[1] Kasner, "Conformal Classification of Analytic Arcs or Elements, Poincaré's Local Problem of Conformal Geometry," *Trans. Am. Math. Soc.*, **16**, 333–349 (1915). The theory of a pair of regular arcs, including the horn angle, is given in "Conformal Geometry," *Proceedings* Cambridge International Congress Mathematicians, 1912, and a paper appearing in *Scripta Mathematica*, 1945.

[2] Kasner and De Cicco, "The General Invariant Theory of Irregular Analytic Arcs or Elements," *Ibid.*, **51**, 232-254 (1942). Also in Publications of the Illinois Institute of Technology (1943.)

[3] See for regular curves Halhhen's dissertation 1878, and his collected works Vol. 2. Also Lane, "Projective Differential Geometry," Chicago Press, 1942.

EULER'S THREE BIQUADRATE PROBLEM

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1. Euler's problem of whether the sum of three biquadrates can be a biquadrate; that is, whether the diophantine equation

$$x^4 + y^4 + z^4 = w^4 \qquad (1)$$

has any (non-trivial) integer solutions, has never been solved.[1] The problem is a hard one; indeed, a modern investigator has stated: ". . . it would be difficult to mention any other [problem] which has yielded so little to the efforts of those who have attempted its solution."

The most that is known to date is that there is no solution of (1) with[3] $w < 1024$. I have recently proved that *there is no solution of* (1) *with*

$$w < 10{,}000. \qquad (2)$$

This result makes it appear probable that there are no solutions of (1) whatever, especially since several closely allied soluble diophantine equations such as $x^4 + y^4 + z^4 + t^4 = w^4$, $x^4 + y^4 = w^4 + t^4$, $x^4 + 2y^4 + 2z^4 = w^4$ are known to have comparatively small solutions.[4]

2. The first step of the proof is to reduce the solution of (1) to the solution of another diophantine equation containing more variables but with the variables subjected to a number of restrictive conditions which it is unnecessary to state here:

$$u^4 + v^4 = 2\epsilon kl(e^8l^2 + 2^{18+8\sigma+2\epsilon}d^8k^2). \qquad (3)$$

The old variables are easily expressed in terms of the new; for example,

$$w = 2^{4\sigma+9+\epsilon}d^4k + e^4l. \qquad (4)$$

Equation (1) has a solution if and only if equation (3) has a solution with d, e, k, l, u and v positive integers. The exponent σ is a positive integer or zero, and the exponent ϵ is either zero or one.

The inequality (2) in conjunction with (4) immediately restricts σ, d, e, k and l to a finite number of choices; in fact,

$$\sigma \leqq 1, d \leqq 1, e \leqq 9, k \leqq 17 \text{ and } l \leqq 9488. \tag{5}$$

3. The second step of the proof is to discuss (3) for each of the cases given by (5). The most difficult case turns out to be when $\sigma = 0$, $\epsilon = 1$ and $d = e = k = 1$. (3) then becomes

$$u^4 + v^4 = 2l(l^2 + 1024^2) \tag{6}$$

with

(i) $l < 8976$.

The restrictions on the variables in (3) alluded to in Section 2 tell us that

(ii) Every prime factor of l and $l^2 + 1024^2$ is congruent to one modulo eight.

On considering (6) modulo 5 and modulo 13, we find that

(iii) $l \equiv 4 \pmod 5$,
(iv) $l \equiv 3, 4, 5, 7, 10$ or $12 \pmod{13}$.

The conditions (i)–(iv) reduce the possible choices of l to twenty-nine numbers: 289, 449, . . ., 8689.

The other cases lead to even fewer choices of l and the other variables in (5).

4. The third step of the proof is to dispose of the cases which survive after all conditions of the type (i)–(iv) just described have been applied. For example, in the case given by (6), we have to show by the composition formulae for products of sums of squares that $2l(l^2 + 1024^2)$ is not a sum of two biquadrates for twenty-nine numerical values of l. This last step is easily carried out, and the proof is complete.

5. The most laborious feature in the proof is the necessity for factoring several numbers greater than ten million, the extent of the present factor tables. For example, in the case discussed in Section 3, it is necessary to factor the number $l^2 + 1024^2$ not only in order that condition (ii) may be applied, but also in order to apply the final restrictions by composition of sums of squares. This work was performed with the aid of a calculating machine by the factor stencil method of D. N. Lehmer and J. D. Elder.[5] Whenever the stencils indicated that the number was a prime, the fact was confirmed by D. H. Lehmer's[6] method based on the converse of Fermat's theorem.

In order to insure accuracy, all the attendant numerical work and the algebra of determining the cases in step two was checked twice at different times. Complete details of the proof will appear elsewhere.

[1] In L. E. Dickson's *History of the Theory of Numbers*, vol. 2, p. 648—there is a statement that might lead one to infer that the impossibility of (1) was proved by A. Werebrusow (*L'Intérmediare des Mathématiciens*, **21**, 161 (1914)). A fatal lacuna in Werebrusow's proof was pointed out by E. T. Bell (*Mathematics Student*, **4**, 78 (1936)).

[2] Mordell, L. J., "The Present State of Some Problems in the Theory of Numbers," *Nature*, **121**, 138 (1928).

[3] Aubry, L., *Sphinx-Oedipe*, **7**, 45–46 (March, 1912).

[4] For example, we have Norrie's well-known result that

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4.$$

[5] Lehmer, D. N., and Elder, J. D., "Factor Stencils," Carnegie Institution, Washington (1939).

[6] Lehmer, D. H., *Amer. Math. Monthly*, **43**, 347–354 (1936).

ERRATUM

In the article, "Dominance Modification and Physiological Effects of Genes," by L. C. Dunn and S. Gluecksohn-Schoenheimer, *Proc. Nat. Acad. Sci.*, **31**, 82 (1945), the formula in the middle of line 1, page 83, should read "138 Sd + (χ^2 = 5.26, p = 0.02)" instead of "138 Sd + $X^{2-5.26}_{p=0.2}$."