THE CALCULATION OF THE COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

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1. Introduction. The difficulty of computing an elliptic integral of the third kind is well known; even a computation of the complete integral

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 + \nu \sin^2 \phi)\sqrt{(1 - k^2 \sin \phi)}} \tag{1.1}$$

for numerical values of its parameters ν and k is a time-consuming operation. Apparently it is only very recently that any systematic tabulation has been made. I develop here a very efficient way to compute this integral when the parameters ν and k are real. The procedure is well adapted to a digital computer, and is so easily programmed that no tables should be necessary in the future.

When ν is zero, the procedure reduces to Gauss' method for computing the complete elliptic integral of the first kind by the use of Landen's transformation. However no knowledge of elliptic functions or integrals is required for the developments which follow. Any reader who has had a first course in function theory can easily understand the proofs.

2. The computation procedure. The process of computation is iterative, but takes slightly different forms according as the parameter ν is positive or negative. We let $\nu = \epsilon\mu^2$, $\epsilon = \pm 1$.

If $\epsilon = +1$, we speak of the positive case; if $\epsilon = -1$, of the negative case. In either case, we denote the integral (1.1) by $L(\mu, k)$. If $\mu = 0$, the integral reduces to the complete integral of the first kind. We denote it then by $L(k)$:

$$L(k) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}}\,. \tag{2.1}$$

The parameters μ and k are assumed to be real, and

$$0 \leqq k < 1, \qquad 0 \leqq \mu \quad \text{if} \quad \epsilon = 1; \qquad 0 \leqq \mu < 1 \quad \text{if} \quad \epsilon = -1. \tag{2.2}$$

Write for brevity

$$k' = \sqrt{(1 - k^2)}, \qquad \mu' = \sqrt{(1 + \epsilon\mu^2)}, \tag{2.3}$$

and let $k^* = (1-k')/(1+k')$ and $\mu^* = \mu k/\{(1+\mu')(1+k')\}$. We shall prove in Section 4 of this paper that

$$L(\mu, k) = \frac{2}{1 + k'}\, \mu'^{-1}(2L(\mu^*, k^*) - L(k^*)). \tag{2.4}$$

Now $k^* = k^2/(1+k')^2 < k^2$ so that if we iterate (2.4), the successive values of μ and k obtained tend rapidly to zero. Since $L(0, 0) = L(0) = \frac{1}{2}\pi$, the following procedure suggests itself for obtaining an approximation $A(\mu, k)$ to $L(\mu, k)$.

First step. Compute $k_1, \cdots, k_r$ and $\mu_1, \cdots, \mu_r$ by the formulas

$$(2.5) \qquad \mu_1 = \mu,\ k_1 = k; \qquad k_{n+1} = \frac{1 - k_n'}{1 + k_n'}, \qquad \mu_{n+1} = \frac{\mu_n k_n}{(1 + \mu_n')(1 + k_n')}.$$

The algorithm is continued until k_{r+1} and μ_{r+1} are sufficiently small. More specifically for $n = 1, 2, \cdots$, let

$$(2.6) \qquad K_n = \prod_{s=1}^{n} 2/(1 + k_s'), \qquad M_n = \prod_{s=1}^{n} \mu_s'^{-1},$$

and let

$$(2.7) \qquad E_n = \begin{cases} 2^n K_n M_n k_{n+1}^2 & \text{in the positive case;} \\ 2^n K_n M_n (\mu_{n+1}^2 + \frac{1}{2} k_{n+1}^2) & \text{in the negative case.} \end{cases}$$

We require that E_r be negligible.

Second step. Compute the approximations $A(\mu_{r+1}, k_{r+1})$, $A(\mu_r, k_r)$, $A(\mu_{r-1}, k_{r-1})$, $\cdots A(\mu_1, k_1) = A(\mu, k)$ and $A(k_{r+1})$, $A(k_r)$, $A(k_{r-1})$, $\cdots A(k_2)$ by the formulas

$$A(\mu_{r+1}, k_{r+1}) = A(k_{r+1}) = \tfrac{1}{2}\pi,$$

$$(2.8) \qquad A(\mu_n, k_n) = \frac{2}{1 + k_n'} \frac{1}{\mu_n'} (2A(\mu_{n+1}, k_{n+1}) - A(k_{n+1})),$$

$$A(k_n) = \frac{2}{1 + k_n'} A(k_{n+1}), \qquad n = r, r - 1, \cdots, 1.$$

Third step. Take $A(\mu, k)$ as an approximation to $L(\mu, k)$; the error in doing so is negligible, for

$$(2.9) \qquad 0 < L(\mu, k) - A(\mu, k) < E_r.$$

It will be observed that the first two steps of the computation are iterative processes very well adapted to a digital computer or even to an ordinary desk calculator.

3. The rapidity of convergence. The convergence of the process is extremely rapid. It is evident from (2.2) and (2.5) that if μ and k are positive, then $0 < k_{n+1} < k_n^2$ and $0 < \mu_{n+1} < \frac{1}{2}(\mu_n^2 + k_n^2)$. We shall show later that *if $0 < k_t, \mu_t < \frac{2}{3}$ and $n \geqq 1$, then*

$$(3.1) \qquad 0 < k_{n+t} < (\tfrac{1}{3}k_t)^{2^n}, \qquad 0 < \mu_{n+t} < \mu_t(\tfrac{1}{3}k_t)^{2^{n-1}}.$$

We shall also show that if $0 < k_n, \mu_n < .1$, then

$$(3.2) \qquad 2/(1 + k_n') = 1 + \theta_n(\tfrac{1}{4}k_n^2), \qquad 1/\mu_n' = 1 - \epsilon\phi_n(\tfrac{1}{2}\mu_n^2),$$

with $1 < \theta_n < 1.006$, $1 < \phi_n < 1.008$.

It follows from (3.2) that the infinite products $K=\prod_1^\infty \{2/(1+k_n')\}$, $M=\prod_1^\infty (1/\mu_n')$, converge and, from (3.1), that the partial products M_r and K_r in the error term E_r change very little as r increases when μ_r and k_r are small. It is also evident from the inequalities (3.1) that E_r may be made arbitrarily small by choosing r large enough; that is, the procedure converges in the usual sense. The following numerical data for the negative case illustrate the rapidity of convergence.

The data is taken from a sytematic computation of E_r to twenty places for μ and k ranging independently over the values .1, .2, $\cdots$, .8, .9. The computation was programmed for the California Institute Datatron by Mr. George W. Logemann, and the complete results will be given elsewhere. Evidently the most unfavorable case is when $\mu=k=.9$. Mr. Logemann found that $E_3=.000174$, $\cdots$, $E_4=.00000\ 00000\ 5197$, $\cdots$, $E_5<10^{-18}$. We conclude then that four or five iterations will suffice for most ordinary purposes.

4. Derivation of the iterative formula. We here derive formula (2.4) on which the computation procedure rests.

The functions $(1+\epsilon\mu^2\sin^2\phi)^{-1}$ and $(1-k^2\sin^2\phi)^{-1/2}$ may evidently be expanded in Fourier series which converge uniformly to them in the closed interval $[-\pi,\pi]$. Let these expansions be

$$\frac{1}{1+\epsilon\mu^2\sin^2\phi}=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos n\phi, \tag{4.1}$$

$$\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}b_n\cos n\phi. \tag{4.2}$$

Then

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos n\phi d\phi}{\sqrt{(1-k^2\sin^2\phi)}}\,. \tag{4.3}$$

It will be shown at the close of this section that $a_{2n+1}=0$ and that

$$a_{2n}=2\mu'^{-1}\epsilon^n\cdot\{\mu/(1+\mu')\}^{2n}. \tag{4.4}$$

Thus by (4.1)

$$\frac{1}{(1+\epsilon\mu^2\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}}$$
$$=\mu'^{-1}\left(\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}+2\sum_1^\infty\epsilon^n\left(\frac{\mu}{1+\mu'}\right)^{2n}\frac{\cos 2n\phi}{\sqrt{(1-k^2\sin^2\phi)}}\right).$$

The series on the right is uniformly convergent with respect to ϕ. Hence integrating term-wise over $[-\pi,\pi]$ we obtain from (2.1) and (4.3) the formula

$$L(\mu, k) = \tfrac{1}{4}\pi\mu'^{-1}\left(b_0 + 2\sum_1^\infty \epsilon^n\{\mu/(1+\mu')\}^{2n}b_{2n}\right). \tag{4.5}$$

We next obtain another expression for the coefficients b_{2n} in this series. Let $w=\phi+i\zeta$ be a complex variable. Then there exists a positive number δ depending on k alone such that $\Delta(w)=(1-k^2\sin^2 w)^{-1/2}$ is regular in the strip $|\text{Im } w| < \delta$. Since $\Delta(w)$ is also periodic with period 2π in the whole plane, it is representable everywhere in the strip by the Fourier series

$$\Delta(w) = \tfrac{1}{2}b_0 + \sum_1^\infty b_n \cos nw, \tag{4.6}$$

with the b_n given by formula (4.3). From now on, we confine ourselves to the rectangle $|\text{Re } w| \leqq \pi$, $|\text{Im } w| < \delta$.

Let T stand for the mapping of the w-plane induced by letting $z=e^{iw}$. Then T transforms the rectangle above into a circular ring R in the z plane: $R=\{z \mid 0<r<|z|<R\}$, where $e^{-\delta}=r<1<R=e^{\delta}$. Furthermore if

$$F(z) = T\Delta(w), \tag{4.7}$$

then $F(z)$ is regular in R. But $T\cos nw=\frac{1}{2}(z^n+z^{-n})$. Hence if we let $b_{-n}=b_n$, we have by (4.6) and (4.7) the Laurent expansion of $F(z)$ in R: $F(z)=\sum_{-\infty}^{\infty}\frac{1}{2}b_n z^n$.

Consequently by Laurent's theorem,

$$b_n = \frac{1}{\pi i}\int_\Gamma z^{n-1}F(z)dz, \qquad n = 0, 1, \cdots. \tag{4.8}$$

Here Γ is the unit circle described once counterclockwise. Now let

$$k = \sin \tfrac{1}{2}\beta, \qquad 0 \leqq \beta < \pi. \tag{4.9}$$

Then we find from (4.7) after a little algebra that

$$F(z) = \frac{2\csc\frac{1}{2}\beta z}{\sqrt{\{(z^2+\tan^2\frac{1}{4}\beta)(z^2+\cot^2\frac{1}{4}\beta)\}}}\,. \tag{4.10}$$

Hence the only singularities of $F(z)$ within Γ are branchpoints of order two at $z=\pm i\tan\frac{1}{4}\beta$. Join these points by a cut along the axis of imaginaries. Then the contour Γ in (4.8) may be shrunk down to a path consisting of two small circles about the branch points and the two sides of the cut. If the radius of each of these circles is ρ, their contribution to the line integral is of order $\sqrt{\rho}$. If therefore we set $z=i\tan\frac{1}{4}\beta\sin\phi$ along the cut and pass to the limit by letting ρ tend to zero, we find that

$$b_n = \frac{4}{\pi}i^n\csc\tfrac{1}{2}\beta\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin^n\phi\tan^{n+1}\frac{1}{4}\beta d\phi}{\sqrt{(1-\tan^4\frac{1}{4}\beta\sin^2\phi)}}\,.$$

Hence b_n is zero when n is odd, and when n is even,

$$b_{2n} = \frac{8}{\pi}\,\frac{(-k^*)^n}{1+k'}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\phi\,d\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}}\,. \tag{4.11}$$

Here we have written k^* for $\tan^2\frac{1}{4}\beta$ and replaced the multiplier $\csc\frac{1}{2}\beta\tan\frac{1}{4}\beta$ by $(1+\cos\frac{1}{2}\beta)^{-1}$ which equals $(1+k')^{-1}$ by (4.9) and (2.3). We easily find from the same formulas that

$$k^* = (1-k')/(1+k'). \tag{4.12}$$

(4.11) is the new expression for b_{2n} which we were seeking. Introduce it into the series (4.5) for $L(\mu, k)$, and let

$$\mu^* = \frac{\mu}{1+\mu'}\sqrt{k^*} = \frac{\mu}{1+\mu'}\,\frac{k}{1+k'}\,. \tag{4.13}$$

Then (4.5) becomes

$$L(\mu,k) = \frac{2\mu'^{-1}}{1+k'}\left(\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}} + 2\sum_1^\infty(-\epsilon)^n\mu^{*2n}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}}\right). \tag{4.14}$$

Now the series $(1-k^*\sin^2\phi)^{-1/2}+2\sum_1^\infty(-\epsilon\mu^{*2})^n\sin^{2n}\phi(1-k^*\sin^2\phi)^{-1/2}$ is uniformly convergent in ϕ, and is essentially a geometric progression whose sum may be written

$$\frac{2}{(1+\epsilon\mu^{*2}\sin^2\phi)\sqrt{(1-k^{*2}\sin^2\phi)}} - \frac{1}{\sqrt{(1-k^{*2}\sin^2\phi)}}\,.$$

Hence on integrating term-wise from 0 to $\frac{1}{2}\pi$ and comparing it with (4.14) and the integral of its sum with (1.1) and (2.1), we obtain the iteration formula (2.4), with the values of k^* and μ^* given in Section 2.

It remains to verify the values given for the Fourier coefficients a_n in the series (3.1) for $(1+\epsilon\mu^2\sin^2\phi)^{-1}$. Let $\mu=\tan\frac{1}{2}\alpha$, $0\leqq\alpha<\pi$, in the positive case; $\mu=\sin\frac{1}{2}\alpha$, $0\leqq\alpha<\pi$, in the negative case. Then we find that

$$T(1+\epsilon\mu^2\sin^2\phi)^{-1} = \begin{cases}4\cot^2\frac{1}{2}\alpha z^2[(z^2-\tan^2\frac{1}{4}\alpha)(z^2-\cot^2\frac{1}{4}\alpha)]^{-1} & \text{if } \epsilon=+1,\\ 4\csc^2\frac{1}{2}\alpha z^2[(z^2+\tan^2\frac{1}{4}\alpha)(z^2+\cot^2\frac{1}{4}\alpha)]^{-1} & \text{if } \epsilon=-1.\end{cases}$$

This result may be written

$$T(1+\epsilon\mu^2\sin^2\phi)^{-1} = \frac{4\mu^{-2}z^2}{(z^2-\epsilon\tan^2\frac{1}{4}\alpha)(z^2-\epsilon\cot^2\frac{1}{4}\alpha)}\,.$$

Hence by analogy with (4.8)

$$a_n = \frac{4\mu^{-2}}{\pi i}\int_\Gamma\frac{z^{n+1}dz}{(z^2-\epsilon\tan^2\frac{1}{4}\alpha)(z^2-\epsilon\cot^2\frac{1}{4}\alpha)}\,.$$

Thus by the residue theorem a_n equals $8\mu^{-2}$ times the sum of the residues of the integrand at its two poles $\pm\sqrt{\epsilon}\tan\frac{1}{4}\alpha$ within Γ. On evaluating these residues, we find that $a_{2n+1}=0$ and that a_{2n} has the value stated in (4.4), so that the proof is complete.

5. Proofs of the basic inequalities. The inequalities (3.1) are proved as follows. We have $k_{t+1}=(1-k_t')/(1+k_t')=k_t^2/(1+k_t')^2$. Hence $k_{t+1}<\frac{1}{3}k_t^2$ if and only if $1+k_t'>\sqrt{3}$. This works out to $k_t^2<\sqrt{(3+2\sqrt{3})}=.681$. Hence: *If* $k_t<\frac{2}{3}$, *then* $k_{t+1}<\frac{1}{3}k_t^2$. This statement is the case $n=1$ of the inequality (3.1) for k_{t+n}. The general inequality follows by an easy induction. The inequality for μ_{t+n} is proved similarly.

To prove that (3.2) is valid, assume that $0<k_n<.1$. Then

$$\frac{2}{1+k_n'}=\frac{2(1-k_n')}{1-k_n'^2}=\frac{2}{k_n^2}\sqrt{(1-k_n^2)}).$$

Now on expanding the function $1-\sqrt{(1-x)}$, $0<x<1$ by Taylor's theorem with remainder and then substituting k_n^2 for x, we obtain the formulas

$$\frac{2}{1+k_n'}=\theta_n(\tfrac{1}{4}k_n),\quad \theta_n=1+\tfrac{1}{2}k_n^2+\frac{5}{16}\frac{k_n^4}{(1-\theta k_n^2)^{7/2}},\qquad 0<\theta<1.$$

Hence

$$1<\theta_n<1+\tfrac{1}{2}(.01)+\frac{5}{16}\frac{.0001}{(.99)^{7/2}}<1.006.$$

The formula for μ_n' in (3.2) is proved in a similar fashion.

We preface the proof of the inequality (2.9) by two lemmas.

LEMMA 5.1. *If* $0<\mu$, $k<.1$, *then*

$$0<L(\mu,k)-\tfrac{1}{2}\pi<\tfrac{4}{5}k^2 \qquad \textit{if } \epsilon \textit{ is positive,} \tag{5.1}$$

$$0<L(\mu,k)-\tfrac{1}{2}\pi<\tfrac{4}{5}(\mu^2+\tfrac{1}{2}k^2) \qquad \textit{if } \epsilon \textit{ is negative.} \tag{5.2}$$

Proof. By Taylor's theorem with remainder, where $0<\theta<1$,

$$\begin{aligned}(1-k^2\sin^2\phi)^{-1/2}&=1+\tfrac{1}{2}k^2\sin^2\phi+\frac{3}{8}k^4\frac{\sin^4\phi}{(1-\theta k^2\sin^2\phi)^{5/2}}\\&>1+\tfrac{1}{2}k^2\left(1+\frac{3}{4}\frac{k^2}{(1-k^2)^{5/2}}\right)\sin^2\phi.\end{aligned}$$

Since $k<.1$,

$$1+\frac{3}{4}\frac{k^2}{(1-k^2)^{5/2}}<1+\frac{3}{4}\frac{.01}{(.99)^{5/2}}=1.0077.$$

Hence

$$\frac{1}{(1-k^2\sin^2\phi)^{1/2}} < 1 + \alpha(\tfrac{1}{2}k^2)\sin^2\phi, \qquad \alpha = 1.0077. \tag{5.3}$$

Similarly

$$\frac{1}{1-\mu^2\sin^2\phi} < 1 + \beta\mu^2\sin^2\phi, \qquad \beta = 1.01031.$$

Therefore

$$\begin{aligned} 0 &< \frac{1}{(1-\mu^2\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}} - 1 \\ &< \tfrac{1}{2}\alpha k^2\sin^2\phi + \beta\mu^2\sin\phi + \alpha\beta(\tfrac{1}{2}\mu^2k^2)\sin^4\phi. \end{aligned}$$

On integrating these inequalities from 0 to $\frac{1}{2}\pi$ we obtain when ϵ is negative the inequality

$$0 < L(\mu, k) - \tfrac{1}{2}\pi < \tfrac{1}{4}\pi\alpha(1 + \tfrac{3}{8}\beta\mu^2)(\tfrac{1}{2}k^2) + \tfrac{1}{4}\pi(1 + \tfrac{3}{16}k^2\alpha)\mu^2.$$

The coefficients of $\frac{1}{2}k^2$ and μ^2 turn out to be .79446 and .794996 which are both less than $\frac{4}{5}$. This completes the proof of the inequality (5.2).

If ϵ is positive, it follows from (5.3) that

$$L(\mu, k) - \tfrac{1}{2}\pi < L(k) - \tfrac{1}{2}\pi - (\tfrac{1}{2}\alpha k^2)(\tfrac{1}{4}\pi).$$

Since $\frac{1}{4}\alpha\pi < \frac{4}{5}$, (5.1) follows, completing the proof of the lemma.

LEMMA 5.2. *If*

$$A(\mu_n, k_n) = \frac{2\mu_n'^{-1}}{1+k_n'}(2a(\mu_{n+1}, k_{n+1}) - A(k_{n+1})),$$

$$A(k_n) = \frac{2}{1+k_n'}A(k_{n+1}),$$

for $n = 1, 2, \cdots$, *then*

$$\begin{aligned} L(\mu, k) - A(\mu, k) &= 2^nM_nK_n(L(\mu_{n+1}, k_{n+1}) - A(\mu_{n+1}, k_{n+1})) \\ &\quad - \sum_{s=1}^{n} 2^{s-1}M_sK_s(L(k_{s+1}) - A(k_{s+1}), \end{aligned} \tag{5.2}$$

where K_n *and* M_n *are given for all* n *by* (2.6).

Proof. We know by the iteration formula that for all n,

$$L(\mu_n, k_n) = \frac{2}{1+k_n'}\mu_n'^{-1}(2L(\mu_{n+1}, k_{n+1}) - L(k_{n+1}),$$

$$L(k_n) = \frac{2}{1+k_n'} L(k_{n+1}).$$

Hence since $\mu=\mu_1$ and $k=k_1$,

$$\begin{aligned} L(\mu, k) - A(\mu, k) & \\ &= \frac{2}{1-k_n'}\mu_1'^{-1}(2L(\mu_2, k_2) - L(k_2) - \frac{2}{1+k_1'}\mu_1'^{-1}(2A(\mu_2, k_2) - A(k_2)) \\ &= 2K_1M_1(L(\mu_2k_2) - A(\mu_2, k_2) - K_1M_1(L(k_2) - A(k_2)). \end{aligned}$$

Hence (5.2) is true when $n=1$. But is easy to show that if it is true for n, it is true for $n+1$. Hence it is true for all n by mathematical induction. Now

$$L(k_{s+1}) = \frac{2}{1+k_{s+1}'}\frac{2}{1+k_{s+2}'}\cdots\frac{2}{1+k_n'}L(k_{n+1}),$$
$$A(k_{s+1}) = \frac{2}{1+k_{s+1}'}\frac{2}{1+k_{s+2}'}\cdots\frac{2}{1+k_n'}A(k_{n+1}).$$

Hence (5.2) may be written

$$\begin{aligned} L(\mu, k) - A(\mu, k) = 2^nK_nM_n(L(\mu_{n+1}k_{n+1}) - A(\mu_{n+1}, k_{n+1}) \\ - K_n\left(\sum_{s=1}^{n} 2^{s-1}M_s\right)(L(k_{n+1}) - A(k_{n+1}). \end{aligned} \tag{5.3}$$

Now suppose that $A(k_{n+1}, \mu_{n+1}) = A(k_{n+1}) = \frac{1}{2}\pi$. Then if ϵ is positive, $L(\mu_{n+1}, k_{n+1}) < L(k_{n+1})$. Hence when ϵ is positive, we obtain from (5.3) the inequalities.

$$\begin{aligned} L(\mu, k) - A(\mu, k) &< 2^nK_nM_n(L(k_{n+1}) - \tfrac{1}{2}\pi) \\ &> K_n\left(2^nM_n - \sum_{s=1}^{n} 2^{s-1}M_s\right)(L(K_{n+1}, \mu_{n+1}) - \tfrac{1}{2}\pi). \end{aligned}$$

Now $M_1 < M_2 < \cdots < M_n$. Hence $\sum_{s=1}^{n} 2^{s-1}M_s < 2^nM_n$. Hence $L(\mu, k) - A(\mu, k)$ is positive from the second inequality. And by the first inequality and Lemma 5.1, it is less than $2^nK_n(\frac{2}{5}k_{n+1}^2)$.

If ϵ is negative, $L(\mu_{n+1}, k_{n+1}) > L(k_{n+1})$. Hence we obtain from (5.3) the inequalities

$$L(\mu, k) - A(\mu, k) > K_n\left(2^nM_n - \sum_{s=1}^{n} 2^{s-1}M_s\right)(L(k_{n+1}) - \tfrac{1}{2}\pi) > 0,$$
$$\begin{aligned} L(\mu, k) - A(\mu, k) &< 2^nK_nM_n(L(\mu_{n+1}, k_{n+1}) - \tfrac{1}{2}\pi) \\ &< 2^nK_nM_n\tfrac{4}{5}(\mu_{n+1}^2 + \tfrac{1}{2}k_{n+s}^2), \end{aligned}$$

by Lemma 5.1, and this completes the proof of the inequality (2.9).

In conclusion, it is worth noting that the method may evidently be extended to compute a complete integral of the third kind of the form

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1+\nu\sin^2\phi)\sqrt{(1+k^2\sin^2\phi)}}$$

for ν and k real and ν greater than -1.

ANALYTICAL EXPRESSIONS AND ELEMENTARY FUNCTIONS

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The title is very nearly meaningless. When Professor Ritt discusses integration in finite terms an elementary function is one (a not too elementary) thing. In common classroom usage it is another—roughly, any differentiable function frequently used by Euler. When an author presents us with an analytic expression for a function, the expression usually has little or nothing to do with analytic functions in any strict sense. The adjective has become a synonym for "nonverbal". As examples we give*

$$\operatorname{sgm}\sin^2\frac{(n-1)!+1}{n}\pi \quad\text{and}\quad \operatorname{sgm}\sin^2\frac{(n-1)!^2}{n}\pi,$$

characteristic functions for the sets of nonprime and prime integers. The spirit of the formula constructing game is clear. Its rules are not.

Consider $[x]$,† the characteristic of x regarded as a logarithm, the greatest integer not greater than x. We find

$$[x] = x - \tfrac{1}{2} + (1/\pi)\tan^{-1}\cot\pi x \qquad x \neq 0, \pm 1, \pm 2, \cdots.$$

How do we fill in the gaps at integral x? Using limits is too much like hunting quail with an elephant gun. It's difficult to find functions which are not expressible as limits of limits.

The admission of one simple additional function somehow seems more sporting. Logical candidates are null $x=\lim_{n\to\infty}(1+|x|)^{-n}$, the characteristic function of the set (0), or sgm $x=\lim_{n\to\infty}x^{1/(2n+1)}=x/(|x|+\text{null } x)$. Thus, for $T(x)=(2/\pi)\sin^{-1}\sin\frac{1}{2}\pi x \operatorname{sgm}\cos\frac{1}{2}\pi x$, we have $[x]=x-\max(T(x), T(x+1))$. Even this procedure may be circumvented if we adopt the convention that an expression meaningless for $x=a$, but suitably defined in a neighborhood, defines $f(a)=\frac{1}{2}\{f(a-)+f(a+)\}$. Then sgm $x=x/|x|$ and $T(x)=(2/\pi)\tan^{-1}\tan\frac{1}{2}\pi x$.

It is interesting to see how far one may go with modest assumptions. If we agree $\cos x$, $\cos^{-1} x$, x, and real constants are "common" functions of x and

* Cf. Louis Brand, Advanced Calculus, New York, 1955, p. 84.

† Cf. Arthur Porges, An analytical expression for $[X]$, this MONTHLY, vol. 66, 1959, pp. 706–707.