Notes On k-Means Clustering

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Abstract

k-Means is a popular clustering algorithm that reduces sum of minimum distances from data points to the centers. Now a days most of the times seeded centers are used instead of choosing all uniformly at random. k-Means++ chooses the centers with a so called D^2 weighting. In this paper, we explore various dimensions of clustering. First of all, we discuss the fact that the primary assumption in k-Means++ algorithm by Arthur and Vassilvitskii is wrong. This does not mean anything about the algorithm itself, just that their fundamental assumption was false rendering the mathematical analysis that follows false as well. We introduce a way to choose the first center with probability instead of choosing uniformly at random for k-Means++, a new way of initializing centers like k-Means++. Empirical evidence shows that our proposed algorithm performs better than available methods of center initialization and that our way of choosing first center for k-Means++ performs better than original k-means++. We also explore various properties regarding a set of points that helps us understand how the centers should be chosen. These properties help us analyze the expected value of inertia function as well. We also provide an upper bound for inertia when centers are initialized under certain assumption.

1. Introduction

Widely regarded as the most popular clustering techniques, k-Means remains a humble interesting topic in machine learning as well as computational geometry. Roughly the problem is: given a set of points $\mathbf{X}$ in $\mathbb{R}^d$. Find a set of centers $\mathcal{C}$ such that the function *inertia*

$$\mathcal{I} = \sum_{\mathbf{x}\in\mathbf{X}} \min_{\mathbf{c}\in\mathcal{C}}(\|\mathbf{c}-\mathbf{x}\|^2)$$

is minimum where $\|\cdot\|$ is the L_2 norm[1].

Default k-Means algorithm starts with random centers and then converge based on minimum distances of the centers from the data points. New centers are calculated based on the centroid. This is known as Lloyd's algorithm (Lloyd, 1982). We repeat this process until no more change is possible. k-Means++ (Arthur et al., 2007) takes it one step further by choosing the initial centers carefully. Only the first center is chosen at random. Then

1. $\|c-x\|$ or L_2 norm of $\mathbf{c}-\mathbf{x}$ is the distance between the center $\mathbf{c}$ and point $\mathbf{x}$ or the magnitude of the vector $\mathbf{c}-\mathbf{x}$.

the rest of the $k-1$ centers are chosen using D^2 *weighting* as the corresponding authors call it.

We will briefly show why we should not look into center updating any further. The reason is that the centroid is a point that minimizes the sum of squared distances.

$$
\begin{aligned}
E &= \sum_{i=1}^{n} \|\mathbf{x}_i - \mathbf{x}\|^2 \\
\implies E &= \sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{x}) I (\mathbf{x}_i - \mathbf{x})^T \\
\implies \frac{\partial E}{\partial \mathbf{x}} &= -2 \sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{x}) \\
\frac{\partial E}{\partial \mathbf{x}} &= 0 \\
\iff \mathbf{x} &= \frac{\sum_{i=1}^{n} \mathbf{x}_i}{n}
\end{aligned}
$$

where I is the identity matrix of the same rank as $\mathbf{x}$.

2. Related Work

There has been multiple surveys on k-Means in the literature. Probably the most relevant work in this regard is done by Celebi et al. (2013). However, most of the algorithms used in that paper are not used practically very much. It was also noted by the authors themselves that k-Means++ and its greedy version work better than most. They also mention that probabilistic algorithms perform better than deterministic ones. Moreover, another highly influential center initialization algorithm (Ostrovsky et al., 2006) was not considered in their experiments. There is no mention of Ostrovsky's algorithm in their paper whatsoever.

We suspect that standard procedures were not followed for comparison among algorithms in Arthur et al. (2007). While the authors provided theoretical limits, data sets were not normalized or standardized as their high inertia values indicate. Even without following those procedures, our results seem to differ quite a little bit from theirs. This may not be any technical error in implementation. Most likely this is due to different ways of implementations coupled with the fact that these are probabilistic algorithms.

We would also like to point out that to our knowledge no surveys were done after removing linear dependency prior to running the experiments. This is a very important step if we are to get a meaningful clustering out of k-Means algorithm. More specifically, PCA decomposes the existing variables into orthogonal[2] ones. If we use PCA (principal component analysis) before running a clustering algorithm, we can redefine the variables into linearly independent ones. For our experiment, we have used PCA on every dataset before running cluster algorithm.

While experimenting on such algorithms, it is of utmost importance to run the same experiment more than once under the same parameters and conditions. For example, assume that we want to compare k-Means++ and Forgy's algorithm (Forgy, 1965) for $k = 5$ clusters.

2. It is well known that orthogonal vectors are linearly independent.

We should run this experiment at least m times where $m > 1$ in order to eliminate bias and account for randomness. We run each experiment a total of 20 times and take the average and minimum values of inertia. The number of iterations was set to 300.

3. Proposed Initialization

First we describe k-Means, Ostrovsky's and k-Means++ algorithm. Then we describe our proposed method of center initialization. For a set of points S and a point x, we use $\min(\|x - S\|)$ to denote the minimum of distances from x to the points of S that is $\min(\|x - S\|) = \min_{a \in S}(\|x - a\|)$. Set $D(\mathbf{x}) = \min(\|\mathbf{x} - \mathcal{C}\|)$ for a point x and a set of centers $\mathcal{C}$. Let us denote the centroid of S by μ_S that is $\mu_S = \frac{1}{|S|} \sum_{x \in S} x$.

3.1 Lloyd

Lloyd's algorithm is the simplest way of solving the k-Means problem.

i Choose k centers $\mathcal{C} = \{c_1, c_2, \cdots, c_k\}$ at random uniformly.

ii For each $1 \leq i \leq k$, set $\mathcal{C}_i = \{\mathbf{x} \in \mathbf{X} : \|\mathbf{x} - c_i\| = \min(x - \mathcal{C})\}$.

iii Set $c_i = \bar{\mathcal{C}_i}$.

iv Repeat step 2 and 3 until convergence is reached or the number of iterations is reached.

3.2 ORSS

ORSS algorithm (Ostrovsky et al., 2006) chooses two initial centers instead of one.

i Choose two points $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with probability proportional to $\|\mathbf{x} - \mathbf{y}\|^2$. Set $\mathcal{C}_2 = \{\mathbf{x}, \mathbf{y}\}$.

ii For a set of $i \geq 2$ existing centers $\mathcal{C}_i$, choose a random point $\mathbf{x} \in \mathbf{X}$ with probability $\frac{D(\mathbf{x})^2}{\sum_{\mathbf{y} \in \mathbf{X}} D(\mathbf{y})^2}$. Set $c_{i+1} = \mathbf{x}$ and $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{c_{i+1}\}$.

iii Repeat step (ii) until $i = k$.

iv For each $1 \leq i \leq k$, set $\mathcal{C}_i = \{\mathbf{x} \in \mathbf{X} : \|\mathbf{x} - c_i\| = \min(x - \mathcal{C})\}$.

v Set $c_i = \bar{\mathcal{C}_i}$.

vi Repeat (iv) and (v) until convergence or number of iteration is reached.

3.3 k-Means++

k-Means++ (Arthur et al., 2007) can be considered as an improvement on ORSS algorithm.

i Choose a point $\mathbf{x} \in \mathbf{X}$ at random uniformly and set $\mathcal{C}_1 = \{\mathbf{x}\}$.

ii For a set of $i \geq 1$ existing centers $\mathcal{C}_i$, choose a random point $\mathbf{x} \in \mathbf{X}$ with probability $\frac{D(\mathbf{x})^2}{\sum_{\mathbf{y} \in \mathbf{X}} D(\mathbf{y})^2}$. Set $c_{i+1} = \mathbf{x}$ and $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{c_{i+1}\}$.

iii Repeat step (ii) until $i = k$.

iv For each $1 \leq i \leq k$, set $\mathcal{C}_i = \{\mathbf{x} \in \mathbf{X} : \|\mathbf{x} - c_i\| = \min(x - \mathcal{C})\}$.

v Set $c_i = \bar{\mathcal{C}}_i$.

vi Repeat (iv) and (v) until convergence or number of iteration is reached.

3.4 Variance Based

Roughly our idea is that we want to minimize the variance of distances from the new center to existing ones. The intuition behind this is to make the clusters as balanced as possible, hence possibly eliminating some anomalies within a cluster and reducing inertia. However, since the new centers are taken based on variances, we need two initial centers. We chose to initialize the first two centers as in ORSS algorithm. Assume that for a new point $\mathbf{x}$ and a set of centers $\mathcal{C}_k = \{c_1, \cdots, c_k\}$, the variance of the squared distances $\{\|c_1 - \mathbf{x}\|^2, \|c_2 - \mathbf{x}\|^2 \cdots, \|c_k - \mathbf{x}\|^2\}$ is $\nu_{\mathbf{x}}(\mathcal{C}_k)$.

i Choose two centers $\mathbf{x}, \mathbf{y}$ with probability proportional to $\|\mathbf{x} - \mathbf{y}\|^2$. Set $\mathcal{C}_2 = \{\mathbf{x}, \mathbf{y}\}$.

ii For already existing set of i centers $\mathcal{C}_i = \{c_1, \cdots, c_i\}$, choose a new center $\mathbf{x} \in \mathbf{X}$ with probability $\rho(\mathbf{x}) = 1 - \dfrac{\nu_{\mathbf{x}}(\mathcal{C}_i)}{\sum_{\mathbf{y} \in \mathbf{X}} \nu_{\mathbf{y}}(\mathcal{C}_i)}$.

iii Repeat step (ii) until $i = k$.

iv For each $1 \leq i \leq k$, set $\mathcal{C}_i = \{\mathbf{x} \in \mathbf{X} : \|\mathbf{x} - c_i\| = \min(x - \mathcal{C})\}$.

v Set $c_i = \bar{\mathcal{C}}_i$.

vi Repeat (iv) and (v) until convergence or number of iteration is reached.

3.5 First center for k-Means++

In k-Means++ algorithm, the first center is chosen uniformly at random. However, not all points have the same contribution to inertia. We choose x as a first center in a way that is analogous to explained variance ratio by x.

i Choose $\mathbf{x}$ with probability $\dfrac{f(\mathbf{x})}{\sum_{\mathbf{y} \in \mathbf{X}} f(\mathbf{y})}$ where $f(x) = \sum_{\mathbf{y} \in \mathbf{X}} \|\mathbf{x} - \mathbf{y}\|^2$. Set $\mathcal{C}_1 = \{\mathbf{x}\}$.

ii Repeat steps (ii) to (vi) in k-Means++.

4. Analysis

First, we will analyze the choosing of first center in k-Means++. Denote the sum of k-th powers of modulus of $\mathbf{x} \in \mathbf{X}$ as $\mathcal{S}_k(\mathbf{X})$ or in short $\mathcal{S}_k$ when the context is clear. That is,

$$\mathcal{S}_k(\mathbf{X}) = \sum_{\mathbf{x} \in \mathbf{X}} \|\mathbf{x}\|^k$$

We also write $\sum_{\mathbf{x}\in\mathbf{X}} \mathbf{x}$ as $\mathcal{S}$. Then we can express the probability of a point $\mathbf{x} \in \mathbf{X}$ being chosen as the first center $p(\mathbf{x})$ in terms of $\mathcal{S}$ and $\mathcal{S}_2$.

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\mathbf{y}\in\mathbf{X}} \|\mathbf{x}-\mathbf{y}\|^2 \\ &= \sum_{\mathbf{y}\in\mathbf{X}} \left(\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2\right) \\ &= n\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \sum_{\mathbf{y}\in\mathbf{X}} \mathbf{y}\rangle + \sum_{\mathbf{y}\in\mathbf{X}} \|\mathbf{y}\|^2 \\ &= n\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathcal{S}\rangle + \mathcal{S}_2(\mathbf{X}) \\ &= n\|x\|^2 - 2\langle \mathbf{x}, n\mu_{\mathbf{X}}\rangle + \mathcal{S}_2(\mathbf{X}) \end{aligned}$$

where $\mu_{\mathbf{X}}$ is the centroid of $\mathbf{X}$.

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x}) &= \sum_{\mathbf{x}\in\mathbf{X}} \left(n\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathcal{S}\rangle + \mathcal{S}_2\right) \\ &= n\sum_{\mathbf{x}\in\mathbf{X}} \|\mathbf{x}\|^2 - 2\langle \sum_{\mathbf{x}\in\mathbf{X}} \mathbf{x}, \mathcal{S}\rangle + \sum_{\mathbf{x}\in\mathbf{X}} \mathcal{S}_2 \\ &= n\mathcal{S}_2 - 2\langle \mathcal{S}, \mathcal{S}\rangle + n\mathcal{S}_2 \\ &= 2(n\mathcal{S}_2 - \langle \mathcal{S}, \mathcal{S}\rangle) \end{aligned}$$

Notice that this looks similar to $2n^2\sigma^2(\mathbf{X})$, except we have $\langle \mathbf{x}, \mathbf{x}\rangle = \|\mathbf{x}\|^2$ instead of $\|\mathbf{x}-\mu\|^2$ and $\left(\frac{\langle \mathcal{S}, \mathcal{S}\rangle}{n}\right)^2$ which is basically $\frac{\|\mathcal{S}\|^2}{n^2}$. Analogously[3], we can say that $p(\mathbf{x})$ is equivalent to the ratio of variance explained by $\mathbf{x}$.

$$\begin{aligned} p(x) &= \frac{f(\mathbf{x})}{\sum_{\mathbf{y}\in\mathbf{X}} f(\mathbf{y})} \\ &= \frac{n\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathcal{S}\rangle + \mathcal{S}_2}{2(n\mathcal{S}_2 - \langle \mathcal{S}, \mathcal{S}\rangle)} \end{aligned}$$

From the definition of variance, for a set of points S,

$$\begin{aligned} \sigma^2(S) &= \frac{\sum_{x\in S} \|x-\mu_S\|^2}{|S|} - \|\mu_S\|^2 \\ \sum_{x\in S} \|x-\mu_S\|^2 &= |S|(\sigma^2 + \|\mu_S\|^2) \end{aligned} \tag{1}$$

3. This is not an exact interpretation of explained variance ratio. See below for an exact analysis of what this actually denotes.

We can use equation (1) to express $f(\mathbf{x})$ above in another way. For any arbitrary point a and μ as the centroid of $\mathbf{X}$,

$$\begin{aligned}
\sum_{\mathbf{x}\in\mathbf{X}} \|\mathbf{x}-a\|^2 &= \sum_{\mathbf{x}\in\mathbf{X}} \|\mathbf{x}-\mu+\mu-a\|^2 \\
&= \sum_{\mathbf{x}\in\mathbf{X}} \left(\|\mathbf{x}-\mu\|^2 + 2\langle \mathbf{x}-\mu, \mu-a\rangle + \|\mu-a\|^2\right) \\
&= \sum_{\mathbf{x}\in\mathbf{X}} \|\mathbf{x}-\mu\|^2 + 2\left\langle \sum_{\mathbf{x}\in\mathbf{X}} \mathbf{x}-n\mu, \mu-a \right\rangle + n\|\mu-a\|^2 \\
&= n(\sigma^2 + \|\mu\|^2) + 2\langle n\mu - n\mu, \mu-a\rangle + n\|\mu-a\|^2 \\
&= n(\sigma^2 + \|\mu\|^2 + \|\mu-a\|^2) \qquad (2)
\end{aligned}$$

Using equation (2), we can rewrite $p(x)$ in another way.

$$\begin{aligned}
\sum_{\mathbf{x}\in\mathbf{X}}\sum_{\mathbf{y}\in\mathbf{X}} \|\mathbf{x}-\mathbf{y}\|^2 &= \sum_{\mathbf{x}\in\mathbf{X}} n(\sigma^2 + \|\mu\|^2 + \|\mu-\mathbf{x}\|^2) \\
&= n(n\sigma^2 + n\|\mu\|^2 + \sum_{\mathbf{x}\in\mathbf{X}} \|\mu-\mathbf{x}\|^2) \\
&= n(n\sigma^2 + n\|\mu\|^2 + n(\sigma^2 + \|\mu\|^2)) \\
&= 2n^2(\sigma^2 + \|\mu\|^2)
\end{aligned}$$

Thus the probability becomes

$$\begin{aligned}
p(x) &= \frac{\sum_{\mathbf{y}\in\mathbf{X}} \|\mathbf{x}-\mathbf{y}\|^2}{\sum_{\mathbf{y}\in\mathbf{X}}\sum_{\mathbf{x}'\in\mathbf{X}} \|\mathbf{x}'-\mathbf{y}\|^2} \\
&= \frac{n(\sigma^2 + \|\mu\|^2 + \|\mu-\mathbf{x}\|^2)}{2n(\sigma^2 + \|\mu\|^2)} \\
&= \frac{\sigma^2 + \|\mu\|^2 + \|\mu-\mathbf{x}\|^2}{2(\sigma^2 + \|\mu\|^2)} \\
&= \frac{1}{2} + \frac{\|\mu-\mathbf{x}\|^2}{2(\sigma^2 + \|\mu\|^2)}
\end{aligned}$$

Now it becomes more clear how this is related to explained variance ratio. However, to make things smoother, one can also choose to use the following as the probability of x being chosen as the first center.

$$p(x) = \frac{\|\mu-\mathbf{x}\|^2}{2(\sigma^2 + \|\mu\|^2)}$$

Also, notice that computationally this version of $p(x)$ is much cheaper than the one we mentioned above.

We will now analyze the center initialization algorithms using inertia value. Since these algorithms are probabilistic, we are going to take a look at the expected value of inertia. For a set of points $\mathbf{X}$ and a set of centers $\mathcal{C}$, we denote the inertia by $\mathcal{I}_{\mathcal{C}}(\mathbf{X})$.

$$\mathcal{I}_{\mathcal{C}}(\mathbf{X}) = \sum_{\mathbf{x}\in\mathbf{X}} \min_{\mathbf{c}\in\mathcal{C}}(\|c-x\|^2)$$

If the context is clear, we may omit $\mathcal{C}$ and $\mathbf{X}$. For a fixed set of points $\mathbf{X}$, if the probability of $\mathbf{x} \in \mathbf{X}$ being chosen to be a center is $p(x)$ with respect to a set of centers $\mathcal{C}$, then the expected value of inertia $E[\mathcal{I}(\mathbf{X})]$ is

$$E[\mathcal{I}(\mathbf{X})] = \sum_{\mathbf{x} \in \mathbf{X}} p(x) \sum_{\mathbf{y} \in \mathbf{X}} \min(D(x), \|y - x\|)^2$$

If we use seeding with $p(x)$ proportional to $f(x) = \|x - \mu_{\mathcal{C}}\|^2$ where $\mu_{\mathcal{C}}$ is the centroid of $\mathcal{C}$, then for any point $y \in S$,

$$\begin{aligned} f(x) &= \|x - \mu_{\mathcal{C}}\|^2 \\ \sum_{x \in S} f(x) &= \sum_{x \in S} \|x - \mu_{\mathcal{C}}\|^2 \\ &= \sum_{x \in S} \left(\|x\|^2 - 2\langle x, \mu_{\mathcal{C}} \rangle + \|\mu_{\mathcal{C}}\|^2 \right) \\ &= \mathcal{S}_2(S) - 2\langle n\mu_{\mathbf{X}}, \mu_{\mathcal{C}} \rangle + n\|\mu_{\mathcal{C}}\|^2 \end{aligned}$$

Then the probability of x being chosen as a center is

$$p(x) = \frac{f(x)}{\sum_{y \in S} f(y)}$$

Since $D(y) \leq \|y - \mu_{\mathcal{C}}\|$, we have the following

$$\begin{aligned} \sum_{y \in S} \|y - \mu_{\mathcal{C}}\|^2 &\geq D(y)^2 \\ \frac{1}{\sum_{y \in S} \|y - \mu_{\mathcal{C}}\|^2} &\leq \frac{1}{\sum_{y \in S} D(y)^2} \\ \frac{\sum_{y \in S} D(y)^2}{\sum_{y \in S} \|y - \mu_{\mathcal{C}}\|^2} &\leq 1 \end{aligned}$$

Our expected value of inertia would be

$$\begin{aligned} E[\phi(S)] &= \sum_{x \in S} p(x) \sum_{y \in S} \min(D(y), \|x - y\|)^2 \\ &= \sum_{x \in S} \frac{\|x - \mu_{\mathcal{C}}\|^2}{\sum_{y \in S} \|y - \mu_{\mathcal{C}}\|^2} \sum_{y \in S} \min(D(y), \|x - y\|)^2 \\ &\leq \sum_{x \in S} \frac{\|x - \mu_{\mathcal{C}}\|^2}{\sum_{y \in S} \|y - \mu_{\mathcal{C}}\|^2} \sum_{y \in S} D(y)^2 \\ &\leq \sum_{x \in S} \|x - \mu_{\mathcal{C}}\|^2 \end{aligned}$$

Using equation (2), we have

$$E[\phi(S)] \leq n(\sigma^2 + \|\mu_{\mathbf{X}}\|^2 + \|\mu_{\mathbf{X}} - \mu_{\mathcal{C}}\|^2)$$

5. Experiment Setup

We ensure that all algorithms are run under the same conditions. All of them share the same environment and no special optimizations were made for any particular algorithm. Only CPU was used to determine the values we are interested in and no parallelism mechanism was in place for speeding up the process. This way, we can get an idea about the raw performance metrics of the algorithms involved.

Python is used as the programming language to write necessary codes. Some common auxiliary packages such as *scipy, scikit-learn, numpy* etc are used to help with the code. All the algorithms are simply different methods of the same class, so they share the same fitting and prediction function. Only the initialization differs for different algorithm. It should be mentioned that even though some packages have native support for k-Means implementation, we did not use them to run the experiments. Not all algorithms we want to test are available in those packages. Therefore, in order to ensure same environment and optimizations for every algorithm, we wrote them all from scratch so that we could be sure they are tested under the same settings. In order to check performance bias, we have used both normalization and standardization.

The data sets we have used are some popular ones: Iris data set, Mall customers data set, Airline clustering data set, Wine testing dataset. We did not duel too much on using too many data sets. The number of datasets does not really mean very much for k-Means. It is the quality of clustering we are interested in. And a good clustering algorithm should be able to handle all type of data sets.

For a particular data set, we have first normalized the data using min max scalar. Miligan et al. (1988) shows that using z-score to standardize the data is not favorable for clustering because it loses between-cluster variation. Then we have used PCA to remove linear dependency among variables. For every data set, we run the experiment a total of 20 times as mentioned before.

6. Results

We present the results on those four algorithms based on inertia and time required to runt them. These are average values of inertia after 20 iterations on each data set, as mentioned before. Instead of checking how many iterations it take for the centers to converge, we have let each algorithm run exactly the same number of times (300) which is more than enough for convergence. The time registered here is the time taken by each algorithm for those 300 iterations. As expected, default k-Means is the fastest algorithm. However, if we were to define convergence, that leads to a separate problem which we think other implementations suffer from. We also mentioned something similar before. This might also be one of the reasons why our results of k-Means++ differ from the authors implementation. Our argument is also complemented by the experiments. All algorithms seem to reach the same minimum inertia. Therefore, it is the average of inertia that we should focus on.

Here are the results for $k = 5$ clusters.

For comparing k-Means++ against k-Means++ improvement, here are the results.

Algorithm	Average	Minimum	Time
k-Means	1603.09	1577.05	9.27s
k-Means++	1563.71	1518.9	9.4s
ORSS	1569.41	1530.46	9.44s
Var-based	1552.37	1510.91	9.69s

Table 1: Results on Cloud data set

Algorithm	Average	Minimum	Time
k-Means	57.74	43.89	2.26s
ORSS	48.25	43.8	2.28s
k-Means++	49.19	43.83	2.28s
Var-based	47.05	44.79	2.32s

Table 2: Results on IRIS data set

Algorithm	Average	Minimum	Time
k-Means	180.37	174.92	0.95s
ORSS	183.13	174.89	0.96s
k-Means++	181.23	0.96	1.72s
Var-based	175.4	174.85	0.97s

Table 3: Results on mall customers data set

Algorithm	Average	Minimum	Time
k-Means	1120.07	1015.19	1.49s
k-Means++	1103.17	1014.54	1.49s

Table 4: Results on Wine dataset

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