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A CHARACTERIZATION OF BOOLEAN ALGEBRAS

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Introduction

When is a lattice a Boolean algebra—that is, isomorphic with a field of sets? The classical conditions are:[1] (1) The distributive law holds, (2) every element has a complement. Now it is well known[2] that (1) and (2) imply (3) no element has more than one complement. We are thus led to conjecture that (2) *and* (3) *imply* (1); in other words, that a necessary and sufficient condition that a lattice be a Boolean algebra is that each of its elements have a unique complement.

The only published result bearing on this question is G. Bergmann's[3] theorem that the distributive law holds if and only if *relative* complements are unique; that is, if and only if given $a \leqq x \leqq b$, there exists one and only one y with $x \cap y = a, x \cup y = b$.

We prove here the truth of our conjecture for all complete atomistic lattices.[4] These include lattices of finite length and of finite order. We do not know whether or not our conjecture is unrestrictedly true.

Exact statement of theorem

Let L be a complete lattice which is "atomistic" in the sense that if $O < a < I$, then $p_\alpha \leqq a \leqq q_\beta$ where p_α covers O and q_β is covered by I. Let us further define a "point" as an element p covering O.

THEOREM 1: *If each element of L has one and only one complement, then L is isomorphic with the Boolean algebra of all subsets of its points.*

THEOREM 2: *In order for L to be a Boolean algebra, it is necessary and sufficient that each element have one and only one complement.*

The second theorem follows from the first, and the known results stated in the introduction.

PROOF OF FIRST THEOREM. To each set S of points p_α, make correspond the join $x(S)$ of the p_α in S. Dually, associate with each set S of elements q_β covered by I, the meet $y(S)$ of the q_β in S. It will follow from generalized

[1] E. V. Huntington "Postulates for the algebra of logic," Trans. Am. Math. Soc. 5 (1904), pp. 288–309. His hypotheses (1) — (7) define lattices.

[2] See for example, A. N. Whitehead's "Universal Algebra" Cambridge (1898) p. 36. The result is due to R. Grassmann.

[3] "Zur Axiomatik der Elementargeometrie" Monatschr. f. Math. u. Phys. 36 (1929), pp. 269–84.

[4] The terminology of the present paper is that of G. Birkhoff's "Lattice theory and its applications" Bull. Am. Math. Soc. 44 (1938), pp. 793–800. By "a covers b," we mean that $a > b$ while $a > x > b$ has no solution.

associativity, that $x(S \cup T) = x(S) \cup x(T)$ and $y(S \cup T) = y(S) \cap y(T)$. Again, the complement of $x(I)$ can contain no point, since $x(I)$ contains every point;[5] hence it is O, and $x(I) = I$. Dually, $y(I) = O$.[5]

Again, given α, β, either $p_\alpha \leqq q_\beta$, or $p_\alpha \cap q_\beta = O$ and $p_\alpha \cup q_\beta = I$—that is, p_α and q_β are complementary. But not every q_β contains p_α, since $y(I) = O < p_\alpha$. Hence (by the existence of unique complements) a suitable subscript notation will make $p'_\alpha = q_\alpha$ and $p_\alpha \leqq q_\beta$ if $\alpha \neq \beta$.

This notation will further identify subsets of p_α with subsets of q_α, and, since every p_α $[\alpha \in S]$ is less than or equal to every q_β $[\beta \in S']$, it will make $x(S) \leqq y(S')$. (Here S' denotes the set complementary to S.) From this important inequality we infer that

$$x(S) \cup y(S) \geqq x(S) \cup x(S') = x(S \cup S') = I,$$

$$x(S) \cap y(S) \leqq y(S') \cap y(S) = y(S \cup S') = O.$$

Consequently $x(S)$ and $y(S)$ are complementary.

Also, $x(S) \cap x(S') \leqq y(S') \cap y(S) = y(S \cup S') = O$ and $x(S) \cup x(S') = x(S \cup S') = I$; hence $x(S)$ and $x(S')$ are complementary. But since $x(S)$ and $x(S')$ are complementary, $x(S)$ contains no p_α not in S, distinct sets S determine distinct $x(S)$, and the partially ordered system of the $x(S)$ is isomorphic with the algebra of all subsets of the p_α.

It remains to show that every member a of L is an $x(S)$. But denote by S the set of $p_\alpha \leqq a$. Evidently $x(S) \leqq a$; moreover $a \cap x(S')$ will by the last paragraph contain no points; hence $a \cap x(S') = O$. On the other hand, $a \cup x(S') \geqq x(S) \cup x(S') = I$; hence a is the unique complement $x(S)$ of $x(S')$, completing the proof.

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[5] We are letting I denote simultaneously: the biggest element in L, the set of all p_α, and the set of all q_β.