The Euler-Lagrange Equation

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[**Summary**]

The Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}$$

Any function y which satisfies this equation for a given function f will cause the following integral S to be locally minimum, maximum, or at least stable relative to other functions in the neighborhood of y

$$S = \int_{t_1}^{t_2} f(y, \dot{y}, t)\, dt$$

[**Discovery**]

Let us examine the following integral

$$S = \int_{t_1}^{t_2} f(y, \dot{y}, t)\, dt$$

Consider the set Y of all functions y which have the property that $y(t_1) = Q$ and $y(t_2) = P$. We seek to find a specific subset of functions within Y which put S in a minimum position, maximum position, or at least in a stable position with respect to other functions in their immediate vicinity. If we perturb a function in this subset and then make use of the integral S to examine the perturbed version we should find that there has been no significant change in S relative to the size of the perturbation.

Let us now carry out this perturbation analysis. We will say that the shape of the perturbation is some function $\eta(t)$, and we will denote the differential magnitude of the perturbation by the symbol ε. The combined shape and magnitude are thus written as $\varepsilon\eta(t)$. We will also specify that the perturbation is such that $\eta(t_1) = 0$ and $\eta(t_2) = 0$, so that even once the perturbation is applied the resulting function still goes from Q to P, which is required for it to be in the set Y. The integral S over the perturbed function will be

$$S = \int_{t_1}^{t_2} f(y + \varepsilon\eta, \dot{y} + \varepsilon\dot{\eta}, t)\, dt$$

In order to shorten our notation we will call the perturbed path $\xi(\varepsilon, t)$, and we will set $u(\varepsilon, t) = \Big[\xi(\varepsilon, t), \dot{\xi}(\varepsilon, t), t\Big]$. We can now write the integral over the perturbed path as

$$S = \int_{t_1}^{t_2} f(u(\varepsilon, t))\, dt$$

We can easily calculate the exact increase in the integral's value relative to the magnitude of the perturbation.

$$\begin{aligned}
\frac{dS}{d\varepsilon} &= \int_{t_1}^{t_2} \frac{d}{d\varepsilon}\left[f(u(\varepsilon,t))\right]\,dt \\
&= \int_{t_1}^{t_2} \eta\frac{\partial f}{\partial y}\Big(u(\varepsilon,t)\Big) + \dot{\eta}\frac{\partial f}{\partial \dot{y}}\Big(u(\varepsilon,t)\Big)\,dt \\
&= \int_{t_1}^{t_2} \eta\frac{\partial f}{\partial y}\Big(u(\varepsilon,t)\Big)\,dt + \left[\eta\frac{\partial f}{\partial \dot{y}}\Big(u(\varepsilon,t)\Big) - \int \eta\frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\Big(u(\varepsilon,t)\Big)\,dt\right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \eta\frac{\partial f}{\partial y}\Big(u(\varepsilon,t)\Big)\,dt + \left[\eta\frac{\partial f}{\partial \dot{y}}\Big(u(\varepsilon,t)\Big)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta\frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\Big(u(\varepsilon,t)\Big)\,dt \\
&= \int_{t_1}^{t_2} \eta\left[\frac{\partial f}{\partial y}\Big(u(\varepsilon,t)\Big) - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\Big(u(\varepsilon,t)\Big)\right]\,dt
\end{aligned}$$

We wish to examine the rate of change of S with respect to ε as the perturbation is only just starting to be applied, and so we will set $\varepsilon = 0$. This will result in $u(\varepsilon, t)$ reducing back to the ideal path

$$u(0,t) = \Big[\xi(0,t), \dot{\xi}(0,t), t\Big] = \Big[y(t), \dot{y}(t), t\Big]$$

Our derivative is now

$$\frac{dS}{d\varepsilon}\Big(0\Big) = \int_{t_1}^{t_2} \eta\left[\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\right]\,dt = 0$$

We have set this all to 0 because the integral is now being evaluated over one of the ideal paths y from the set Y, which by construction is a path that causes S to be stabilized. We now notice that the magnitude ε of the perturbation has been eliminated from our equation, but the term η denoting the perturbation's shape still remains. This shape might be anything, and everything. It certainly need not be 0 for all t, and it certainly need not be well behaved enough to to render the integral always equal to 0. The only way to guarantee that the integral comes out to 0 for all η is to ensure that

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} = 0$$

or in other words that

$$\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}$$

This is called the Euler-Lagrange equation. Any function y which satisfies this equation for a given function f will cause the integral S to be locally minimum, maximum, or at least stable.