The slopes of the two tangents of inflection are given by the expression

$$q \pm \left(\frac{-2p}{3}\right)^{3/2}.$$

If the slope of one inflection tangent is zero, then the slope of the other is $2q$.

MATHEMATICAL NOTES

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A GENERALIZED INTEGRAL TEST FOR CONVERGENCE OF SERIES

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The following useful generalization of the familiar Maclaurin-Cauchy integral test for convergence of real series deserves to be better known. It is apparently due to G. H. Hardy,* who made a redundant hypothesis on $f(t)$. The integrals may be taken either in the sense of Riemann or in the sense of Lebesgue.

THEOREM. *Let $f(t)$ be a complex-valued function of the real variable in the interval $1 \leqq t < \infty$, such that $f'(t)$ exists and is integrable to $f(t)$ over any finite interval $1 \leqq t \leqq T$. Then if $\int_1^\infty f'(t)dt$ is absolutely convergent, the series $\sum_1^\infty f(n)$ and the integral $\int_1^\infty f(t)dt$ converge and diverge together.*

Proof: By Abel's partial summation formula, we have

$$\sum_{r=1}^{n} a_r b_r = A_n B_n - \sum_{r=1}^{n-1} A_r(b_{r+1} - b_r),$$

where $A_r = a_1 + a_2 + \cdots + a_r$, $(r = 1, 2, \cdots, n)$.

Let $s_n = \sum_{r=1}^{n} f(r)$. Then on taking $a_r = 1$ and $b_r = f(r)$ in the summation formula, we find that

$$s_n = nf(n) - \sum_{r=1}^{n-1} r(f(r+1) - f(r)).$$

Now if $[t]$ denotes as usual the greatest integer in t, then

* G. H. Hardy: Proc. London Math. Soc. (2), vol. 9, 1910, pp. 126–144.

$$r(f(r+1)-f(r)) = \int_r^{r+1} [t] f'(t)dt.$$

Also

$$nf(n) - 1\cdot f(1) = \int_1^n \frac{d}{dt}(tf(t))dt,$$

or

$$nf(n) = f(1) + \int_1^n f(t)dt + \int_1^n tf'(t)dt.$$

On substituting these expressions into the formula for s_n, simplifying and transposing, we obtain the formula

$$s_n - \int_1^n f(t)dt = f(1) + \int_1^n (t-[t])f'(t)dt.$$

Now $|(t-[t])f'(t)| < |f'(t)|$. Hence the infinite integral $\int_1^\infty (t-[t])f'(t)dt$ is convergent, and

$$\lim_{n\to\infty}\left(s_n - \int_1^n f(t)dt\right) \text{ exists.} \tag{1}$$

Now assume that the integral $\int_1^\infty f(t)dt$ is convergent. Then $\lim_{n\to\infty}\int_1^n f(t)dt$ exists. Hence by (1), $\lim_{n\to\infty} s_n$ exists; that is, the series $\sum_1^\infty f(n)$ is convergent.

The converse result is a little more troublesome. Assume that $\sum_1^\infty f(n)$ converges. Then

$$\lim_{n\to\infty} f(n) = 0, \tag{2}$$

and by (1),

$$\lim_{n\to\infty}\int_1^n f(t)dt \text{ exists.} \tag{3}$$

Now $f(T)=f(1)+\int_1^T f'(t)dt$. But since $\int_1^\infty f'(t)dt$ converges, $\lim_{T\to\infty}\int_1^T f'(t)dt$ exists. Hence $\lim_{T\to\infty} f(T)$ exists, so that by (2),

$$\lim_{t\to\infty} f(t) = 0. \tag{4}$$

Now

$$\left|\int_1^T f(t)dt - \int_1^{[T]} f(t)dt\right| = \left|\int_{[T]}^T f(t)dt\right| \leqq \max_{[T]\leqq t\leqq T} |f(t)|\,(T-[T]) < \max_{t\geqq [T]} |f(t)|.$$

Hence by (4)

$$\lim_{T\to\infty}\left(\int_1^T f(t)dt - \int_1^{[T]} f(t)dt\right) = 0.$$

But

$$\lim_{T\to\infty}\int_1^{[T]} f(t)dt$$

exists by (3). Hence $\lim_{T\to\infty}\int_1^T f(t)dt$ exists; that is $\int_1^\infty f(t)dt$ is convergent.

As an example, suppose that $f(t)=t^{-1}e^{-i\mu\log t}$, μ real. Then $f'(t)=0(1/t^2)$ and the conditions of the theorem are met. But

$$\int_1^T f(t)dt = \frac{i}{\mu}\left(e^{-i\mu\log T}-1\right).$$

Hence $\int_1^\infty f(t)dt$ diverges. Therefore $\sum_1^\infty 1/n^{1+i\mu}$ diverges.

Again, suppose that $f(t)=e^{it^\alpha\theta}/t^\beta$, where α and θ are real, and Rl $\beta>\alpha>0$, $\theta\neq 0$. Then $f'(t)$ is continuous and of order $t^{-1-\mu}$, where $\mu=\text{Rl }\beta-\alpha$, in the range $1\leqq t<\infty$. Hence the conditions of the theorem are met. Now the infinite integral $\int_1^\infty f(t)dt$ is easily seen to converge on making the change of variable $s=t^\alpha$. Hence the infinite series $\sum_1^\infty e^{in^\alpha\theta}/n^\beta$ converges. In particular then, if β is real, we see that the two real series

$$\sum_1^\infty \frac{\cos n^\alpha\theta}{n^\beta} \quad\text{and}\quad \sum_1^\infty \frac{\sin n^\alpha\theta}{n^\beta}$$

both converge if $\beta>\alpha>0$ and $\theta\neq 0$.

The ordinary integral test is included as a special case if we use Lebesgue integrals; for if $f(t)$ is real, continuous and tends to zero steadily, $f'(t)$ exists almost everywhere and $f(t)=\int_1^t f'(s)ds+f(1)$. Since $|f'(t)|=-f'(t)$, the hypotheses of the theorem are evidently satisfied.

GEOMETRY OF THE SQUARE ROOT OF THREE

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That the diagonal of a square is incommensurable with its side and the quotient is representable by the continued fraction

$$1+\frac{1}{2+}\frac{1}{2+}\frac{1}{2+\cdots}$$

is easy to prove geometrically. The corresponding fact that the altitude of an equilateral triangle and half its side are incommensurable and the quotient representable by