ANNALS OF MATHEMATICS
Vol. 40, No. 2, April, 1939

# EVALUATIONS OVER RESIDUATED STRUCTURES

BY MORGAN WARD AND R. P. DILWORTH

(Received June 6, 1938)

## I. INTRODUCTION

**1.** In previous papers,[1] we have developed a theory of structures over which auxiliary operations of multiplication and residuation may be defined with properties analogous to the like-named operations in polynomial ideal theory. We have applied our results to generalize the various decomposition theorems of ideal theory (van der Waerden [1]) to extensive classes of structures.[2]

We give here some results on the evaluations of such structures. By an evaluation we mean a homomorphism (Ore [1], Chapter II) between the structure and a set of real numbers ordered by the relation $\leqq$ under which multiplication in the structure corresponds to addition of the reals. If we are willing to assume that the structure homomorphism preserves residuation, we can obtain results of a simplicity and finality comparable with the classic evaluation theory for domains of integrity and fields. (van der Waerden [1], Albert [1]). But this assumption unduly restricts the kinds of structures which may be evaluated and complicates the arithmetical interpretation of the "discrete evaluations" (part III of paper) which are our main concern. We shall accordingly assume it only incidentally. (Part IV of paper.)

**2.** Our main result is an arithmetical characterization of all discrete evaluations of a residuated structure with the ascending chain condition in terms of certain chains of primary elements belonging to the structure. No appeal is necessary to the Dedekind modular axiom or to the special decomposition theorems which ensue on assuming that every irreducible is primary (Ward-Dilworth [1], [2]).

We also discuss briefly some interesting topological questions suggested by the evaluation.

---

[1] See the references Ward-Dilworth [1], Dilworth [1], Ward [1], Ward [2] at the close of the paper. The idea goes back to Dedekind (Dedekind [1], but our only immediate predecessor seems to have been W. Krull (Krull [1]). We have expanded and elaborated the results summarized in Ward-Dilworth [1] in a paper "Residuated Lattices" (Ward-Dilworth [2]) which has been submitted for publication elsewhere. We take this occasion to correct some errors in Ward-Dilworth [1] section 4, page 163. In condition D 1, the exponent $r$ should be one. In the fourth theorem, the words "and sufficient" should be struck out. The fifth theorem should be struck out in toto. We may add that we have greatly extended the results of this section in Ward-Dilworth [2] and largely freed them of their dependence on the modular axiom.

[2] Distributive structures are studied in detail in the paper Ward [2] in this journal.

The following examples show that evaluations are of frequent occurrence. Here $\pi x$ denotes the evaluation function, and $\mathfrak{S}$ the basic structure.

(i) $\mathfrak{S}$, rational integers with union and cross-cut G.C.D. and L.C.M. and multiplication ordinary multiplication. $\pi a = 0$, $a$ odd; $\pi a = 1$, $a$ even, defines an evaluation.

(ii) $\mathfrak{S}$, ideals of a principal ideal ring, multiplication ordinary multiplication. Evaluation is essentially the same as the ordinary evaluation.

(iii) $\mathfrak{S}$, Boolean algebra or distributive structure, multiplication identified with cross-cut (Ward [2]). $\mathfrak{p}$ any prime ideal of the structure (Stone [1], Birkhoff [1]). $\pi a = 1$ if $a$ is in $\mathfrak{p}$, $\pi a = 0$ otherwise, defines an evaluation.

(iv) $\mathfrak{S}$ a chain structure of finite length $n$:

$$i = a_1 > a_2 > \cdots > a_n .$$

Let $n = n_1 + n_2 + \cdots + n_l$ be any fixed partition of $n$ into positive summands. Effect a class separation of $\mathfrak{S}$, $\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}_1 + \cdots + \mathfrak{S}_l$ by the rule $a_i \in \mathfrak{S}_k$ if $n_1 + n_2 + \cdots + n_{k-1} < i \leqq n_1 + n_2 + \cdots + n_k$. Define a multiplication over $\mathfrak{S}$ as follows. Let $b_k = a_{n_1+n_2+\cdots+n_k}$, $(k = 1, 2, \cdots, l)$. Then if $a_u \in \mathfrak{S}_i$ and $a_v \in \mathfrak{S}_j$, $a_u a_v = b_{i+j}$ if $i + j \leqq l$, $a_u a_v = b_l = a_n$ if $i + j \geqq l$. Then $\pi a = k$ if $a \in \mathfrak{S}_k$ is an evaluation.

(v) $\mathfrak{S}$, finite arithmetical lattice. Since such a lattice is a direct cross-cut of chains of finite length, the procedure of (iv) allows us to construct evaluations at will. The case when each $n_i = 1$ in (iv) gives the ordinary evaluations. One of us plans to discuss the residuation of such a lattice elsewhere.

(vi) $\mathfrak{S}$, an arbitrary chain structure. $a_1 \supset a_2 \supset a_3 \supset \cdots$ any selection of elements of $\mathfrak{S}$ where each $a_i$ properly divides $a_{i+1}$. Define $\mathfrak{S}_i$ as the set of all elements $x$ of $\mathfrak{S}$ such that $a_{i-1} \supset x \supset a_i$, $x \neq a_i$. Define a multiplication by the rule if $x \in \mathfrak{S}_i$, $y \in \mathfrak{S}_j$ then $xy = a_{i+j}$. The evaluation is defined then by $\pi x = k$ if $x \in \mathfrak{S}_k$.

(vii) $\mathfrak{S}$, a residuated structure with ascending chain condition. $p$ any prime of $\mathfrak{S}$. $\pi a = 1$ if $p \supset a$, $\pi a = 0$ otherwise defines an evaluation. (See part III of paper.) This example applies to the ideals of any commutative ring with chain condition, but no modular condition need be assumed.

**3.** We assumarize here the notations and definitions we shall employ. We denote our structures by German capitals $\mathfrak{S}$, $\mathfrak{S}_i$, $\mathfrak{S}'$, $\cdots$. The letters $\mathfrak{U}$, $\cdots$, $\mathfrak{Y}$ are reserved to denote sub-sets of elements of our basic structure $\mathfrak{S}$ which are not necessarily structures. We use small latin letters $a$, $b$, $\cdots$ for the elements of our structure, and write $x \in \mathfrak{X}$ ($x \in \mathfrak{S}$) for the set $\mathfrak{X}$ (the structure $\mathfrak{S}$) contains the element $x$. $x \supset y$ or $y \subset x$, $x \not\supset y$, $x = y$ denote as usual $x$ divides $y$, $x$ does not divide $y$, $x$ equals $y$. We use $(x, y)$ for union and $[x, y]$ for cross-cut, reversing Ore's usage. (Ore [1], Ward [1], [2].) We assume that $\mathfrak{S}$ has a unit element $i$ dividing every other element. The null element $z$ divisible by every other element need not exist. Multiplication $x \cdot y$ or $xy$ and residuation $x:y$ are one-valued operations on $\mathfrak{S}$ to $\mathfrak{S}$ defined by the following conditions:

R 1. *$a, b \in \mathfrak{S}$ implies $a:b \in \mathfrak{S}$.*
R 2. *$a:b = i$ if and only if $a \supset b$.*
R 3. *$a \supset b$ implies $a:c \supset b:c$ and $c:b \supset c:a$.*
R 4. *$(a:b):c = (a:c):b$.*
R 5. *$[a, b]:c = [a:c, b:c]$.*
R 6. *$c:(a, b) = [c:a, c:b]$.*

M 1. *$a, b \in \mathfrak{S}$ implies $ab \in \mathfrak{S}$.*
M 2. *$a = b$ implies $ac = bc$.*
M 3. *$ab = ba$.*
M 4. *$(ab)c = a(bc)$.*
M 5. *$ai = a$.*
M 6. *$a(b, c) = (ab, ac)$.*

In addition, the two operations are assumed to be interconnected by the formulas

$$a \supset (a:b)b; \quad \text{if } a \supset xb \quad \text{then } a:b \supset x. \tag{3.1}$$

$$ab:a \supset b; \quad \text{if } y:a \supset b \text{ then } y \supset ab. \tag{3.2}$$

A structure over which both a residuation and a multiplication may be defined satisfying R 1–M 6 and (3.1)–(3.2) will be said to be *residuated* (Ward-Dilworth [1]). We shall assume that the reader is familiar with the elementary properties of residuation and multiplication such as $a:b = a:(a, b) = [a, b]:b$; $a:bc = (a:b):c$; $a:(a:b) \supset (a, b)$ and so on. (Ward [1], [2], Dilworth [1], Ward-Dilworth [2].)

If $\mathfrak{X}$, $\mathfrak{Y}$ are any two subsets of $\mathfrak{S}$, we denote by $(\mathfrak{X}, \mathfrak{Y})$ $[\mathfrak{X}, \mathfrak{Y}]$ and $\mathfrak{X}\mathfrak{Y}$ the sets consisting respectively of all unions, cross-cuts or products of elements of $\mathfrak{X}$ with elements of $\mathfrak{Y}$ (Ward [2]). We write $\mathfrak{X} \supseteq \mathfrak{Y}$ if every element of $\mathfrak{Y}$ lies in $\mathfrak{X}$. For example, $\mathfrak{X} \supseteq \mathfrak{X}^2$ means $\mathfrak{X}$ is closed under multiplication. We use $\mathfrak{X} + \mathfrak{Y}$ for the set-theoretic sum of $\mathfrak{X}$ and $\mathfrak{Y}$.

**4.** An element $p$ of $\mathfrak{S}$ is said to be a *prime* if $p \supset ab$ implies $p \supset a$ or $p \supset b$, and *primary* if $p \supset ab$, $p \not\supset a$ implies $p \supset b^t$ for some $t$. The following lemmas are true in any residuated structure in which the ascending chain condition holds. They are readily proved by transcribing their analogues for commutative ideal theory (von der Waerden [1] chapter 12) into the language of structure theory.

LEMMA 4.1. *If $q$ is primary, there exists a prime $p$ such that $p \supset q \supset p^t$.*

$p$ will be said to *correspond* to $q$.

LEMMA 4.2. *Let $q$ and $p$ be elements of $\mathfrak{S}$ with the properties*

($\alpha$) *$q \supset ab$ and $q \not\supset a$ imply $p \supset b$.*

($\beta$) *$p \supset q$.*

($\gamma$) *$p \supset b$ implies $q \supset b^t$ for some $t$.*

*Then $q$ is primary, and $p$ is the prime element corresponding to $q$.*

The union of any set $\mathfrak{X}$ will on occasion be called the leader of $\mathfrak{X}$.

If $q = [a, b]$ implies $q = a$ or $q = b$, $q$ will be said to be *irreducible*.

## II. Evaluations

**5.** Any set of real numbers closed with respect to addition forms a residuated structure with respect to the division relation "less than or equal to." The union $(\alpha, \beta)$ and cross-cut $[\alpha, \beta]$ of two real numbers $\alpha$ and $\beta$ are respectively their minimum and maximum, while their "product" $\alpha\beta$ and "residual" $\alpha:\beta$ are respectively $\alpha + \beta$ and $\alpha - (\alpha, \beta)$. (Dilworth [1]; Ward-Dilworth [1] section 2, first theorem; Ward [3].) If the set of real numbers is bounded above, they still form a residuated structure provided that we take the product $\alpha\beta$ equal to the least upper bound of the set whenever $\alpha + \beta$ is greater than it. We shall use the letter $\sigma$ to denote the least upper bound of values. If no upper bound exists, we take $\sigma = +\infty$, the ideal null element of the structure of all the reals.

A function $\pi$ on $\mathfrak{S}$ to such a set of reals is called an *evaluation* of $\mathfrak{S}$ if the following four conditions are satisfied:

E 1. *For every element $a$ of $\mathfrak{S}$, $\pi a$ is a uniquely determined real number.*
E 2. *$a = b$ implies $\pi a = \pi b$.*
E 3. (i) $\pi(a, b) = (\pi a, \pi b)$ *and* (ii) $\pi[a, b] = [\pi a, \pi b]$.
E 4. $\pi ab = (\pi a + \pi b, \sigma)$.

We shall call the real number $\pi a$ the *value* of the structure element $a$.

If the values of $\pi a$ are bounded above, we shall say that the evaluation is bounded. Since for any $a$, $a = ai$, $i \supset a$ and $a \supset b$ implies $a = (a, b)$, we have

(5.1) $$\pi i = 0 \qquad \pi x \geqq 0, x \in \mathfrak{S}.$$

(5.2) $$a \supset b \text{ implies } \pi a \leqq \pi b.$$

Since every evaluation is a homomorphism, we may define by means of the evaluation a congruence relation $x \equiv y \pmod{\pi}$ over $\mathfrak{S}$, elements being congruent if and only if they have the same values. This congruence relation has the usual properties; that is, it is an equivalence relation, and if $a \equiv b \pmod{\pi}$ then for any $c$,

$$(a, c) \equiv (b, c), \qquad [a, c] \equiv [b, c], \qquad ac \equiv bc \pmod{\pi}.$$

We call the elements congruent to $i$ the units of $\mathfrak{S}$ modulo $\pi$. They form a dense residuated sub-structure of $\mathfrak{S}$. Moreover if $u$ is any unit

$$au \equiv a \pmod{\pi}, \text{ every } a \text{ of } \mathfrak{S}.$$

**6.** Since we may think of an evaluation as a mapping of the structure onto the metric space of the real numbers, the question arises as to the connection of the evaluation with the topology of the structure. Structures have been topologized in several ways. For instance, Glivenko [1, 2] has shown the identity of normed structures with certain types of metric spaces. Stone [1] has shown that Boolean algebras are mathematically equivalent to locally bicompact totally disconnected topological spaces, and H. Wallman [1] has similarly treated the topology of a distributive structure. In this connection one of us has found that much of Wallman's theory for a distributive structure holds in any structure over which a multiplication is defined.

If we attempt to introduce a metric into the structure by the use of the evaluation following the method of Glivenko, the difficulty arises that we may have $a \supset b$ properly with $\pi a = \pi b$. If we attempt to connect the evaluation with the topology of the structure by the method of Stone and Wallman, it is not clear what the map of a point (that is, a structure ideal) should be in terms of the evaluation. Our results in this direction are incomplete, and belong properly to the general question of the topology of residuated structures which one of us (R. P. Dilworth) will treat elsewhere.

## III. Discrete Evaluations—Cleavages

**7.** If $\alpha$ is any positive value, then the elements of $\mathfrak{S}$ whose values are $0$, $\alpha$, $2\alpha$, $3\alpha$, $\cdots$ obviously form a multiplicatively closed sub-structure of $\mathfrak{S}$. A particularly interesting and important case occurs when this sub-structure coincides with $\mathfrak{S}$ itself. We shall call the evaluation then *discrete*. For a discrete evaluation, there is no loss in generality in taking the values to be $0, 1, 2, \cdots$, the set breaking off or not accordingly as the evaluation is bounded or unbounded. Let $\mathfrak{S}_k$ denote the set of elements of $\mathfrak{S}$ with values $k$. Then we have a set-theoretic separation of $\mathfrak{S}$

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots + \mathfrak{S}_k + \mathfrak{S}_{k+1} + \cdots, \tag{7.1}$$

where each $\mathfrak{S}_i$ is a structure, and $\mathfrak{S}_0$ is multiplicatively closed. We denote the set $\mathfrak{S}_1 + \mathfrak{S}_2 + \cdots$ by $\mathfrak{S}'$, so that

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}'. \tag{7.2}$$

We shall now introduce the important notion of a *cleavage*[3] of a residuated structure. Let $\mathfrak{S}$ be a structure. From now on we assume explicitly

N 1. *$\mathfrak{S}$ is residuated.*

N 2. *The ascending chain condition holds in $\mathfrak{S}$.*

Definition of a Prime Cleavage. *A separation of $\mathfrak{S}$*

(7.3) $\mathfrak{S} = \mathfrak{U} + \mathfrak{V}$, *$\mathfrak{U}$, $\mathfrak{V}$ no elements in common, $\mathfrak{U}$ non-empty, is called a prime cleavage of $\mathfrak{S}$ provided that*

$$\mathfrak{U} \supset \mathfrak{U}^2, \qquad \mathfrak{U} \supset (\mathfrak{U}, \mathfrak{S}), \qquad \mathfrak{V} \supset (\mathfrak{V}, \mathfrak{V}). \tag{7.31}$$

Theorem 7.1. *Every cleavage of $\mathfrak{S}$ determines a prime, and with every prime is associated a cleavage.*

[3] The idea goes back to Krull [1]. For the special case when the multiplication of the structure is the cross-cut operation (but no chain condition is assumed, so that the cleavage does not necessarily define a structure element), the notion was applied to Boolean algebras by M. H. Stone [1], and extended by G. Birkhoff [2] to any distributive structure. The "primary cleavages" we introduce here seem to be new.

PROOF. By N 2 and (7.31)(iii), the union of $\mathfrak{B}$ exists, and lies in $\mathfrak{B}$ (Ore [1] §2). Denote it by $p$. *$p$ cannot divide an element of* $\mathfrak{U}$. For then by (7.31) (ii) $p = (p, u) \in \mathfrak{U}$ contrary to (7.3). Assume that

$$p \supset ab, \qquad p \not\supset a, \qquad a, b \text{ in } \mathfrak{S}.$$

Then $ab \in \mathfrak{B}$, $a \in \mathfrak{U}$. If $b \in \mathfrak{U}$, $ab \in \mathfrak{U}$ by (7.31) (i), and $p \not\supset ab$. Hence $b \in \mathfrak{B}$ and $p \supset b$. Hence $p$ is a prime by definition. The conceivable case $p = i$ is excluded because then since $i \supset x$, every $x$, $\mathfrak{B}$ would contain $\mathfrak{S}$ and $\mathfrak{U}$ be empty.

We observe that the separation (7.2) induced by the evaluation of $\mathfrak{S}$ is a cleavage. Since both $\mathfrak{U}$ and $\mathfrak{B}$ in (7.3) are easily shown to be structures, we shall use (7.2) henceforth to denote any prime cleavage. Conversely, given (7.2) with associated prime $p$, we may define a bounded evaluation over $\mathfrak{S}$ by $\pi x = 0$ if $p \not\supset x$; $\pi x = 1$ if $p \supset x$, $x \in \mathfrak{S}$. We have thus proved

THEOREM 7.2. *Every discrete evaluation of $\mathfrak{S}$ determines a prime, and every prime determines at least one evaluation.*

**8.** We shall next extend the notion of a cleavage so as to characterize the primary elements of $\mathfrak{S}$.

With the notation of the previous section, let

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}' \tag{7.2}$$

be a cleavage with associated prime $p$, so that

$$\mathfrak{S}_0 \supset \mathfrak{S}_0^2, \qquad \mathfrak{S}_0 \supset (\mathfrak{S}_0, \mathfrak{S}), \qquad \mathfrak{S}' \supset (\mathfrak{S}', \mathfrak{S}'), \tag{8.1}$$

while

$$p \in \mathfrak{S}'; x \in \mathfrak{S}' \text{ implies } p \supset x. \tag{8.11}$$

DEFINITION OF A PRIMARY CLEAVAGE. *A separation of* $\mathfrak{S}'$
(8.2) $\mathfrak{S}' = \mathfrak{U}^* + \mathfrak{B}$, $\mathfrak{U}^*$, $\mathfrak{B}$ *no elements in common, is called a primary cleavage of $\mathfrak{S}$ provided that*

$$\mathfrak{U}^* \supset \mathfrak{S}_0\mathfrak{U}^*; \mathfrak{B} \supset \mathfrak{S}'^k \textit{ for some positive integer } k; \mathfrak{B} \supset (\mathfrak{B}, \mathfrak{B}). \tag{8.21}$$

THEOREM 8.1. *The leader of $\mathfrak{B}$ is a primary element $q$ of $\mathfrak{S}$ whose corresponding prime is $p$.*

PROOF. The result is trivial if $\mathfrak{U}^*$ is empty, as then $\mathfrak{B} = \mathfrak{S}'$. Assume henceforth that $\mathfrak{U}^*$ is non-empty. N 2 and (8.21) (iii) guarantee that the leader $q$ exists and lies in $\mathfrak{B}$. Assume that $q \supset ab$, $q \not\supset a$; $a, b$ in $\mathfrak{S}$. Then either $a \in \mathfrak{S}_0$ or $a \in \mathfrak{U}^*$. In either case $b \in \mathfrak{S}'$. For if $b \in \mathfrak{S}_0$, $ab \in \mathfrak{S}_0$ or $ab \in \mathfrak{U}^*$ by (8.1) (i) and (8.21) (i). But if $b \in \mathfrak{S}'$, then $p \supset b$ by (8.11). We have thus shown that

($\alpha$) $q \supset ab$ and $q \not\supset a$ imply $p \supset b$.

By (8.11) and (8.21) (ii), we have

($\beta$) $p \supset b$. ($\gamma$) If $p \supset b$, then $q \supset b^k$.

Hence by lemma 4.2, $q$ is primary, and $p$ is the prime corresponding to $q$.

THEOREM 8.2. *Every primary element of* $\mathfrak{S}$ *determines a primary cleavage.*

PROOF. Let $q$ be any primary, and let $p$ be its corresponding prime. Then by lemma 4.1

$$p \supset q \supset p^k \tag{8.3}$$

where we may assume that $k > 1$. Let (7.2) as before be the cleavage associated with $p$. We now separate $\mathfrak{S}'$ into two disjoint classes $\mathfrak{U}^*$ and $\mathfrak{B}$ as in (8.2) by the rule $x \in \mathfrak{U}^*$ if $q \not\supset x$; $x \in \mathfrak{B}$ if $q \supset x$, $x$ any element of $\mathfrak{S}'$. We shall show that (8.21) holds.

First if $b \in \mathfrak{S}'$, then $p \supset b$. Hence $p^k \supset b^k$, so that by (8.3), $q \supset b^k$ or $b^k \in \mathfrak{B}$. Hence (8.21) (ii) holds. Also, if $q \supset b$, $q \supset c$ then $q \supset (b, c)$ so that (8.21) (iii) holds. Finally, assume that $a \in \mathfrak{S}_0$ and $b \in \mathfrak{U}^*$. Then since $p \supset ab$, either $ab \in \mathfrak{U}^*$ or $ab \in \mathfrak{B}$. If $ab \in \mathfrak{B}$, then $q \supset ab$. Since $b \in \mathfrak{U}^*$, $q \not\supset b$. Hence since $q$ is primary, $q \supset a^t$ for some $t$. Then by (8.3), $p \supset a^t$ or $p \supset a$, contradicting $a \in \mathfrak{S}_0$. Hence if $a \in \mathfrak{S}_0$ and $b \in \mathfrak{U}^*$, then $ab \in \mathfrak{U}^*$ giving (8.21) (i).

It is important to observe that the set $\mathfrak{U}^*$ is in general not a sub-structure of $\mathfrak{S}$.

**9.** We return now to the evaluation $\pi$ and the associated separation of $\mathfrak{S}$ into residue classes:

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots + \mathfrak{S}_k + \mathfrak{S}_{k+1} + \cdots \tag{7.1}$$

Let us define

$$\mathfrak{S}^* = \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots + \mathfrak{S}_{k-1}, \qquad \mathfrak{S}'' = \mathfrak{S}_k + \mathfrak{S}_{k+1} + \cdots$$

so that

$$\mathfrak{S}' = \mathfrak{S}^* + \mathfrak{S}''.$$

Then it is easy to see that a primary cleavage is defined, for the conditions (8.2), (8.21) are all satisfied. Hence we may state

THEOREM 9.1. *Let* $\pi$ *be a discrete evaluation of a residuated structure in which the ascending chain condition holds, and let* (7.1) *be the corresponding separation into residue classes modulo* $\pi$. *Then the leader of the substructure* $\mathfrak{S}_k$ *of all elements of* $\mathfrak{S}$ *with the value* $k \geqq 1$ *is a primary element* $q^{(k)}$ *of* $\mathfrak{S}$ *which divides all elements of* $\mathfrak{S}$ *with values* $\geqq k$ *and belongs to the prime element* $p = q^{(1)}$ *leading* $\mathfrak{S}_1$.

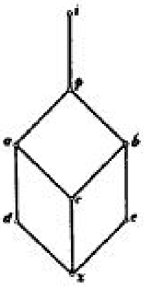
THEOREM 9.2. *The primaries* $q^{(1)}, q^{(2)}, \cdots$ *are all irreducible.*

PROOF. If $q$ leads $\mathfrak{S}_k$ and $q = [a, b]$, then by E 3, either $\pi q = \pi a$ or $\pi q = \pi b$. Hence by Theorem 9.1, either $q \supset a$ or $q \supset b$. Hence $q = [a, b]$ implies $q = a$ or $q = b$ so that $q$ is irreducible.

Although a primary cleavage can be associated with an arbitrary primary $q$, an evaluation cannot in general be determined having $q$ as one of its leaders. For $\mathfrak{S}^*$ and $\mathfrak{S}''$ must both be structures and admit of special multiplication rules not required in our general definition.

To give a simple illustration, consider the non-modular structure $\mathfrak{L}$ pictured.

A residuation and multiplication may be defined over $\mathfrak{L}$ by the rule given in theorem 8.1 of Ward-Dilworth [2]; viz. $xy = y$ if $x = i$; $xy = x$ if $y = i$; $xy = z$,



otherwise. $i$ is the unit and $p$ is a prime while all other elements are primary. We may define three distinct evaluations over $\mathfrak{L}$ by the separations

(1) $\qquad \mathfrak{L}_0 = \{i\}; \mathfrak{L}_1 = \{p, a, b, c, d, e, z\}.$

(2) $\qquad \mathfrak{L}_0 = \{i\}; \mathfrak{L}_1 = \{p, a, d\}; \mathfrak{L}_2 = \{b, c, e, z\}.$

(3) $\qquad \mathfrak{L}_0 = \{i\}; \mathfrak{L}_1 = \{p, b, e\}; \mathfrak{L}_2 = \{a, c, d, z\}.$

Thus the primaries $a$ and $b$ both have evaluations associated with them. But no other primary determines an evaluation.

Consider $c$ for example. Its primary cleavage is $\mathfrak{L}_0 = \{i\}$, $\mathfrak{L}^* = \{p, a, b, e, d\}$, $\mathfrak{L}'' = \{c, z\}$. But $\mathfrak{L}^*$ is not a structure, so that no evaluation is defined. Similar results hold for $d$, $e$ and $z$.

Thus while any discrete evaluation of a structure satisfying N 1 and N 2 determines a chain of primaries all associated with the same prime, not every such chain determines an evaluation. On the other hand, every prime determines at least one evaluation; namely that determined by $\pi a = 1$ if $p \supset a$; $\pi a = 0$ if $p \not\supset a$.

If the evaluation $\pi$ is bounded, so that $\mathfrak{S}$ separates into a finite number of residue classes modulo $\pi$,

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}_1 + \cdots + \mathfrak{S}_n,$$

then the evaluation may be defined in a manner strictly analogous to the evaluations of a finite principal ideal ring. Namely, let the leader of $\mathfrak{S}_n$ be $q$, and let $p$ be its associated prime. For any other element $a$ of $\mathfrak{S}$, there is then a least power of $p$ such that $q \supset ap^s$. We then may define

$$\pi a = n - s \text{ if } q \supset ap^s,\ q \not\supset ap^{s-1}.$$

In particular, this definition applies to any discrete evaluation of a residuated lattice of finite order.

## IV. Evaluations Preserving Residuation

**10.** We shall conclude by giving a few properties of bounded evaluations under which residuation is preserved. Consider an evaluation satisfying E 1, E 2 and

E 3 (i) $\pi(a, b) = (\pi a, \pi b)$.
E 5 $\pi a:b = \pi a:\pi b = \pi a - (\pi a, \pi b)$.
E 6 *The evaluation is bounded.*

Let $\sigma$ as before denote the least upper bound of the values of $\pi$. If $\mathfrak{S}$ does not contain a null element $z$, we may adjoin $z$ to $\mathfrak{S}$ without destroying the residuation by defining $z:z = i$, $zi = z$, $z:x = zx = z$, $x \neq i$, $x \in \mathfrak{S}$. (This fact is a special instance of theorem 8.2 of Ward-Dilworth [2].) Clearly $\pi z = \sigma$.

THEOREM 10.1. *If $a$, $b$ are any two elements of $\mathfrak{S}$, then*

E 4. $\pi ab = (\pi a + \pi b, \sigma)$.

PROOF. Adjoin $z$ to $\mathfrak{S}$. Then by E 5, $\pi z:\pi ab = \pi(z:ab) = \pi((z:a):b) = (\pi z:\pi a):\pi b$ or $\sigma - \pi ab = \sigma - \pi a - (\sigma - \pi a, \pi b)$. Hence since $\sigma$ is finite, $\pi ab = \pi a + (\sigma - \pi a, \pi b)$, giving E 4.

Let $\lambda$ denote the greatest lower bound of all positive values of $\pi$.

THEOREM 10.2. *If $\lambda = 0$, every real number in the interval $(0, \sigma)$ is a limit point of values. If $\lambda > 0$, then the evaluation is discrete.*

This theorem does not require the evaluation to be bounded. The first part of the theorem uses only E 4, and is true for any evaluation. But the second part of the theorem depends essentially on E 5, as may be shown by simple examples.

PROOF. If $\lambda = 0$, we may select a sequence of elements $a_1, a_2, \cdots, a_n, \cdots$ of $\mathfrak{S}$ such that $\alpha_n = \pi a_n > 0$, $\alpha_{n+1} \leqq \alpha_n$, $\lim \alpha_n = 0$. Since $\alpha_n$ is positive, for any positive $\beta$ there exists an integer $r_n$ such that $r_n\alpha_n < \beta \leqq (r_n + 1)\alpha_n$ for all sufficiently large $n$. Then if $b_n = a_n^{r_n}$, by E 4, $\pi b_n = r_n\alpha_n$ and $\lim \pi b_n = \beta$.

Suppose that $\lambda > 0$. *Then there exists an element $l$ of $\mathfrak{S}$ such that $\pi l = \lambda$.* For otherwise, we may pick a sequence of elements $x_i$ of $\mathfrak{S}$ such that $\pi x_n > \pi x_{n+1} > \lambda$; $\lim \pi x_n = \lambda$. Choose $m$ so that $\pi x_m < 2\lambda$. Then by E 5, $\pi x_m:x_{m+1} = \pi x_m - \pi x_{m+1} < 2\lambda - \lambda < \lambda$. Hence $\pi x_m = \pi x_{m+1}$, giving a contradiction.

Let $b$ be any other element of $\mathfrak{S}$ with a positive value $\pi b$. Then we can choose a positive integer $r$ such that $r\lambda \leqq \pi b < (r + 1)\lambda$. Then by E 5 and E 4 $\pi b:l^r = \pi b - \pi(b, l^r) = \pi b - r\lambda < \lambda$. Hence $\pi b = r\lambda$, and the evaluation is discrete.

THEOREM 10.3. *If a discrete evaluation satisfies the conditions* E 1, E 2, E 3 (i) *and* E 5, *then it satisfies* E 3 (ii); *that is*

$$\pi[a, b] = [\pi a, \pi b] \qquad \text{for any elements } a, b \text{ of } \mathfrak{S}.$$

PROOF. We may assume that $\pi b \geqq \pi a \geqq 0$, so that we need only prove that $\pi[a, b] = \pi b$ if $\pi a \leqq \pi b$.

Since the evaluation is discrete, we may assume that $\pi a = r$, $\pi b = s$ where $r$ and $s$ are positive integers. Furthermore, there exists an element $l$ of $\mathfrak{S}$ such that $\pi l = 1$.

LEMMA 1. $\pi[l^r, l^s] = \pi l^s$.

For since $r \leqq s$, $l^r \supset l^s$ so that $[l^r, l^s] = l^s$. The result now follows from E 2.

LEMMA 2. *If* $\pi a = \pi c$ then $\pi[a, b] = \pi[c, b]$.

For $\pi[a, b]:[c, b] = \pi[a:[c, b], b:[c, b]] = \pi a:[c, b]$ by E 2. Since $c \supset [c, b]$, $\pi c \leqq \pi[c, b]$. Hence $\pi a \leqq \pi[c, b]$, so that $\pi a:[c, b] = 0$ by E 5. Thus $\pi[a, b]: [c, b] = 0$. Hence by E 5, $\pi[a, b] \leqq \pi[c, l]$. Similarly, $\pi[c, b] \leqq \pi[a, b]$, giving the lemma.

The theorem now follows easily. For since $\pi a = \pi l^r$ and $\pi b = \pi l^s$ by E 4, the lemmas give

$$\pi[a, b]^1 = \pi[l^r, b] = \pi[l^r, l^s] = \pi l^s = \pi b.$$

If the evaluation preserves residuation, the class separation

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots \tag{7.1}$$

is subject to very stringent conditions; for if $a \in \mathfrak{S}_i$ and $b \in \mathfrak{S}_j$, then we must have $a:b \in \mathfrak{S}_0$ or $a:b \in \mathfrak{S}_{i-j}$ according as $i - j \leqq 0$.

The reader can easily show thereby that the structure discussed in section 9 admits of no such residuation. The interesting correspondence we have developed between primes and evaluations is thus destroyed.

CALIFORNIA INSTITUTE OF TECHNOLOGY,
PASADENA, CALIFORNIA

REFERENCES

A. A. ALBERT
1. *Modern Higher Algebra*, Chicago, 1937.

GARETT BIRKHOFF
1. *Rings of sets.* Duke Math. Jour., vol. 3 (1937), pp. 443-454.

R. DEDEKIND
1. *Gesammelte Werke*, vol. 2, Braunschweig (1931), paper XXI.

R. P. DILWORTH
1. *Abstract residuation over lattices.* Bulletin Am. Math. Soc., vol. 44 (1938) pp. 262-268.

V. GLIVENKO
1. *Géométrie des systèmes de choses normées.* Am. Jour. of Math., vol. 58 (1936), pp. 799-828.
2. *Contribution à l'étude des systèmes de choses normées.* Am. Jour. of Math., vol. 59 (1937), pp. 941-956.

W. KRULL
1. *Axiomatische Begründung der allgemein en Idealtheorie.* Erlanger Sitzungsberichte, vol. 36 (1924), pp. 47-63.

O. ORE
1. *On the foundations of abstract algebra.* These Annals, vol. 36 (1935), pp. 406-437.

M. H. STONE
1. *The theory of representations of Boolean algebras.* Trans. Am. Math. Soc., vol. 40 (1936), pp. 37-111.

B. L. VAN DER WAERDEN
1. *Moderne Algebra*, vol. 2. Berlin (1931).

H. WALLMAN
1. *Lattices and topological spaces.* These Annals, vol. 39, January 1938, pp. 112-126.