Taylor Series

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[Summary]

The Taylor series for a function $f(x)$ centered around a point x_0 has the coefficients

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

[Discovery]

Let us begin by noticing that an infinite polynomial in $x - x_0$ will converge when x is close to x_0. We will write such a polynomial, and then take its derivatives.

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

The first few derivatives are

$$\begin{aligned}
p'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_3(x - x_0)^3 + \cdots \\
p''(x) &= 2!\, a_2 + 3!\, a_3(x - x_0) + 4!/2\, a_4(x - x_0)^2 + 5!/3!\, a_5(x - x_0)^3 + \cdots \\
p'''(x) &= 3!\, a_3 + 4!\, a_4(x - x_0) + 5!/2\, a_5(x - x_0)^2 + 6!/3!\, a_6(x - x_0)^3 + \cdots \\
p^{(4)}(x) &= 4!\, a_4 + 5!\, a_5(x - x_0) + 6!/2\, a_6(x - x_0)^2 + 7!/3!\, a_7(x - x_0)^3 + \cdots \\
p^{(5)}(x) &= 5!\, a_5 + 6!\, a_6(x - x_0) + 7!/2\, a_7(x - x_0)^2 + 8!/3!\, a_8(x - x_0)^3 + \cdots \\
p^{(6)}(x) &= 6!\, a_6 + 7!\, a_7(x - x_0) + 8!/2\, a_8(x - x_0)^2 + 9!/3!\, a_9(x - x_0)^3 + \cdots
\end{aligned}$$

Notice that in general we will have

$$p^{(n)}(x_0) = n!\, a_n$$

Let us now consider some function $f(x)$, where f might be some familiar function like cos, sin, log, exp, $1/x$, and so on. We will set the derivatives of p at $x = x_0$ equal to the derivatives of $f(x)$ at $x = x_0$. This will serve to determine the value of a_n for every n, and it will make $p(x)$ a good approximation to $f(x)$ on and near the point $x = x_0$.

$$\begin{aligned}
p^{(n)}(x_0) &= f^{(n)}(x_0) \\
n!\, a_n &= f^{(n)}(x_0) \\
a_n &= \frac{f^{(n)}(x_0)}{n!}
\end{aligned}$$