Type I′. *$D(\rho)=0$ has one real root and two complex roots.* Corresponding to the real root ρ_1 there is a single real fixed point P_1 and a single real fixed line p_1, where, by Theorem VII, p_1 does not pass through P_1. If now we choose P_1 as the vertex $(1, 0, 0)$ and the line p_1 as the side $x_1=0$ of our triangle of reference, the collineation assumes the canonical form

$$\begin{aligned} \tau x_1' &= \rho_1 x_1 \\ \tau x_2' &= \quad a_{22}x_2 + a_{23}x_3 \\ \tau x_3' &= \quad a_{32}x_2 + a_{33}x_3, \end{aligned}$$

where the quadratic

$$F(\rho) = \rho^2 - (a_{22} + a_{33})\rho + a_{22}a_{33} - a_{23}a_{32} = 0,$$

has imaginary roots, i.e.,

$$(a_{22} - a_{33})^2 + 4a_{23}a_{32} < 0.$$

A CORRECTION

Professor Morgan Ward has kindly called my attention to an unintentional misstatement of Theorem I of my note "*On a Certain Transformation of Infinite Series*" in the April number of this MONTHLY (vol. 40, p. 226). It should read as follows:

If $\lim_{n\to\infty} nu_n = l$ exists, then the two series (U) and (V) both diverge, if $l\neq 0$. If $l=0$, the convergence of one implies that of the other, and the two series have the same sum.

J. A. SHOHAT

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A CERTAIN CLASS OF TRIGONOMETRIC INTEGRALS

By MORGAN WARD, California Institute of Technology

1. In the December issue of the MONTHLY[1] Professor Uhler has raised some questions about the functions defined by the indefinite integrals

$$\int \genfrac{}{}{0pt}{}{\cos}{\sin}(\cot\theta)\genfrac{}{}{0pt}{}{\cos}{\sin}\theta d\theta$$

[1] American Mathematical Monthly, vol. 39 (1932), p. 589.

which I propose to answer here. Let us define four functions of the real variable θ:

$$K_1(\theta) = \int_0^\theta \cos(\tan\phi)\cos\phi d\phi, \quad K_2(\theta) = \int_0^\theta \cos(\tan\phi)\sin\phi d\phi,$$

$$K_3(\theta) = \int_0^\theta \sin(\tan\phi)\cos\phi d\phi, \quad K_4(\theta) = \int_0^\theta \sin(\tan\phi)\sin\phi d\phi.$$

The integrals under discussion are immediately expressible in terms of these functions; for example,

$$\int \cos(\cot\theta)\sin\theta d\theta = \text{const.} - K_1\left(\frac{\pi}{2} - \theta\right).$$

We shall show that the functions $K(\theta)$ are not expressible in finite terms by any simple known functions. However, in the range $-\pi/2<\theta<\pi/2$, they are representable by convergent series of which

$$(1.1) \quad K_1(\theta) = \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{1}{6}\cos\frac{3\theta}{2}\left(2\sin\frac{\theta}{2}\right)^3 - \frac{1}{8}\sin 2\theta\left(2\sin\frac{\theta}{2}\right)^4$$
$$+ \frac{1}{30}\cos\frac{5\theta}{2}\left(2\sin\frac{\theta}{2}\right)^5 - \frac{1}{36}\sin 3\theta\left(2\sin\frac{\theta}{2}\right)^6$$
$$+ \frac{2}{45}\cos\frac{7\theta}{2}\left(2\sin\frac{\theta}{2}\right)^7 + \frac{1}{30}\sin 4\theta\left(2\sin\frac{\theta}{2}\right)^8 + \cdots$$

may be quoted as typical. I shall give recursion formulas by which the numerical coefficients in these series may be calculated, and an estimate of the error terms. It turns out that the convergence is fairly good in the range $-\pi/4 \leqq \theta \leqq \pi/4$.

There also exist asymptotic expansions giving the behavior of the functions near $\pi/2$ and $-\pi/2$ which limitations of space forbid my developing here.

2. We begin by observing that from our defining relations, it follows that[1]

$$K_1(\theta+\pi) = -K_1(\theta); \quad K_2(\theta+\pi) = 2K_2\left(\frac{\pi}{2}\right) - K_2(\theta);$$

$$K_3(\theta+\pi) = 2K_3\left(\frac{\pi}{2}\right) - K_3(\theta); \quad K_4(\theta+\pi) = -K_4(\theta).$$

We may therefore assume that $-\pi/2 \leqq \theta \leqq \pi/2$.

If we let

$$P(\theta) = K_1(\theta) + iK_2(\theta), \quad Q(\theta) = K_3(\theta) + iK_4(\theta),$$

we obtain immediately from (1.1) the integral formulas

[1] The constants $K_2(\pi/2)$, $K_3(\pi/2)$ may be expressed as infinite integrals by writing $\tan\phi = x$, and these integrals may be evaluated in terms of Bessel functions, and related expressions.

$$P(\theta) + iQ(\theta) = \int_0^\theta \cos(\tan\phi)e^{i\phi}d\phi + i\int_0^\theta \sin(\tan\phi)e^{i\phi}d\phi = \int_0^\theta e^{i(\tan\phi+\phi)}d\phi,$$

$$P(\theta) - iQ(\theta) = \int_0^\theta \cos(\tan\phi)e^{i\phi}d\phi - i\int_0^\theta \sin(\tan\phi)e^{i\phi}d\phi = \int_0^\theta e^{-i(\tan\phi-\phi)}d\phi.$$

We shall now introduce two functions of a complex variable in terms of which these last integrals are easily expressible.

3. Let Z denote a complex variable. Consider the two functions

$$(3.1)\qquad F(Z) = \int_1^Z \exp\left(\frac{1-z^2}{1+z^2}\right)dz, \quad G(Z) = \int_1^Z \exp\left(\frac{z^2-1}{z^2+1}\right)dz.$$

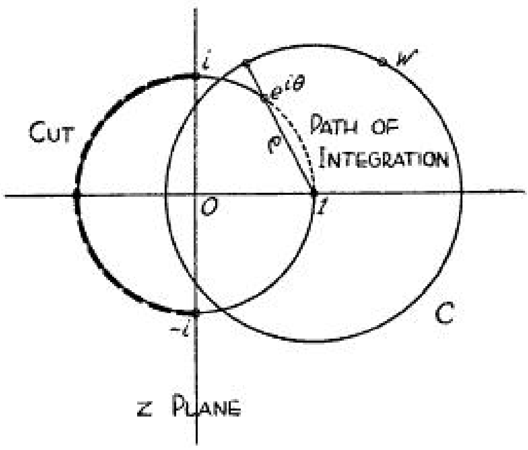
Then if we join the points $z=i$ and $z=-i$ by a cut which it is convenient to take along the left half of the unit circle (see figure), the reader may verify that

(a) The functions $F(Z)$ and $G(Z)$ are one-valued and analytic at all points Z of the cut z-plane save the point at infinity where each has a pole of order one, and their value at any point Z is independent of the path of integration joining 1 and Z.

(*b*) In particular, both functions are one-valued and analytic in the interior of a circle of radius $\sqrt{2}$ about the point $Z=1$.

(c) Both functions have essential singularities and logarithmic branch points at $z=i$ and $z=-i$, but remain finite as we approach these points along the right half of the unit circle.

(*d*) Upon a circle C with centre 1 and radius $\rho<\sqrt{2}$,



$$|F(Z)| \leqq \rho \exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)}, \qquad |G(Z)| \leqq \rho\exp\frac{\sqrt{\rho^4+4}-\rho^2}{2(2-\rho^2)}.$$

4. Now let $Z=e^{i\theta}$, $-\pi/2<\theta<\pi/2$, be a point on the right half of the unit circle. Then if in the integrals (3.1) we choose for our path of integration the arc of the unit circle from 0 to θ, we obtain on writing $e^{i\phi}$ for z and reducing,

$$F(e^{i\theta}) = i\int_0^\theta e^{-i(\tan\phi-\phi)}d\phi, \quad G(e^{i\theta}) = i\int_0^\theta e^{i(\tan\phi+\phi)}d\phi.$$

On combining these results with the formulas of sections 1 and 2, we find that

$$\begin{aligned}
K_1(\theta) &= \text{Real part of } \frac{F(e^{i\theta}) + G(e^{i\theta})}{2i},\\
K_2(\theta) &= \text{Imaginary part of } \frac{F(e^{i\theta}) + G(e^{i\theta})}{2i},\\
K_3(\theta) &= \text{Real part of } \frac{F(e^{i\theta}) - G(e^{i\theta})}{2},\\
K_4(\theta) &= \text{Imaginary part of } \frac{F(e^{i\theta}) - G(e^{i\theta})}{2}.
\end{aligned} \tag{4.1}$$

It is clear from these formulas that the function-theoretic nature of the $K(\theta)$ is determined by that of $F(Z)$ and $G(Z)$.

5. The nature of the functions $F(Z)$ and $G(Z)$ may be best seen from the differential equations which they satisfy. From formula (3.1)[1]

$$\frac{dF}{dz} = \exp\frac{1-z^2}{1+z^2}; \qquad \frac{dG}{dz} = \exp\frac{z^2-1}{z^2+1}.$$

Hence differentiating logarithmically, we see that

$$(z^2+1)^2\frac{d^2F}{dz^2} + 4z\frac{dF}{dz} = 0; \; (z^2+1)^2\frac{d^2G}{dz^2} - 4z\frac{dG}{dz} = 0. \tag{5.1}$$

Consider the differential equation for $F(z)$. If we make the substitution $\cdot\cdot = z^2/(z^2+1)$, this differential equation becomes

$$w(1-w)\frac{d^2F}{dw^2} - (2w^2 - w - \tfrac{1}{2})\frac{dF}{dw} = 0. \tag{5.3}$$

The equation has 0 and 1 for regular points and ∞ for an irregular point. Now drawing upon the results of the theory of second order linear differential equations,[2] we see that if (5.3) is regarded as obtained by confluence from a differential equation with only regular points, the initial equation must have had five or more regular points. By actual trial, we find it is impossible to derive (5.3) from a differential with exactly five regular points. But all of the elementary functions of mathematical physics may be derived as solutions of confluent

[1] For convenience in printing, we hereafter write z for Z.

[2] See for example Whittaker and Watson, *Modern Analysis* Chap. X; or Ince, *Ordinary Differential Equations*, Chap. XX.

forms of such an equation. Hence $F(z)$ cannot be expressed in finite form by means of such functions.

Neither can $F(z)$ be expressed by means of elliptic integrals of the first or second kinds, for such integrals have no essential singularity in any part of the plane. A precisely similar argument holds for $G(z)$.

6. We are thus driven to seek series representations of $F(z)$ and $G(z)$ in the range in which we are interested. One such series is immediately obvious; namely an expansion about the point $z=1$ in ascending powers of $z-1$.

We have by Taylor's theorem

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!}(z-1)^n, \quad G(z) = \sum_{n=0}^{\infty} \frac{G^{(n)}(1)}{n!}(z-1)^n,$$

the radius of convergence of both series being $\sqrt{2}$ in accordance with section 3, (b), (c). On writing $z=e^{i\theta}$, we find that $z-1=2e^{(\pi+\theta)i/2}\sin(\theta/2)$. Hence

$$\begin{aligned} F(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!} e^{ni/2(\pi+\theta)}\left(2\sin\frac{\theta}{2}\right)^n, \\ G(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{G^{(n)}(1)}{n!} \alpha^{ni/2(\pi+\theta)}\left(2\sin\frac{\theta}{2}\right)^n, \end{aligned} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{6.1}$$

Since $F(z)$ and $G(z)$ are real when z is real and greater than -1, the constants $F^{(n)}(1)$ and $G^{(n)}(1)$ in (6.1) are all real. Hence on combining these formulas with (4.1), we find that

$$K_1(\theta) = \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{F^{(n)}(1)+G^{(n)}(1)}{n!}\sin\frac{n}{2}(\pi+\theta)\left(2\sin\frac{\theta}{2}\right)^n, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \tag{6.2}$$

with similar formulas for $K_2(\theta)$, $K_3(\theta)$ and $K_4(\theta)$.

To calculate the constants $F^{(n)}(1)$ and $G^{(n)}(1)$, we differentiate the equations (5.1) $n+2$ times and set $z=1$. We thus obtain the recursion formulas

$$\begin{aligned} F^{(n+4)}(1) &= -(2n+5)F^{(n+3)}(1) - (n+2)(2n+3)F^{(n+2)}(1) \\ &\quad - (n+2)(n+1)nF^{(n+1)}(1) - \frac{(n+2)(n+1)n(n-1)}{4}F^{(n)}(1); \\ G^{(n+4)}(1) &= -(2n+3)G^{(n+3)}(1) - (n+2)(2n+1)G^{(n+2)}(1) \\ &\quad - (n+2)(n+1)nG^{(n+1)}(1) - \frac{(n+2)(n+1)n(n-1)}{4}G^{(n)}(1). \end{aligned} \tag{6.3}$$

On setting $n=-2, -1, 0, 1, \cdots$ in these formulas, we find from (5.2) that[1]

[1] These coefficients have been checked from (3.1) by expanding $\exp\pm(z^2-1)/(z^2+1)$ in ascending powers of $z-1$ up to the terms of order $(z-1)^7$, and integrating term by term.

$F^{(0)}(1) = 0$; $F^{(1)}(1) = 1$; $F^{(2)}(1) = -1$; $F^{(3)}(1) = 2$; $F^{(4)}(1) = -4$; $F^{(5)}(1) = 4$;
$F^{(6)}(1) = 34$; $F^{(7)}(1) = -374$; $F^{(8)}(1) = 2498$.
$G^{(0)}(1) = 0$; $G^{(1)}(1) = 1$; $G^{(2)}(1) = 1$; $G^{(3)}(1) = 0$; $G^{(4)}(1) = -2$; $G^{(5)}(1) = 4$;
$G^{(6)}(1) = 6$; $G^{(7)}(1) = -74$; $G^{(8)}(1) = 190$.

On substituting these values in the first nine terms of (6.2), we obtain the series for $K_1(\theta)$ given in section 1.

7. Let $z=e^{i\theta}$ be a fixed point on the unit circle to the right of the y axis, and C a circle of radius $\rho<\sqrt{2}$ about the point $z=1$ including the point $e^{i\theta}$. Then

$$F(e^{i\theta}) = \frac{1}{2\pi i}\int_C \frac{F(w)dw}{w-z}, \tag{7.1}$$

where w denotes a complex current co-ordinate upon the circle C.

From the identity

$$\frac{1}{w-z} = \frac{1}{w-1} + \frac{z-1}{(w-1)^2} + \cdots + \frac{(z-1)^n}{(w-1)^{n+1}} + \frac{(z-1)^{n+1}}{(w-1)^{n+1}(w-z)},$$

we obtain

$$F(z) = c_0 + c_1(z-1) + \cdots + c_n(z-1)^n + \mathfrak{R}_n$$

where $c_k = F^{(k)}(1)/k!$ $(k=0, \cdots, n)$ and

$$\mathfrak{R}_n = \frac{(z-1)^{n+1}}{2\pi i}\int_C \frac{F(w)dw}{(w-1)^{n+1}(w-z)}. \tag{7.2}$$

Now

$$|z-1| = 2\sin\frac{|\theta|}{2}, \quad |w-1| = \rho, \quad |w-z| \geqq \rho - |z-1| = \rho - 2\sin\frac{|\theta|}{2}$$

and by 3 (d),

$$|F(w)| \leqq \rho \exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)}.$$

Hence from (7.2),

$$|\mathfrak{R}_n| \leqq \left(\frac{2\sin\dfrac{|\theta|}{2}}{\rho}\right)^{n+1} \frac{\rho^2}{\rho - 2\sin\dfrac{|\theta|}{2}} \exp\frac{\sqrt{\rho^4+4+\rho^2}}{2(2-\rho^2)}.$$

The inequality for the remainder in the series for $G(z)$ is precisely the same, save that the numerator of the exponential is replaced by $\sqrt{\rho^2+4}-\rho^4$.

A somewhat better inequality when n is large may be obtained by integrating the right side of (7.2) by parts before obtaining the dominant. It gives

$$|\Re_n| \leqq \frac{1}{n}\left(\frac{2\sin\frac{|\theta|}{2}}{\rho}\right)^{n+1} \frac{2\rho^2\left(\rho+\sin\frac{|\theta|}{2}\right)}{\left(\rho-2\sin\frac{|\theta|}{2}\right)^2}\exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)}.$$

If we take $\rho^2=3/2$, $\theta=\pi/4$ in the first inequality, we obtain

$$|\Re_n| < (\tfrac{5}{8})^{n+1}\times 51.08 = .0042 \text{ for } n = 19.$$

The second inequality gives

$$|\Re_n| < \frac{1}{n}\left(\frac{5}{8}\right)^{n+1} 438 = .0019 \text{ for } n = 19.$$

If θ is quite small, we may take $\rho=1$, obtaining

$$|\Re_n| < \left(2\sin\frac{|\theta|}{2}\right)^{n+1}\frac{e^{(\sqrt{5}+1)/2}}{1-2\sin\frac{|\theta|}{2}} \text{ for } F(z)$$

and

$$|\Re_n| < \left(2\sin\frac{|\theta|}{2}\right)^{n+1}\frac{e^{(\sqrt{5}-1)/2}}{1-2\sin\frac{|\theta|}{2}} \text{ for } G(z).$$

Homogeneous Linear Differential Equations of the Second Order

By T. C. Benton, Pennsylvania State College

1. *Introduction.* After teaching the subject of linear differential equations to sophomore students a number of times, it has seemed to the author that the customary methods of assuming the correct answers and then verifying them are highly unsatisfactory from a pedagogical viewpoint. The student always asks how the form of solution used was obtained in the first place. Also the letter student is left with the feeling, that except for a lucky guess, there is no way to obtain the solution of similar problems. It is the purpose of this development of the subject to present a method in which every step is forced—a method in which there is no guesswork at all. The actual work is all of well known character but the fact that the general methods of the higher theory of differential equations work out in such a simple way for the elementary cases seems worthy of attention

2. *General Theory* of the solution of:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0, \tag{A}$$

P, Q being functions of x or constants.