A CALCULUS OF SEQUENCES.

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I. *Introduction.*

1. I propose in this paper to give a generalization of a large portion of the formal parts of algebraic analysis and the calculus of finite differences. The generalization consists in systematically replacing the ordinary binomial coefficient $n(n-1)\cdots(n-r+1)/1\cdot 2\cdots r$ by a "binomial coefficient to the base (u)," $[n,r]=u_n\cdot u_{n-1}\cdots u_{n-r+1}/u_1\cdot u_2\cdots u_r$ where (u): $u_0, u_1, u_2, \cdots$ is a fixed sequence of complex numbers subject to the restrictions $u_0=0$; $u_1=1$; $u_n\neq 0$, $n>1$. The exponential function for example is replaced by the formal series $1+\sum_{n=1}^{\infty} x^n/u_1\cdot u_2\cdots u_n$, differentiation by an operation which throws x^n into $[n,1]x^{n-1}$, and differencing by an operation which throws x^n into $\sum_{r=1}^{n}[n,r]x^{n-r}$.

The formula $n^r=(1+1+\cdots+1)^r=\sum_{(s)} r!/s_1!s_2!\cdots s_n!$ guides us in replacing the powers of rational integers where necessary by sums of multinomial coefficients to the base (u); for example, $\sum_{s=0}^{r}[n,r]$ may replace 2^r. We are thus enabled to generalize successfully a great variety of formulas involving the exponential functions, the Bernoulli numbers and polynomials.

2. In a series of papers which have appeared during the past thirty years [1],[1] F. H. Jackson has developed a somewhat similar extension of elementary analysis for the particular sequence (u) in which $u_n=(q^n-1)/(q-1)$ q a fixed complex number, $|q|\neq 1$. His results are based essentially on an identity of Euler's [2]:

$$(2.1)\qquad (x+y)(x+qy)\cdots(x+q^{n-1}y)=\sum_{r=0}^{n}[n,r]q^{r(r-1)/2}x^{n-r}y^r.$$

In effect he replaces the ordinary binomial coefficient by $[n,r]q^{r(r-1)/2}$. But the presence of this power of q introduces a lack of symmetry in his formulas,

[1] The numbers in square brackets refer to references at the end of the paper. Jackson wrote in all over thirty papers on this subject. We have listed only those directly connected with the present paper.

and leads to certain complications in defining the exponential functions.[2] He was not led furthermore to consider the developments of the calculus of finite differences which we give here. These appear to be among the most striking results of the entire theory.

The present work originated in an (unsuccessful) attempt to frame a definition of the Bernoulli numbers in Jackson's calculus to which the Staudt-von Klausen theorem might apply.

II. *Formal theory.*

3. Let

$$(u): \quad u_0 = 0, \quad u_1 = 1, \quad u_2, \cdots, u_n, \cdots$$

be a fixed sequence of complex numbers subject for the present to the single restriction $u_n \neq 0$, $n > 1$. For convenience, we shall write $[n]$ for u_n. We define:

$$\begin{aligned}&[n]! \ \text{ to be } 1 \text{ if } n = 0, \text{ and } [n][n-1]\cdots[1] \text{ if } n > 0;\\ &[n, r] \ \text{ to be } [n]!/[r]![n-r]!\end{aligned}$$

where n, r are positive integers,[3] and $n \geqq r$. Then

$$[n, 0] = 1, \quad [n, 1] = [n], \quad [n, n-r] = [n, r].$$

We shall call $[n, r]$ a binomial coefficient to the base (u), or simply a *basic*[4] binomial coefficient.

We write $(x+y)^n$ for the polynomial $\sum_{r=0}^{n} [n, r] x^{n-r} y^r$. It is evident that

$$\begin{aligned}&(x+y)^0 = 1, \quad (x+y)^1 = x+y, \quad (x+0)^n = x^n\\ &(cx+cy)^n = c^n(x+y)^n, \quad (x+y)^n = (y+x)^n.\end{aligned}$$

From the identities[5]

$$(x-y)^{2n+1} = \sum_{r=0}^{n} (-1)^r [2n+1, r] x^r y^r (x^{2n+1-2r} - y^{2n+1-r})$$

$$(x-y)^{2n} = \sum_{r=0}^{n-1} (-1)^r [2n, r] x^r y^r (x^{2n-2r} + y^{2n-2r}) + (-1)^n [2n, n] x^n y^n$$

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[2] It is necessary to consider not only the series $1 + \sum_{1}^{\infty} x^n/u_1 u_2 \cdots u_n$ as an analogue of the exponential, but also the series $1 + \sum_{1}^{\infty} q^{n(n-1)/2} x^n/u_1 u_2 \cdots u_n$ with a corresponding complexity in the theory of the trigonometric functions.

[3] We count zero as a positive integer.

[4] This convenient terminology is due to F. H. Jackson.

[5] We write $(x-y)^n$ for $(x + (-y))^n = \sum_{r=0}^{n} [n, r] x^{n-r} (-y)^r$.

we see that

$$(x - x)^{2n+1} = 0; \quad (n = 0, 1, 2, \cdots).$$

On the other hand,

$$(x - x)^{2n} = x^{2n}(1 - 1)^{2n} = x^{2n}\{2 \sum_{r=0}^{n-1} (-1)^r [2n, r] + (-1)^n [2n, n]\}$$

generally does *not* vanish. A sequence (u) such that

$$(1 - 1)^{2n} = 0, \ (n = 1, 2, 3, \cdots)$$

will be said to be *normal*.

4. More generally, we define

$$(x_1 + x_2 + \cdots + x_k)^n \text{ to be } \sum_{(s)} \frac{[n]!}{[s_1]! \cdots [s_k]!} x_1^{s_1} \cdots x_k^{s_k}$$

where the summation is over all integers s satisfying the conditions

$$s_1 + s_2 + \cdots + s_k = n, \qquad 0 \leqq s_i \leqq n.$$

If we denote this polynomial by $P_{kn}(x) = P_{kn}(x_1, x_2, \cdots, x_k)$ then it is a symmetric function of its k arguments, and if c is any constant, then

$$P_{kn}(cx_1, cx_2, \cdots, cx_k) = c^n P_{kn}(x_1, x_2, \cdots, x_k).$$

Furthermore,

$$(4.1) \qquad P_{k+1n}(x) = P_{kn}(x_1, x_2, \cdots, x_{k-1}, x_k + x_{k+1}).$$

For consider $P_{2n}(x) = (x_1 + x_2)^n$. We have

$$\begin{aligned}
(x_1 + x_2)^n &= \sum_{t=0}^{n} \frac{[n]!}{[n-t]!\,[t]!} x_1^{n-t} x_2^{t} \\
(x_1 + (x_2 + x_3))^n &= \sum_{t=0}^{n} \frac{[n]!}{[n-t]!\,[t]!} x_1^{n-t} (x_2 + x_3)^{t} \\
&= \sum_{t=0}^{n} \sum_{r=0}^{t} \frac{[n]!\,[t]!}{[n-t]!\,[t]!\,[t-r]!\,[r]!} x_1^{n-t} x_2^{t-r} x_3^{r} \\
&= \sum_{(s)} \frac{[n]!}{[s_1]!\,[s_2]!\,[s_3]!} x_1^{s_1} x_2^{s_2} x_3^{s_3}, \\
&\qquad\qquad s_1 + s_2 + s_3 = n, \ 0 \leqq s_i \leqq n \\
&= (x_1 + x_2 + x_3)^n = P_{3n}(x).
\end{aligned}$$

Hence (4.1) is true for $k = 2$. Its validity for any value of k follows by an easy induction.

It is evident that formula (4.1) can be extended so as to express $P_{k+ln}(x)$ in terms of $P_{kn}(x)$ in various ways. For example,

$$P_{4n}(x_1, x_2, x_3, x_4) = P_{2n}(x_1 + x_2, x_3 + x_4).$$

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5. The numerical values of the polynomials $P_{kn}(x)$ when all of the arguments x_i are equal to plus one play an important rôle in the developments which are to follow. We shall write $\bar{k}^n$ for the number

$$P_{kn}(1, 1, \cdots, 1) = (1 + 1 + \cdots + 1)^n = \sum_{(s)} \frac{[n]!}{[s_1]! \cdots [s_k]!}.$$

We see from the formulas of section 4 that

$$\bar{3}^n = (\bar{2} + \bar{1})^n, \quad \bar{4}^n = (\bar{2} + \bar{2})^n = (\bar{3} + \bar{1})^n.$$

It is easily shown by induction that we have quite generally

$$\overline{r + s}^n = (\bar{r} + \bar{s})^n \tag{5.1}$$

where r and s are any positive integers.

If furthermore the sequence (u) is normal (section 3) then we can show by induction that (5.1) holds for any integral values of r and s. A somewhat longer induction establishes the formula

$$\overline{m_1 + m_2 + \cdots + m_t}^n = (\bar{m}_1 + \bar{m}_2 + \cdots + \bar{m}_t)^n \tag{5.2}$$

where if (u) is normal, $m_1, \cdots, m_t$ are any integers, but if (u) is not normal, the integers are to be positive.

In case (u) is normal, there is no gain in generality in replacing some of the plus signs in formula (5.2) by minus signs because we can show that

$$(\bar{r} - \bar{s})^n = (\bar{r} + \overline{-s})^n. \tag{5.3}$$

6. If $F(x)$ denotes the formal power series

$$F(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{6.1}$$

we define $F(x + y)$ to mean the series

$$\sum_{n=0}^{\infty} c_n (x + y)^n = \sum_{n=0}^{\infty} \sum_{r=0}^{n} c_n [n, r] x^{n-r} y^r.$$

In like manner

$$\begin{aligned} F(x_1 + x_2 + \cdots + x_k) &= \sum_{n=0}^{\infty} c_n (x_1 + x_2 + \cdots + x_k)^n \\ &= \sum_{n=0}^{\infty} c_n P_{kn}(x). \end{aligned} \tag{6.2}$$

We have furthermore formal identities of the type

$$F(x_1+x_2)=F(x_2+x_1),$$
$$F(x_1+x_2+x_3)=F(x_1+(x_2+x_3))$$
$$F(x_1+x_2+x_3+x_4)=F((x_1+x_2)+(x_3+x_4))$$

since the like identities hold for the polynomials $P_{kn}(x)$.

If in the series (6.2) we make all the arguments x_i equal to x, the right side becomes $\sum_{n=0}^{\infty} c_n(x+x+\cdot\cdot\cdot+x)^n = \sum_{n=0}^{\infty} c_n\bar{k}^n x^n$. We shall accordingly denote the resulting series by $F(\bar{k}x)$. It is obvious then from the formula (5.2) that

$$(6.3)\qquad F(\overline{m_1+m_2+\cdot\cdot\cdot+m_t}x)=F(\bar{m}_1x+\bar{m}_2x+\cdot\cdot\cdot+\bar{m}_tx)$$

for suitably restricted integers m_i.

Let $F(x)$, $G(x)$, $H(x)$ be three formal power series in x. Then the following theorem is easily seen to be true.

THEOREM 6.1. *If $F(x)=G(x)\pm H(x)$ and m,n are any positive integers, then $F(\bar{m}x)=G(\bar{m}x)\pm H(\bar{m}x)$ and*

$$F(\bar{m}x+\bar{n}y)=G(\bar{m}x+\bar{n}y)\pm H(\bar{m}x+\bar{n}y).$$

7. We next define an operator $D=D_x$ which transforms the formal power series (6.1) into

$$(7.1)\qquad F'(x)=DF(x)=\sum_{n=0}^{\infty}[n]c_nx^{n-1}.$$

In particular then, $Dx^n=[n]x^{n-1}$. The operator D is easily shown to be linear and distributive, and it converts a polynomial of degree n in x into one of degree $n-1$.

If we define $F^{(r)}(x)=D^rF(x)$ recursively by $F^{(r+1)}(x)=DF^{(r)}(x)$; $F^{(0)}(x)=F(x)$, it easily follows that

$$\frac{F^{(r)}(x)}{[r]!}=\sum_{n=r}^{\infty}[n,r]c_nx^{n-r}.$$

The expansion $F(x+y)=\sum_{n=0}^{\infty}c_n(x+y)^n$ is formally replaceable by

$$(7.2)\qquad F(x+y)=\sum_{n=0}^{\infty}\frac{F^{(n)}(x)}{[n]!}y^n.$$

We shall refer to (7.2) as *Taylor's formula* for the base (u).

Finally, we note that

$$D_x{}^rF(x+y)=F^{(r)}(x+y).$$

8. As a simple concrete example of such an operator D, let us assume that the sequence (u) is a linear recurring series of order k whose associated polynomial $x^k - a_1x^{k-1} - \cdots - a_k$ has k distinct roots $\alpha_1, \alpha_2, \cdots, \alpha_k$. Then u_n is of the form

$$u_n = \beta_1\alpha_1{}^n + \beta_2\alpha_2{}^n + \cdots + \beta_k\alpha_k{}^n$$

where the constants α and β are subject to the conditions

$$\beta_1 + \beta_2 + \cdots + \beta_k = 0, \quad \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_k\alpha_k = 1, \quad u_n \neq 0, n > 1.$$

It is obvious then that

$$DF(x) = (\beta_1F(\alpha_1x) + \cdots + \beta_kF(\alpha_kx))/(\beta_1\alpha_1x + \cdots + \beta_k\alpha_kx).$$

This operator can therefore be applied to any function of x regular at $x = 0$, and transforms it into another function regular at $x = 0$.

In particular, if $k = 2$, $\alpha_1 = q$, $\alpha_2 = 1$, $\beta_1 = (q - 1)^{-1}$, $\beta_2 = -\beta_1$, where q is not a root of unity,

$$DF(x) = \frac{F(qx) - F(x)}{qx - x}$$

is the operation of q-differencing.

In case some of the roots α of the polynomial associated with the recurrence relation are repeated, a similar but more complicated formula for $DF(x)$ may be given which involves both $F(x)$ and its ordinary derivatives. For example, if $k = 2$ and $\alpha_1 = \alpha_2 \neq 0$, $u_n = n\alpha_1{}^{n-1}$ and $DF(x) = \dfrac{1}{\alpha_1}\,\dfrac{dF(\alpha_1x)}{dx}$.

III. *The exponential and trigonometric functions.*

9. We shall now assume that the sequence (u) is chosen in such a manner that the series

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{9.1}$$

is convergent in the neighborhood of $x = 0$. It accordingly is an element of an analytic function of x which we shall call the *basic exponential*. There exists then a positive number ρ such that the series (9.1) converges absolutely within the circle $|x| = \rho$.

The basic exponential has the following properties for sufficiently small absolute values of its arguments x, y, x_i:

$$D\mathcal{E}(x) = \mathcal{E}(x), \quad \mathcal{E}^{(n)}(cx) = c^n\mathcal{E}(cx), c \quad \text{a constant}, \tag{9.2}$$

$$\mathcal{E}(x + y) = \mathcal{E}(x)\mathcal{E}(y), \tag{9.21}$$

$$\mathcal{E}(x_1 + x_2 + \cdots + x_k) = \mathcal{E}(x_1)\mathcal{E}(x_2)\cdots\mathcal{E}(x_k). \tag{9.22}$$

Consider for example the formula (9.21). That it is formally true is immediately obvious from the basic Taylor's formula (7.2). For

$$\mathcal{E}(x+y)=\sum_{n=0}^{\infty}\frac{\mathcal{E}^{(n)}(x)y^n}{[n]!}=\mathcal{E}(x)\mathcal{E}(y)$$

since by (9.2), $\mathcal{E}^{(n)}(x)=\mathcal{E}(x)$.

But the series

$$\sum_{n=0}^{\infty}\frac{(x+y)^n}{[n]!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{x^{n-r}y^r}{[n-r]!\,[r]!}$$

is in fact the Cauchy product of the series $\sum\frac{x^n}{[n]!}, \sum\frac{y^n}{[n]!}$ so that the formula is actually true provided the latter two series are both absolutely convergent. And by our initial hypothesis, both series converge absolutely if $|x|<\rho$, $|y|<\rho$.

10. The trigonometric and hyperbolic functions are defined by Euler's formulas:

$$\begin{aligned}&\sin(x)=\frac{\mathcal{E}(ix)-\mathcal{E}(-ix)}{2i}, \quad \cos(x)=\frac{\mathcal{E}(ix)+\mathcal{E}(-ix)}{2},\\ &\sinh(x)=-i\sin(ix), \quad \cosh(x)=\cos(ix).\end{aligned} \tag{10.1}$$

Among the many formal analogies with the ordinary trigonometric functions, we shall merely note here:

$$\begin{aligned}&\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y),\\ &\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y),\\ &D\sin(x)=\cos(x), \qquad D\cos(x)=-\sin(x).\end{aligned}$$

As a consequence of the last two formulas, we see that both $\sin(x)$ and $\cos(x)$ satisfy the basic differential equation $y^{(2)}(x)+y(x)=0$.

On the other hand

$$\sin^2(x)+\cos^2(x)=\mathcal{E}(ix)\mathcal{E}(-ix) \tag{10.2}$$

and in general, $\mathcal{E}(ix)\mathcal{E}(-ix)\neq 1$.

The remaining trigonometric and hyperbolic functions are defined in terms of the basic sine and cosine as in the ordinary case.

11. It is possible to give analogues of De Moivre's and Simpson's formulas. For in formula (9.22), take $k=n$ and let $x_1=x_2=\cdots=x_n=i\theta$. Then with the notation explained in section 6,

$$\mathcal{E}(\bar{n}i\theta)=(\mathcal{E}(i\theta))^n. \tag{11.1}$$

Hence we obtain from the formulas (10.1) and theorem 6.1 the basic form of De Moivre's formula,[6] $\cos(\bar{n}\theta) + i\sin(\bar{n}\theta) = (\cos(\theta) + i\sin(\theta))^{n}$.

THEOREM 11.1. *The sequence* (u) *is normal when and only when* $\mathcal{E}(x)\mathcal{E}(-x) = 1$ *or when and only when* $\sin^2(x) + \cos^2(x) = 1$.

For by formula (9.21), and the previous definitions, if $|x| < \rho$,

$$\mathcal{E}(x)\mathcal{E}(-x) = \mathcal{E}(x - x) = \sum_{n=0}^{\infty} \frac{(x-x)^n}{[n]!} = \sum_{n=0}^{\infty} \frac{(1-1)^n}{[n]!} x^n.$$

Now we have seen in section 3 that $(1-1)^{2n+1} = 0$, $(n = 0, 1, 2, \cdots)$. Therefore

$$\mathcal{E}(x)\mathcal{E}(-x) = 1 + \sum_{n=0}^{\infty} \frac{(1-1)^{2n}}{[2n]!} x^{2n}.$$

Hence the first part of the theorem follows. The second part of the theorem is an immediate consequence of formula (10.2).

Let us assume now that (u) is normal. We see from formula (11.1) that

$$\begin{aligned} \mathcal{E}(\overline{n+2}i\theta) &= \mathcal{E}(i\theta)\mathcal{E}(\overline{n+1}i\theta), \\ \mathcal{E}(i\theta)\mathcal{E}(\bar{n}i\theta) &= \mathcal{E}(\overline{n+1}i\theta). \end{aligned} \tag{11.3}$$

But by theorem 11.1, $\mathcal{E}(-i\theta)\mathcal{E}(i\theta) = 1$. Therefore this last equation may be written

$$\mathcal{E}(\bar{n}i\theta) = \mathcal{E}(-i\theta)\mathcal{E}(\overline{n+1}i\theta). \tag{11.31}$$

On adding and subtracting the two formulas (11.3), (11.31) and applying (10.1) and theorem (6.1), we obtain the basic Simpson's formulas:

$$\begin{aligned} \cos(\overline{n+2}\theta) &= 2\cos(\theta)\cos(\overline{n+1}\theta) - \cos(\bar{n}\theta), \\ \sin(\overline{n+2}\theta) &= 2\cos(\theta)\sin(\overline{n+1}\theta) - \sin(\bar{n}\theta). \end{aligned}$$

12. In order that the results of the previous section may have more than a purely formal significance, it is necessary to show that we can choose the sequence (u) so that (u) is normal and so that $E(\bar{n}x)$ is an entire function of x for any integer n. Since $E(\overline{-n}x) = E(-\bar{n}x)$, $E(\bar{n}x) = (E(\bar{1}x))^n$, $E(\bar{1}x) = E(x)$, we need only consider the case when $n = +1$.

Now it is easy to show that the most general solution of the functional equation

$$\Phi(x)\Phi(-x) = 1 \tag{12.1}$$

[6] It should be noted here that $(\cos(\theta) + i\sin(\theta))^n$ stands for the product $(\cos(\theta) + i\sin(\theta))(\cos(\theta) + i\sin(\theta))\cdots$ taken to n factors, and not for the result of substituting $\cos(\theta)$ for x_1, and $i\sin(\theta)$ for x_2 in the polynomial $P_{2n}(x) = (x_1 + x_2)^n$.

which is regular at the origin is of the form

$$\Phi(x) = \pm \exp(x\Psi(x^2)) \tag{12.2}$$

where $\Psi(x)$ is regular at the origin. But by theorem 11.1, (u) is normal when and only when $\mathcal{E}(x)$ is a solution of (12.1). Since $\mathcal{E}(x)$ was assumed to be regular at the origin, $\mathcal{E}(x)$ must be of the form (12.2) where $\Psi(x)$ is an entire function of x. We must also satisfy the conditions[7]

$$\mathcal{E}(0) = 1, \quad \mathcal{E}'(0) = 1, \quad \frac{\mathcal{E}^{(n)}(0)}{n!} = \frac{1}{u_1 u_2 \cdots u_n} \neq 0,$$

as then $u_n = n\mathcal{E}^{(n-1)}(0)/\mathcal{E}^{(n)}(0) \neq 0 \ (n = 1, 2, \cdots)$ and $u_1 = 1$.

It will therefore suffice to choose for $\Psi(x)$ an entire function $G(x)$ with a series expansion of the form $G(x) = 1 + \sum_{n=1}^{\infty} g_n x^n$ where the quantities g_n are all real and non-negative. The ordinary case ensues on taking all the quantities g_n equal to zero.

13. If we assume that $\mathcal{E}(x)$ is an entire function satisfying the condition $\mathcal{E}(x)\mathcal{E}(-x) = 1$, we can generalize the periodic properties of the exponential function. For since $\mathcal{E}(x)$ never vanishes, by Picard's theorem there exists a complex number $\lambda \neq 0$ such that $\mathcal{E}(\lambda) = 1$. But then if n is a positive integer,

$$\begin{aligned}
\mathcal{E}(x + \overline{n}\lambda) &= \mathcal{E}(x)\mathcal{E}(\overline{n}\lambda) = \mathcal{E}(x)(\mathcal{E}(\lambda))^n = \mathcal{E}(x),\\
\mathcal{E}(x) &= \mathcal{E}(x - \overline{n}\lambda + \overline{n}\lambda) = \mathcal{E}(x - \overline{n}\lambda)\mathcal{E}(\overline{n}\lambda) = \mathcal{E}(x - \overline{n}\lambda).
\end{aligned}$$

We have therefore proved the following theorem.

THEOREM 13.1. *If (u) is a normal sequence so chosen that the basic exponential function $\mathcal{E}(x)$ is an entire function of x, and if $\lambda \neq 0$ is any zero of the function $\mathcal{E}(x) - 1$, and m any integer, then*

$$\mathcal{E}(x + \overline{m}\lambda) = \mathcal{E}(x).$$

Furthermore one such zero λ always exists.

On utilizing the formulas of section 10, we can easily show that under the hypotheses of theorem 13.1, we also have

$$\sin(x + \overline{m}i\lambda) = \sin(x), \quad \cos(x + \overline{m}i\lambda) = \cos(x).$$

[7] The superscripts here denote ordinary differentiation.

IV. *The calculus of finite differences.*

14. Let (u) now be subject only to the conditions $u_0 = 0$, $u_1 = 1$, $u_n \neq 0$, $n \neq 0$. We shall denote by $\mathfrak{R}$ the ring of all polynomials in x with coefficients in the field of all complex numbers.

If

$$\phi = \phi(x) = \sum_{r=0}^{n} a_{n-r} x^r, \qquad a_0 \neq 0 \tag{14.1}$$

is any element of $\mathfrak{R}$ of degree n, we define the basic displacement symbol E by

$$\begin{aligned} E\phi(x) &= \phi(x \dotplus 1) = \sum_{r=0}^{n} \sum_{s=0}^{r} a_{n-r}[r, s] x^{r-s} \\ E^{t+1}\phi(x) &= E(E^t \phi(x)), \quad E^0\phi(x) = \phi(x) \end{aligned} \tag{14.2}$$

where t is any positive integer.

It is obvious that E is a linear and distributive operator over $\mathfrak{R}$, and it may readily be shown that

$$E^t \phi(x) = \phi(x \dotplus \bar{t}). \tag{14.3}$$

If (u) is normal, formula (14.3) holds for all integral values of t.

15. The *basic difference operator* Δ is defined to be $E - 1$, where 1 stands for the identity operator over $\mathfrak{R}$. The following properties of Δ may be mentioned.

(i) Δ is linear and distributive over $\mathfrak{R}$, and converts an element of $\mathfrak{R}$ of degree n into one of degree $n - 1$. Moreover E, Δ and D are commutative over $\mathfrak{R}$.

(ii) The only solutions of $\Delta\phi = 0$ lying in $\mathfrak{R}$ are $\phi =$ a constant.

(iii) $\Delta^t \phi(x) = \sum_{s=0}^{t} (-1)^s \binom{t}{s} \phi(x \dotplus \bar{s})$.

(iv) We have the operational identity over $\mathfrak{R}$

$$\Delta = \mathcal{E}(D) - 1 \tag{15.1}$$

where formally $$\mathcal{E}(D) = \sum_{n=0}^{\infty} \frac{D^n}{[n]!}.$$

The last one of these properties is the only one requiring comment. If ϕ of formula (14.1) is operated on by D of section 7, then

$$\frac{D^s \phi(x)}{[s]!} = 0, \quad s > n; \qquad = \sum_{r=s}^{n} [r, s] a_{n-r} x^{r-s}, \quad s \leqq n.$$

Hence

$$\mathcal{E}(D)\phi(x) = \sum_{s=0}^{n} \sum_{r=s}^{n} [r, s] a_{n-r} x^{r-s} = \sum_{r=0}^{n} \sum_{s=0}^{r} [r, s] a_{n-r} x^{r-s} = E\phi(x)$$

by formula (14.2), so that (15.1) follows.

16. The *basic Bernouilli numbers* $B_0, B_1, \cdots, B_n, \cdots$ are defined by the recurrences

$$B_0 = 1; \quad (B+1)^n - B^n = 0,\ n > 1; \quad (B+1)^1 - B^1 = 1.$$

Here after expansion the exponents of B are to be degraded into suffices as in the usual theory [3].

The *basic Bernoulli polynomials* $B_n(z)$ may then be defined by

$$B_n(z) = (z+B)^n, \qquad (n = 0, 1, \cdots)$$

or non-symbolically,

$$B_n(z) = \sum_{r=0}^{n} [n, r] B_r z^{n-r}.$$

The following results [4] may be established precisely as in the ordinary theory.

$$(16.1) \qquad B_n(0) = B_n, \quad B_n(1) = B_n, \quad n \neq 1; \quad B_1(1) = B_1 + 1.$$

$$(16.2) \qquad B_n(x+y) = \sum_{r=0}^{n} [n, r] x^r B_{n-r}(y).$$

THEOREM 16.1. *If $\phi'(x) = D\phi(x)$ denotes the basic derivative of the polynomial $\phi(x)$, then a polynomial solution of the difference equation*

$$\Delta\Psi(x) = \phi'(x)$$

is given by

$$(16.3) \qquad \Psi(x) = \phi(x+B).$$

$$(16.31) \qquad \phi(x+B) = \sum_{r=0}^{n} \frac{\phi^{(r)}(x)}{[r]!} B_r = \sum_{r=0}^{n} \frac{\phi^{(r)}(0)}{[r]!} B_r(x).$$

$$(16.32) \qquad \Delta B_n(x) = [n] x^{n-1}.$$

THEOREM 16.2. *If the sequence (u) be chosen so that the series (9.1) for $\mathcal{E}(x)$ is convergent near $x = 0$ then for sufficiently small values of $|t|$ and $|x|$*

$$(16.4) \qquad \sum_{n=0}^{\infty} \frac{B_n t^n}{[n]!} = \frac{t}{\mathcal{E}(t) - 1}, \quad \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{[n]!} = \frac{t\mathcal{E}(xt)}{\mathcal{E}(t) - 1}.$$

$$(16.5) \qquad \bar{1}^r + \bar{2}^r + \cdots + \overline{n-1}^r = \frac{B_{r+1}(\bar{n}) - B_{r+1}}{[r+1]},$$

if r is a positive integer $\geqq 1$.

To prove the last written formula for example, we observe by theorem 6.1 that (16.32) implies that $B_{r+1}(\overline{s+1}) - B_{r+1}(\bar{s}) = [r+1]\bar{s}^r$, s a positive integer. On summing this equation with respect to s from 0 to $n-1$, we obtain (16.5).

THEOREM 16.3. *$B_{2n} = 0$ $(n = 1, 2, 3, \cdots)$ when and only when*

$$B_n(1-z) = (-1)^n B_n(z), \quad (n = 2, 3, \cdots).$$

If moreover the series (9.1) for $\mathcal{E}(x)$ converges for some $x \neq 0$, then

$$B_{2n} = 0, \ n \leqq 1$$

when and only when (u) is normal.

The equivalences stated follow immediately from formulas (16.1), (16.2) and (16.4).

We plan to give elsewhere a detailed treatment of the basic analogues for the numbers of Euler, Genocchi, Lucas and Stirling and their associated polynomials and difference operators.

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