The Dot Product

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[Summary]

The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

The dot product has the property that

$$\mathbf{a} \cdot \mathbf{b} = ab \cdot \cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

[Discovery]

We know from the law of cosines that for a triangle with sides a, b, and c we will hold to the relation

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where θ is the angle between sides a and b. Yet, if we allow this triangle to live in an n-dimensional space where $n \geq 2$ then we can represent it with the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c} = \mathbf{b} - \mathbf{a}$. The squared length of $\mathbf{c}$ will be

$$\begin{aligned}
\mathbf{c}^2 = (\mathbf{b} - \mathbf{a})^2 &= (b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2 \\
&= \left(b_x^2 - 2b_x a_x + a_x^2\right) + \left(b_y^2 - 2b_y a_y + a_y^2\right) + \left(b_z^2 - 2b_z a_z + a_z^2\right) \\
&= a^2 + b^2 - 2\left(a_x b_x + a_y b_y + a_z b_z\right)
\end{aligned}$$

Setting this result equal to c^2 from before shows us that

$$ab \cdot \cos\theta = a_x b_x + a_y b_y + a_z b_z$$

We define this quantity to be the dot product of $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} \cdot \mathbf{b} = ab \cdot \cos\theta = a_x b_x + a_y b_y + a_z b_z$$