Hence

$$\rho \geqq p_1 + \ldots + p_r.$$

The limit given here is therefore always as good as Vranceanu's. That it is sometimes better is seen from the following system whose species is two:

$$\omega^1 = dx^5 + x^1dx^2,\ \omega^2 = dx^6 - x^3dx^1 + x^2dx^4,$$
$$\omega'^1 = dx^1dx^2,\ \omega'^2 = dx^1dx^3 + dx^2dx^4.$$

We have $p_1 = 1$, $p_2 = 0$, $p_1 + p_2 = 1$, whereas $\rho = 2$.

Had the equations been written in the opposite order, we should have found $p_1 = 2$, $p_2 = 0$, $p_1 + p_2 = 2$. This illustrates the fact that $p_1 + \ldots + p_r$, unlike the rank, is not an invariant.

[1] Species is defined in the author's paper "Pfaffian Systems of Species One," *Trans. Amer. Math. Soc.*, **35**, 356–71 (1933).

[2] Cf. Goursat, E., *Problème de Pfaff*, Paris, 291 (1922).

[3] Cartan, E., *Invariants Intégraux*, Paris, 59 (1922).

[4] Vranceanu, G., *Comptes Rendus*, Paris, **196**, 1859–61 (1933).

A PROPERTY OF RECURRING SERIES

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1. If

$$(U): \qquad U_1, U_2, \ldots, U_n, \ldots$$

denotes a sequence of rational numbers satisfying a linear difference equation of order k with rational coefficients, then if a group of k consecutive terms of (U) ever repeats itself, all the roots of the polynomial associated with the difference equation are roots of unity, and the sequence (U) is periodic.[1] I show here that the like occurs, generally speaking, if *one* term of the sequence repeats itself at regular intervals a sufficient number of times. More precisely, I shall show that

If in any particular rational solution (U) of the difference equation

$$\Omega_{n+k} = P_1\Omega_{n+k-1} + P_2\Omega_{n+k-1} + \ldots + P_k\Omega_n,\ P_i \text{ rational},\ i = 1, \ldots, k, \quad (1)$$

we have

$$U_a = U_{a+b} = U_{a+2b} = \ldots = U_{a+kb} \neq 0, \quad (2)$$

where a, b are fixed positive integers, and if the associated polynomial

$$x^k - P_1x^{k-1} - P_2x^{k-2} - \ldots - P_k \qquad (3)$$

is irreducible in the field of rationals, then the polynomial is cyclotomic, and every solution of the difference equation is periodic.

2. The necessity for the restrictive hypotheses of the theorem is shown by the following two examples.

For the difference equation $\Omega_{n+3} = \Omega_{n+2} + 4\Omega_{n+1} - 4\Omega_n$ the associated polynomial $x^3 - x^2 - 4x + 4$ factors into $(x - 1)(x - 2)(x + 2)$. Therefore, if c is any rational number $\neq 0$, the particular solution $U_n = 2^n + (-2)^n + c\,1^n$ has all terms with odd subscripts equal to c.

On the other hand, for the difference equation $\Omega_{n+4} = \Omega_{n+2} + \Omega_n$, the associated polynomial $x^4 - x^2 - 1$ is irreducible and not cyclotomic, while any particular solution (U) with $U_2 = U_4 = 0$ has all terms with even subscripts equal to zero.

3. The theorem itself may be proved as follows. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the roots of the polynomial (2). Then U_n is of the form

$$U_n = A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_k\alpha_k^n$$

where the A_i are fixed non-vanishing algebraic numbers. If the common rational value of $U_a, U_{a+b}, \ldots, U_{a+kb}$ in (2) is denoted by c, we have then $k + 1$ homogeneous linear equations in $A_1\alpha_1^a, \ldots, A_k\alpha_k^a$ and c:

$$(A_1\alpha_1^a)\alpha_1^{rb} + (A_2\alpha_2^a)\alpha_2^{rb} + \ldots + (A_k\alpha_k^a)\alpha_k^{rb} - c = 0, r = 0, 1, \ldots, k. \quad (4)$$

Since $c \neq 0$, the determinant of this system must vanish. But this determinant is of Vandermonde's type; we obtain, therefore

$$V(\alpha_1^b, \alpha_2^b, \ldots, \alpha_k^b, 1) = \prod_{i \neq j} (\alpha_i^b - \alpha_j^b) \prod (\alpha_i^b - 1) = 0.$$

Hence for some i, j, we must have either $\alpha_i^b = 1$ or $\alpha_i^b = \alpha_j^b$.

In the first case, $\alpha_1^b = \alpha_2^b = \ldots = \alpha_k^b = 1$ for (3) was assumed irreducible. Therefore every root of (3) is a root of unity, and the theorem follows.

In the second case, on appealing again to the irreducibility of (3), we see that the b^{th} powers of the roots of (3) can be grouped into m sets of l equal powers each, say

$$\alpha_{rl+1}^b = \alpha_{rl+2}^b = \ldots = \alpha_{rl+l}^b = \zeta_{r+1}, r = 0, 1, \ldots, m - 1, ml = k,$$

where $\zeta_1, \zeta_2, \ldots, \zeta_m$ are distinct algebraic numbers.

If $m = 1$, ζ_1 is rational, and on comparing U_a and U_{a+b}, we see that ζ_1 is unity, giving the first case again. The assumption that $m > 1$ leads to a contradiction. For then we obtain from (4) the equations

$$(A_1\alpha_1^a + \ldots + A_l\alpha_l^a)\zeta_1^r + (A_{l+1}\alpha_{l+1}^a + \ldots + A_{2l}\alpha_{2l}^a)\zeta_2^r + \ldots -c = 0,$$
$$r = 0, 1, \ldots, m.$$

Since $c \neq 0$, we infer as before that

$$V(\zeta_1, \zeta_2, \ldots, \zeta_m, 1) = \prod_{i \neq j} (\zeta_i - \zeta_j) \prod (\zeta_i - 1) = 0.$$

This last relation is impossible, for the ζ are all distinct and all irrational.

[1] See E. T. Bell, these PROCEEDINGS, 16, 750–752 (1930). Such a repetition will certainly occur if the difference equation has rational integral coefficients and the sequence (U) is bounded provided that $U_1, U_2, \ldots, U_k$ are integers, a result which is due to Laguerre.

INTENSITIES IN ATOMIC SPECTRA

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Introductory.—In the theory of complex spectra a number of methods have been developed for calculating the relative energies of the states arising from an electronic configuration. In these methods various schemes of coupling together the orbital and spin angular momenta play an important rôle. Thus a common procedure is to find the matrix of the energy in LS coupling and to obtain the eigenvalues as the roots of the corresponding secular determinant. The advantage of this and other schemes is principally due to the fact that the total angular momentum $\mathbf{J}$, which is an integral of the motion, is given specified values. The energy matrix then takes on an especially simple form (factored according to J values). There is a further advantage in that the energy as well as other dynamical quantities can often be calculated rather simply in definite coupling schemes. The particular choice of coupling may be dictated by convenience for the calculation and occasionally by physical consideration, for the actual states of an atom sometimes very nearly correspond to definite coupling arrangement among the angular momentum vectors.

In this paper we are concerned with the matrix of the electric moment whose components determine the intensity of the lines radiated by the atom. The problem naturally divides into two parts: I, the determination of the electric moment in a definite coupling scheme; II, the electric moment matrix in intermediate coupling. The procedure for determining the latter from the former is quite straightforward and will be illustrated by an example.

Intensities in a Definite Coupling Scheme.—Let us suppose the atomic states are obtained by coupling the vectors $\mathbf{l}_1 \ldots \mathbf{l}_N$ $\mathbf{s}_1 \ldots \mathbf{s}_N$ together