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THE CLOSURE OPERATORS OF A LATTICE

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I. INTRODUCTION

1. If $\mathfrak{S}$ is a lattice of elements $A, B, \cdots$, the class of all operators of $\mathfrak{S}$ (that is, one-valued functions $\phi X = \phi(X)$ on $\mathfrak{S}$ to $\mathfrak{S}$) may be made into a lattice by defining the union δ and cross-cut κ of any set Φ of operators ϕ by[1]

$$\delta X = (\cdots \phi X \cdots), \qquad \kappa X = [\cdots \phi X \cdots], \qquad \phi \in \Phi.$$

The union and cross-cut here are taken over all the values ϕX of the operators in Φ for any given X of $\mathfrak{S}$.

It is easily verified that the operators of $\mathfrak{S}$ form a lattice in which $\phi \supset \psi$ if and only if $\phi X \supset \psi X$ for every X of $\mathfrak{S}$; furthermore this lattice is closed, modular, or distributive according as $\mathfrak{S}$ is closed, modular or distributive.[2]

The operator lattice of a lattice is a concept comparable in generality to the Boolean algebra of all subsets of a lattice. As in the algebra, it is certain distinguished sets of operators which are useful in investigating the given lattice rather than the operator lattice itself.

One obviously important distinguished type is the linear operator. An operator ϕ is said to be linear if for any subset $\mathfrak{A}$ of elements A of $\mathfrak{S}$, it has one or more of the four properties

$$(1.1)\qquad \begin{array}{ll} \text{(i) } \phi(\cdots A \cdots) = (\cdots \phi A \cdots), & \text{(iii) } \phi[\cdots A \cdots] = [\cdots \phi A \cdots], \\ \text{(ii) } \phi(\cdots A \cdots) = [\cdots \phi A \cdots], & \text{(iv) } \phi[\cdots A \cdots] = (\cdots \phi A \cdots). \end{array}$$

Here the unions and cross-cuts are taken over all the elements of $\mathfrak{A}$, and $\mathfrak{A}$ is finite if $\mathfrak{S}$ is not closed. Lattice homomorphisms and homomorphisms with respect to union with properties (i), (iii) and (i) respectively are familiar examples. (Ore 1).

The linear operators and certain associated lattices are important in the study of residuated lattices (Ward-Dilworth 1) as I plan to show in detail elsewhere.[3]

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[1] If $\mathfrak{S}$ is not closed, Φ is assumed to contain only a finite number of operators. A lattice is said to be closed (or "complete" or "continuous") if it contains the union and cross-cut of any subset of elements in it.

[2] Chain conditions in $\mathfrak{S}$ do not usually carry over to the operator-lattice.

[3] The product $\phi\psi$ of two operators ϕ and ψ defined by $\phi\psi X = \phi(\psi(X))$ immediately gives us an associative multiplication over the operator lattice. On the other hand if B is any fixed element of a residuated lattice $\mathfrak{S}$, the operators μ and ρ defined by $\mu X = BX$, $\rho X = B:X$ have the linear properties $\mu(\cdots A \cdots) = (\cdots \mu A \cdots)$, $\rho(\cdots A \cdots) = [\cdots \rho A \cdots]$.

I have discussed elsewhere (Ward 2) a type of operator associated with a point lattice,[4] which may be used to classify all such lattices of finite order.

2. I develop here the properties of a type of operator which is of fundamental importance in the study of certain imbedding problems of ring theory and semi-group theory.[5] A typical problem of this class is to imbed a system I of elements over which a commutative and associative multiplication is defined in a residuated lattice $\mathfrak{S}$ so as to preserve the multiplication in I and thus to study the arithmetical properties of I. (Clifford 2, Ward-Dilworth 2). The imbedding is effected by defining a suitable type of "ideal" (distinguished subset) of I in the Boolean algebra of its subsets.[6] A closely related problem is to imbed a semi-ordered set in a closed lattice. (Mac Neille 1).

The "closure operators" introduced here enable us to view all these problems from a unified standpoint, and explain why in all extant theories of ideals as distinguished subsets, the cross-cut of two ideals is the set-theoretic cross-cut of their elements.

II. Closure Operators

3. Let $\mathfrak{S}$ be a closed lattice. An operator ϕ of $\mathfrak{S}$ is said to be a closure operator if it satisfies the following three conditions:[7]

I 1. *$A \supset B$ implies that $\phi A \supset \phi B$.*

I 2. $\phi \supset \iota$.

I 3. $\phi^2 = \phi$.

Here ι is the identity operator leaving every element of $\mathfrak{S}$ unchanged.

If $\mathfrak{T}$ is any set of elements T of $\mathfrak{S}$, it may be proved that every closure operator ϕ has the quasi-linear properties

$$\phi[\cdots \phi T \cdots] = [\cdots \phi T \cdots], \tag{3.1}$$

$$\phi(\cdots T \cdots) = \phi(\cdots \phi T \cdots), \qquad T \in \mathfrak{T}. \tag{3.2}$$

No actual linearity is assumed.

Theorem 3.1. *The cross-cut[8] of any set of closure operators is again a closure operator.*

[4] A lattice is called a point lattice if every element in it save the null element is a union of points. Here a point is any element covering the null element. Point lattices include important types of projective geometries, exchange lattices, and Boolean algebras.

[5] For a discussion of these problems, the reader is referred to Clifford 1, 2 where references are given to the work of Prüfer and others.

[6] Several definitions are usually possible. See Ward-Dilworth 2.

[7] These axioms are satisfied by Kuratowski's closure operator over a Boolean algebra with points. (Kuratowski 1). But they are essentially weaker, as Kuratowski's operator is linear with respect to union. Compare also Birkhoff 1.

[8] In general, no closure properties hold for the union and product of (closure) operators. It may be shown that if ϕ and ψ are operators, then (ϕ, ψ) is an operator if and only if $(\phi, \psi) = \phi\psi = \psi\phi$. Commutativity is thus a necessary condition for the union (ϕ, ψ) to be an operator. It is evidently a sufficient condition for the product $\phi\psi$ to be an operator.

PROOF. Let Φ be a set of closure operators ϕ, and let $\kappa = [\cdots \phi \cdots]$ be their cross-cut. We shall show that κ satisfies I 1, I 2, I 3.

I 1 *is satisfied.* For $A \supset B$ implies $\phi A \supset \phi B$ for every $\phi \in \Phi$. Hence $[\cdots \phi A \cdots] \supset [\cdots \phi B \cdots]$, $\kappa A \supset \kappa B$. I 2 *is satisfied.* For since $\phi A \supset A$ for every $\phi \in \Phi$, $[\cdots \phi A \cdots] \supset A$ or $\kappa \supset \iota$. I 3 *is satisfied.* For $\mathrm{K}^2 A = [\cdots \phi\kappa A \cdots]$, $\phi \in \Phi$. Now $\phi \supset \kappa$. Hence $\phi A \supset \kappa A$, $\phi^2 A \supset \phi\kappa A$, $\phi A \supset \phi\kappa A$. Accordingly $\kappa \supset \kappa^2$. By I 1 and I 2, $\kappa^2 \supset \kappa$. Hence $\kappa^2 = \kappa$, completing the proof.

4. Let ϕ be a given closure operator, and let $\mathfrak{S}' = \phi\mathfrak{S}$ be the set of all its values $X' = \phi X$ in $\mathfrak{S}$. By formula (3.1) any subset $\mathfrak{T}'$ of the X' is closed under cross-cut. We may express this fact by writing

$$(4.1) \qquad [\cdots T' \cdots]_{\mathfrak{S}'} = [\cdots T' \cdots]_{\mathfrak{S}}, \qquad T' \in \mathfrak{T}', \qquad \mathfrak{T}' \subsetneq \mathfrak{S}'.$$

If I is the unit element of $\mathfrak{S}$, then $I' = I$ divides all elements A' of $\mathfrak{S}'$. Hence for any subset $\mathfrak{L}'$ of elements L' of $\mathfrak{S}'$, the class $\mathfrak{K}'$ of all K' such that $K' \supset L'$ is non-empty. We define the union of the L' to be the cross-cut of the K':

$$(4.2) \qquad (\cdots L' \cdots)_{\mathfrak{S}'} = [\cdots K' \cdots]_{\mathfrak{S}'}, \quad L' \supset K' \quad \text{every} \quad L' \text{ of } \mathfrak{L}'.$$

We obtain by a familiar argument:

THEOREM 4.1. *The set $\mathfrak{S}'$ of values of a given closure operator forms a closed lattice within $\mathfrak{S}$ with respect to the operations of union and cross-cut defined by* (4.2) *and* (4.1).

To each closure operator ϕ we may accordingly assign a lattice $\mathfrak{S}' = \phi\mathfrak{S}$. In particular, $\mathfrak{S} = \iota\mathfrak{S}$. We shall establish a converse result.

Let $\mathfrak{S}'$ now denote a fixed subset of $\mathfrak{S}$ closed under cross-cut and containing the unit element I. We make $\mathfrak{S}'$ into a lattice within $\mathfrak{S}$ by assigning to any subset $\mathfrak{L}'$ of elements of $\mathfrak{S}'$ as in (4.2) a union defined as the cross-cut of the set of all multiples of the elements of $\mathfrak{L}'$.

We next define an operator ϕ on $\mathfrak{S}$ to $\mathfrak{S}'$ as follows: If A is any element of $\mathfrak{S}$, then ϕA is the cross-cut of all elements B' of $\mathfrak{S}'$ such that $B' \supset A$. Then ϕ is a closure operator, for I 1, I 2, I 3 are evidently satisfied. Furthermore, $\phi\mathfrak{S} = \mathfrak{S}'$.

We have thus established a one-to-one correspondence between the closure operators of $\mathfrak{S}$ and subsets of $\mathfrak{S}$ closed under cross-cut and containing I. The lattice $\mathfrak{S}' = \phi\mathfrak{S}$ and the operator ϕ will be said to belong to one another.

It is also easily proved from formula (3.2) that $\mathfrak{S}'$ is a sublattice of $\mathfrak{S}$ if and only if the closure operator belonging to $\mathfrak{S}'$ is linear with respect to union.

THEOREM 4.2. *The closure operators of any closed lattice themselves form a lattice within the operator lattice of $\mathfrak{S}$.*

PROOF. Let Σ denote the set of all closure operators of $\mathfrak{S}$. By formula (3.1), the cross-cut of any set of such operators is again a closure operator. Furthermore the operator ω defined by $\omega A = I$, every A of $\mathfrak{S}$, is obviously a closure operator dividing every other closure operator. Hence we may define the union of any set Φ of such operators as the cross-cut of the non-empty set of closure operators containing every operator of Φ.

We may evidently define lattice operations on the set of all subsets $\mathfrak{S}'$ of $\mathfrak{S}$ closed under cross-cut and containing I by the rules

$$(4.3)\qquad \begin{aligned} &[\cdots \mathfrak{S}' \cdots] = [\cdots \phi \cdots]\mathfrak{S}, \qquad \phi \in \Phi, \qquad \mathfrak{S}' = \phi\mathfrak{S} \subset \Phi\mathfrak{S} \\ &(\cdots \mathfrak{S}' \cdots) = (\cdots \phi \cdots)\mathfrak{S}. \end{aligned}$$

The lattices $\mathfrak{S}'$ thus form a lattice simply isomorphic with the lattice Σ of closure operators. We shall return to these operations at the close of the next section.

5. Consider an operator ϕ belonging to a set consisting of two elements I and T of $\mathfrak{S}$. It follows from the previous theorems that ϕ is characterized by

$$(5.1)\qquad \phi A = I \quad \text{if} \quad T \not\supset A, \quad \phi A = T \quad \text{if} \quad T \supset A, \quad A \text{ any element of } \mathfrak{S}.$$

We call ϕ the two-valued operator belonging to T. Since $\phi\mathfrak{S}$ is a sub-lattice of $\mathfrak{S}$, ϕ is linear with respect to union, as is directly evident from (5.1). It is easy to prove

THEOREM 5.1. *The ideal operator ϕ belonging to any set $\mathfrak{S}'$ of elements of $\mathfrak{S}$ which is closed under cross-cut and contains I is the operator cross-cut of all the two-valued operators belonging to elements of $\mathfrak{S}'$.*

If $\mathfrak{T}$ is any set of elements T of $\mathfrak{S}$ containing I, we obtain a lattice $\mathfrak{S}'$ within $\mathfrak{S}$ containing $\mathfrak{T}$ by adjoining to $\mathfrak{T}$ the cross-cuts of all sets of its elements. $\mathfrak{S}'$ is evidently the smallest such lattice containing $\mathfrak{T}$. We call $\mathfrak{S}'$ the imbedding lattice of $\mathfrak{T}$, and its corresponding closure operator the "imbedding operator" of $\mathfrak{T}$. We shall use the letter θ to denote an imbedding operator.

THEOREM 5.2. *If θ is the imbedding operator of a set $\mathfrak{T}$ of elements of $\mathfrak{S}$ containing I, then the value of θ for any element A of $\mathfrak{S}$ is given by the formula*

$$(5.2)\qquad \theta A = [\cdots T \cdots], \qquad T \supset A, \qquad T \in \mathfrak{T}.$$

PROOF. We have $\theta A = S'$ where S' lies in $\mathfrak{S}' = \theta\mathfrak{S}$. Hence S' is the cross-cut of a certain set of the T in $\mathfrak{T}$. Now since $\theta A \supset A$, every such T divides A. But since $\theta T = T$ if $T \in \mathfrak{T}$, $T \supset A$ implies that $T \supset \theta A = S'$. Hence (5.2) follows.

THEOREM 5.3. *Let ϕ and ψ be any two closure operators of $\mathfrak{S}$. Then $\phi \supset \psi$ if and only if the lattice belonging to ψ contains the lattice belonging to ϕ in the set-theoretic sense.*

PROOF. Assume that $\phi \supset \psi$ and let $A \in \phi\mathfrak{S}$. Then $\phi A = A$. By I 1, $\phi A \supset \psi A$. Hence $A \supset \psi A$. Therefore by I 2, $A = \psi A$ or $A \in \psi\mathfrak{S}$. Since ϕ and ψ are the imbedding operators of their respective lattices, the converse follows from Theorem 5.2.

The following corollaries are immediate:

COROLLARY 5.31. *Let θ be the imbedding operator belonging to any set $\mathfrak{T}$ of elements of $\mathfrak{S}$ containing I, and let ψ be any closure operator such that ψ leaves every element of $\mathfrak{T}$ invariant. Then ψ divides θ.*

COROLLARY 5.32. *The imbedding operator of any set is the union of all closure operators which leave every element of the set invariant.*

COROLLARY 5.33. *The union operation on the lattices which belong to closure operators defined by* (4.3) *is the operation of taking the set-theoretic cross-cut of their elements.*

It is this correspondence between operator union and set-theoretic cross-cut which makes the ideal operators of importance in imbedding problems.

III. Applications to Imbedding Problems

6. Let I be a set of elements $a, b, \cdots$ semi-ordered with respect to a division relation $x \mid y$ and containing a unit element ι dividing every other element. The following problem has been considered by Mac Neille: (Mac Neille 1). To construct a closed lattice $\mathfrak{S}'$ such that: (i) $\mathfrak{S}'$ contains a subset of elements $A', B', \cdots$ which may be set in a one-to-one correspondence $x \leftrightarrow X'$ with $a, b, \cdots$; (ii) If $a \leftrightarrow A'$ and $b \leftrightarrow B'$, then

$$(6.1) \qquad a \mid b \text{ in } I \quad \text{implies} \quad A' \supset B' \text{ in } \mathfrak{S}'.$$

$$(6.2) \qquad A' \supset B' \text{ in } \mathfrak{S}' \quad \text{implies} \quad a \mid b \text{ in } I.$$

We call such a construction an "isomorphic imbedding" of the set I. If we do not require (6.2), we speak of a "homomorphic imbedding" of I.

We shall solve these problems by determining suitable ideal operators in the lattice $\mathfrak{B}$ (Boolean algebra) of all subsets of I. In other words, we shall determine all ideal operators ϕ of $\mathfrak{B}$ such that $\mathfrak{S}' = \phi\mathfrak{B}$ will be a suitable lattice.

Consider first the condition (6.1). Let $T = (t)$ be a subset of I consisting of the single element t. Since $T' = \phi T \supset T$, we must have $t \in \phi T$. But by (6.1), if $t \mid y$ in I, $\phi T \supset \phi(y)$. Hence if $t \mid y$, $y \in \phi T$. Thus (6.1) implies that ϕT *must contain all elements y of I such that $t \mid y$.*

For a homomorphic imbedding, no further conditions are imposed on the values of ϕT. But if the imbedding is isomorphic and $\phi T \supset \phi(X)$, then (6.2) requires that $t \mid x$. Hence ϕT *must consist only of elements x of I such that $t \mid x$.*

We let $\mathfrak{T}$ denote the set of all $T' = \phi(t)$, $t \in I$ for any ideal operator ϕ. We call the elements of $\mathfrak{T}$ the principal ideals of I.

It is evident from the preceding section that any ideal operator of $\mathfrak{B}$ leaving every element of $\mathfrak{T}$ invariant will solve our initial imbedding problem, and that the simplest of these operators is the imbedding operator of the set $\mathfrak{T}$ itself; for its lattice $\theta\mathfrak{B}$ is the smallest lattice in the set-theoretic sense in which the imbedding can be made in $\mathfrak{B}$. The isomorphism between I and $\mathfrak{T}$ with respect to division shows that this same minimal property of $\theta\mathfrak{B}$ will apply to any isomorphic imbedding of I in any closed lattice $\mathfrak{S}'$ whatever; within the lattice $\mathfrak{S}'$ there must lie a lattice simply isomorphic to $\theta\mathfrak{B}$. $\theta\mathfrak{B}$ is the lattice defined in Mac Neille 1 by "Dedekind cuts."

A similar situation occurs for homomorphic imbeddings. For a homomorphic imbedding, the "principal ideals" $A', B', \cdots$ which make up the set $\mathfrak{T}$ are not uniquely determined by the corresponding elements $a, b, \cdots$ of I; for if $a \leftrightarrow A'$, A' may contain elements of I not divisible by a. But once the set $\mathfrak{I}$ of principal ideals is chosen, the imbedding operator of I gives the smallest lattice in which the particular homomorphic imbedding can be performed.

7. If A is any subset of I, let A be the subset of all elements l such that $l \mid k$ for every k in A, and let A' be the subset of all elements a such that $l \mid a$ for every l in A. Then the operator

$$A' = \theta A \tag{7.1}$$

is the isomorphic imbedding operator of the set I discussed above. This result follows easily from Theorem 5.3. For a detailed discussion, the reader may consult Ward-Dilworth 2 or Clifford 1, to whom this definition of θ is originally due.[9]

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[9] The identity of this operator and Mac Neille's operator was pointed out to me by Dr. A. H. Clifford in a letter. The definition (7.1) is used in Ward-Dilworth 2 to imbed any ovum (semi-group) in a residuated lattice of ideals.