THE ALGEBRA OF RECURRING SERIES.*

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1. Introduction. It is well known that if the function

$$F(x) \equiv x^3 - Px^2 + Qx - R, \qquad R \neq 0 \tag{1.1}$$

is irreducible in the field $\mathfrak{F}$ of its coefficients, then the properties of those solutions of the linear difference equation

$$\Omega_{n+3} = P\,\Omega_{n+2} - Q\,\Omega_{n+1} + R\,\Omega_n \tag{1.2}$$

which lie in $\mathfrak{F}$ are ultimately based on the algebra of the field $\mathfrak{F}(\alpha)$ obtained by adjoining to $\mathfrak{F}$ a root α of $F(x) = 0$.[1]

The object of this paper is to develop a general method for obtaining formal properties of the solutions of (1.2) from simple algebraic identities in $\mathfrak{F}(\alpha)$. The process is as follows:

We set up a one-to-one correspondence between the field $\mathfrak{F}(\alpha)$ and a certain class of square matrices of order three with elements lying in $\mathfrak{F}$. We then group these matrices into sets which are particular solutions of a matric difference equation of order one. Finally, we associate with each set a number of particular solutions of the scalar difference equation (1.2). Our method then consists of translating identities in $\mathfrak{F}(\alpha)$ into identities between matrices, and these in turn into relations between solutions of (1.2).[2] The treatment is simple and direct, and leads to a number of interesting formulas.[2]

The method may easily be extended to a difference equation of any order whose characteristic function is irreducible. We confine ourselves to the case of a third order equation, both for its interest in view of Lucas' claim to have discovered a remarkable connection between (1.2) and the theory of the elliptic functions,[3] and for simplicity of notation.

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[1] See for example, Bell, Tohoku Mathematical Journal, vol. 24, Numbers 1, 2 (1924), page 168. This paper gives a concise exposition of the algebraic theory of (1.2). We shall refer to it as "Bell", giving page reference. For the elementary theory of the linear difference equation of order r, see Bachmann, Niedere Zahlentheorie. The equation of order three is treated with considerable detail by Draeger, Thesis, Jena 1919. For other references, see Dickson's History, vol. I.

[2] See Section 5.

[3] In this connection, see Bell, p. 168.

2. Basic definitions. The most general solution of the difference equation (1.2) lying in the field $\mathfrak{F}$ of its coefficients is given by

$$\begin{aligned}\Omega_n = (K_0 + K_1\alpha + K_2\alpha^2)\alpha^n + (K_0 + K_1\beta + K_2\beta^2)\beta^n \\ + (K_0 + K_1\gamma + K_2\gamma^2)\gamma^n, \qquad (n = 0, \pm 1, \cdots).\end{aligned} \tag{2.1}$$

Here α, β, γ are the roots of the irreducible equation $F(x) = 0$, and K_0, K_1, K_2 are arbitrary elements of $\mathfrak{F}$.

If

$$\Omega_n = A_n, \qquad (n = 0, \pm 1, \cdots)$$

is any particular solution of (1.2) obtained by giving the constants K_0, K_1, K_2 in (2.1) definite values, A_0, A_1, A_2 are called the *initial values* of the *sequence* $(A)_n$. We write

$$(A)_n \sim [A_0, A_1, A_2].$$

Any sequence $(A)_n$ is completely determined as soon as we have specified its initial values.

There are four particular solutions of (1.2) of sufficient importance to have a special notation; we shall invariably write $(X)_n, (Y)_n, (Z)_n, (S)_n$ for the sequences defined by[4]

$$\begin{aligned}&(X)_n \sim [1, 0, 0]; \qquad (Y)_n \sim [0, 1, 0]; \\ &(Z)_n \sim [0, 0, 1]; \qquad (S)_n \sim [3, P, P^2 - 2Q].\end{aligned} \tag{2.2}$$

Finally, we have the well known formulas

$$\begin{aligned}&P = \alpha + \beta + \gamma, \qquad Q = \alpha\beta + \beta\gamma + \gamma\alpha, \qquad R = \alpha\beta\gamma; \\ &S_n = \alpha^n + \beta^n + \gamma^n, \qquad R^n S_{-n} = \alpha^n\beta^n + \beta^n\gamma^n + \gamma^n\alpha^n.\end{aligned} \tag{2.3}$$

3. Introduction of matrices. Let $\boldsymbol{M}_n$ denote the square matrix of order three

$$\boldsymbol{M}_n = \begin{pmatrix} X_n, & Y_n, & Z_n \\ X_{n+1}, & Y_{n+1}, & Z_{n+1} \\ X_{n+2}, & Y_{n+2}, & Z_{n+2} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots). \tag{3.1}$$

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[4] For properties of the first three, see Bell's paper. The solution

$$(Z)_n : 0, 0, 1, P, P^2 - Q, P^3 - 2PQ + R, \cdots$$

is important in Combinatory Analysis; in fact, Z_{n+2}, $n > 0$ is the homogeneous product sum of α, β, γ of degree n. See MacMahon, Combinatory Analysis, Cambridge, (1915), vol. I, p. 3. S_n is the familiar sum of the nth powers of the roots of $F(x) = 0$.

Then by direct calculation from (2.2) and (1.2), we find that

$$(3.2)\quad \boldsymbol{M}_0 = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \boldsymbol{M}_1 = \begin{pmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ R, & -Q, & P \end{pmatrix}, \boldsymbol{M}_2 = \begin{pmatrix} 0, & 0, & 1 \\ R, & -Q, & P \\ PR, & R-PQ, & P^2-Q \end{pmatrix}.$$

Thus $\boldsymbol{M}_0 = \boldsymbol{I}$, the identity matrix. We shall often write $\boldsymbol{M}$ for $\boldsymbol{M}_1$, omitting the subscript one.

The following properties of the matrices (3.1) are easily proved by induction:

$$(3.3)\quad \begin{aligned} \boldsymbol{M}_{n+1} &= \boldsymbol{M}\cdot\boldsymbol{M}_n = \boldsymbol{M}^{n+1}, \\ \boldsymbol{M}_n\cdot\boldsymbol{M}_m &= \boldsymbol{M}_m\cdot\boldsymbol{M}_n = \boldsymbol{M}_{m+n}, \\ \boldsymbol{M}_{n+3} &= P\boldsymbol{M}_{n+2} - Q\boldsymbol{M}_{n+1} + R\boldsymbol{M}_n, \\ \boldsymbol{M}_n &= X_n\boldsymbol{M}_0 + Y_n\boldsymbol{M}_1 + Z_n\boldsymbol{M}_2. \end{aligned} \qquad (m, n = 0, \pm 1, \cdots).$$

The first of these formulas shows that $\boldsymbol{M}_n$ is a particular solution of the matric difference equation of order one

$$(3.4)\qquad \boldsymbol{\Omega}_{n+1} = \boldsymbol{M}\cdot\boldsymbol{\Omega}_n$$

and the third formula shows that $\boldsymbol{M}_n$ is a particular solution of (1.2). We shall hereafter refer to the matrices (3.1) as the *sequence* $(\boldsymbol{M})_n$.

Combining the second and fourth of the formulas (3.3) gives the more general formula

$$\boldsymbol{M}_{n+m} = X_n\boldsymbol{M}_m + Y_n\boldsymbol{M}_{m+1} + Z_n\boldsymbol{M}_{m+2}, \quad (m, n = 0, \pm 1, \cdots).$$

By transforming $\boldsymbol{M}$ to the diagonal form and applying (3.3), (2.3), we can prove[5]

THEOREM 3.1. *The characteristic function of the matrix* $\boldsymbol{M}_n$ *is* $\lambda^3 - S_n\lambda^2 + R^n S_{-n}\lambda - R^n$.

The most general solution of (3.4) is

$$\boldsymbol{\Omega}_n = \boldsymbol{M}_n\cdot\boldsymbol{\Omega}_0, \qquad (n = 0, \pm 1, \cdots)$$

where the elements of the matrix $\boldsymbol{\Omega}_0$ are arbitrary. Let

$$\boldsymbol{\Omega}_n = \boldsymbol{P}_n$$

[5] We may note in passing a useful formula immediately obtainable from Theorem 3.1 and (3.1); namely,

$$S_n = X_n + Y_{n+1} + Z_{n+2}, \qquad (n = 0, \pm 1, \cdots).$$

be a particular solution obtained by letting

$$(3.41) \qquad \boldsymbol{\Omega}_0 = \boldsymbol{P}_0 = \begin{pmatrix} U_0, & V_0, & W_0 \\ U_1, & V_1, & W_1 \\ U_2, & V_2, & W_2 \end{pmatrix},$$

where $U_0, \cdots, W_2$ are fixed elements of $\mathfrak{F}$. Then from (3.3),

$$(3.5) \qquad \begin{aligned} \boldsymbol{P}_{m+n} &= \boldsymbol{M}_n \cdot \boldsymbol{P}_m, \\ \boldsymbol{P}_{n+3} &= P\boldsymbol{P}_{n+2} - Q\boldsymbol{P}_{n+1} + R\boldsymbol{P}_n, \end{aligned} \qquad (m, n = 0, \pm 1, \cdots).$$

From (3.5), we obtain the following theorem:

THEOREM 3.2. *If the sequence of matrices* $(\boldsymbol{P})_n$ *is a particular solution of the matric difference equation* (3.4), *where the value of* $\boldsymbol{P}_0$ *is given by* (3.41), *then*

$$\boldsymbol{P}_n = \begin{pmatrix} U_n, & V_n, & W_n \\ U_{n+1}, & V_{n+1}, & W_{n+1} \\ U_{n+2}, & V_{n+2}, & W_{n+2} \end{pmatrix}, \quad (n = 0, \pm 1, \cdots),$$

where $(U)_n \sim [U_0, U_1, U_2]$, $(V)_n \sim [V_0, V_1, V_2]$, $(W)_n \sim [W_0, W_1, W_2]$ *are particular solutions of the scalar difference equation* (1.2).

It is easily shown that the converse of this theorem is also true.

4. **Associated fields.** We shall now establish an isomorphism between $\mathfrak{F}(\alpha)$ and a certain class of matrices with elements in $\mathfrak{F}$.

THEOREM 4.1. *The class* $\mathfrak{M}$ *of all matrices of the form*

$$\boldsymbol{P} = U\boldsymbol{I} + V\boldsymbol{M} + W\boldsymbol{M}^2$$

where U, V, W *are any elements of* $\mathfrak{F}$ *forms a field which is simply isomorphic with the field* $\mathfrak{F}(\alpha)$ *obtained by adjoining a root* α *of* $F(x) = 0$ *to* $\mathfrak{F}$.

Proof. It is clear from formulas (3.2), (3.3), that any matrix $\boldsymbol{P}$ of $\mathfrak{M}$ can vanish when and only when U, V and W vanish.

$\mathfrak{M}$ is obviously closed under addition and subtraction; by (3.3), $\boldsymbol{M}^3 = P\boldsymbol{M}^2 - Q\boldsymbol{M} + R\boldsymbol{I}$; consequently, $\mathfrak{M}$ is also closed under multiplication. Furthermore, multiplication is commutative, and distributive with respect to addition.

Any element π of the field $\mathfrak{F}(\alpha)$ may be put in the unique canonical form

$$\pi = U + V\alpha + W\alpha^2$$

where U, V, W are elements of $\mathfrak{F}$. Set $\mathfrak{M}$ and $\mathfrak{F}(\alpha)$ into one-to-one correspondence by pairing the elements π and $\boldsymbol{P}$ for which U, V, W have the same values; we write in this case $\pi \sim \boldsymbol{P}$.

Then if $\pi_1 \sim \boldsymbol{P}_1$, $\pi_2 \sim \boldsymbol{P}_2$, $\pi_3 \sim \boldsymbol{P}_3$, it is easily verified that

$$\pi_1 \pm \pi_2 \sim \boldsymbol{P}_1 \pm \boldsymbol{P}_2; \ \pi_1 \cdot \pi_2 \sim \boldsymbol{P}_1 \cdot \boldsymbol{P}_2; \ \pi_1 (\pi_2 \pm \pi_3) \sim \boldsymbol{P}_1 (\boldsymbol{P}_2 \pm \boldsymbol{P}_3).$$

Furthermore, if $\pi \pi' = 1$, $\pi \sim \boldsymbol{P}$, $\pi' \sim \boldsymbol{P}'$, then $\boldsymbol{P} \cdot \boldsymbol{P}' = \boldsymbol{I}$.

Hence $\mathfrak{M}$ forms a field simply isomorphic with $\mathfrak{F}(\alpha)$.

As a corollary to this theorem, we have

THEOREM 4.11. *The characteristic equation of any matrix $\boldsymbol{P}$ of $\mathfrak{M}$ is the same as the equation which the corresponding element π of $\mathfrak{F}(\alpha)$ satisfies in $\mathfrak{F}$.*

We shall use the notations Det $\boldsymbol{P}$ and Adj $\boldsymbol{P}$ for the determinant and adjoint of any matrix $\boldsymbol{P}$, and $N(\pi)$ for the norm of any number π of $\mathfrak{F}(\alpha)$. It is easily seen from Theorem 4.11 that

(4.12) If $\boldsymbol{P} \sim \pi$, then $\operatorname{Det} \boldsymbol{P} = N(\pi)$;

(4.13) If $\boldsymbol{P} \sim \pi \neq 0$, then $\operatorname{Adj} \boldsymbol{P} \sim N(\pi)/\pi$.

THEOREM 4.2. *The necessary and sufficient condition that any matrix of order three with elements in $\mathfrak{F}$ be commutative with $\boldsymbol{M}$ is that it lies in the field $\mathfrak{M}$.*

Proof. The sufficiency of the condition follows from Theorem 4.1. To establish the necessity, suppose that $\boldsymbol{L}$ is a matrix of order three over $\mathfrak{F}$ commutative with $\boldsymbol{M}$. Then

$$\boldsymbol{L} \cdot \boldsymbol{M} = \boldsymbol{M} \cdot \boldsymbol{L}; \qquad \boldsymbol{L} \cdot \boldsymbol{M}^2 = \boldsymbol{M}^2 \cdot \boldsymbol{L}. \tag{4.2}$$

There exists a non-singular matrix $\boldsymbol{T}$ transforming $\boldsymbol{M}$ into the diagonal form $\boldsymbol{M}^*$. By (4.2), $\boldsymbol{T}^{-1} \cdot \boldsymbol{L} \cdot \boldsymbol{T} = \boldsymbol{L}^*$ must also be in the diagonal form. Let $\alpha, \beta, \gamma; \alpha', \beta', \gamma'$ be the diagonal elements in $\boldsymbol{M}^*; \boldsymbol{L}^*$, and consider the traces of $\boldsymbol{L}^*, \boldsymbol{M}^* \cdot \boldsymbol{L}^*, (\boldsymbol{M}^*)^2 \cdot \boldsymbol{L}^*$. They are the same as the traces of $\boldsymbol{L}, \boldsymbol{M} \cdot \boldsymbol{L}, \boldsymbol{M}^2 \cdot \boldsymbol{L}$. Hence

$$\alpha' + \beta' + \gamma' = I, \quad \alpha\alpha' + \beta\beta' + \gamma\gamma' = J, \quad \alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma' = K,$$

where I, J, K are elements of $\mathfrak{F}$. Solving these equations for α', β', γ' we find that

$$\alpha' = U + V\alpha + W\alpha^2, \quad \beta' = U + V\beta + W\beta^2, \quad \gamma' = U + V\gamma + W\gamma^2$$

where U, V, W are elements of $\mathfrak{F}$. Thus

$$\boldsymbol{L}^* = U\boldsymbol{I} + V\boldsymbol{M}^* + W\boldsymbol{M}^{*2},$$
$$\boldsymbol{L} = \boldsymbol{T} \cdot \boldsymbol{L}^* \cdot \boldsymbol{T}^{-1} = U\boldsymbol{I} + V\boldsymbol{M} + W\boldsymbol{M}^2.$$

THEOREM 4.3. *Let $(\boldsymbol{P})_n$ denote the sequence of matrices defined in Theorem 3.2. Then a necessary and sufficient condition that $(\boldsymbol{P})_n$ should ie in $\mathfrak{M}$ is that the sequences $(U)_n$, $(V)_n$, $(W)_n$ be connected by the relations*

$$\begin{aligned} V_n &= W_{n+1} - P W_n, \\ U_n &= W_{n+2} - P W_{n+1} + Q W_n = R W_{n-1}, \end{aligned} \qquad (n = 0, \pm 1, \cdots). \tag{4.3}$$

Proof. We easily find that

$$\boldsymbol{P}_n \cdot \boldsymbol{M} = \boldsymbol{M} \cdot \boldsymbol{P}_n$$

when and only when the relations (4.3) hold. The result now follows from Theorem 4.2.

Let $(\boldsymbol{P})_n$ now denote a sequence of matrices whose elements satisfy the relations[6] (4.3). Then

$$\boldsymbol{P}_n = I\boldsymbol{M}_0 + J\boldsymbol{M}_1 + K\boldsymbol{M}_2$$

where I, J, K lie in $\mathfrak{F}$. By comparing the elements in the first row of both sides of this identity, we find from (3.2) that

$$I = U_n, \qquad J = V_n, \qquad K = W_n$$

so that

$$\boldsymbol{P}_n = U_n\boldsymbol{M}_0 + V_n\boldsymbol{M}_1 + W_n\boldsymbol{M}_2, \qquad (n = 0, \pm 1, \cdots).$$

5. Derivation of formulas. We are now in a position to illustrate the method of translating identities in $\mathfrak{F}(\alpha)$ into relations between solutions of (1.2). With the notation of Theorem 4.1, we write $\pi \sim \boldsymbol{P}_0$ for

$$\pi = U_0 + V_0\alpha + W_0\alpha^2, \qquad \boldsymbol{P}_0 = U_0\boldsymbol{M}_0 + V_0\boldsymbol{M}_1 + W_0\boldsymbol{M}_2.$$

In particular, $\alpha \sim \boldsymbol{M}_1$, and by Theorem 4.1, (3.11), Theorem 3.3,

$$\alpha^n \sim \boldsymbol{M}_n, \qquad \alpha^n \cdot \pi \sim \boldsymbol{P}_n \tag{5.1}$$

where the elements of $\boldsymbol{P}_n$ satisfy the conditions (4.3).

Let us start with the following trivial identities in $\mathfrak{F}(\alpha)$.

$$\begin{aligned} &\text{I.} \quad \alpha^{n+m} \cdot \pi = \pi \cdot \alpha^{n+m} = (\alpha^n \cdot \pi)\,\alpha^m = \alpha^m(\alpha^n \cdot \pi), \\ &\text{II.} \quad (\pi \cdot \alpha^{n+m})\,\pi = \pi(\alpha^{n+m} \cdot \pi) = (\alpha^m \cdot \pi) \cdot (\alpha^n \cdot \pi). \end{aligned}$$

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[6] It is perhaps worth noting that on account of the linearity of (1.2), (4.3) will hold for all values of n if it holds for $n = 0, 1, 2$; i. e. $(\boldsymbol{P})_n$ lies in $\mathfrak{M}$ if $\boldsymbol{P}_0$ lies in $\mathfrak{M}$.

The corresponding matrix identities in $\mathfrak{M}$ are

$$\text{I}'. \quad \boldsymbol{P}_{n+m} = \boldsymbol{P}_0 \cdot \boldsymbol{M}_{n+m} = \boldsymbol{P}_n \cdot \boldsymbol{M}_m = \boldsymbol{M}_m \cdot \boldsymbol{P}_n,$$
$$\text{II}'. \quad \boldsymbol{P}_{n+m} \cdot \boldsymbol{P}_0 = \boldsymbol{P}_0 \cdot \boldsymbol{P}_{m+n} = \boldsymbol{P}_m \cdot \boldsymbol{P}_n.$$

By equating corresponding elements on both sides of I′ and II′, we obtain a number of formulas involving $(U)_n$, $(V)_n$, $(W)_n$, $(X)_n$, $(Y)_n$, $(Z)_n$; for instance, from I′ we obtain

$$\begin{aligned} U_{n+m} &= U_0 X_{n+m} + V_0 X_{n+m+1} + W_0 X_{n+m+2} \\ &= U_n X_m + V_n X_{m+1} + W_n X_{m+2} \\ &= X_m U_n + Y_m U_{n+1} + Z_m U_{n+2}. \end{aligned} \tag{5.2}$$

From II′, we obtain

$$\begin{aligned} U_{n+m} U_0 + V_{n+m} U_1 + W_{n+m} U_2 &= U_0 U_{m+n} + V_0 U_{m+n+1} + W_0 U_{m+n+2} \\ &= U_m U_n + V_m U_{n+1} + W_m U_{n+2}. \end{aligned}$$

If we introduce the number

$$\pi' = N(\pi)/\pi = U_0' + V_0' \alpha + W_0' \alpha^2,$$

we obtain another class of formulas. For by (4.13),

$$\text{Adj}\, \boldsymbol{P} = \boldsymbol{P}_0' = U_0' \boldsymbol{M}_0 + V_0' \boldsymbol{M}_1 + W_0' \boldsymbol{M}_2,$$

and if we let $\boldsymbol{P}_n' = \boldsymbol{M}_n \cdot \boldsymbol{P}_0'$, we find that

$$\text{Adj}\, \boldsymbol{P}_n = R^n \boldsymbol{P}_n'.$$

Hence,

$$V_{n+1} W_{n+2} - W_{n+1} V_{n+2} = R^n U_{-n}', \qquad W_{n+2} U_{n+1} - U_{n+2} W_{n+1} = R^n V_n',$$

and so on.

The identity

$$\begin{aligned} \text{III}. \quad (\alpha^m \cdot \pi') \cdot (\alpha^n \cdot \pi) &= (\alpha^m \cdot \pi) \cdot (\alpha^n \cdot \pi') = \pi \cdot (\alpha^{m+n} \cdot \pi') \\ &= \pi' \cdot (\alpha^{n+m} \cdot \pi) \quad = N(\pi)\, \alpha^{n+m} \end{aligned}$$

gives us

$$\text{III}'. \quad \boldsymbol{P}_m' \cdot \boldsymbol{P}_n = \boldsymbol{P}_m \cdot \boldsymbol{P}_n' = \boldsymbol{P}_0 \cdot \boldsymbol{P}_{m+n}' = \boldsymbol{P}_{m+n} \cdot \boldsymbol{P}_0' = N(\pi) \boldsymbol{M}_{n+m}.$$

From III′, we obtain formulas of the type

$$\begin{aligned} U_m' U_n + V_m' U_{n+1} + W_m' U_{n+2} &= U_m U_n' + V_m U_{n+1}' + W_m U_{n+2}' \\ = U_0 U_{m+n}' + V_0 U_{m+n+1}' + W_0 U_{m+n+2}' &= U_{m+n} U_0' + V_{m+n} U_1' + W_{m+n} U_2' \\ &= N(\pi) X_{n+m}, \quad (m, n = 0, \pm 1, \cdots). \end{aligned}$$

6. Extension of method. We shall conclude by extending the method so as to apply to an important class of matrices not lying in $\mathfrak{M}$.

Let $(\boldsymbol{S})_n$ denote the sequence of matrices

$$\boldsymbol{S}_n = \begin{pmatrix} S_n, & S_{n+1}, & S_{n+2} \\ S_{n+1}, & S_{n+2}, & S_{n+3} \\ S_{n+2}, & S_{n+3}, & S_{n+4} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots)$$

where S_n is given by (2.3).

By the converse to Theorem 3.2, $\boldsymbol{S}_m = \boldsymbol{M}_m \cdot \boldsymbol{S}_0$, so that

$$\boldsymbol{M}_n \cdot \boldsymbol{S}_m = \boldsymbol{S}_{m+n}. \tag{6.1}$$

However, $\boldsymbol{M} \cdot \boldsymbol{S}_0 \neq \boldsymbol{S}_0 \cdot \boldsymbol{M}$, so that $(\boldsymbol{S})_n$ does not lie in $\mathfrak{M}$.

Define a new matrix $\boldsymbol{T}_n$ by

$$\boldsymbol{T}_n = \boldsymbol{P}_0 \cdot \boldsymbol{S}_n. \tag{6.2}$$

If we write T_n for

$$U_0 S_n + V_0 S_{n+1} + W_0 S_{n+2}, \qquad (n = 0, \pm 1, \cdots)$$

we find from (6.2) that

$$\boldsymbol{T}_n = \begin{pmatrix} T_n, & T_{n+1}, & T_{n+2} \\ T_{n+1}, & T_{n+2}, & T_{n+3} \\ T_{n+2}, & T_{n+3}, & T_{n+4} \end{pmatrix}, \qquad (n = 0, \pm 1, \cdots).$$

From (6.2) and (6.1)

$$\boldsymbol{T}_{m+n} = \boldsymbol{P}_0 \cdot \boldsymbol{S}_{m+n} = \boldsymbol{P}_0 \cdot \boldsymbol{M}_n \cdot \boldsymbol{S}_m = \boldsymbol{P}_n \cdot \boldsymbol{S}_m,$$

giving the useful formula

$$T_{m+n} = U_n S_m + V_n S_{m+1} + W_n S_{m+2}. \tag{6.3}$$

Formula (6.3) applies to any sequence satisfying (1.2). For if

$$(T)_n \sim [T_0, T_1, T_2],$$

the three equations

$$T_i = U_0 S_i + V_0 S_{i+1} + W_0 S_{i+2}, \qquad (i = 0, 1, 2)$$

will determine U_0, V_0, W_0. The six remaining elements of $\boldsymbol{P}_0$, $U_1, \cdots, W_2$ are then completely determined by the relations (4.3), and the demonstration given applies.

There is an interesting consequence of formula (6.3). We may use (4.3) to express (6.3) in the form

$$T_{m+n} = (W_{n+2} - PW_{n+1} + QW_n)S_m + (W_{n+1} - PW_n)S_{m+1} + W_n S_{m+2}. \tag{6.31}$$

On interchanging m and n in (6.31) and rearranging the terms, we obtain

$$T_{m+n} = (S_{n+2} - PS_{n+1} + QS_n)W_m + (S_{n+1} - PS_n)W_{m+1} + S_n W_{m+2}. \tag{6.32}$$

Since (6.32) may be derived from (6.31) by simply interchanging the S and the W, we have a parallelism between the expression for T_{m+n} in terms of S_m, S_{m+1}, S_{m+2} and in terms of W_m, W_{m+1}, W_{m+2}. There is a similar parallelism between the expression for T_{m+n} in terms of T_m, T_{m+1}, T_{m+2} and in terms of Z_m, Z_{m+1}, Z_{m+2}. For from (5.2), taking $(U)_n = (T)_n$,

$$T_{m+n} = X_n T_m + Y_n T_{m+1} + Z_n T_{m+2}.$$

On replacing X_n and Y_n by their expressions in terms of Z_n from (4.3),[7] we obtain two formulas analogous to (6.31) and (6.32); namely,[8]

$$T_{m+n} = (Z_{n+2} - PZ_{n+1} + QZ_n)T_m + (Z_{n+1} - PZ_n)T_{m+1} + Z_n T_{m+2},$$
$$T_{m+n} = (T_{n+2} - PT_{n+1} + QT_n)Z_m + (T_{n+1} - PT_n)Z_{m+1} + T_n Z_{m+2}.$$

[7] Bell, p. 173 formula (12). The matrix $\boldsymbol{M}_n$ is of course a special form of $\boldsymbol{P}_n$.

[8] If we take $(T)_n = (Z)_n$, $n = n+1$, $m = m-1$, the last two formulas become equation (34) in section 7 of Bell, p. 179.