Fourier Series

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[Summary]

Any periodic function with period τ can be written as

$$f(t) = \sum_{n=0}^{\infty} \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)$$

where the coefficient are determined by

$$\alpha_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \cos\left(\frac{2\pi}{\tau} n \cdot t\right) \, dt$$

$$\beta_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \sin\left(\frac{2\pi}{\tau} n \cdot t\right) \, dt$$

[Discovery]

Let us examine some function $f(x)$ which is periodic over an interval τ, meaning that

$$f(t+\tau) = (t)$$

We might hypothesize that this function can be approximated as a weighted sum of N other period functions $g_n(t)$:

$$f(t) \approx \sum_{n=0}^{N} a_n g_n(t)$$

But what functions should we use for $g_n(t)$? And how should we determine the weights a_n? The trigonometric functions might be a good place to start. The most general form for a sine wave is

$$A \cos(\omega t - \phi)$$

where ω is some angular frequency, and ϕ is some phase. We can also write this as

$$\alpha \cos(\omega t) + \beta \sin(\omega t)$$

where

$$A = \sqrt{\alpha^2 + \beta^2}$$
$$\phi = \arctan(\beta/\alpha)$$

If we want this sine wave to have the period τ then we should make sure that $\omega\tau$ is a multiple of 2π:

$$\omega\tau \bmod 2\pi = 0$$

or in other words that

$$\frac{\omega\tau}{2\pi} \bmod 1 = 0$$

or in other words that $\omega\tau/2\pi$ evaluates to some integer n

$$\frac{\omega\tau}{2\pi} = n$$

Solving this for ω gives

$$\omega = \frac{2\pi n}{\tau}$$

Now, from this result for ω it is obvious that there are infinitely many sine waves with period τ, because n can take on any positive value. Previously we had said that we were going to take N function $g_n(t)$ to make an approximation for $f(t)$. Yet now we have infinitely many choices for $g_n(t)$. Perhaps if we use all of them with the proper weights we will find an exact representation:

$$f(t) = \sum_{n=0}^{\infty} a_n A_n \cos(\omega_n t - \phi_n)$$

At this point we should notice that encoding the weight as $a_n A_n$ is redundant. We might as well just encode the whole weight in A_n, and instead write

$$f(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t - \phi_n)$$

or alternatively

$$f(t) = \sum_{n=0}^{\infty} \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)$$

Keep in mind that the above equality is still just a speculation. The truth of the above will depend on our ability to find values for A_n and ϕ_n such that that the infinite sum converges to $f(x)$. It would do just as well to find values for α_n and β_n, because we know how they can later be transformed into A_n and ϕ_n.

It turns out that finding α_n and β_n is easier than finding A_n and ϕ_n. The method we are about to present might seem like a leap of genius to those readers who have not studied the mathematics of time-averages. Yet it will not come as surprise to those readers who are familiar with time-averages, and who know that many time-averages can be made to zero out. In order to make conditions conducive to this zeroing let us multiply the whole relation by $\cos(\omega_m t)$, where m is some arbitrary integer:

$$f(t)\cos(\omega_m t) = \sum_{n=0}^{\infty} \alpha_n \cos(\omega_n t)\cos(\omega_m t) + \beta_n \sin(\omega_n t)\cos(\omega_m t)$$

We know that taking a time average of this equation over the period τ will make almost every term drop to 0, save for the term $\alpha_n \cos(\omega_n t) \cos(\omega_m t)$ where $m = n$, which will become $\cos^2(\omega_n t)$ and will time average to $\alpha_n \cdot \tau/2$:

$$\int_{-\tau/2}^{\tau/2} f(t) \cos(\omega_n t)\, dt = \alpha_n \frac{\tau}{2}$$

And so we have

$$\alpha_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \cos\left(\frac{2\pi}{\tau} n \cdot t\right) dt$$

Doing the same analysis with $\sin(\omega_m t)$ then shows that

$$\beta_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cdot \sin\left(\frac{2\pi}{\tau} n \cdot t\right) dt$$