RESIDUATED LATTICES

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1. *Introduction.*—We summarize here our investigations of a lattice $\Sigma: a, b, \ldots, z$ over which a multiplication or a residuation is defined. (Ward 1, Dilworth 1.) We denote division, union and cross-cut by $x \supset y$, (x, y), $[x, y]$. Σ is closed with respect to union (cross-cut) if any set of elements have a union (cross-cut). The unit and null elements i and n are defined by $i \supset x$, $x \supset n$, every $x.a$ covers b (Birkhoff 1) if $a \supset b$, $a \neq b$ and $a \supset x \supset b$ implies $x = a$ or $x = b$. Elements covered by i are called divisor-free. A sub-lattice Λ is dense over Σ if Λ contains l, m and $l \supset x \supset m$ imply Λ contains x. An element a is a node if either $x \supset a$ or $a \supset x$, every x. Any involution of Σ interchanging union and cross-cut is called a negation. An element a is idempotent relative to a binary operation $x \circ y$ if $a \circ a = a$. Two properties P and Q which Σ may possess are completely independent if there exist instances of lattices in which both P and Q hold, neither holds, P holds but not Q, Q holds but not P.

2. *Residuations and Multiplications.*—Assume Σ contains i. A well defined binary operation $x:y$ is called a residuation over Σ provided that

R 1. $\mathrm{a:b}$ *lies in* Σ *whenever* a, b *lie in* Σ.
R 2. $\mathrm{a:b = i}$ *if and only if* $\mathrm{a \supset b}$.
R 3. $\mathrm{a \supset b}$ *implies* $\mathrm{a:c \supset b:c}$ *and* $\mathrm{c:b \supset c:a}$.
R 4. $\mathrm{(a:b):c = (a:c):b}$.
R 5. $\mathrm{[a, b]:c = [a:c, b:c]}$ *and* $\mathrm{c:(a, b) = [c:a, c:b]}$.

This residual has the formal properties of the residual in polynomial ideal theory. (Ward 1, Dilworth 1.)

THEOREM. *A residuated lattice closed with respect to cross-cut is also closed with respect to a well-defined multiplication* $\mathrm{x \cdot y}$ *satisfying the following conditions:*

M 1. $\mathrm{a \cdot b}$ *lies in* Σ *whenever* a, b *lie in* Σ.
M 2. $\mathrm{(a \cdot b) \cdot c = a \cdot (b \cdot c)}$.
M 3. $\mathrm{a \cdot b = b \cdot a}$.
M 4. $\mathrm{a \cdot i = a}$.
M 5. $\mathrm{a \cdot (b, c) = (a \cdot b, a \cdot c)}$.

If a multiplication over Σ satisfies M 1–M 5 and M 6: *The product of the unions of any two sets of elements of* Σ *is the union of the products of all pairs of elements of the sets,* then a residuation exists satisfying R 1–R 5. (Ward 1.) The relationship between the two operations is as follows.

$a \supset (a:b)\cdot b$; if $a \supset x\cdot b$ then $a:b \supset x$.
$(a\cdot b):a \supset b$; if $x:a \supset b$ then $x \supset a\cdot b$.

Both operations may be dualized.

THEOREM. *The Dedekind modular condition and the existence of a residual are completely independent properties of a lattice. The existence of a residual and the existence of a negation are completely independent.*

3. *Conditions for Residuation.*

THEOREM. *Every distributive lattice which is closed with respect to union can be residuated in at least one way.* (Ward 2.)

THEOREM. *Every Boolean algebra can be residuated in only one way.*

The residual in this case is $a:b = a \vee b'$. (Dilworth 1.) A lattice is said to be complemented (Birkhoff 2) if it contains i and n and for every element a an element a' such that $(a, a') = i$, $[a, a'] = n$.

THEOREM. *The only complemented lattices which can be residuated are Boolean algebras.*

COROLLARY. *No non-trivial projective geometry* (Birkhoff 2) *can be residuated.*

THEOREM. *The free modular lattice of order twenty-eight cannot be residuated.*

THEOREM. *Every lattice in which only one divisor free element exists can be residuated in at least one way.*

THEOREM. *A lattice built up out of a set of residuated lattices connected into a chain by nodes can be residuated.*

THEOREM. *A direct product of residuated lattices can be residuated; conversely if a residuated lattice can be expressed as a direct product, each of its factors can be residuated.*

THEOREM. *A necessary condition that a residuated lattice in which an ascending chain condition holds* (Ore 1) *can be residuated is that every Boolean algebra generated by a finite number of divisor free elements be dense over the lattice.*

4. *Noether Lattices.*—We propose here the name "Noether lattice" for any residuated modular lattice in which both the (ascending) chain condition holds and

D 1. *For any two elements* a, b *of* Σ, *there exist exponents* r *and* s *such that* $\mathrm{a}\cdot\mathrm{b} \supset [\mathrm{a}^{\mathrm{r}}, \mathrm{b}^{\mathrm{s}}]$.

For the ideal theory terminology used here see van der Waerden 1.

THEOREM. *In a Noether lattice, every irreducible is primary. Conversely, if in a residuated modular lattice with chain condition every irreducible is primary, then condition* D 1 *holds.*

THEOREM. *The three decomposition theorems and the uniqueness theorems of* E. *Noether for the ideals of a commutative ring in which the chain condition holds are all valid in an abstract Noether lattice.*

THEOREM. *Conditon* D 1 *is completely independent both of the modular condition, the distributive condition and the chain condition.*

THEOREM. *A necessary and sufficient condition that a finite residuated modular lattice be a Noether lattice is that* a·b = [a, b] *for all idempotent elements* a, b *of the lattice.*

THEOREM. *A sufficient condition that a residuated modular lattice in which the ascending chain condition holds be a Noether lattice is*

M 7. a·[b, c] = [a·b, a·c].

The resulting lattice need not be distributive.

5. *Distributive Residuated Lattices.*—Consider a lattice in which one or more of the following conditions hold:

D 2. *If* a $\supset$ b, *there exists at least one element* q *such that* a·q = b.
R 6. (a:b, b:a) = i.
R 7. a:[b, c] = (a:b, a:c).
R 8. (b, c):a = (b:a, c:a).

THEOREM. *Every lattice closed with respect to union in which* D 2 *holds can be residuated, and is distributive.*

THEOREM. *If* Σ *is a residuated lattice, any one of* R 6, R 7, R 8 *implies* Σ *is distributive.* R 6 *and* D 2 *implies* R 7 *and* R 8. R 8 *implies* R 7.

We call a residuated lattice satisfying D 2 and R 6 *semi-arithmetical*. The properties of such lattices are similar to the instance in Ward 2 where multiplication is cross-cut.

6. *Residuated Group Lattices—Dual Operations.*—On assuming Σ is a semi-group and D_2, we may pass to the group Γ of quotients a/b. We have made Γ into a residuated lattice having properties R 1–R 8. However the lattice has no unit element.

The existence of a dual residuation and multiplication is completely independent of the existence of the initial residuation and multiplication. An interesting case arises in a residuated lattice containing n if the correspondence $a \rightarrow n:a$ is a negation. The lattice is then distributive, and multiplication and its dual are also distributive with respect to one another.

Proofs of these results will be published elsewhere.

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