ON THE NUMBER OF VANISHING TERMS IN AN INTEGRAL CUBIC RECURRENCE

MORGAN WARD, California Institute of Technology

Introduction. Let

$$(T):\ T_0,\ T_1,\ T_2,\ \cdots,\ T_n,\ \cdots$$

be an integral cubic recurrence; that is, the initial values T_0, T_1, T_2 of (T) are integers and

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, \cdots).$$

Here P, Q and R are fixed integers and $R \neq 0$. The polynomial

$$f(z) = z^3 - Pz^2 + Qz - R$$

and the recurrence (T) are said to be associated. If

$$g(z) = T_0 z^2 + (T_1 - PT_0)z + (T_2 - PT_1 + QT_0),$$

then for $|z|$ sufficiently large,

$$g(z)/f(z) = \sum_{1}^{\infty} T_{n-1} z^{-n}$$

is a generating function for (T).

The two most interesting cases are when $g(z) = df(z)/dz$ and $g(z) = 1$; we denote the corresponding recurrences by (S) and (H).

If $f(z)$ has distinct roots u, v, w, then S_n is simply the Newtonian sum of the nth powers of the roots, while H_{n+2} as a symmetric function is the homogeneous product sum of the roots of degree n [4]:

$$S_n = u^n + v^n + w^n, \qquad H_{n+2} = \sum u^{r_1} v^{r_2} w^{r_3}.$$

Here the second sum is extended over all non-negative integers r_i such that $r_1 + r_2 + r_3 = n$. (H) is of particular importance in the algebra of recurring sequences [1], [2]; the corresponding recurrences of order two are the well known Lucas functions [3].

An integer $k \geqq 0$ is called a "zero" of the recurrence (T) if $T_k = 0$. For example, 0 and 1 are zeros of (H). We determine here the maximum number of possible zeros of (T) when its associated polynomial has integral roots subject to a restriction to be described presently.

Prior work on this problem has been confined to the cases when $f(z)$ has complex roots and $R = \pm 1$ [6], [7] or when $(T) = (S)$ and $P = 0$ [8]. There is an important general result of Kurt Mahler's [5] in this connection. Let us call both (T) and its polynomial $f(z)$ "degenerate" or "nondegenerate" according as the ratio of any pair of different roots of $f(z)$ is, or is not, a root of unity.

Mahler showed that if (T) is non-degenerate, $|T_n|$ tends to infinity with n. Hence

The total number of zeros of any non-degenerate recurrence (T) *is finite.*

We prove here the following more precise result.

THEOREM 1. *If the associated polynomial of an integral cubic recurrence is both non-degenerate and has integral roots which are co-prime in pairs, then at most three terms of the recurrence can vanish.*

The exact determination of the number of zeros of a given recurrence (T) under these hypotheses may be very difficult. For example, it is easy to see that (S) can have at most one zero; but the assertion that if $S_1 \neq 0$, (S) has no zeros is essentially Fermat's last theorem. On the other hand, it follows from the results of this paper that the sequence (T) with initial values $T_0=0$, $T_1=1$ and $T_2=0$ has no other zeros.

The plan of the paper is sufficiently indicated by the section headings. In the conclusion we mention some unsolved problems concerning the zeros of (H) suggested by the investigation.

2. Preliminary lemmas. We denote the roots of $f(z)$ by u, v and w. Then u, v and w are integers co-prime in pairs with distinct absolute values, since $f(z)$ is non-degenerate.

The general term T_n of (T) is of the form

$$T_n = Uu^n + Vv^n + Ww^n \tag{2.1}$$

where U, V, W are rational and different from zero, since (T) is of order three. Consequently, k and l are zeros of (T) if and only if

$$\begin{aligned} Uu^k + Vv^k + Ww^k &= 0 \\ Uu^l + Vv^l + Ww^l &= 0. \end{aligned} \tag{2.2}$$

On solving (2.2) for the ratios $U:V:W$, we obtain

LEMMA 2.1. *If k and l are distinct integers and $l>k\geqq 0$, a necessary and sufficient condition that k and l be zeros of* (T) *is that U, V and W of formula* (2.1) *satisfy the conditions*

$$\frac{Uu^k}{w^{l-k}-v^{l-k}} = \frac{Vv^k}{u^{l-k}-w^{l-k}} = \frac{Ww^k}{v^{l-k}-u^{l-k}}.$$

We note for future use two simple corollaries of this result.

LEMMA 2.2. *If $T_0=0$, then a necessary and sufficient condition that $T_n=0$ for $n>0$ is that*

$$\frac{U}{w^n-v^n} = \frac{V}{u^n-w^n} = \frac{W}{v^n-u^n}. \tag{2.3}$$

LEMMA 2.3. *If $T_0=0$ and if $l>k>0$, then a necessary condition that $T_l=T_k=0$ is that*

$$\frac{w^k - v^k}{w^l - v^l} = \frac{u^k - w^k}{u^l - w^l} = \frac{v^k - u^k}{v^l - u^l}. \tag{2.4}$$

LEMMA 2.4. *Let t be a real variable and n any real number greater than one. Then the function $(t^n-1)/(t+1)$ increases steadily if $t>0$ and $(t^n+1)/(t+1)$ increases steadily if $t>1$.*

For the derivatives of the functions are positive under the stated conditions.

LEMMA 2.5. *Let r and s be co-prime integers and k and l positive integers. Then*

$$(r^k - s^k, r^l - s^l) = r^d - s^d \qquad \textit{where } d = (k, l). \tag{2.5}$$

Here (x, y) denotes the greatest common divisor of the integers x and y.

For denote the left side of (2.5) by m. Since d divides k and l, r^d-s^d divides r^k-s^k and r^l-s^l. Consequently, r^d-s^d divides m. It suffices then to show that m divides r^d-s^d.

Since m divides r^k-s^k, it is prime to both r and s. Hence there exists a positive integer t with the property that m divides r^t-s^t but m does not divide r^n-s^n if $0<n<t$. Then *m divides r^n-s^n if and only if t divides n*. For let $n=2qt\pm c$, where $0\leqq c<t$ and let $a=qt$, $b=qt\pm c$. Then if $b\geqq a$, $r^n-s^n=(r^a-s^a)(r^b-s^b)+(rs)^a(r^c-s^c)$, and if $a\geqq b$, $r^n-s^n=(r^a-s^a)(r^b-s^b)-(rs)^b(r^c-s^c)$. In either case, since m divides r^n-s^n and r^a-s^a and is prime to r and s, the minimal property of t is contradicted unless $c=0$ or $c=t$.

Now m divides both r^k-s^k and r^l-s^l. Hence t divides both k and l. Therefore, t divides d, and m divides r^d-s^d, completing the proof.

3. Recurrences with three zeros. Let (T) be a recurrence with $T_0=0$ and at least two other zeros, k and l. We may assume that

$$0 < k < l, \qquad T_n \neq 0, \qquad 0 < n < k. \tag{3.1}$$

Then by Lemma 2.3, the equalities (2.4) must hold. Let $d=(k, l)$. Since u, v and w are co-prime in pairs, if we divide the numerator and denominator of each of the fractions in (2.4) by the corresponding integers w^d-v^d, u^d-w^d and v^d-u^d, we obtain by Lemma 2.5 three equal fractions in their lowest terms. Hence corresponding numerators and denominators must be equal up to sign; that is

$$\begin{aligned} \frac{w^k - v^k}{w^d - v^d} &= \pm \frac{u^k - w^k}{u^d - w^d} = \pm \frac{v^k - u^k}{v^d - u^d}, \\ \frac{w^l - v^l}{w^d - v^d} &= \pm \frac{u^l - w^l}{u^d - w^d} = \pm \frac{v^l - u^l}{v^d - u^d}. \end{aligned} \tag{3.2}$$

Consider now the equalities

$$\frac{w^n - v^n}{w - v} = \pm \frac{u^n - w^n}{u - w} = \pm \frac{v^n - u^n}{v - u}, \qquad n \geqq 1. \tag{3.3}$$

Observe that each of the equalities (3.2) may be put into this form by letting $u' = u^d$, $v' = v^d$, $w' = w^d$, taking n equal to k/d or l/d and then dropping the primes.

Let a, b and c be the absolute values of u, v and w respectively. We may evidently assume that

$$w = c > b > a > 0; \qquad v = \pm b, \ u = \pm a. \tag{3.4}$$

For changing the signs of all the roots of $f(z)$ merely multiplies T_n by $(-1)^n$. There are four cases according to the choices of sign of u and v.

Case	*Roots of $f(z)$* u	v	w	*Equalities* (3.3)
1	a	b	c	$\dfrac{c^n - b^n}{c - b} = \dfrac{c^n - a^n}{c - a} = \dfrac{b^n - a^n}{b - a}$
2	$-a$	b	c	$\dfrac{c^n - b^n}{c - b} = \dfrac{c^n - (-a)^n}{c + a} = \dfrac{b^n - (-a)^n}{b + a}$
3	a	$-b$	c	$\dfrac{c^n - (-b)^n}{c + b} = \dfrac{c^n - a^n}{c - a} = \dfrac{b^n - (-a)^n}{b + a}$
4	$-a$	$-b$	c	$\dfrac{c^n - (-b)^n}{c + b} = \dfrac{c^n - (-a)^n}{c + a} = \dfrac{b^n - a^n}{b - a}$

Both case 1 *and case* 2 *are impossible if* $n > 1$. In case 1, this statement is evident, since $c > b > a > 0$. In case 2, if $n > 1$, we have

$$\frac{c^n \pm a^n}{c + a} = \frac{b^n \pm a^n}{b + a}.$$

Now let $c = ax$ and $b = ay$. Then

$$\frac{x^n \pm 1}{x + 1} = \frac{y^n \pm 1}{y + 1} \qquad \text{with } x > y > 1 \text{ and } n > 1$$

contradicting Lemma 2.4.

Both cases 3 *and* 4 *are impossible if n is even. For assume* n is even. Then $n > 1$, and in case 3 we have

$$\frac{b^n - a^n}{b + a} = \frac{c^n - a^n}{c - a}.$$

But $(b^n - a^n)/(b+a) < (b^n - a^n)/(b-a) < (c^n - a^n)/(c-a)$ since $c > b > a > 0$.

In case 4, we have

$$\frac{c^n - b^n}{c + b} = \frac{c^n - a^n}{c + a}.$$

Hence if $b = cx$ and $a = cy$, then

$$\frac{1 - x^n}{1 + x} = \frac{1 - y^n}{1 + y} \qquad \text{with } 1 > x > y > 0.$$

But by Lemma 4, $(1-t^n)/(1+t)$ steadily decreases if $t > 0$.

We now apply these results to the equalities (3.2) in accordance with the remark following (3.3). First, *d must be odd*. For if d is even, u^d, v^d, w^d are positive, and case 1 applies; that is, $n = 1$ so that $k = d$ and $l = d$ contrary to (3.1). Since n must be odd, k and l *must both be odd*. Hence $w^k - v^k$ and $w^d - v^d$ are of like sign. Therefore, the first equality (3.2) may be written as

$$\frac{w^k - v^k}{w^d - v^d} = \frac{u^k - w^k}{u^d - w^d} = \frac{v^k - u^k}{v^d - u^d}.$$

But since $T_0 = T_k = 0$, this equality and lemma 2.2 imply that

$$\frac{U}{w^d - v^d} = \frac{V}{u^d - w^d} = \frac{W}{v^d - u^d}.$$

Hence $T_d = 0$ by Lemma 2.2. But $0 < d \leqq k$. Hence $d = k$ by condition (3.1). We have thus proved:

THEOREM 2. *Let* (T) *be an integral cubic recurrence whose associated polynomial is non-degenerate and has integral roots which are co-prime in pairs, and assume that the first two zeros of* (T) *are* 0 *and* k. *Then if* k *is even,* (T) *has no other zeros. If* k *is odd, any other zero of* (T) *must be an odd multiple of* k.

It follows immediately that the recurrence with initial values 0, 1, 0 has no other zeros. On the other hand, the recurrence (H) of the introduction has $H_3 = P = 0$ if $u + v + w = 0$. *Hence there exist recurrences with three zeros.*

4. Proof of Theorem 1. We will now use Theorem 2 to give a proof by contradiction of Theorem 1. Let (T) and $f(z)$ satisfy the hypotheses of the theorem, and suppose (T) has more than three zeros. Let the first four zeros of (T) be k_1, k_2, k_3, k_4, so that $0 \leqq k_1 < k_2 < k_3 < k_4$.

The recurrence (T') defined by $T'_n = T_{n+k_1}$ is associated with $f(z)$ and its first three zeros are 0, $k_2 - k_1$, and $k_3 - k_1$. Both $k_2 - k_1$ and $k_3 - k_1$ are odd by Theorem

2. Hence their difference k_3-k_2 is even. The recurrence (T'') defined by $T''_n = T_{n+k_2}$ is associated with $f(z)$ and its first three zeros are 0, k_3-k_2, and k_4-k_2. But k_3-k_2 is even, contradicting Theorem 2.

5. Conclusion. It follows from Theorem 2 that if (T) has three zeros, say k_1, k_2 and k_3, then $d=k_2-k_1$ is odd and the zeros lie in a recurrence (T^*) defined by $T_n^*=T_{k_1+nd}$ whose associated polynomial $f^*(z)$ has for its roots the dth powers of the roots of $f(z)$. Since the initial values of (T^*) are 0, 0, and T_2^*, $T_n^*=T_2^*H_n^*$ where (H^*) is the recurrence associated with $f^*(z)$ discussed in the introduction. We are thus led to consider the zeros of the recurrence (H).

We have seen that H_3 can vanish. The next possibility is that $H_5=0$, giving the diophantine equation

$$(u+v+w)(u^2+v^2+w^2)+uvw=0. \tag{5.1}$$

Trivial solutions of (5.1) evidently violate our hypotheses on $f(z)$. Whether or not (5.1) has non-trivial solutions appears to be unknown. The more general question of whether H_{2n+1} can vanish when $n>1$ appears to be of a difficulty comparable to the Fermat problem for the recurrence (S).

The simplest case when $(T)\neq(H)$ would have three zeros would be when $d=3$ and $k_3=k_1+5d$ making $H_5^*=0$. The related diophantine equation is obtained from (5.1) by replacing u, v, w by their cubes. It would imply then that the diophantine system

$$u^3+v^3+w^3=4^\epsilon s^3, \qquad u^6+v^6+w^6=2^\epsilon t^3, \qquad \epsilon=0 \text{ or } 1,$$

has a non-trivial solution, which appears unlikely.

It is tempting to conjecture in view of these remarks that under our hypotheses on $f(z)$, (H) is the *only* recurrence which can have more than two zeros. It follows of course from Lemma 2.2 that there exist recurrences (T) with arbitrarily assigned zeros k and l. But it may be shown by simple examples that neither Theorem 1 or Theorem 2 is true if we change our hypotheses on the roots of $f(z)$.

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