# NOTE ON THE GENERAL RATIONAL SOLUTION OF THE EQUATION $ax^2 - by^2 = z^3$.*

By MORGAN WARD.

---

1. In a recent paper in this journal, E. Fogels (Fogels 1) has utilized the elements of algebraic number theory to determine all rational solutions of the diophantine equation

$$ax^2 - by^2 = z^3, \qquad a,\ b \text{ rational}, \neq 0. \tag{1}$$

I show here that the underlying reason for the success of Fogel's method is an arithmetical relationship between the degrees of the left and right sides of (1). For consider the more general equation

$$a_0x^m + a_1x^{m-1}y + \cdots + a_my^m = z^n \tag{2}$$

where $m, n$ are positive integers and $a_0, \cdots, a_m$ rational. The general rational solution of (2) may be immediately written down provided that we impose the following condition: *The degrees m and n are co-prime.*

In this case we may reduce (2) to the simple equation

$$Y^m = Z^n. \tag{3}$$

2. There are four types of rational solutions of (2) to consider; namely, (i) solutions with $z = 0$, (ii) solutions with $x = 0$, (iii) solutions with $y = 0$ and (iv) solutions with none of $x, y, z$ zero.

The first three types are readily treated.

(i) If $z = 0$, the solution of (2) may be expressed in the form $x = tx'$, $y = ty'$. Here $t$ is an arbitrary rational and $x'$, $y$ are either both zero or co-prime integers satisfying $a_0x'^m + \cdots + a_my'^m = 0$. Thus only a finite number of choices for $x'$ and $y'$ are possible.

(ii) If $x = 0$ and $a_m = 0$, then $y$ is arbitrary, $z$ is zero. If $a_m \neq 0$, then

---

* Received August 31, 1938.

(2) reduces to $a_m y^m = z^n$. Since $m$ and $n$ are co-prime, we may determine integers $k$ and $l$ such that

(4) $$1 + km = ln.$$

On letting $y = a^k{}_m Y$, $z = a^l{}_m Z$ (2) is reduced to (3).

(iii) If $y = 0$, (2) is either trivial or similarly reducible to (3).

(iv) Let $x, y, z$ be a rational solution of (2) with $xyz \neq 0$. Then if we let $x = uy$, $u$ is rational and not zero, and (2) becomes

$$y^m w = z^n, \quad w = a_0 u^m + a_1 u^{m-1} + \;.\;.\; + a_m \neq 0.$$

As in type (ii), we let $y = w^k Y$, $z = w^l Z$. Then $Y, Z$ are rational and non-zero and

(3) $$Y^m = Z^n.$$

**3.** Equation (3) is a very special case of a type of diophantine system which I have already discussed in this journal. (Ward [1]). To solve (3), we let

$$Y = \Pi p^\alpha, \qquad Z = \Pi p^\beta$$

where the products extends over all primes, and the $\alpha$ and $\beta$ are integers having only a finite number of non-zero values. Then (3) yields the condition $m\alpha = n\beta$. Hence $\alpha = (n:m)y$, $\beta = (m:n)y$ where $y$ is an integer, and $m:n$ denotes the residual $m/(m,n)$ of $n$ with respect to $m$. (Ward [2]). Since $m$ and $n$ are co-prime, $n:m = n$, $m:n = m$. Thus

$$Y = v^n, \qquad Z = v^m$$

where $v$ is a rational number. On combining these results, we find that *if $x, y, z$ is any rational solution of* (2) *with $xyz \neq 0$, then there exist rationals $u$ and $v$ such that*

(5)
$$\begin{aligned} x &= u(a_0 u^m + a_1 u^{m-1} + \cdots + a_m)^k v^n, \\ y &= (a_0 u^m + a_1 u^{m-1} + \cdots + a_m)^k v^n, \\ z &= (a_0 u^m + a_1 u^{m-1} + \cdots + a_m)^l v^m. \end{aligned}$$

Here $k$ and $l$ are integers satisfying

(4) $$1 + km = ln.$$

Conversely, if $u$ and $v$ are rational and $a_0u^m + \cdots + a_m \neq 0$, $uv \neq 0$, then (5) always gives a rational solution of (2) with $xyz \neq 0$.

4. In particular, let $m = 2$, $n = 3$, $a_0 = a$, $a_1 = 0$, $a_2 = -b$. (2) then reduces to (1), and we may take $k = l = 1$ in (3). The formulas (5) become

$$\begin{aligned} x &= u(au^2 - b)v^3 \\ y &= \phantom{u}(au^2 - b)v^3 \\ z &= \phantom{u}(au^2 - b)v^2 \end{aligned}$$

If $a \neq 0$, we can make the reversible rational substitution $u = s/a$, $v = at$. We thus obtain

$$\begin{aligned} x &= as(s^2 - ab)t^3 \\ y &= a^2(s^2 - ab)t^3 \\ z &= \phantom{a}a(s - ab)t^2. \end{aligned}$$

These are the formulas for the solution of (1) obtained by Fogels.

PASADENA, CALIFORNIA.

## REFERENCES

E. Fogels, 1. *American Journal of Mathematics*, vol. 60 (July, 1938), pp. 734-736.

M. Ward, 1. *American Journal of Mathematics*, vol. 55 (1933), pp. 67-76; 2. *American Journal of Mathematics*, vol. 59 (1937), pp. 921-926.