CERTAIN EXPANSIONS INVOLVING DOUBLY INFINITE SERIES.*

By Morgan Ward.

1. **Introduction.** The present paper extends the results of a previous article† on the same subject. If

$$P(x, y) = \sum_{r,s=0}^{\infty} P_{rs}\, x^r\, y^s, \qquad Q(x, y) = \sum_{r,s=0}^{\infty} Q_{rs}\, x^r\, y^s$$

are power series in x and y, I showed in R how to express the coefficients of P/Q, $\exp Q$, $\log Q$ as simple determinants in the coefficients P_{rs}, Q_{rs} and here I shall apply these results to the expansion of

$$[P_1(x, y)]^{m_1} \cdot [P_2(x, y)]^{m_2} \cdots [P_t(x, y)]^{m_t}$$

where $m_1, m_2, \cdots, m_t$ are any real numbers.

The analogous expansion problem for singly infinite series has been solved by Mangeot‡; David§ and Segar|| had previously dealt with the simple case of a single power series raised to an arbitrary power.

2. **Notation.** In this paper, if P, Q, P_i, etc. are power series in x, y the corresponding coefficients are denoted by P_{rs}, Q_{rs}, $P_{i,rs}$, etc. The Einstein summation convention is used, so that for instance, we shall write $Q = Q_{ij}\, x^i\, y^j$ instead of $\sum_{i,j=0}^{\infty} Q_{ij}\, x^i\, y^j$. The terms of a series Q will always be taken to be arranged in the usual numerical order; viz.

$$Q_{00} + Q_{10}\, x + Q_{01}\, y + Q_{20}\, x^2 + Q_{11}\, xy + Q_{02}\, y^2 + \cdots. \tag{1}$$

When the series is arranged in this manner, the term Q_{ij} occurs in the $(\frac{1}{2}(i+j)(i+j+1) + (j+1))$ place.¶ We shall call the number $\frac{1}{2}(i+j)(i+j+1) + (j+1)$ the *rank* of the coefficient Q_{ij} and denote it by q_{ij}.

3. **Evaluation of coefficients.** Suppose then that

$$P_\tau = P_\tau(x, y) = P_{\tau,rs}\, x^r\, y^s \qquad (\tau = 1, 2, \cdots, t)$$

$$W(x, y) = W_{ij}\, x^i\, y^j = P_1^{m_1} \cdot P_2^{m_2} \cdot \cdots P_t^{m_t}. \tag{2}$$

* Received October 4, 1927; in revised form, February 4, 1929.

† A Generalization of Recurrents. Bull. Amer. Math. Soc., vol. 33 (1927), pp. 477–492. I shall refer to this paper as R. A correction is given p. 580 in the second footnote (†).

‡ Annales de l'Ec. Norm., vol. 14 (1897), pp. 247–250.

§ Journ. de Math., vol. 8, ser. 3 (1882), pp. 61–72.

|| Messenger of Math., vol. 21, ser. 2 (1892), pp. 177–188. See vol. 4, chapter 8 of Muir's *History* for other references.

¶ R, section 2.

Our object is *to show how to express* W_{ij} *in terms of the coefficients of the P series.* We may assume first of all that

$$P_{\tau,00} \neq 0 \qquad (\tau = 1, 2, \cdots, t).$$

Let

(3) $$[P_\tau(x, y)]^{m_\tau} = \exp Q_\tau(x, y) \qquad (\tau = 1, 2, \cdots, t)$$

so that

(4) $$Q_\tau(x, y) = Q_{\tau,rs}\, x^r y^s = m_\tau \log P_\tau(x, y).$$

On substituting from (3) into (2) we obtain

(5) $$W(x, y) = \exp Q(x, y)$$

where

(6) $$\begin{aligned} Q(x, y) &= Q_1(x, y) + Q_2(x, y) + \cdots + Q_t(x, y), \\ Q_{rs} &= Q_{1,rs} + Q_{2,rs} + \cdots + Q_{t,rs}. \end{aligned}$$

Our next step is to express W_{ij} as a determinant* in the Q's; but we shall explain the formation of this determinant more fully than in R.

4. Fundamental determinants. Consider first of all the determinants

$$\Delta_1 = 1, \quad \Delta_2 = \begin{vmatrix} Q_{10} & -1 \\ Q_{01} & \cdot \end{vmatrix}_2, \quad \Delta_3 = \begin{vmatrix} Q_{10} & -1 & \cdot & \cdot & \cdot \\ Q_{01} & \cdot & -1 & \cdot & \cdot \\ 2\,Q_{20} & Q_{10} & \cdot & -2 & \cdot \\ 2\,Q_{11} & Q_{01} & Q_{10} & \cdot & -2 \\ 2\,Q_{02} & \cdot & Q_{01} & \cdot & \cdot \end{vmatrix}_5,$$

$$\Delta_4 = \begin{vmatrix} Q_{10} & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Q_{01} & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2\,Q_{20} & Q_{10} & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2\,Q_{11} & Q_{01} & Q_{10} & \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ 2\,Q_{02} & \cdot & Q_{01} & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ 3\,Q_{30} & 2\,Q_{20} & \cdot & Q_{10} & \cdot & \cdot & -3 & \cdot & \cdot \\ 3\,Q_{21} & 2\,Q_{11} & 2\,Q_{20} & Q_{01} & Q_{10} & \cdot & \cdot & -3 & \cdot \\ 3\,Q_{12} & 2\,Q_{02} & 2\,Q_{11} & \cdot & Q_{01} & Q_{10} & \cdot & \cdot & -3 \\ 3\,Q_{03} & \cdot & 2\,Q_{02} & \cdot & \cdot & Q_{01} & \cdot & \cdot & \cdot \end{vmatrix}_9$$

and so on.

The dots indicate zeros and the subscripts $2, 5, 9, \cdots$ the orders of $\Delta_2, \Delta_3, \Delta_4, \cdots$. In general, Δ_n is a determinant of order $n(n+1)/2-1$ whose mode of formation is fairly apparent from the examples just given.†

We next introduce a set of N numerical functions‡ somewhat analogous to Kronecker's δ_{ij} symbol:

* R, section 10, p. 489.

† R, pp. 490–91; 481–82.

‡ R, p. 480; p. 489.

$$(q_{t-ss};\, i+s-t,\, j-s) \quad \begin{pmatrix} t = 0, 1, \cdots, i+j-1 \\ s = 0, 1, \cdots, t \\ N = \frac{1}{2}(i+j)(i+j+1) = q_{i+j0}-1 \end{pmatrix}.$$

The relations defining these functions are as follows:

$$(q_{t-ss};\; i+s-t,\, j-s) = 0$$

if either $i+s-t<0$ or $j-s<0$ and

$$(q_{t-ss};\; i+s-t,\, j-s) = (i+j-t)\, Q_{i-t+s,\, j-s}$$

if neither $i+s-t<0$ nor $j-s<0$.

The general coefficient W_{ij} in (5) is then given by*

$$(7) \qquad \begin{aligned} &(i+j)\, e^{-Q_{00}}\, 1^2 . 2^2 . 3^4 \cdots \\ &\quad \cdots (i+j-1)^{i+j}\, W_{ij} = \end{aligned} \left| \begin{array}{c:c} & 0 \\ & 0 \\ & \cdot \\ & \cdot \\ \Delta_{i+j} & \cdot \\ & \cdot \\ \hdashline \cdots & -(i+j-1) \\ \multicolumn{2}{c}{\cdots\cdots\; (q_{t-ss};\; i+s-t,\, j-s)\; \cdots\cdots} \end{array} \right| .$$

The determinant on the right side of (7) consists of the determinant Δ_{i+j} bordered by

$$\begin{array}{cccc} (q_{00},\; i, j), & (q_{10};\; i-1, j), & (q_{01};\; i, j-1), & (q_{20};\; i-2, j), \\ (q_{11};\; i-1, j-1), & (q_{02};\; i, j-2), & \cdots, & \cdots, \\ \cdots, & \cdots, & \cdots, & \cdots, \\ \cdots, & \cdots, & (q_{t-s,s};\; i+s-t,\, j-s), & \cdots, \\ \cdots, & \cdots, & \cdots, & \cdots, \\ \cdots, & \cdots, & \cdots, & (q_{0\,i+j-1};\; i, 1-i) \end{array}$$

in the last row, and by

$$0, \;\; 0, \;\; 0, \qquad \cdots, \qquad -(i+j-1), \qquad (q_{0\,i+j-1};\; i, 1-i)$$

in the last column.†

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* R, p. 491.

† In R, p. 491, the factor $(i+j)\, e^{-Q_{00}}$ was omitted from the left-hand side of (7), and the last row of the determinant on the right-hand side of (7) was given incorrectly as $(q_{00};\, i, j)$, $(q_{10};\, i-1, j+1)$, $(q_{01};\, i-2, j+2)$, $\cdots$ etc. instead of the correct expressions above. Similar corrections should be made on p. 483 (interpreting the numerical functions appearing there as in (9) p. 480), and in the procedure sketched in section 9 R for $\log Q(x, y)$.

5. Illustrative example. To show the ease with which the general formula (7) may be applied, we shall evaluate the coefficient W_{21}. Here $i=2$, $j=1$, $i+j=3$, $N=\frac{3\cdot 4}{2}=6$, so that we must border the determinant Δ_3 with the row

$$(q_{00};2,1),\quad (q_{10};1,1),\quad (q_{01};2,0),\quad (q_{20};0,1),\quad (q_{11};1,0),\quad (q_{02};2,-1)$$

and with the column

$$0,\ 0,\ 0,\ 0,\ -2,\ (q_{02};2,-1).$$

That is, with the row

$$3\,Q_{21},\ 2\,Q_{11},\ 2\,Q_{20},\ Q_{01},\ Q_{10},\ 0$$

and the column

$$0,\ 0,\ 0,\ 0,\ -2,\ 0.$$

Thus*

$$3\,e^{-Q_{00}}\,1^2\cdot 2^3\,W_{21}=\begin{vmatrix} Q_{10} & -1 & 0 & 0 & 0 & 0\\ Q_{01} & 0 & -1 & 0 & 0 & 0\\ 2\,Q_{20} & Q_{10} & 0 & -2 & 0 & 0\\ 2\,Q_{11} & Q_{01} & Q_{10} & 0 & -2 & 0\\ 2\,Q_{02} & 0 & Q_{01} & 0 & 0 & -2\\ 3\,Q_{21} & 2\,Q_{11} & 2\,Q_{20} & Q_{01} & Q_{10} & 0 \end{vmatrix}_6$$

6. Final evaluation of coefficients. We have thus expressed the W_{ij} as determinants in the Q_{rs}. But we can express the Q_{rs} as sums of determinants in the $P_{\tau,rs}$; for since by (4) $Q_\tau(x,y)=m_\tau\log P_\tau(x,y)$ we can apply the method developed in R section 9 to expand a logarithm. We need only to substitute in R (17) p. 483 $m_\tau\log P_{\tau,00}=Q_{\tau,00}$ for Z_{00}; $\frac{i+j}{m_\tau}Q_{\tau,ij}$ for $Z_{ij}\,(i+j>0)$; $i+j\ P_{\tau,ij}$ for P_{ij}; and $P_{\tau,ij}$ for Q_{ij}. Since P_{00} vanishes, we obtain thus†

$$(8)\qquad P_{\tau,00}^{N-1}\,\frac{i+j}{m_\tau}\,Q_{\tau,ij}=\begin{vmatrix} P_{\tau,10} & P_{\tau,00} & 0 & \cdot & 0 & P_{\tau,10}\\ P_{\tau,01} & 0 & P_{\tau,00} & \cdot & 0 & P_{\tau,01}\\ P_{\tau,20} & P_{\tau,10} & 0 & \cdot & 0 & 2\,P_{\tau,20}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ P_{\tau,0\,i+j-1} & \cdot & \cdot & \cdot & \cdot & \cdot\\ P_{\tau ij}, & (2;i-1,j), & \cdots & \cdots & \cdots, & (i+j)\,P_{\tau,ij} \end{vmatrix},$$

* This result may be readily checked from the equations at the foot of p. 490 in R.

† We have applied the correction mentioned in the footnote to section 4 to the last row of the determinant.

where $N = \frac{1}{2}(i+j)(i+j+1)+1$ and $1 \leqq \tau \leqq t$. By combining (8), (6) and (7) we can finally express the coefficients of $W(x, y)$ in terms of the coefficients of $P_1(x, y), P_2(x, y), \cdots, P_t(x, y)$.

7. **A special case.** In case $t = 1$ we may proceed more simply. We have

$$W(x, y) = [P(x, y)]^m, \qquad P_{00} \neq 0. \tag{9}$$

Take the logarithms of both sides of (9) and then operate with

$$W(x, y)\, P(x, y) \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\}.$$

We obtain

$$P_{rs}\, x^r y^s (u+v) W_{uv}\, x^u y^v = m(r+s) P_{rs}\, x^r y^s W_{uv}\, x^u y^v$$

or transposing and equating the coefficient of $x^r y^q$ to zero,

$$\sum_{\tau=0}^{p} \sum_{\sigma=0}^{q} [m(r+q) - (m+1)(\sigma+\tau)] P_{r-\tau\, q-\sigma} W_{\sigma\tau} = 0 \quad (r, q, = 0, 1, 2, \cdots).$$

These equations may be identified with the equations (8) on p. 479 of R on taking in R

$$\begin{aligned} P_{uv} &= 0 \ (u \neq 0,\ v \neq 0), \qquad P_{00} = P_{00}^n, \\ Q_{u-\sigma, v-\tau} &= [m(r+q) - (m+1)(\sigma+\tau)] P_{p-\tau, q-\sigma} \ (u \neq 0,\ v \neq 0), \\ Q_{00} &= 1, \\ W_{\sigma\tau} &= Z_{\sigma\tau}. \end{aligned}$$

It follows that they may be solved in the same manner by introducing a properly defined numerical function.

8. **Conclusion.** The method I have applied here for obtaining simple determinants for the expansions of various elementary functions of doubly infinite series by the introduction of suitably defined numerical functions can theoretically be extended to m-tuply infinite series.* The numerical functions which must be introduced are unfortunately of such complexity* as to reduce the results to a mere jumble of symbols. A successful application of the method to the problem of reverting two doubly infinite series would be of more interest and of some practical importance. It would seem, however, to be rather difficult.†

* R, section 8, p. 487.

† I have treated the problem for singly infinite series in a note which is to appear shortly in the Rendiconti del Circolo di Palermo.

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