THEOREM IV. *The summation on the right referring to all pairs* $(t, \tau)$ *of integers* $t>0$, $\tau>0$, $\tau$ *odd, such that* $n=t\tau$,

$$F(n) = -1/n[\{1+(-1)^n\}\sigma(\tfrac{1}{2}n)f(0) + 2\sum(-1)^t\tau f(t)].$$

To see first that this implies the $Q(n)$ part of theorem I, take $f(x)=1$ for all integer values of $x$, as clearly is permissible under the definition of $f(x)$. A short reduction of the resulting right hand member gives the required relation.

To prove theorem IV, observe that we may take $f(n)\equiv\cos nx$, where $x$ is a parameter, for all integers $n$, and get a true theorem provided theorem IV is true. But conversely, if the cosine form of the theorem is an identity in $x$, we can infer the general form as stated.[1] The cosine form, however, follows by a straightforward reduction from the identity.[2]

$$\log(1+\sum{}' q^{n^2}) = \log\theta_3 + 2\sum\frac{(-1)^n q^n}{n(1-q^{2n})}(1-\cos 2nx),$$

where $\Sigma'$ refers to $n=\pm1, \pm2, \pm3, \cdots$, $\Sigma$ to $n=1, 2, 3, \cdots$, and

$$\log\theta_3 = \sum[\log(1-q^{2n}) + 2\log(1+q^{2n-1})],$$

where $\Sigma$ refers to $n=1, 2, 3, \cdots$. The expansions are valid for $q$, $x$ suitably restricted, and similarly for the series obtained by expanding the logarithms by the logarithmic series. Comparison of coefficients of like powers of $q$ in the result gives the stated cosine identity, as can be easily verified.

There is a similar but more complicated generalization of the $T(n)$ part of theorem I. Omitting this, we need only state the classic identity which implies the theorem as stated:

$$1+\sum_{n=1}^{\infty} q^{n(n+1)/2} = \Pi(1-q^n)\Pi(1+q^n)^2,$$

from which, by taking logarithms and expanding, the result follows.

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# ON CERTAIN FUNCTIONAL RELATIONS

By MORGAN WARD, California Institute of Technology

1. *Introductory problem.* If $y=f(x)$ is an analytic function of $x$ for $0\leqq|x|<r$ and if $f(0)=0, f'(0)\neq0$ so that

$$y = a_1x + a_2x^2 + a_3x^3 + \cdots \quad (a_1\neq 0), \tag{1}$$

then the inverse function $x=f^{-1}(y)$ is also analytic for $0\leqq|y|<\rho=\rho(r)$ and vanishes with $y$.

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[1] This is a simple instance of what was called paraphrase in several previous papers, e.g., Transactions of the American Mathematical Society, vol. 22 (1921), p. 1.

[2] See almost any text on elliptic functions, e.g., Tannery-Molk, vol. 3, p. 116.

Suppose that the function $f^{-1}(x)$ is identical with[1] $f(x)$, so that

$$x = a_1 y + a_2 y^2 + a_3 y^3 + \cdots . \tag{2}$$

The coefficients $a_1, a_2, a_3, \cdots$, must then satisfy certain algebraic conditions. These conditions express the fact that the result of substituting for the successive powers of $x$ in (1) their expressions in terms of $y$ from (2) must reduce to the identity $y=y$. In particular, we see that $a_1^2=1$. There are two totally different cases according as $a_1=+1$ or $a_1=-1$. If $a_1=+1$, the remaining coefficients $a_2, a_3, \cdots$, all vanish; if however $a_1=-1$, the situation is more complicated. We find that

$$a_3 = -a_2^2, \quad a_5 = 2a_2^4 - 3a_2a_4$$

$$a_7 = -13a_2^6 + 18a_2^3a_4 - 4a_2a_6 - 2a_4^2$$

$$a_9 = 145a_2^8 - 221a_2^5a_4 + 35a_2^3a_6 + 50a_2^2a_4^2 - 5a_2a_8 - 5a_4a_6$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x^n = (-y)^n \left\{ 1 - na_2y + \frac{n(n+1)}{2!}a_2^2y^2 - \left(\frac{n(n-1)(n+4)}{3!}a_2^3 + na_4\right)y^3 + \left(\frac{n(n-3)(n+2)(n+7)}{4!}a_2^4 + n(n+2)a_2a_4\right)y^4 - \cdots \right\}$$

By a somewhat lengthy induction, we can establish the following theorem:

Theorem 1: *Given*

$$y = f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$x = f(y) = a_1y + a_2y^2 + a_3y^3 + \cdots \qquad (a_1^2 = 1)$$

*Then if* $a_1=1$,

$$a_n = 0 \quad (n = 2,3,4, \cdots, )$$

*But if* $a_1=-1$,

$$a_{2n+1} = P_n(a_2, a_4, \cdots, a_{2n}) = P_n(-a_2, -a_4, \cdots, -a_{2n}),$$

*where* $P_n$ *is a uniquely determined polynomial in* $a_2, a_4, \cdots, a_{2n}$ *with integral coefficients.*[2]

The following result is immediate.

Theorem 2: *The necessary and sufficient condition that* $x$ *and* $y$ *be related as in theorem* 1 *is that there exist an analytic function* $F(x, y)$ *of* $x$ *and* $y$ *satisfying the conditions,*

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[1] A simple example of such a function is $y=x/(x-1)=-x-x^2-x^3-\cdots$.

[2] I have not yet succeeded in obtaining the explicit expression for $P_n$.

$$F(x,y) = F(y,x,) = F(0,0) = 0\ ;\quad F_x(0,0) \neq 0.$$

2. *Extended problem.* Suppose that $y=\Phi(a, b; x)$ is a function of $x$ and the two real parameters $a$ and $b$ which satisfies the following conditions:
(i) $\Phi(a, b; x)$ *is an analytic function of* $x$ *on and within the square* $\mathfrak{S}$ *bounded by the lines* $a=\pm h$, $b=\pm h$ *in the* $a\cdot b$ *plane for* $0\leqq |x| < r = r(h)$;
(ii) $\Phi(a, b; x)$ *vanishes with* $x$ *throughout* $\mathfrak{S}$.

We may consequently write

$$y = \Phi(a,b\ ;\ x) = \phi_1(a,b)x + \frac{\phi_2(a,b)x^2}{2!} + \frac{\phi_3(a,b)}{3!}x^3 + \cdots,$$

where

$$\phi_n(a,b) = (\partial^n\Phi/\partial x^n).$$

If, moreover, $\phi_1(a, b)\neq 0$ in $\mathfrak{S}$, the inverse function $x=\Phi^{-1}(y)$ exists for $0\leqq |y| < \rho(h)$. Let us assume finally that
(iii) $\Phi^{-1}(x)=\Phi(b, a; x)$ for $0\leqq |x| < r(h)$ throughout $\mathfrak{S}$. We shall then have

$$x = \Phi(b,a\ ;y) = \phi_1(b,a)y + \frac{\phi_2(b,a)y^2}{2!} + \frac{\phi_3(b,a)y^3}{3!} + \cdots$$

and as in section 1 it is necessary that

$$\phi_1(a,b)\cdot\phi_1(b,a) = 1. \tag{3}$$

Let us determine some of the properties of functions which satisfy the conditions (i), (ii), and (iii). We shall refer to such functions as "$\Phi$-functions."

3. *Canonical form for functions.* From (3) we see that $\phi_1(a, b)$ and $\phi_1(b, a)$ can never vanish in $\mathfrak{S}$ and must both be of the same sign in $\mathfrak{S}$. If we write

$$y = |\phi_1(a,b)|^{1/2}v,\quad x = |\phi_1(b,a)|^{1/2}u,$$

the series defining $\Phi$ in section 2 become either

$$\text{(I)}\qquad \begin{aligned} v &= u + \psi_2u^2 + \psi_3u^3 + \cdots \\ u &= v + \psi_2'v^2 + \psi_3'v^3 + \cdots \end{aligned}$$

if $\phi_1(a, b)$ and $\phi_1(b, a)$ are positive in $\mathfrak{S}$, or

$$\text{(II)}\qquad \begin{aligned} v &= -u + \psi_2u^2 + \psi_3u^3 + \cdots \\ u &= -v + \psi_2'v^2 + \psi_3'v^3 + \cdots \end{aligned}$$

if $\phi_1(a, b)$ and $\phi_1(b, a)$ are negative in $\mathfrak{S}$; where in both cases

$$\text{(4)}\qquad \begin{aligned} \psi_n = \psi_n(a,b) &= \frac{\phi_n(a,b)}{n!}|\phi_1(b,a)|^{(n+1)/2} \\ \psi_n' = \psi_n(b,a) &= \frac{\phi_n(b,a)}{n!}|\phi_1(a,b)|^{(n+1)/2} \end{aligned}$$

$$(n = 2,3,\cdots,).$$

If we write $a=b$ in (I) and (4), we see from the first part of theorem 1 that

$$\phi_n(a,a) = 0. \qquad (n = 2,3,\cdots,).$$

From (3) and (4) follows

Theorem 3. *If $y=F(a, b; x)$ is any $\Phi$-function whose coefficients are polynomials in $a$ and $b$, then the coefficient of every power of $x$ in $F$ save the first is divisible by $a-b$.*

The two canonical forms of $\Phi$-function in (I) and (II) show us that we have a correspondence with the two types of solution of $y=f(x)$, $x=f^{-1}(y)$ in theorem 1. Let us call $\Phi$-functions of the first type "proper functions" and $\Phi$-functions of the second type "improper functions." $y=x$ is the simplest proper function, but in contrast to the first part of theorem 1, we have a theorem analagous to theorem 2 for both proper and improper functions. We shall confine our statement to the former type of function in the canonical form (I).

Theorem 4. *The necessary and sufficient condition that $v$ be a proper $\Phi$-function of $u$ is that $u$ and $v$ be connected by an implicit relation of the form*

$$u - v + F(a,b\,;u,v) - F(b,a\,;v,u) = 0, \tag{5}$$

*where for sufficiently small positive values of $|u|$ and $|v|$, $F(a, b; u, v)$ is an analytic function of both $u$ and $v$ in some region $\mathfrak{R}$ in the ab-plane which includes the origin, and where*

$$F(a,b\,;0,0) = F(b,a\,;0,0),$$
$$F_x(a,b\,;0,0) = F_x(b,a\,;0,0),$$
$$F_y(a,b\,;0,0) = F_y(b,a\,;0,0),$$

*for all values of $a$ and $b$ in $\mathfrak{R}$.*

In fact these conditions allow us to substitute for $v$ a series in $u$ with undetermined coefficients $\Psi_n$ which we know will be convergent, and to determine the $\Psi_n=\Psi_n(a, b)$ by equating the coefficients of $u$, $u^2$, $u^3$, $\cdots$, to zero in the resulting identity; in particular, we shall have $\Psi_1=1$. Now if instead we substitute for $u$ a series in $v$ with undetermined coefficients $\Psi_n'$, the equations determining $\Psi_n'$ are obtained from those determining $\Psi_n$ by merely inter-changing $a$ and $b$, so that $\Psi_n'(a, b)=\Psi_n(b, a)$ and $v$ is a proper function of $u$. Conversely, if (I) holds, by halving and subtracting the two series we obtain

$$u - v + \tfrac{1}{2}\sum_{n=2}^{\infty}\psi_n(a,b)u^n - \tfrac{1}{2}\sum_{n=2}^{\infty}\psi_n(b,a)v^n = 0,$$

where, by our definition of a $\Phi$-function, $\frac{1}{2}\sum_{n=2}^{\infty}\psi_n(a, b)\, u^n$ satisfies all the conditions imposed upon $F(a, b, u, v)$ in the theorem.

5. *Example.* As an illustration of a proper $\Phi$-function, suppose temporarily that $a$, $b$ are real, but never zero. Consider the relation

$$(1 + bx)^a = (1 + ay)^b. \tag{6}$$

By the binomial theorem, if $|x|<1/|b|$, $|y|<1/|a|$

$$x - y + \frac{(a-1)bx^2}{1\cdot 2} + \frac{(a-1)(a-2)b^2x^3}{1\cdot 2\cdot 3} + \cdots$$
$$- \frac{(b-1)\cdot ay^2}{1\cdot 2} - \frac{(b-1)(b-2)a^2y^3}{1\cdot 2\cdot 3} - \cdots = 0,$$

so that

$$x - y + F(a,b\,;x,y) - F(b,a\,;y,x) = 0,$$

where

$$F(a,b\,;x,y) = \sum_{n=1}^{\infty} \frac{(a-1)(a-2)\cdots}{(n+1)!}(a-n)b^n x^{n+1}.$$

Now $F(a, b; x, y)$ satisfies all the conditions of theorem 4, even when $a$, $b$ are zero, so that we have

$$y = x + J(a,b\,;x)\,; \quad x = y + J(b,a\,;y),$$

where we easily see that

$$(7) \qquad J(a,b\,;x) = \sum_{n=1}^{\infty} \frac{(a-b)(a-2b)\cdots(a-nb)x^{n+1}}{(n+1)!}.$$

Moreover if $\lambda\neq 0$,

$$\lambda J\left(\lambda a,\lambda b\,;\frac{x}{\lambda}\right) = J(a,b\,;x)$$

and if $ab\neq 0$,

$$1 + ax + aJ(a,b\,;x) = (1+bx)^{a/b}.$$

We can define $J(a, b; x)$ by the series (7) and then prove that $x+J(a, b; x)$ is actually a proper $\Phi$-function. This has been done by O. Jezek.[1] We conclude with a few easily proved but curious properties of the function $J(a, b; x)$. If

$$x = t + J(b,ab\,;t) \qquad \text{and} \qquad y = t + J(a,ab\,;t),$$

then

$$x + J(ab,b\,;x) = y + J(ab,a\,;y)\,;$$
$$y - x = J(a,b\,;x)\,; \quad x - y = J(b,a\,;y).$$

If

$$x = t - J(a,abc\,;t) + J(b,abc\,;t) + J(c,abc\,;t),$$
$$y = t + J(a,abc\,;t) - J(b,abc\,;t) + J(c,abc\,;t),$$
$$z = t + J(a,abc\,;t) + J(b,abc\,;t) - J(c,abc\,;t),$$

then

[1] O. Jezek, *Ueber die Reihenumkehrung*, Wiener Sitzungsberichte Zweite Abteilung, vol. XCIX (1890), pp. 191–203.—See also Whittaker and Watson, *Modern Analysis*, 3rd edition, p. 147, example 14.

$$x - y = J(a, b ; y + z), \; y - x = J(b, a ; x + z)$$
$$y - z = J(b,c ; z + x), \qquad z - y = J(c,b ; y + x),$$
$$z - x = J(c,a ; x + y), \qquad x - z = J(a,c ; z + y),$$
$$J(abc,c ; x + y) = J(abc,a ; y + z) = J(abc,b ; z + x).$$

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# GENERALIZATIONS IN GEOMETRY AS SEEN IN THE HISTORY OF DEVELOPABLE SURFACES

By FLORIAN CAJORI, University of California

"The mathematicians of the eighteenth century would have been astonished to a high degree, had they been told that there exist developable surfaces which are not ruled surfaces." Perhaps this passage from the pen of Picard[1] surprises many mathematicians even of the present time; it challenges the historian to endeavor to trace the evolution of ideas. The result alluded to is no less surprising to us than was to Euler in the eighteenth century the fact that $i^i$, where $i=\sqrt{(-1)}$, has a real value. In a letter to Goldbach, Euler showed his interest by computing this value to ten decimal places. Picard's statement is no ess surprising than the declaration about integral numbers made by Galileo in the seventeenth century: "Neither is the number of squares less than the totality of all numbers, nor the latter greater than the former."

## *Period of Primitive Intuition*

Aristotle remarked that "a line by its motion produces a surface."[2] When this line was a straight line, ruled surfaces would result, which clearly included the cone and cylinder. But Aristotle's statement does not necessarily carry the implication that there are ruled surfaces which can be spread out upon a plane. Nevertheless, early students of geometry must have recognized as intuitionally evident the fact that, without stretching or tearing, the curved surface of cylinders and cones could be unbent upon a plane. Explanations of this property are not generally given. We have found the developed surface of a right cone drawn as the sector of a circle, in a practical work on mensuration,[3] without any novelty being claimed for it. In the same treatise the cylinder is described as being "in form of a Rolling stone used in Gardens," an expression conveying the picture of a surface rolled over a plane so that all its points are brought into coincidence with the plane.

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[1] Émile Picard, *La science moderne et son état actuel*, Paris, p. 53.

[2] Aristotle, *De Anima*, I, 4, 409, a4; T. L. Heath's *Thirteen Books of Euclid*, vol. 1, 2nd edition (1926), p. 170.

[3] William Hawney, *The Complete Measurer*, ninth Edition (1755), p. 159. See also p. 154. (First edition, London, 1717).