A TYPE OF MULTIPLICATIVE DIOPHANTINE SYSTEM.

By Morgan Ward.

1. Consider the system of M equations in the $K+L$ unknowns $x_1, \cdots, x_K, y_1, \cdots, y_L$

$$\text{(S)} \qquad A_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_K^{a_{iK}} = B_i y_1^{b_{i1}} y_2^{b_{i2}} \cdots y_L^{b_{iL}}, \qquad (i=1,2,\cdots,M).$$

The exponents a, b are assumed to be positive integers or zero, while the constants A, B are positive integers.

The problem of determining all the real positive * solutions of (S) is a trivial one; for if we let

$$z_1 = \log x_1, \cdots, z_K = \log x_K, \; w_1 = \log y_1, \cdots, w_L = \log y_L, \; e_i = \log(A_i/B_i), \qquad (i=1,\cdots,M),$$

then on taking the logarithm of both sides of each equation in (S) we obtain the linear system

$$\text{(E)} \quad a_{i1}z_1 + \cdots + a_{iK}z_K - b_{i1}w_1 - \cdots - b_{iL}w_L = e_i, \quad (i=1,\cdots,M).$$

The solution of (S) is thus effectively reduced to a mere inspection of the matrix of the coefficients of (E).

On the other hand, the problem of determining all positive integral solutions of (S) is distinctly non-trivial, and offers several interesting and unexpected features.† To give an idea of the difficulties involved, if we seek to replace (S) by the linear system (E), we must add the restrictions that $z_1, \cdots, w_L$ be non-negative, and that $e^{z_1}, \cdots, e^{w_L}$ be rational integers. But to select from the totality of solutions of (E) the particular solutions which meet these restrictions appears to be as difficult as to solve the original system (S).

For a direct attack upon this problem, the reader may consult the paper of Bell's already referred to. The method I develop here is indirect. It is, however, strictly arithmetical, being based upon the fundamental theorem of rational arithmetic—unique decomposition into prime factors. It accordingly would not be applicable if we were attempting to find all solutions of (S) in an arbitrary domain of integrity.‡

* The negative solutions may be immediately obtained from the positive on considering the parity of the a and b.

† E. T. Bell, "Reciprocal arrays and diophantine analysis," this Journal, Vol. 55 (1933), pp. 50-66. In this paper a general non-tentative method for solving the system (M) is developed.

‡ van der Waerden, *Algebra*, Part I, Berlin (1931), p. 39.

The essentials of the method are as follows. We consider along with (S) a more special system (M) obtained on setting all the constants A and B equal to unity:

$$\text{(M)} \qquad x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_K^{a_{iK}} = y_1^{b_{i1}} y_2^{b_{i2}} \cdots y_L^{b_{iL}}, \qquad (i = 1, 2, \cdots, M).$$

We then show that there exists a correspondence between the solutions of (M) in positive integers x and y and the solutions of the linear system

$$\text{(A)} \qquad a_{i1}z_1 + a_{i2}z_2 + \cdots + a_{iK}z_K = b_{i1}w_1 + b_{i2}w_2 + \cdots + b_{iL}w_L, \qquad (i = 1, 2, \cdots, M),$$

in non-negative integers z and w. This correspondence is of a dual character, so that any theorem about the solutions of (A) yields a theorem about the solutions of (M) and vice-versa. Since the broad outlines of the theory of the solution of (A) are known,* we obtain without effort considerable information about the solutions of (M). A slight extension of the method allows us finally to discuss the general system (S).

2. We must first lay down a few definitions. The systems (M) and (A) will be said to be *associated*. By a solution of (S) or (M) we shall mean a solution in positive integers, and by a solution of (A) a solution in non-negative integers. To avoid trivialities, we shall furthermore assume that (S) actually has solutions.

We shall find it convenient to represent a solution $\xi_1, \xi_2, \cdots, \xi_K, \eta_1, \eta_2, \cdots, \eta_L$ of any one of the three systems (S), (M) or (A) under consideration as a one-rowed matrix,†

$$[\xi; \eta] = [\xi_1, \xi_2, \cdots, \xi_K, \eta_1, \eta_2, \cdots, \eta_L].$$

If

$$[\xi'; \eta'] = [\xi'_1, \xi'_2, \cdots, \xi'_K, \eta'_1, \eta'_2, \cdots, \eta'_L]$$

is a second such solution, then the matrix

$$[\xi + \xi'; \eta + \eta'] = [\xi_1 + \xi'_1, \cdots, \eta_L + \eta'_L]$$

is called the sum of the solutions $[\xi; \eta]$, $[\xi'; \eta']$ and expressed as usual by the notation

$$[\xi + \xi'; \eta + \eta'] = [\xi; \eta] + [\xi'; \eta'].$$

In like manner, the product of two solutions is expressed by

$$[\xi\xi'; \eta\eta'] = [\xi; \eta] \cdot [\xi'; \eta'],$$

* See for example, Grace and Young, *Algebra of Invariants*, Cambridge (1903), pp. 102-106.

† Bell, *Algebraic Arithmetic*, pp. 15-16.

and the identity of any two solutions by

$$[\xi;\eta]=[\xi';\eta'].$$

Finally, if t is any integer,

$$\begin{aligned} t[\xi;\eta] &= [t\xi_1,\cdots,t\eta_L],\\ [\xi;\eta]^t &= [\xi_1{}^t,\cdots,\eta_L{}^t]. \end{aligned}$$

We shall on occasion denote matrices of solutions by German capitals. It is immediately evident that

the product of a solution of (S) *and a solution of* (M) *is a solution of* (S);
the product of two solutions of (M) *is a solution of* (M);
the sum of two solutions of (A) *is a solution of* (A).

The solution $x_1=x_2=\cdots=x_K=y_1=y_2=\cdots=y_L=1$ of (M) will be called the trivial solution of (M) and denoted by

$$\mathfrak{J}=[1;1].$$

The trivial solution of (A) is defined in an analogous manner as

$$\mathfrak{O}=[0;0].$$

A solution of (A) is said to be irreducible if it cannot be expressed as the sum of two non-trivial solutions *; similarly, a solution of (M) is said to be irreducible if it cannot be expressed as the product of two non-trivial solutions. Lastly, a solution of (S) is said to be irreducible if it cannot be expressed as the product of a solution of (S) and a non-trivial solution of (M).

The Greek letters α and β appearing as sub-scripts or super-scripts will have the ranges $1,2,\cdots,K$ and $1,2,\cdots,L$ respectively. Thus we write $x_\alpha=P^{u_\alpha}$ for $x_1=P^{u_1}$, $x_2=P^{u_2},\cdots,x_K=P^{u_K}$,

$$\sum_{(\beta)} v_\beta \text{ for } v_1+v_2+\cdots+v_L, \qquad \prod_{(\alpha)} P^{u_\alpha} \text{ for } P^{u_1}P^{u_2}\cdots P^{u_K},$$

and so on.

3. We shall first give some properties of the system (M).

THEOREM 3.1. *Every primitive solution of* (M) *is of the form*

$$x_\alpha=P^{u_\alpha}, \qquad y_\beta=P^{v_\beta}$$

where P is a prime, and $[u;v]$ is a primitive solution of (A).

* Grace and Young, p. 102.

Proof. Assume that (M) has a primitive solution $[\xi;\eta]$. Then there exists a prime P dividing at least one of the numbers ξ, η. Write

$$\xi_\alpha = P^{u_\alpha}\xi'_\alpha, \qquad \eta_\beta = P^{v_\beta}\eta'_\beta$$

where the ξ', η' are prime to P. Substituting these numbers in (M), we obtain

$$\prod_{(\alpha)} P^{a_{i\alpha}u_\alpha} \prod_{(\alpha)} \xi'^{a_{i\alpha}}_\alpha = \prod_{(\beta)} P^{b_{i\beta}v_\beta} \prod_{(\beta)} \eta'^{b_{i\beta}}_\beta, \qquad (i = 1, \cdots, M).$$

Therefore

$$\prod_{(\alpha)} P^{a_{i\alpha}u_\alpha} = \prod_{(\beta)} P^{b_{i\beta}v_\beta}, \qquad \prod_{(\alpha)} \xi'^{a_{i\alpha}}_\alpha = \prod_{(\beta)} \eta'^{b_{i\beta}}_\beta, \qquad (i = 1, \cdots, M), \tag{3.1}$$

and $[P^u; P^v]$, $[\xi';\eta']$ are solutions of (M). Since the first is non-trivial, the second must be trivial, and

$$[\xi;\eta] = [P^u; P^v].$$

$[u;v]$ must be a primitive solution of (A). For from the first set of equations in (3.1)

$$\sum_{(\alpha)} a_{i\alpha}u_\alpha = \sum_{(\beta)} b_{i\beta}v_\beta, \qquad (i = 1, \cdots, M),$$

so that $[u;v]$ is a solution of (A). But if it were the sum of two non-trivial solutions of (A), $[P^u; P^v]$ would be the product of two non-trivial solutions of (M).

COROLLARY. *Both the systems* (A) *and* (M) *have non-trivial solutions, or both have only trivial solutions.*

The primitive solution $[P^u; P^v]$ of (M) will be said to be of type $[u;v]$. There are an infinite number of primitive solutions of (M) of a given type; namely, one for each rational prime P. However *the number of types of primitive solutions of* (M) *is finite,* for the number of primitive solutions of (A) is known to be finite.*

Suppose that (A) has in all the R distinct primitive solutions

$$\mathfrak{U}_i = [\xi_i;\eta_i], \qquad (i = 1, 2, \cdots, R).$$

THEOREM 3.2. *Every solution of* (M) *is of the form*

$$\begin{aligned} x_\alpha &= T_1^{\xi_{1\alpha}} T_2^{\xi_{2\alpha}} \cdots T_R^{\xi_{R\alpha}} \\ y_\beta &= T_1^{\eta_{1\beta}} T_2^{\eta_{2\beta}} \cdots T_R^{\eta_{R\beta}} \end{aligned} \tag{3.2}$$

where the parameters $T_1, T_2, \cdots, T_R$ are positive integers. Conversely, every such expression is a solution of (M).

* Grace and Young, p. 103.

Proof. From the proof of theorem 3.1, it is evident that any solution $[\lambda;\mu]$ of (M) is of the form

$$\prod_{\sigma=1}^{S}[P_\sigma^{u_{\sigma1}}, P_\sigma^{u_{\sigma2}}, \cdots, P_\sigma^{u_{\sigma K}};\ P_\sigma^{v_{\sigma1}}, P_\sigma^{v_{\sigma2}}, \cdots, P_\sigma^{v_{\sigma L}}],$$

where $P_1, P_2, \cdots, P_S$ are the distinct primes dividing $\lambda_1\lambda_2\cdots\lambda_K\,\mu_1\mu_2\cdots\mu_L$, and the $[u_\sigma;\ v_\sigma]$ are solutions of (A). Now *

$$[u_\sigma;\ v_\sigma] = k_1^{(\sigma)}\mathfrak{U}_1 + k_2^{(\sigma)}\mathfrak{U}_2 + \cdots + k_R^{(\sigma)}\mathfrak{U}_R$$

where the $k^{(\sigma)}$ are non-negative integers. Therefore

$$P_\sigma^{u_{\sigma\alpha}} = \prod_{\tau=1}^{R} P^{k_\tau^{(\sigma)}\xi_{\tau\alpha}}, \qquad P_\sigma^{v_{\sigma\beta}} = \prod_{\tau=1}^{R} P_\sigma^{k_\tau^{(\sigma)}\eta_{\tau\beta}}.$$

Accordingly,

$$\lambda_\alpha = \prod_{(\sigma)} P_\sigma^{u_{\sigma\alpha}} = \prod_{(\sigma)}\prod_{(\tau)} P_\sigma^{k_\tau^{(\sigma)}\xi_{\tau\alpha}} = \prod_{(\tau)}\prod_{(\sigma)} P_\sigma^{k_\tau^{(\sigma)}\xi_{\tau\alpha}} = \prod_{(\tau)} T_\tau^{\xi_{\tau\alpha}},$$

$$\mu_\beta = \prod_{(\sigma)} P_\sigma^{v_{\sigma\beta}} = \prod_{(\sigma)}\prod_{(\tau)} P_\sigma^{k_\tau^{(\sigma)}\eta_{\tau\beta}} = \prod_{(\tau)}\prod_{(\sigma)} P_\sigma^{k_\tau^{(\sigma)}\eta_{\tau\beta}} = \prod_{(\tau)} T_\tau^{\eta_{\tau\beta}},$$

where

$$T_\tau = \prod_{(\sigma)} P_\sigma^{k_\tau^{(\sigma)}} = P_1^{k_\tau^{(1)}} P_2^{k_\tau^{(2)}} \cdots P_S^{k_\tau^{(s)}}, \qquad (\tau = 1, 2, \cdots, R),$$

so that the T are positive integers. The converse of the theorem is obvious from the relations just given.

4. Since for each fixed value of α there must be at least one value of τ for which $\xi_{\tau\alpha} \neq 0$, and for each fixed value of β one value of τ for which $\eta_{\tau\beta} \neq 0$, none of the parameters T in (3.2) can be equal to unity for all solutions of (M) unless all solutions of (M) are trivial. In other words, *the number of primitive solutions of* (A) *gives the minimum number of parameters T necessary to express every solution of* (M) *in the form* (3.2).

The question naturally arises whether we can determine this number *a priori* without actually exhibiting all the primitive solutions of (A). In general, this appears to be impossible, but there are certain fairly general special systems (M) for which such a determination can be made. We give in this connection the following two theorems.

THEOREM 4.2. *The total number of parameters T necessary to express all solutions of the system*

$$(\mathrm{M}') \qquad x_1^{a_1}x_2^{a_2}\cdots x_K^{a_K} = y_{11}y_{12}\cdots y_{1L_1} = \cdots = y_{n1}y_{n2}\cdots y_{nL_n}$$

is given by the formula

$$\sum_{\alpha=1}^{K}\prod_{\tau=1}^{n}\binom{L_\tau + a_\alpha - 1}{a_\alpha}.$$

* Grace and Young, pp. 104, 103.

Here $\binom{m}{n}$ denotes as usual the number of combinations of m things taken n at a time.

THEOREM 4. 2. *The total number of parameters T necessary to express all solutions of the system*

$$(\mathrm{M}'') \quad (x_{11}x_{12}\cdots x_{1K_1})^{a_1} = (x_{21}x_{22}\cdots x_{2K_2})^{a_2} = \cdots = (x_{n1}x_{n2}\cdots x_{nK_n})^{a_n}$$

is given by the formula

$$\prod_{\tau=1}^{n} \binom{a'_\tau + K_\tau - 1}{a'_\tau},$$

where $a'_\tau = a/a_\tau$, $(\tau = 1, 2, \cdots, n)$, and a is the least common multiple of integers $a_1, a_2, \cdots, a_n$.

To illustrate these theorems,* consider the three systems

$$\begin{aligned}
&(\mathrm{i}) & x^2y^3z^3 &= uv = wrst,\\
&(\mathrm{ii}) & x^3y^3z^3 &= u^2v^2 = wrst,\\
&(\mathrm{iii}) & x^9 = y^5 &= u^4v^4 = wrst.
\end{aligned}$$

For the first system, we apply theorem 4. 1 with $K = 3$, $a_1 = 2$, $a_2 = a_3 = 3$, $n = 2$, $L_1 = 2$, $L_2 = 4$,

$$\sum_{\iota=1}^{3} \prod_{\tau=1}^{2} \binom{L_\tau + a_a - 1}{a_a} = \sum_{a=1} \binom{a_a + 1}{a_a}\binom{a_a + 3}{a_a} = \binom{3}{2}\binom{2}{5} + 2\binom{4}{3}\binom{6}{3} = 190$$

For the second system, we apply theorem 4. 2 wth $n = 3$, $a_1 = 3$, $a_2 = 2$, $a_3 = 1$, $K_1 = 3$, $K_2 = 2$, $K_3 = 4$, $a = 6$, $a'_1 = 2$, $a'_2 = 3$, $a'_3 = 6$,

$$\prod_{\tau=1}^{3} \binom{a'_\tau + K_\tau - 1}{a'_\tau} = \binom{4}{2}\binom{4}{3}\binom{9}{6} = 2{,}016.$$

For the third system, which involves only five algebraically independent variables, theorem 4. 2 gives

$$\prod_{\tau=1}^{4} \binom{K_\tau + a'_\tau - 1}{a'_\tau} = \binom{20}{20}\binom{36}{36}\binom{46}{45}\binom{183}{180} = \binom{46}{1}\binom{183}{3} = 46{,}217{,}626.$$

From these illustrations it is clear that even for rather simple looking

* In the last section of the paper will be found a simple system for which a verification of the theorems is feasible. If we take in (M') $a_1 = a_2 = \cdots = a_k = 1$ or in (M'') $a_1 = a_2 = \cdots = a_n = 1$, we find that the total number of parameters necessary to express all solutions of the system

$$x_{11}\,x_{12}\cdots x_{1k_1} = x_{21}\,x_{22}\cdots x_{2k_2} = \cdots = x_{n1}\,x_{n2}\cdots x_{nk_n}$$

is $\displaystyle\sum_{a=1}^{k_1} \prod_{\tau=a}^{n} \binom{K_\tau}{1} = \prod_{\tau=1}^{n} \binom{K_\tau}{1} = k_1 \cdot k_2 \cdots k_n$, a result obtained by Bell in the paper already cited by an entirely different argument.

systems, the number of parameters may be extraordinarily large, and that the actual exhibition of the solutions of a given system in the form (3.2) is usually impracticable.

The proof of theorem 4.1 is as follows. Consider the additive system associated with (M′),

$$(\mathrm{A}') \quad a_1 z_1 + \cdots + a_K z_K = w_{11} + \cdots + W_{1L_1} = \cdots = w_{n1} + \cdots + w_{nL_n}.$$

We have seen that the number of parameters T necessary for the solution of (M′) is the number of primitive solutions of (A′).

There exist solutions of (A′) with one of the z equal to one and all the remaining z equal to unity, and every such solution is primitive. Let us consider those solutions in which $z_a = 1$ and $z_1 = z_2 = \cdots = z_{a-1} = z_{a+1} = \cdots = z_K = 0$.

For such a solution we must have from (A′) n relations of the type

$$(4.1) \qquad a_a = w_1 + w_2 + \cdots + w_L$$

where the w are non-negative integers. But the total number of ways that we can choose such numbers w to satisfy (4.1) equals the coefficient of t^{a_a} in the product $(1 + t + t^2 + \cdots)^L$, which is $\binom{L + a_a - 1}{a_a}$.

Therefore the total number of solutions under consideration is

$$\prod_{\tau=1}^{n} \binom{L_\tau + a_a - 1}{a_a} .$$

On taking $\alpha = 1, 2, \cdots, K$ it follows that *there are at least*

$$\sum_{a=1}^{k} \prod_{\tau=1}^{n} \binom{L_\tau + a_a - 1}{a_a} \text{ \textit{primitive solutions of} (A}').$$

To show that there are exactly this number, it suffices to show that no solution of (A′) not of the special form considered can be primitive.

Let the values of z in such a solution be $\eta_1, \eta_2, \cdots, \eta_K$ where $\eta_i \neq 0$ and let $N = a_1\eta_1 + a_2\eta_2 + \cdots + a_K\eta_K$, $M = a_i$. Then by our hypothesis, $N > M$.

It follows as for (4.1) that the values of w in any one of the sums in (A′) must form a partition of N into L or fewer parts. But for every such partition of N,

$$N = \gamma_1 + \gamma_2 + \cdots \gamma_{L'},$$

where $\gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_{L'} > 0$, $(L' \leqq L)$ we can find a partition of M

$$M = \partial_1 + \partial_2 + \cdots + \partial_{K'}$$

such that $K' \leqq L'$, $\partial_j \leqq \gamma_j$, $(j = 1, 2, \cdots, K')$.

Therefore by assigning the proper w to the γ and ∂, we exhibit our solution as the sum of a primitive solution of (A′) and a non-trivial solution of (A′) associated with a certain set of partitions of $N - M$.

The proof of theorem 4.2 follows similar lines. With an obvious extension of our matrix notation, let

$$[\xi^{(1)}; \xi^{(2)}; \cdots; \xi^{(n)}]$$

be a solution of the additive system associated with (M″),

$$(\mathrm{A}'') \qquad a_1(z_{11} + \cdots + z_{1K_1}) = \cdots = a_n(z_{n1} + \cdots + z_{nK_n})$$

and let

$$N_\tau = \xi_1^{(\tau)} + \xi_2^{(\tau)} + \cdots + \xi_{K_\tau}^{(\tau)}, \qquad (\tau = 1, 2, \cdots, n).$$

Then

$$(4.2) \qquad a_1N_1 = a_2N_2 = \cdots = a_nN_n = N, \quad \text{say.}$$

Now for integral $N_1, \cdots, N_n$ the least positive value of N which can satisfy a relation of the form (4.2) is the least common multiple of $a_1, a_2, \cdots, a_n$. Denote this number by a, and let

$$a'_\tau = a/a_\tau, \qquad (\tau = 1, 2, \cdots, n).$$

Then if

$$(4.3) \qquad a'_\tau = \eta_1^{(\tau)} + \eta_2^{(\tau)} + \cdots + \eta_{K_\tau}^{(\tau)}$$

is a partition of a'_τ into K_τ parts, zero counting as a part,

$$[\eta^{(1)}; \eta^{(2)}; \cdots; n^{(n)}]$$

will be a primitive solution of (A″). There are $\binom{a'_\tau + K_\tau - 1}{a'_\tau}$ distinct ways of selecting non-negative $\eta^{(\tau)}$ to satisfy (4.3), and hence in all

$$\prod^{n} \binom{a'_\tau + K_\tau - 1}{a'_\tau}$$

such primitive solutions. The proof that there are no other primitive solutions is almost exactly the same as for Theorem 4.1.

5. The results of section three allow us to complete the discussion of the general system (S).

Let $P_1, P_2, \cdots, P_H$ be the distinct prime factors of the $2M$ integers $A_1, \cdots, B_M$ so that

$$A_i = P_1^{c_{i1}} P_2^{c_{i2}} \cdots P_H^{c_{iH}}, \quad B_i = P_1^{d_{i1}} P_2^{d_{i2}} \cdots P_H^{d_{iH}}, \quad (i = 1, \cdots, M)$$

where the c and d are non-negative integers, and for a fixed k at least one of the $2M$ numbers $c_{1k}, c_{2k}, \cdots, c_{Mk}, d_{1k}, d_{2k}, \cdots, d_{Mk}$ is positive.

Consider the system

$$(M^{(k)}) \quad P_k^{c_{ik}} x_1^{a_{i1}} \cdots x_K^{a_{iK}} = P_k^{d_{ik}} y_1^{b_1} \cdots y_L^{b_{iL}}, \qquad (i = 1, \cdots, M)$$

and the associated additive system

$$(A^{(k)}) \quad c_{ik} + a_{i1} z_1 + \cdots + a_{iK} z_K = d_{ik} + b_{i1} w_1 + \cdots + b_{iL} w_L, \qquad (i = 1, \cdots, M).$$

Then if a primitive solution of $(A^{(k)})$ is defined as one which cannot be expressed as the sum of a solution of $(A^{(k)})$ and a non-trivial solution of (A), it follows as in the proof of theorem 3.1 that every primitive solution of $(M^{(k)})$ is of the form $[P_k^{\lambda}; P_k^{\mu}]$ where $[\lambda; \mu]$ is a primitive solution of $(A^{(k)})$.

Consider in connection with $(A^{(k)})$ the additive system

$$(B^{(k)}) \quad c_{ik} z_0 + a_{i1} z_1 + \cdots + a_{iK} z_K = d_{ik} w_0 + b_{i1} w_1 + \cdots + b_{iL} w_L, \qquad (i = 1, \cdots, M).$$

Then the number of primitive solutions of $(B^{(k)})$ is finite. If among these primitive solutions there are l_0 with $z_0 = w_0 = 1$, l_1 with $z_0 = 0$, $w_0 = 1$ and l_2 with $z_0 = 1$, $w_0 = 0$ then $(A^{(k)})$ and hence $(M^{(k)})$ has exactly $\nu_k = l_0 + l_1 l_2$ primitive solutions. If $l_0 + l_1 l_2 = 0$, $(M^{(k)})$ has no primitive solutions, and hence no solutions whatever. We shall see in a moment that this would entail (S) having no solutions contrary to our hypothesis. Hence $\nu_k > 0$ and the primitive solutions of $(M^{(k)})$ may be exhibited, since the primitive solutions of $(B^{(k)})$ can be found by trial in a finite number of steps.*

If we denote such a primitive solution of $(M^{(k)})$ by $[\xi^{(k)}; \eta^{(k)}]$, then

$$(5.1) \qquad [\xi; \eta] = [\xi^{(1)}; \eta^{(1)}] \cdot [\xi^{(2)}; \eta^{(2)}] \cdots [\xi^{(H)}; \eta^{(H)}]$$

is a primitive solution of (S), and there are in all exactly $\nu = \nu_1 \nu_2 \cdots \nu_H$ such solutions. Conversely, if (S) has solutions, and hence primitive solutions, a decomposition such as (5.1) is possible, so that each $(M^{(k)})$ must have primitive solutions. We summarize our results in the following theorem.

THEOREM 5.1. *If* (S) *has solutions, every solution is of the form*

$$(5.2) \qquad \begin{aligned} x_a &= C_a T_1^{\xi_{1a}} T_2^{a\xi_2} \cdots T_s^{\xi_{sa}} \\ y_\beta &= D_\beta T_1^{\eta_{1\beta}} T_2^{\eta_{2\beta}} \cdots T_s^{\eta_{s\beta}} \end{aligned}$$

where the T, ξ and η are as in Theorem 3.2, and the pairs of integers C_a, D_β may assume at most ν sets of values, where ν is given in the discussion above.

6. We have not treated here the important problem of what restrictions

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* Grace and Young, p. 104.

it is necessary to impose upon the parameters T so that the formulas (5.1) shall give the solutions of (S) once and once only.* This question is bound up in a highly interesting manner with the co-primality of sets of the parameters and their restriction to be numbers of a special form; e.g. square free. I hope to give some results connected with this problem subsequently.

I conclude by solving by the additive method the system used by Bell to illustrate his general process of solution,†

$$x_1 x_2^2 = y_1 y_2 = z_1 z_2. \tag{iv}$$

The additive dual of (iv) is

$$X_1 + 2X_2 = Y_1 + Y_2 = Z_1 + Z_2. \tag{6.1}$$

By inspection we can write down the following thirteen primitive solutions of (6.1):

$$\begin{aligned}
\mathfrak{U}_1 &= [1,0;\ 1,0;\ 1,0], & \mathfrak{U}_7 &= [0,1;\ 0,2;\ 2,0],\\
\mathfrak{U}_2 &= [1,0;\ 1,0;\ 0,1], & \mathfrak{U}_8 &= [0,1;\ 0,2;\ 0,2],\\
\mathfrak{U}_3 &= [1,0;\ 0,1;\ 1,0], & \mathfrak{U}_9 &= [0,1;\ 1,1;\ 2,0],\\
\mathfrak{U}_4 &= [1,0;\ 0,1;\ 0,1], & \mathfrak{U}_{10} &= [0,1;\ 1,1;\ 0,2],\\
\mathfrak{U}_5 &= [0,1;\ 2,0;\ 2,0], & \mathfrak{U}_{11} &= [0,1;\ 2,0;\ 1,1],\\
\mathfrak{U}_6 &= [0,1;\ 2,0;\ 0,2], & \mathfrak{U}_{12} &= [0,1;\ 0,2;\ 1,1],
\end{aligned}$$
$$\mathfrak{U}_{13} = [0,1;\ 1,1;\ 1,1].$$

By theorem (4.1), the solution of (iv) will contain $\binom{2}{1}\binom{2}{1} + \binom{3}{2}\binom{3}{2} = 13$ parameters.

Hence $\mathfrak{U}_1, \cdots, \mathfrak{U}_{13}$ are all the primitive solutions of (6.1), so that by theorem (3.2) the solution of (iv) is

$$\begin{aligned}
x_1 &= T_1T_2T_3T_4, & x_2 &= T_5T_6T_7T_8T_9T_{10}T_{11}T_{12}T_{13},\\
y_1 &= T_1T_2T_5^2T_6^2T_9T_{10}T_{11}^2T_{13}, & y_2 &= T_3T_4T_7^2T_8^2T_9T_{10}T_{12}^2T_{13},\\
z_1 &= T_1T_3T_5^2T_7^2T_9^2T_{11}T_{12}T_{13}, & z_2 &= T_2T_4T_6^2T_8^2T_{10}^2T_{11}T_{12}T_{13}.
\end{aligned}$$

On making the change of variables

$$\begin{matrix} T_1, & T_2, & T_3, & T_4, & T_5, & T_6, & T_7, & T_8, & T_9, & T_{10}, & T_{11}, & T_{12}, & T_{13} & \text{into}\\ \phi_1, & \phi_2, & \phi_3, & \phi_4, & \psi_9, & \psi_5, & \psi_4, & \psi_8, & \psi_7, & \psi_6, & \psi_2, & \psi_1, & \psi_3, & \end{matrix}$$

this solution agrees with that obtained by Bell.

The additive method gives no information about the co-primeness of the parameters T, and it is to some extent tentative. In compensation, it is usually shorter that the multiplicative method.

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* See Elliott, *Quarterly Journal of Mathematics*, Vol. 34 (1903), pp. 348-377 for a discussion of the similar problem for (A) in the case $M = 1$, with considerable detail for the sub-case $K + L = 3$.

† Paper cited, § 15.