Vector Projection

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[**Summary**]

The perpendicular vector projection operator applied to $\mathbf{r}$ is

$$\mathbf{r}_\perp = -\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r})$$

Where $\hat{\boldsymbol{\omega}}$ is a unit vector relative to which we take the perpendicular component of $\mathbf{r}$. This operator can be written alternatively as follows, though this form is less preferable because it contains the operand $\mathbf{r}$ twice:

$$\mathbf{r}_\perp = \mathbf{r} - \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r})$$

The matrix form of the operator is

$$\mathbf{r}_\perp = \frac{1}{\omega^2} \begin{bmatrix} \omega_y^2 + \omega_z^2 & -\omega_x\omega_y & -\omega_x\omega_z \\ -\omega_y\omega_x & \omega_x^2 + \omega_z^2 & -\omega_y\omega_z \\ -\omega_z\omega_x & -\omega_z\omega_y & \omega_x^2 + \omega_y^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $\boldsymbol{\omega}$ need not necessarily be of unit length, so long as its squared length is divided out after the matrix is applied.

[**Perpendicular Projection**]

Let us find the component of $\mathbf{r}$ that is perpendicular to some unit vector $\hat{\boldsymbol{\omega}}$. We will call this component $\mathbf{r}_\perp$. To begin, we know that $\hat{\boldsymbol{\omega}} \times \mathbf{r}$ will have the correct length $r \sin\theta$, which is what we are looking for. However, it will point perpendicular to the plane where $\hat{\boldsymbol{\omega}}$ and $\mathbf{r}$ live, and so it cannot satisfy

$$\mathbf{r} = \mathbf{r}_\parallel + \mathbf{r}_\perp$$

Fortunately it is easy to project $\hat{\boldsymbol{\omega}} \times \mathbf{r}$ back onto the proper plane by crossing it with $\hat{\boldsymbol{\omega}}$:

$$\mathbf{r}_\perp = (\hat{\boldsymbol{\omega}} \times \mathbf{r}) \times \hat{\boldsymbol{\omega}}$$

This is better written in operator form

$$\mathbf{r}_\perp = -\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r})$$

An alternative method for finding $\mathbf{r}_\perp$ is by subtracting $\mathbf{r}_\parallel$ from $\mathbf{r}$. We already know that

$$\mathbf{r}_\parallel = \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r})$$

and so we can construct $\mathbf{r}_\perp$ as

$$\mathbf{r}_\perp = \mathbf{r} - \hat{\boldsymbol{\omega}}\,(\hat{\boldsymbol{\omega}} \cdot \mathbf{r})$$

It is not difficult to verify that these definitions are the equivalent

$$\begin{aligned}
\mathbf{r} - \hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right) &= -\hat{\boldsymbol{\omega}} \times \left(\hat{\boldsymbol{\omega}} \times \mathbf{r}\right) \\
\mathbf{r} - \hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right) &= -\left[\hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right) - \mathbf{r}\left(\hat{\boldsymbol{\omega}} \cdot \hat{\boldsymbol{\omega}}\right)\right] \\
\mathbf{r} - \hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right) &= -\left[\hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right) - \mathbf{r}\right] \\
\mathbf{r} - \hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right) &= \mathbf{r} - \hat{\boldsymbol{\omega}}\left(\hat{\boldsymbol{\omega}} \cdot \mathbf{r}\right)
\end{aligned}$$

[Perpendicular Projection (Matrix Form)]

Let us find the matrix form for the projection

$$\mathbf{r}_\perp = -\hat{\boldsymbol{\omega}} \times \left(\hat{\boldsymbol{\omega}} \times \mathbf{r}\right)$$

Our result will come out more general if we don't assume ω to be to be a unit vector. We will thus opt to deal with

$$\omega^2 \mathbf{r}_\perp = -\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right)$$

First we must expand out the expression component wise. Let us start by finding $\boldsymbol{\omega} \times \mathbf{r}$

$$\boldsymbol{\omega} \times \mathbf{r} = \left(\omega_y z - \omega_z y\right)\hat{\boldsymbol{\imath}} + \left(\omega_z x - \omega_x z\right)\hat{\boldsymbol{\jmath}} + \left(\omega_x y - \omega_y x\right)\hat{\boldsymbol{k}}$$

Now we can cross this again with ω:

$$\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right) = \begin{bmatrix} \omega_y\left(\omega_x y - \omega_y x\right) - \left(\omega_z x - \omega_x z\right)\omega_z \\ \omega_z\left(\omega_y z - \omega_z y\right) - \left(\omega_x y - \omega_y x\right)\omega_x \\ \omega_x\left(\omega_z x - \omega_x z\right) - \left(\omega_y z - \omega_z y\right)\omega_y \end{bmatrix}$$

The negation of this is then

$$-\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right) = \begin{bmatrix} \left(\omega_z x - \omega_x z\right)\omega_z - \omega_y\left(\omega_x y - \omega_y x\right) \\ \left(\omega_x y - \omega_y x\right)\omega_x - \omega_z\left(\omega_y z - \omega_z y\right) \\ \left(\omega_y z - \omega_z y\right)\omega_y - \omega_x\left(\omega_z x - \omega_x z\right) \end{bmatrix}$$

When we expand things out a bit we get

$$\omega^2 \mathbf{r}_\perp = \begin{bmatrix} \omega_z\omega_z x - \omega_x\omega_z z - \omega_y\omega_x y + \omega_y\omega_y x \\ \omega_x\omega_x y - \omega_y\omega_x x - \omega_z\omega_y z + \omega_z\omega_z y \\ \omega_y\omega_y z - \omega_z\omega_y y - \omega_x\omega_z x + \omega_x\omega_x z \end{bmatrix}$$

Next we group terms on x, y, and z:

$$\omega^2 \mathbf{r}_\perp = \begin{bmatrix} x\left(\omega_y\omega_y + \omega_z\omega_z\right) - y\left(\omega_y\omega_x\right) - z\left(\omega_x\omega_z\right) \\ y\left(\omega_x\omega_x + \omega_z\omega_z\right) - x\left(\omega_y\omega_x\right) - z\left(\omega_z\omega_y\right) \\ z\left(\omega_x\omega_x + \omega_y\omega_y\right) - y\left(\omega_z\omega_y\right) - x\left(\omega_x\omega_z\right) \end{bmatrix}$$

The matrix equation for perpendicular projection is therefore

$$\mathbf{r}_\perp = \hat{\boldsymbol{\omega}} \times \left(\hat{\boldsymbol{\omega}} \times \mathbf{r}\right) = \frac{1}{\omega^2}\begin{bmatrix} \omega_y^2 + \omega_z^2 & -\omega_x\omega_y & -\omega_x\omega_z \\ -\omega_y\omega_x & \omega_x^2 + \omega_z^2 & -\omega_y\omega_z \\ -\omega_z\omega_x & -\omega_z\omega_y & \omega_x^2 + \omega_y^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$