The Derivative of arctan

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[Summary]

The derivative of arctan is

$$\arctan' = \frac{1}{1+x^2}$$

[Derivation]

Consider that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

If we set $\theta = \arcsin(x)$ we then have

$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$$

At this point it is easy to isolate x in terms of $\tan(\arcsin x)$

$$\begin{aligned}
\tan(\arcsin x)\sqrt{1-x^2} &= x \\
\tan^2(\arcsin x)\left(1-x^2\right) &= x^2 \\
\tan^2(\arcsin x) &= x^2\left(1+\tan^2(\arcsin x)\right) \\
x &= \left[\frac{\tan^2(\arcsin x)}{1+\tan^2(\arcsin x)}\right]^{1/2}
\end{aligned}$$

Now we can set $x = \sin(\theta)$ to recover the previous relationship between the arguments:

$$\sin\theta = \left[\frac{\tan^2\theta}{1+\tan^2\theta}\right]^{1/2}$$

Next we can apply arcsin to both sides

$$\theta = \arcsin\left(\left[\frac{\tan^2\theta}{1+\tan^2\theta}\right]^{1/2}\right)$$

Finally we can set $\theta = \arctan(x)$:

$$\arctan(x) = \arcsin\left(\left[\frac{x^2}{1+x^2}\right]^{1/2}\right)$$

This function can then be differentiated directly. This is a bit laborious, but it is worth the effort

$$\begin{aligned}
\arctan'(x) &= \arcsin'\left(\left[\frac{x^2}{1+x^2}\right]^{1/2}\right) \cdot \frac{1}{2}\left[\frac{x^2}{1+x^2}\right]^{-1/2} \cdot \frac{d}{dx}\left[\frac{x^2}{1+x^2}\right] \\
&= \frac{1}{\sqrt{1-x^2/(1+x^2)}} \cdot \frac{1}{2}\left[\frac{x^2}{1+x^2}\right]^{-1/2} \cdot \frac{d}{dx}\left[\frac{x^2}{1+x^2}\right] \\
&= \frac{1}{\sqrt{1-x^2/(1+x^2)}} \cdot \frac{1}{2}\left[\frac{x^2}{1+x^2}\right]^{-1/2} \cdot \frac{2x(1+x^2)-x^2(2x)}{\left(1+x^2\right)^2} \\
&= \frac{1}{\sqrt{1-x^2/(1+x^2)}} \cdot \frac{1}{2}\left[\frac{x^2}{1+x^2}\right]^{-1/2} \cdot \frac{2x}{\left(1+x^2\right)^2} \\
&= \frac{1}{\sqrt{1-x^2/(1+x^2)}} \cdot \frac{\sqrt{1+x^2}}{2x} \cdot \frac{2x}{\left(1+x^2\right)^2} \\
&= \frac{1}{\sqrt{1-x^2/(1+x^2)}} \cdot \frac{\sqrt{1+x^2}}{\left(1+x^2\right)^2} \\
&= \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1-x^2/(1+x^2)}} \cdot \frac{\sqrt{1+x^2}}{\left(1+x^2\right)^2} \\
&= \sqrt{1+x^2} \cdot \frac{1}{\sqrt{1+x^2-x^2}} \cdot \frac{\sqrt{1+x^2}}{\left(1+x^2\right)^2} \\
&= \frac{1}{1+x^2}
\end{aligned}$$

This completes the proof. We have found that

$$\arctan' = \frac{1}{1+x^2}$$