HW1

November 14, 2022

2. Let $U \in \mathbb{R}^{n\times r}$ be an $n \times r$ matrix with orthonormal columns $U^\top U = I_r$. Let V be an $m \times r$ matrix.

- (2 pts). Propose an algorithm for the computation of $\|A\|_F^2$, where $A = UV^\top$, and estimate its complexity with respect to n, m, r. How orthogonalization can be used?
- (3 pts). Consider the matrix $B = A \circ A$, where $\circ$ is the elementwise product of matrices (i.e., the elements of the matrix B are squares of the elements of the matrix A). What is the maximal possible rank of the matrix B?
- (5 pts). Propose an algorithm for the computation of $\|B\|_F^2$ and estimate its complexity with respect to n, m, r.
- (2 pts). Let $O \in \mathbb{R}^{n\times n}$ be orthogonal matrix. Characterise explicitly all orthogonal matrices that are positive definite. You may start with $n = 2$.
- (2 pts). Show that any unitary matrix from $\mathbb{C}^{n\times n}$ can be represented as a product of at most n Householder reflectors.

3. We define approximate ϵ−rank of matrix A as $\operatorname{rank}_\epsilon A = \min\left\{\operatorname{rank}(X) : \|A - X\|_{\max} \le \epsilon\right\}$.

- (6 pts). Let $f(x, y)$, $x, y \in [0, 1]$ is analytic function with $\partial_y^{(k)} f(x, y)\Big|_{y=0} \le M$ for all k. Show that approximate rank of matrix $A_{ij} = f(x_i, y_j)$ where x_i, $i = 1, \dots, p$, y_j, $j = 1, \dots, q$ are arbitrary points inside $[0, 1]^2$, is bounded independent of p, q and the choice of points.
- (6 pts). Extend result from the previous point (a.) to $x, y \in \mathbb{R}^N$. Observe how the bound depends on N.