The Double Pendulum

by Patrick Rutkowski, Columbia University www.patrick-rutkowski.com

[Summary]

Consider a double pendulum composed of masses m_1 and m_2. Let us say that mass m_1 has polar coordinates (α, θ) with respect to the origin, and that mass m_2 has polar coordinates (β, ϕ) with respect to mass m_1.

The system of differential equations governing the motion of double pendulum is as follows (note that we use the shorthand $M = m_1 + m_2$).

$$\begin{aligned}
\left[M\alpha^2\right]\ddot{\theta} + \left[\alpha\beta\cos(\theta-\phi)\right]\ddot{\phi} &= -m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) - Mg\alpha\cos\theta \\
\left[m_2\alpha\beta\cos(\theta-\phi)\right]\ddot{\theta} + \left[m_2\beta^2\right]\ddot{\phi} &= m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi) - m_2 g\beta\cos\phi
\end{aligned}$$

From the above equations can derive the separate generalized forces acting on each mass. These forces are complicated to such a degree that they are beyond intuitive human understanding. Nonetheless, we give them here:

$$\begin{aligned}
\ddot{\theta} = &\Bigg[\Big(-m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) - Mg\alpha\cos\theta\Big)\left(m_2\beta^2\right) \\
&- \left(m_2\alpha\beta\cos(\theta-\phi)\right)\Big(m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi) - m_2 g\beta\cos\phi\Big)\Bigg] \\
&\cdot 1/\left[\left(M\alpha^2\right)\left(m_2\beta^2\right) - \left(m_2\alpha\beta\cos(\theta-\phi)\right)\left(m_2\alpha\beta\cos(\theta-\phi)\right)\right] \\
\ddot{\phi} = &\Bigg[\left(M\alpha^2\right)\Big(m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi) - m_2 g\beta\cos\phi\Big) \\
&- \Big(-m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) - Mg\alpha\cos\theta\Big)\left(m_2\alpha\beta\cos(\theta-\phi)\right)\Bigg] \\
&\cdot 1/\left[\left(M\alpha^2\right)\left(m_2\beta^2\right) - \left(m_2\alpha\beta\cos(\theta-\phi)\right)\left(m_2\alpha\beta\cos(\theta-\phi)\right)\right]
\end{aligned}$$

[Discovery]

Consider a double pendulum composed of masses m_1 and m_2. Let us say that mass m_1 has polar coordinates (α, θ) with respect to the origin, and that mass m_2 has polar coordinates (β, ϕ) with respect to mass m_1. Our goal is to find a formula for the forces on each body, such that we can numerically integrate the forces and produce a computer simulation of a swinging double-pendulum.

To begin, we say that the location of the first body is

$$\begin{aligned}
x_1 &= \alpha\cos(\theta) \\
y_1 &= \alpha\sin(\theta)
\end{aligned}$$

Next we say that the location of the second body is

$$\begin{aligned} x_2 &= x_1 + \beta\cos\phi \\ &= \alpha\cos\theta + \beta\cos\phi \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + \beta\sin\phi \\ &= \alpha\sin\theta + \beta\sin\phi \end{aligned}$$

The squared velocity of the first body is then

$$v_1^2 = \alpha^2\dot{\theta}^2$$

and the squared velocity of the second body is

$$\begin{aligned} \dot{x}_1^2 &= \left(-\alpha\dot{\theta}\sin\theta - \beta\dot{\phi}\sin\phi\right)^2 \\ &= \alpha^2\dot{\theta}^2\sin^2\theta + 2\alpha\beta\dot{\theta}\dot{\phi}\sin\theta\sin\phi + \beta^2\dot{\phi}^2\sin^2\phi \end{aligned}$$

$$\begin{aligned} \dot{y}_1^2 &= \left(+\alpha\dot{\theta}\cos\theta + \beta\dot{\phi}\cos\phi\right)^2 \\ &= \alpha^2\dot{\theta}^2\cos^2\theta + 2\alpha\beta\dot{\theta}\dot{\phi}\cos\theta\cos\phi + \beta^2\dot{\phi}^2\cos^2\phi \end{aligned}$$

$$\begin{aligned} v_2^2 &= \dot{x}_1^2 + \dot{y}_1^2 \\ &= \alpha^2\dot{\theta}^2 + \beta^2\dot{\phi}^2 + 2\alpha\beta\dot{\theta}\dot{\phi}\left[\sin\theta\sin\phi + \cos\theta\cos\phi\right] \\ &= \alpha^2\dot{\theta}^2 + \beta^2\dot{\phi}^2 + 2\alpha\beta\dot{\theta}\dot{\phi}\cos(\theta-\phi) \end{aligned}$$

The kinetic energy of the system can then be written as follows. Note that we use the shorthand $M = m_1 + m_2$

$$\begin{aligned} T &= \frac{1}{2}m_1\alpha^2\dot{\theta}^2 + \frac{1}{2}m_2\alpha^2\dot{\theta}^2 + \frac{1}{2}m_2\beta^2\dot{\phi}^2 + m_2\alpha\beta\dot{\theta}\dot{\phi}\cos(\theta-\phi) \\ &= \frac{1}{2}M\alpha^2\dot{\theta}^2 + \frac{1}{2}m_2\beta^2\dot{\phi}^2 + m_2\alpha\beta\dot{\theta}\dot{\phi}\cos(\theta-\phi) \end{aligned}$$

The total potential energy of the system is just the usual gravitational potential

$$\begin{aligned} U &= m_1 g\alpha\sin\theta + m_2 g\left[\alpha\sin\theta + \beta\sin\phi\right] \\ &= Mg\alpha\sin\theta + m_2 g\beta\sin\phi \end{aligned}$$

The Lagrangian of the system is then simply

$$\mathcal{L} = T - U$$

The system's generalized momenta are then

$$\begin{aligned} p_\theta &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = M\alpha^2\dot{\theta} + m_2\alpha\beta\dot{\phi}\cos(\theta-\phi) \\ p_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m_2\beta^2\dot{\phi} + m_2\alpha\beta\dot{\theta}\cos(\theta-\phi) \end{aligned}$$

and the system's generalized forces are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= -\, m_2\alpha\beta\dot{\theta}\dot{\phi}\sin(\theta-\phi) - Mg\alpha\cos\theta \\
\frac{\partial \mathcal{L}}{\partial \phi} &= +\, m_2\alpha\beta\dot{\theta}\dot{\phi}\sin(\theta-\phi) - m_2 g\beta\cos\phi
\end{aligned}$$

In order to apply the Euler-Lagrange equations we must differentiate the momenta with respect to time

$$\begin{aligned}
\dot{p}_\theta &= M\alpha^2\ddot{\theta} + m_2\alpha\beta\ddot{\phi}\cos(\theta-\phi) - m_2\alpha\beta\dot{\phi}\sin(\theta-\phi)\left[\dot{\theta}-\dot{\phi}\right] \\
\dot{p}_\phi &= m_2\beta^2\ddot{\phi} + m_2\alpha\beta\ddot{\theta}\cos(\theta-\phi) - m_2\alpha\beta\dot{\theta}\sin(\theta-\phi)\left[\dot{\theta}-\dot{\phi}\right]
\end{aligned}$$

Recall that the Euler-Lagrange equations for this system are

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left[\partial\mathcal{L}/\partial\dot{\theta}\right] \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left[\partial\mathcal{L}/\partial\dot{\phi}\right]$$

We have already calculated all of these terms, and so we can immediately proceed

$$\begin{aligned}
-\, m_2\alpha\beta\dot{\theta}\dot{\phi}\sin(\theta-\phi) - Mg\alpha\cos\theta &= M\alpha\ddot{\theta} + m_2\alpha\beta\ddot{\phi}\cos(\theta-\phi) \\
&\quad - m_2\alpha\beta\dot{\phi}\sin(\theta-\phi)\left[\dot{\theta}-\dot{\phi}\right] \\
m_2\alpha\beta\dot{\theta}\dot{\phi}\sin(\theta-\phi) - m_2 g\beta\cos\phi &= m_2\beta\ddot{\phi} + m_2\alpha\beta\ddot{\theta}\cos(\theta-\phi) \\
&\quad - m_2\alpha\beta\dot{\theta}\sin(\theta-\phi)\left[\dot{\theta}-\dot{\phi}\right]
\end{aligned}$$

Thankfully, some simplification here is possible

$$\begin{aligned}
-\, Mg\alpha\cos\theta &= M\alpha^2\ddot{\theta} + m_2\alpha\beta\ddot{\phi}\cos(\theta-\phi) + m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) \\
-\, m_2 g\beta\cos\phi &= m_2\beta^2\ddot{\phi} + m_2\alpha\beta\ddot{\theta}\cos(\theta-\phi) - m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi)
\end{aligned}$$

We can view the above result as a pair of linear equations in $\ddot{\theta}$ and $\ddot{\phi}$.

$$\begin{aligned}
\left[M\alpha^2\right]\ddot{\theta} + \left[m_2\alpha\beta\cos(\theta-\phi)\right]\ddot{\phi} &= -\, m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) - Mg\alpha\cos\theta \\
\left[m_2\alpha\beta\cos(\theta-\phi)\right]\ddot{\theta} + \left[m_2\beta^2\right]\ddot{\phi} &= m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi) - m_2 g\beta\cos\phi
\end{aligned}$$

These equations can be solved for $\ddot{\theta}$ and $\ddot{\phi}$ by using Cramer's rule.

$$\begin{aligned}
\ddot{\theta} = \Big[&\Big(-\, m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) - Mg\alpha\cos\theta\Big)\left(m_2\beta^2\right) \\
&- \left(m_2\alpha\beta\cos(\theta-\phi)\right)\Big(m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi) - m_2 g\beta\cos\phi\Big)\Big] \\
&\cdot 1/\left[\left(M\alpha^2\right)\left(m_2\beta^2\right) - \left(m_2\alpha\beta\cos(\theta-\phi)\right)\left(m_2\alpha\beta\cos(\theta-\phi)\right)\right]
\end{aligned}$$

$$\begin{aligned}
\ddot{\phi} = \Big[&\left(M\alpha^2\right)\Big(m_2\alpha\beta\dot{\theta}^2\sin(\theta-\phi) - m_2 g\beta\cos\phi\Big) \\
&- \Big(-\, m_2\alpha\beta\dot{\phi}^2\sin(\theta-\phi) - Mg\alpha\cos\theta\Big)\left(m_2\alpha\beta\cos(\theta-\phi)\right)\Big] \\
&\cdot 1/\left[\left(M\alpha^2\right)\left(m_2\beta^2\right) - \left(m_2\alpha\beta\cos(\theta-\phi)\right)\left(m_2\alpha\beta\cos(\theta-\phi)\right)\right]
\end{aligned}$$

These solutions via Cramer's rule can easily be plugged into a computer simulation and numerically integrated in order to show a visualization of the chaotic motion of a double pendulum, a chaos which arises from the tremendous complexity of the equations which we have derived here in this paper.