THE REPRESENTATION OF STIRLING'S NUMBERS AND STIRLING'S POLYNOMIALS AS SUMS OF FACTORIALS.

By Morgan Ward.

1. I give here a new representation of the Stirling numbers and the associated Stirling polynomials * as sums of factorials, and use the formulas to deduce various arithmetical and algebraic properties of the numbers. My fundamental formula for the Stirling polynomial † $\psi_{p-1}(x)$ reads as follows:

$$(3.31)\quad \psi_{p-1}(x) = \frac{(-1)^{p-1}}{(p+1)!}\Big[H_p^{p-1} - \frac{x+2}{p+2}H_p^{p-2} + \frac{(x+2)(x+3)}{(p+2)(p+3)}H_p^{p-3} - \cdots + (-1)^{p-1}\frac{(x+2)(x+3)\cdots(x+p)}{(p+2)(p+3)\cdots\ 2p}H_p^0\Big].$$

The constants H_p^r appearing here are positive integers defined recursively by

$$(4.1)\qquad H^r_{p+1} = (2p+1-r)H_p^r + (p-r+1)H_p^{r-1},$$

with the initial values

$$(4.11)\qquad H_0^0 = 1,\quad H^0_{p+1} = 1\cdot 3\cdot 5\cdots(2p+1),\quad H^p_{p+1} = 1.$$

Nielsen ‡ has expressed the Stirling polynomial $\psi_{p-1}(x)$ in the form

$$\psi_{p-1}(x) = \sigma_{p-1,0}x^{p-1} + \sigma_{p-1,1}x^{p-2} + \cdots + \sigma_{p-1,p-2}x + \sigma_{p-1,p-1}.$$

Unfortunately, the numbers $\sigma_{p,r}$ are not integers, and the recursion formulas for them are very complicated, so that it is difficult both to ascertain their form,§ and to obtain properties of the Stirling polynomial from such a representation. In contrast, the numbers H_p^r are integers of comparatively simple

* We use here freely the notation and formulas for the Stirling numbers given by Nielsen in his well known *Handbuch der Theorie der Gammafunction* (Leipzig, 1906), Chapter V. We shall refer to this source as Nielsen, *Handbuch*, giving page reference. A recent paper by C. Tweedie, *Proceedings of the Edinburgh Mathematical Society*, vol. 37 (1918), pp. 2-25, contains many interesting new results on these numbers. Since this paper was in press, C. Jordan (*Tohoku Journal*, vol. 37 (1933), pp. 254-278) has given an expression for the Stirling number as a sum of factorials. See especially pp. 264-265 of his paper, where his numbers $\bar{C}_{mi}$ are my $H_m{}^i$.

† Nielsen, *Handbuch*, p. 72.

‡ *Handbuch*, pp. 72-73. See also *Annali di Matematica, III*, vol. 10, pp. 309-316.

§ Nielsen, *Annali di Matematica, III*, vol. 10, p. 313; Tweedie, paper cited, Section 11.

form, while (3. 31) leads directly to interesting congruential properties of the Stirling polynomials and Stirling numbers.

To give an example of such congruences, let P be any prime greater than $2p$, and r any positive integer. Then if C_n^p denotes the Stirling number,

$$C^p_{n+1} \equiv 1 \pmod{P^r} \qquad \text{if } n+2 \equiv 0 \pmod{P^r},$$
$$C^p_{n+1} \equiv 2^{p+1} - 1 \pmod{P^r} \qquad \text{if } n+3 \equiv 0 \pmod{P^r}.$$

As a numerical illustration, take $p=3$, $P=7$, $r=1$. Then from Glaisher's table * of C_n^p, $C_5^3 = 225$, $C_4^3 = 50$, and

$$225 \equiv 1 \pmod 7, \qquad 50 \equiv 2^4 - 1 \pmod 7.$$

2. We begin with the Stirling numbers of the first kind defined by

$$x(x+1)\cdots(x+n-1) = C_n^0 x^n + C_n^1 x^{n-1} + \cdots + C_n^p x^{n-p} + \cdots + C_n^{n-1} x.$$

We call n the rank of C_n^p and p its order. We have the immediate relations

$$(2.1) \qquad C^p_{n+1} = C_n^p + nC_n^{p-1},$$

$$(2.2) \qquad C_n^0 = 1, \quad C_n^{n-1} = (n-1)\,!, \quad (n = 1, 2, \cdots;\ p = 0, 1, \cdots, n-1).$$

We now define C_n^p for all integral values of n and p, positive or negative, by the recursion formula (2. 1) with the initial values (2. 2). Then it is readily shown that

$$(2.3) \qquad C_n^{n+r-1} = 0, \qquad (n = 0, 1, \cdots;\ r = 1, 2, \cdots).$$

Furthermore, if

$$F_p(z) = \sum_{n=0}^{\infty} C_n^p z^n$$

is the generating function for the Stirling numbers

$$C_0^p,\ C_1^p,\ C_2^p, \cdots$$

of fixed order p, then an easy induction shows us that

$$(2.4) \qquad F_p(z) = [z^{p+1}/(1-z)^{2p+1}]H_p(z), \qquad (p = 0, 1, 2, \cdots)$$

where $H_p(z)$ is a polynomial in z of degree $p-1$ with positive integral coefficients, and, by convention, we take

$$(2.41) \qquad H_0(z) = 1.$$

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* *Quarterly Journal*, vol. 31 (1900), pp. 26-28. This Table extends as far as $n = 20$. C_n^p is denoted in Glaisher's notation by $S_p(1, 2, \cdots, n-1)$.

The polynomials $H_p(z)$ appearing in (2.4) satisfy the recursion relation

$$(2.5) \qquad H_{p+1}(z) = (pz + p + 1) H_p(z) + (1 - z) z (d/dz) H_p(z)$$

which with (2.41) determines them completely.

3. We next put the polynomial $H_p(z)$ in the form

$$(3.1) \qquad H_p(z) = H_p{}^0 - H_p{}^1(1-z) + H_p{}^2(1-z)^2 - \cdots + (-1)^{p-1} H_p{}^{p-1}(1-z)^{p-1}.$$

Before studying the constants $H_p{}^r$, we shall deduce our main formulas. On substituting (3.1) into (2.4) and then expanding in ascending powers of z, we find that

$$F_p(z) = z^{p+1} \sum_{r=0}^{p-1} \sum_{s=0}^{\infty} (-1)^r H_p{}^r \frac{(s+1)(s+2)\cdots(s+2p-r)}{1 \cdot 2 \cdot 3 \cdots (2p-r)} z^s.$$

Therefore by comparing the coefficient of z^n on both sides of this expression, we find that

$$C_n{}^p = \sum_{r=0}^{p-1} (-1)^r H_p{}^r (n-p)(n+1-p) \cdots (n+p-r-1)/(2p-r)\,!\,.$$

On replacing n by $n+1$ in this expression and removing the common factor $(-1)^{p-1}(n+1)n \cdots (n+1-p)/(p+1)\,!$ from the right side, we obtain finally the formula

$$(3.2) \quad C^p{}_{n+1} = \frac{(n+1)\,!\,(-1)^{p-1}}{(n-p)\,!\,(p+1)\,!} \times \Big[H_p{}^{p-1} - \frac{n+2}{p+2} H_p{}^{p-2} + \frac{(n+2)(n+3)}{(p+2)(p+3)} H_p{}^{p-3} - \cdots + (-1)^{p-1} \frac{(n+2)(n+3)\cdots(n+p)}{(p+2)(p+3)\cdots(2p)} H_p{}^0 \Big].$$

Now *

$$C^p{}_{n+1} = [(n+1)\,!/(n-p)\,!]\, \psi_{p-1}(n)$$

where $\psi_{p-1}(x)$ is the Stirling polynomial of order $p-1$. Hence

$$\psi_{p-1}(n) = \frac{(-1)^{p-1}}{(p+1)\,!} \Big[H_p{}^{p-1} - \cdots + (-1)^{p-1} \frac{(n+2)\cdots(n+p)}{(p+2)\cdots 2p} H_p{}^0 \Big].$$

Since this formula holds for all positive integral values of n, we deduce that

* Nielsen, *Handbuch*, p. 14, formula (15).

$$(3.21)\quad \psi_{p-1}(x) = \frac{(-1)^{p-1}}{(p+1)!}\Big[H_p^{p-1} - \frac{x+2}{p+2}H_p^{p-2} + \frac{(x+2)(x+3)}{(p+2)(p+3)}H_p^{p-3} - \cdots + (-1)^{p-1}\frac{(x+2)(x+3)\cdots(x+p)}{(p+2)(p+3)\cdots 2p}H_p^0\Big]$$

for all values of the variable x.

We may use this result to obtain a formula similar to (3.2) for the Stirling numbers $\mathfrak{C}_n^s$ of the second kind * defined by the expansion

$$1/x(x+1)\cdots(x+n-1) = \sum_{s=0}^{\infty}(-1)^s\,\mathfrak{C}_n^s/x^{n+s}.$$

For †

$$\mathfrak{C}_n^p = [(-1)^{p-1}(n+p-1)/(n-2)!]\,\psi_{p-1}(-n),$$

so that by (3.21),

$$(3.3)\quad \mathfrak{C}_n^p = \frac{(n+p-1)!}{(n-2)!\,(p+1)!}\Big[H_p^{p-1} + \frac{n-2}{p+2}H_p^{p-2} + \frac{(n-2)(n-3)}{(p+2)(p+3)}H_p^{p-3} + \cdots + \frac{(n-2)(n-3)\cdots(n-p)}{(p+2)(p+3)\cdots 2p}H_p^0\Big].$$

These formulas have immediate arithmetical consequences. For suppose that P denotes a fixed prime greater than $2p$, and r any positive integer. Then we deduce from (3.21) that

$$\psi_{p-1}(n) \equiv [(-1)^{p-1}/(p+1)!]H_p^{p-1} \pmod{P^r} \quad \text{if } n+2 \equiv 0 \pmod{P^r},$$
$$\psi_{p-1}(n) \equiv [(-1)^{p-1}/(p+1)!]\{H_p^{p-1} + [1/(p+2)]H_p^{p-2}\} \pmod{P^r} \quad \text{if } n+3 \equiv 0 \pmod{P^r},$$

and so on. There are analogous congruences for the Stirling numbers deducible from (3.2) and (3.3); namely,

$$(3.4)\quad \begin{aligned} C^p_{n+1} &\equiv H_p^{p-1} \pmod{P^r} && \text{if } n+2 \equiv 0 \pmod{P^r},\\ C^p_{n+1} &\equiv (p+2)H_p^{p-1} + H_p^{p-2} \pmod{P^r} && \text{if } n+3 \equiv 0 \pmod{P^r},\\ \mathfrak{C}_n^p &\equiv H_p^{p-1} \pmod{P^r} && \text{if } n-2 \equiv 0 \pmod{P^r},\\ \mathfrak{C}_n^p &\equiv (p+2)H_p^{p-1} + H_p^{p-2} \pmod{P^r} && \text{if } n-3 \equiv 0 \pmod{P^r}, \end{aligned}$$

and so on.

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* Nielsen, *Handbuch*, p. 68.

† Nielsen, *Handbuch*, p. 74.

We may note in passing an interesting consequence of the form of the generating function $F_p(z)$ given in (2.4). For if

$$\Delta C_n^p = C_n^p - C^p_{n-1}\ \Sigma C_n^p = C_0^p + C_1^p + \cdots + C_n^p$$

denote the usual operations of the calculus of finite differences applied to the rank of the Stirling number C_n^p, the generating functions of the numbers ΔC_n^p and ΣC_n^p are $(1-z)F_p(z)$ and $(1-z)^{-1}F_p(z)$ respectively. But with $H_p(z)$ in the form (3.1), each of these functions may be immediately expanded in ascending powers of z. We obtain in this manner the formulas

$$\Delta C^p_{n+1} = (-1)^{p-1}\binom{n}{p} \times \left[H_p^{p-1} - \frac{n+1}{p+1} H_p^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_p^{p-3} - \cdots\right], \tag{3.5}$$

$$\Sigma C^p_{n+1} = (-1)^{p-1}\binom{n+2}{p+2} \times \left[H_p^{p-1} - \frac{n+3}{p+3} H_p^{p-2} + \frac{(n+3)(n+4)}{(p+3)(p+4)} H_p^{p-3} - \cdots\right],$$

and it is easy to write down analogous formulas for the higher differences and summations of C^p_{n+1}. The method by which we obtained the congruences (3.4) yields then an unlimited number of congruences involving sums and differences of Stirling numbers of the same order.

If we compare (3.5) (i) with (2.1), we see that

$$nC_n^{p-1} = (-1)^{p-1}\binom{n}{p} \times \left[H_p^{p-1} - \frac{n+1}{p+1} H_p^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_p^{p-3} - \cdots\right].$$

On the other hand, if we put $n = n-1$, $p = p-1$ in (3.2), we find that

$$C_n^{p-1} = (-1)^{p-2}\binom{n}{p} \times \left[H_{p-1}^{p-2} - \frac{n+1}{p+1} H_{p-1}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_{p-1}^{p-3} - \cdots\right].$$

Therefore, for all integral values of n, we have the fundamental formula

$$\begin{aligned}(3.6)\quad &\left[H_p^{p-1} - \frac{n+1}{p+1} H_p^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_p^{p-3} - \cdots + (-1)^{p-1}\frac{(n+1)(n+2)\cdots(n+p-2)}{(p+1)(p+2)\cdots(2p-2)} H_p^{p-1}\right] = -n \\ &\left[H_{p-1}^{p-2} - \frac{n+1}{p+1} H_{p-1}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_{p-1}^{p-3} - \cdots + (-1)^{p-2}\frac{(n+1)(n+2)\cdots(n+p-3)}{(p+1)(p+2)\cdots(2p-3)} H_{p-1}^{p-2}\right].\end{aligned}$$

We may if we please replace n here by a continuous variable x as in formula (3.21).

In particular, if we let $n = p$, we have

$$H_p^{p-1} - H_p^{p-2} + H_p^{p-3} - \cdots = -(H_{p-1}^{p-2} - H_{p-1}^{p-3} + H_{p-2}^{p-3} - \cdots).$$

From this result and the fact that $H_1^0 = 1$, we deduce that

$$H_p^0 - H_p^1 + H_p^2 - \cdots + (-1)^{p-1} H_p^{p-1} = p\,!, \tag{3.7}$$

a formula which affords a convenient check when computing the numerical values of the integers H_p^r.

4. We now proceed with the study of the numbers H_p^r. If we assume that (3.1) holds for all positive integral values of p, we obtain by substituting in (2.5) and comparing the coefficients of the various powers of $1 - z$, the recursion relations

$$\begin{gathered} H^0{}_{p+1} = (2p+1) H_p^0, \quad H^p{}_{p+1} = H_p^{p-1} \quad \text{and} \\ H^r{}_{p+1} = (2p+1-r) H_p^r + (p-r+1) H_p^{r-1}. \end{gathered} \tag{4.1}$$

Since $H_0^0 = 1$, we deduce from the first two relations that

$$H^0{}_{p+1} = 1 \cdot 3 \cdot 5 \cdots 2p+1, \quad H^p{}_{p+1} = 1, \qquad (p = 0, 1, \cdots). \tag{4.12}$$

The first few numbers H_p^r are given in the following table: *

p	$r = 0$	1	2	3	4	5	6	7	8	9
1	1									
2	3	1								
3	15	10	1							
4	105	105	25	1						
5	945	1260	490	56	1					
6	10395	17325	9450	1918	119	1				
7	135135	270270	190575	56980	6825	246	1			
8	2027025	4729725	4099095	1636635	302995	22935	501	1		
9	34459425	91891800	94594500	47507460	12122110	1487200	74316	1012	1	
10	654729075	1964187225	2343240900	1422280860	466876410	81431350	6914908	235092	2035	1

Here the number in the p-th row and r-th column is H_p^r; thus $H_4^2 = 25$.

We next extend the definition of H_p^r to all integral values of p and r by (4.1) and (4.12). By a brief induction, we find that

$$H_p^{-r} = 0 \quad (r \geqq 1;\ p = 0, 1, 2, \cdots) \tag{4.13}$$

$$H_p^{p+r} = 0 \quad (r = 0,\ p = 1, 2, 3, \cdots;\ r \geqq 1,\ p = r, r+1, \cdots). \tag{4.14}$$

* The table has been checked by the use of formula (3.7).

Now replace r by $p-r$ in (4.1). We obtain

$$(4.2) \qquad H_{p+1}^{p-r}=(p+r+1)H_p^{p-r}+(r+1)H_p^{p-r-1}.$$

Let the generating function of the numbers

$$H_0^{-r}, H_1^{1-r}, H_2^{2-r}, \cdots, H_p^{p-r}, \cdots$$

be denoted by $\mathcal{H}_r(x)$, so that

$$(4.3) \qquad \mathcal{H}_r(x)=\sum_{p=0}^{\infty} H_p^{p-r}x^p=\sum_{p=r}^{\infty} H_p^{p-r}x^p$$

since by (4.13), $H_0^{-r}=H_1^{1-r}=\cdots=H_{r-1}^{-1}=0$.

On replacing r by $r+1$ in (4.3), changing the summation variable from p to $p+1$, and reducing by (4.2), we obtain the formula

$$(4.4) \qquad (1-(r+1)x)\mathcal{H}_{r+1}(x)=(r+1)x\mathcal{H}_r(x)+x^2(d/dx)\mathcal{H}_r(x), \quad (r=0,1,2,\cdots).$$

Since by (4.14), $H_p^p=0$ and $H_0^0=1$, we have

$$(4.41) \qquad \mathcal{H}_0(x)=1.$$

These two formulas serve then to define the functions $\mathcal{H}_r(x)$ completely, and $\mathcal{H}_r(x)$ is seen to be a rational function of x. It is easy to determine its form. For by direct calculation from (4.4), we find that

$$\mathcal{H}_1(x)=\frac{x}{1-x}, \quad \mathcal{H}_2(x)=\frac{x^2[3-2x]}{(1-x)^2(1-2x)},$$

$$(4.5) \qquad \mathcal{H}_3(x)=\frac{x^3[15-45x+40x^2-12x^3]}{(1-x)^3(1-2x)^2(1-3x)},$$

$$\mathcal{H}_4(x)=\frac{x^4[105-840x+2625x^2-4130x^3+3500x^4-1560x^5+288x^6]}{(1-x)^4(1-2x)^3(1-3x)^2(1-4x)}.$$

We are therefore led to infer that the generating function $\mathcal{H}_r(x)$ is of the form

$$(4.51) \qquad \mathcal{H}_r(x)=x^r\Phi_r(x)/(1-x)^r(1-2x)^{r-1}\cdots(1-rx)$$

where $\Phi_r(x)$ is a polynomial in x with integral coefficients of degree $r(r-1)/2$. The proof is a straightforward induction from (4.41) and (4.4) and will be omitted here.*

If we put the right-hand side of (4.51) into partial fractions, we see that $\mathcal{H}_r(x)$ may be written as

* The relationship between $\Phi_{r+1}(x)$ and $\Phi_r(x)$ deduced in the course of the induction is unfortunately too complicated to be of much service.

$$\mathcal{H}_r(x) = A_0 + \frac{B_1}{1-rx} + \frac{C_1}{1-(r-1)x} + \frac{C_2}{(1-(r-1)x)^2} + \cdots + \frac{U_1}{1-x} + \cdots + \frac{U_r}{(1-x)^r},$$

where the numbers $A_0, \cdots, U_r$ are all rational. If we now expand the right-hand side of this expression in ascending powers of x and collect the coefficient of x^p, we find that $H_p{}^{p-r}$ is of the form

$$H_p{}^{p-r} = b_0 r^p + (c_0 + c_1 p)(r-1)^p + \cdots + (u_0 + u_1 p + u_2 p^2 + \cdots + u_{r-1} p^{r-1})$$

where the numbers $b_0, \cdots, u_{r-1}$ are again all rational. We can however assert more than this. For if we apply the process just described to the expressions in (4.5), we find that *

$$\begin{aligned} &H_p{}^{p-1} = 1, \quad H_p{}^{p-2} = [2^{p+1} - (p+3)], \\ (4.6) \quad &H_p{}^{p-3} = \tfrac{1}{2}[3^{p+2} - (2p+10)2^{p+1} + (p^2+7p+13)], \\ &H_p{}^{p-4} = \tfrac{1}{6}[4^{p+3} - (3p+21)3^{p+2} \\ &\qquad + (3p^2+33p+96)2^{p+1} - (p^3+12p^2+50p+73)]. \end{aligned}$$

We infer therefore that $H_p{}^{p-r}$ is actually of the form

$$(4.61) \qquad H_p{}^{p-r} = [1/(r-1)\,!\,] \sum_{l=0}^{r-1} (-1)^l \theta_l(p)(r-l)^{p+r-1-l}$$

where $\theta_l(p)$ is a polynomial in p of degree l with positive integral coefficients, and $\theta_0(p) = 1$. I cannot however prove this statement.†

5. We conclude by giving a method for calculating the polynomials $\theta_l(p)$ in (4.61) recursively. We begin by assuming that

$$(5.1) \qquad H_p{}^{p-(r+1)} = (1/r\,!\,) \sum_{l=0}^{r} (-1)^l \Theta_l(p)(r+1-l)^{p+r-l},$$

where $\Theta_l(p)$ is a polynomial of the same form as $\theta_l(p)$. On setting $p = p+1$ in (5.1), we find that

$$H_{p+1}^{\,p-r} = (1/r\,!\,) \sum_{l=0}^{r} (-1)^l \Theta_l(p+1)(r+1-l)^{p+r-1-l}.$$

* All of these formulas have been checked numerically, and are believed to be correct. The two congruences mentioned in the introduction are obtained by substituting for $H_p{}^{p-1}$ and $H_p{}^{p-2}$ from (4.6) into (3.4).

† As additional support for it, I have found that
$$H_p{}^{p-5} = (1/24)[5^{p+4} - (4p+36)4^{p+3} + (6p^2+90p+354)3^{p+1} - (4p^3+72p^2+452p+992)2^p + (p^4+18p^3+125p^2+400p+501)].$$

If we now substitute these expressions for H_{p+1}^{p-r}, H_p^{p-r}, $H_p^{p-(r+1)}$ into (4.2) and express the fact that the resulting expression must be an identity in p, we obtain the formula

$$(5.2) \qquad l\Theta_l(p+1)-(r+1)(\Theta_l(p+1)-\Theta_l(p)) \\ = r(p+r+1)\theta_{l-1}(p), \qquad (r \geqq l \geqq 1),$$

which determines $\Theta_l(p)$ if $\theta_{l-1}(p)$ is known.

If we attempt to determine $\Theta_l(p)$ by writing it as a polynomial in p with undetermined coefficients, we find that we can express the coefficients only as determinants in the coefficients of $\theta_{l-1}(p)$. We therefore assume instead that $\theta_{l-1}(p)$ and $\Theta_l(p)$ are expressed as sums of factorials:

$$(5.3) \quad \begin{aligned} \theta_{l-1}(p) &= y_0+y_1p+y_2p(p+1)+\cdots+y_{l-1}p(p+1)\cdots(p+l-2), \\ \Theta_l(p) &= x_0+x_1p+x_2p(p+1)+\cdots+\ \ x_lp(p+1)\cdots(p+l-1), \end{aligned}$$

and seek to determine the x in terms of the y. Needless to say, the x and y are all integers, when and only when all the coefficients in the ordinary polynomial expressions for $\theta_{l-1}(p)$ and $\Theta_l(p)$ are integers.

If for convenience we set

$$x_{l+1}=y_{l+1}=y_l=y_{-1}=0,$$

we obtain on substituting from (5.3) into (5.2) the difference relation

$$(5.4) \quad lx_s-(s+1)(r+1)x_{s+1}=r(y_{s-1}+(r-2s)y_s-(s+1)(r-s)y_{s+1}), \\ (s=0,1,\cdots,l).$$

As a numerical verification of this formula, take $r=3$ and $l=2$ so that we have to do with H_p^{p-4} and H_p^{p-3}. From the formulas (4.6), we have $\Theta_2(p)=3p^2+33p+96$, $\theta_1(p)=2p+10$, so that $x_0=96$, $x_1=30$, $x_2=3$, $y_0=10$, $y_1=2$. The formula (5.6) with $s=0,1,2$ then gives

$$2x_0-4x_1=3(3y_0-3y_1); \quad 2x_1-8x_2=3(y_0+y_1); \quad 2x_2=3y_1;$$

or $$192-120=3(30-6); \quad 60-24=3(10+2); \quad 6=6.$$

Since (5.4) is effectively a linear difference equation of the first order for x_s, the explicit form of x_s may be written down, but the general result is too complicated to be of interest.