphi}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial \varphi}{\partial z}\,\hat{\boldsymbol{k}} \\
&= \frac{\partial}{\partial x}\arctan\left(\frac{y}{x}\right)\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\arctan\left(\frac{y}{x}\right)\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\arctan\left(\frac{y}{x}\right)\hat{\boldsymbol{k}} \\
&= \frac{1}{1+y^2/x^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)\hat{\boldsymbol{\imath}} + \frac{1}{1+y^2/x^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\hat{\boldsymbol{\jmath}} \\
&= -\frac{1}{1+y^2/x^2}\left(\frac{y}{x^2}\right)\hat{\boldsymbol{\imath}} + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)\hat{\boldsymbol{\jmath}} \\
&= -\frac{y}{x^2+y^2}\,\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\,\hat{\boldsymbol{\jmath}} \\
&= -\frac{y}{\rho}\,\hat{\boldsymbol{\imath}} + \frac{x}{\rho}\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

$$= -\frac{\sin\varphi\sin\theta}{r\sin\theta}\,\hat{\boldsymbol{\imath}} + \frac{\cos\varphi\sin\theta}{r\sin\theta}\,\hat{\boldsymbol{\jmath}}$$
$$= -\frac{\sin\varphi}{r}\,\hat{\boldsymbol{\imath}} + \frac{\cos\varphi}{r}\,\hat{\boldsymbol{\jmath}}$$

In summary the gradients are:

$$\nabla r = \cos\varphi\sin\theta\,\hat{\boldsymbol{\imath}} + \sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}}$$
$$\nabla\theta = \frac{\cos\varphi\cos\theta}{r}\,\hat{\boldsymbol{\imath}} + \frac{\sin\varphi\cos\theta}{r}\,\hat{\boldsymbol{\jmath}} - \frac{\sin\theta}{r}\,\hat{\boldsymbol{k}}$$
$$\nabla\varphi = -\frac{\sin\varphi}{r}\,\hat{\boldsymbol{\imath}} + \frac{\cos\varphi}{r}\,\hat{\boldsymbol{\jmath}}$$

As mentioned earlier we must normalize these gradients to get the associated basis vectors:

$$\hat{\mathbf{r}} = \cos\varphi\sin\theta\,\hat{\boldsymbol{\imath}} + \sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}}$$
$$\hat{\boldsymbol{\theta}} = \cos\varphi\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\varphi\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{k}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\boldsymbol{\imath}} + \cos\varphi\,\hat{\boldsymbol{\jmath}}$$

We could have perhaps found these vectors by circumscribing circles in a sphere and working out the trigonometry, but it is nonetheless satisfying to see that the formal gradient definitions really do work.

[**The Velocity**]

Now that we have our basis vectors we can work out an expression for velocity in spherical coordinates. The position vector in spherical coordinates is just $\mathbf{r} = r\hat{\mathbf{r}}$. The velocity is therefore

$$\begin{aligned}
\mathbf{v} = \frac{d\mathbf{r}}{dt} &= \frac{d}{dt}\Big[r\,\hat{\mathbf{r}}\Big] \\
&= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\
&= \dot{r}\hat{\mathbf{r}} + \frac{d}{dt}\Big[\cos\varphi\sin\theta\,\hat{\boldsymbol{\imath}} + \sin\varphi\sin\theta\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}}\Big] \\
&= \dot{r}\hat{\mathbf{r}} + \dot{\theta}\Big[\cos\varphi\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\varphi\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{k}}\Big] \\
&\qquad + \dot{\varphi}\Big[-\sin\varphi\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\varphi\sin\theta\,\hat{\boldsymbol{\jmath}}\Big] \\
&= \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\varphi}\hat{\boldsymbol{\varphi}}
\end{aligned}$$

In summary this is

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\varphi}\hat{\boldsymbol{\varphi}}$$

Before we go on take note that that the velocity was especially easy to formulate in spherical coordinates because the position vector was just $\mathbf{r} = r\,\hat{\mathbf{r}}$. In cylindrical coordinates the position vector is $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, and in other systems the position might be a yet more complicated mixture of the basis vectors and their coordinates.

[The Gradient]

Consider a scalar function ψ of euclidean space and some associated gradient $\nabla\psi$. Recall that the projection of some vector $\mathbf{b}$ onto some unit vector $\hat{\mathbf{a}}$ is defined as $\hat{\mathbf{a}}\,(\hat{\mathbf{a}}\cdot\mathbf{b})$. We can use this projection method to split $\nabla\psi$ into its component parts on the spherical basis vectors:

$$\nabla\psi = \Big(\nabla\psi\cdot\hat{\mathbf{r}}\Big)\,\hat{\mathbf{r}} + \Big(\nabla\psi\cdot\hat{\boldsymbol{\theta}}\Big)\,\hat{\boldsymbol{\theta}} + \Big(\nabla\psi\cdot\hat{\boldsymbol{\varphi}}\Big)\,\hat{\boldsymbol{\varphi}}$$

For brevity let us write this as

$$\nabla\psi = G_r\,\hat{\mathbf{r}} + G_\theta\,\hat{\boldsymbol{\theta}} + G_\varphi\,\hat{\boldsymbol{\varphi}}$$

where $G_r = \nabla\psi\cdot\hat{\mathbf{r}}$, $G_\theta = \nabla\psi\cdot\hat{\boldsymbol{\theta}}$, and $G_\varphi = \nabla\psi\cdot\hat{\boldsymbol{\varphi}}$. The values of G_r, G_θ, and G_φ are still unknown to us. It is the goal of this section to figure out what they should be. We will find these terms by moving down some path in space and computing $d\psi/dt$ as we go. We will make this computation of $d\psi/dt$ twice, each time using a different method. We will then say that both methods must give the same answer, because physically there is only one $d\psi/dt$ down any given path. Formally equating these two methods will give us the information we seek about G_r, G_θ, and G_φ.

For the first method consider moving along some path in Cartesian coordinates and measuring ψ as you go. We can take time derivative of ψ over our path in the usual way:

$$\frac{d\psi}{dt} = \frac{d}{dt}\psi(x,y,z) = \frac{\partial\psi}{\partial x}\frac{dx}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dt} + \frac{\partial\psi}{\partial z}\frac{dz}{dt} = \nabla\psi\cdot\mathbf{v}$$

In summary we have:

$$\frac{d\psi}{dt} = \nabla\psi\cdot\mathbf{v}$$

In the previous section we found a spherical expression for the velocity $\mathbf{v}$. We can apply this to our expression for $d\psi/dt$:

$$\begin{aligned}\frac{d\psi}{dt} = \nabla\psi\cdot\mathbf{v} &= \Big(G_r\,\hat{\mathbf{r}} + G_\theta\,\hat{\boldsymbol{\theta}} + G_\varphi\,\hat{\boldsymbol{\varphi}}\Big)\cdot\Big(\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\varphi}\hat{\boldsymbol{\varphi}}\Big)\\ &= \dot{r}\,G_r + r\dot{\theta}G_\theta + r\sin\theta\,\dot{\varphi}G_\varphi\end{aligned}$$

In summary we have

$$\frac{d\psi}{dt} = \dot{r}\,G_r + r\dot{\theta}G_\theta + r\sin\theta\,\dot{\varphi}G_\varphi$$

That concludes the first method. Now for the second method. If we are working in spherical coordinates then presumably we have an expression for ψ in terms of r, θ, and φ.

We can use spherical coordinates to express both ψ and the path along which we measure ψ. When we do so we find that

$$\begin{aligned}\frac{d\psi}{dt} &= \frac{d}{dt}\psi(r,\theta,\varphi)\\ &= \frac{\partial\psi}{\partial r}\frac{dr}{dt} + \frac{\partial\psi}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\psi}{\partial\varphi}\frac{d\varphi}{dt}\\ &= \frac{\partial\psi}{\partial r}\dot r + \frac{\partial\psi}{\partial\theta}\dot\theta + \frac{\partial\psi}{\partial\varphi}\dot\varphi\end{aligned}$$

These two lines of calculus are hardly necessary to summarize, but let us write a summary anyway:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial r}\dot r + \frac{\partial\psi}{\partial\theta}\dot\theta + \frac{\partial\psi}{\partial\varphi}\dot\varphi$$

That concludes the second method. The form of the answer in this second method is different from the form in the first because it does not involve the G terms, nor does it involve the expression for $\mathbf{v}$ in spherical coordinates. Rather, this new form looks almost like a dot product between

$$\left(\frac{\partial\psi}{dr}, \frac{\partial\psi}{d\theta}, \frac{\partial\psi}{d\psi}\right) \quad \text{and} \quad \left(\dot r, \dot\theta, \dot\varphi\right)$$

These look almost like a gradient vector and a velocity vector. Yet the left one is not actually a gradient vector, and the right one is plainly not a velocity vector. Some of its terms do not even have units of meters per second!

We have now computed $d\psi/dt$ in two separate ways. Both methods must give the same answer, numerically speaking, and so we must have

$$\dot r\,G_r + r\dot\theta\,G_\theta + r\sin\theta\,\dot\varphi\,G_\varphi = \frac{\partial\psi}{\partial r}\dot r + \frac{\partial\psi}{\partial\theta}\dot\theta + \frac{\partial\psi}{\partial\varphi}\dot\varphi$$

After some shuffling and factoring we can express this relation as

$$\dot r\left(G_r - \frac{\partial\psi}{\partial r}\right) + \dot\theta\left(r\,G_\theta - \frac{\partial\psi}{\partial\theta}\right) + \dot\varphi\left(r\sin\theta\,G_\varphi - \frac{\partial\psi}{\partial\varphi}\right) = 0$$

The derivatives $\dot r$, $\dot\theta$, and $\dot\varphi$ are characteristic of the path along which we are measuring $d\psi/dt$. For example, if we are measuring $d\psi/dt$ along a path moving radially outward then we have only $\dot r$, while $\dot\theta$ and $\dot\varphi$ vanish. If we are measuring $d\psi/dt$ while moving in an azimuthal circle then perhaps we have only $\dot\varphi$, while $\dot r$ and $\dot\theta$ vanish. The point is that the above sum must come out to 0 for any and all paths, and thus we must have

$$G_r - \frac{\partial\psi}{\partial r} = 0 \qquad r\,G_\theta - \frac{\partial\psi}{\partial\theta} = 0 \qquad r\sin\theta\,G_\varphi - \frac{\partial\psi}{\partial\varphi} = 0$$

These relations let us compute the G values without too much more effort:

$$G_r = \frac{\partial\psi}{\partial r} \qquad G_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad G_\varphi = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}$$

The gradient in spherical coordinates is therefore

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\,\hat{\boldsymbol{\varphi}}$$