possesses an approximate functional equation, cf. reference 1, where this last result was used in connection with the mean value of $\zeta(s)$ on the critical line. The case $k = 1$ is due to Wigert.

It seems very difficult to establish corresponding results for

$$f_k(x, y) = \sum_{n=1}^{\infty} d_k(n) V_k(n^2x^2) e^{iny} \tag{5}$$

[1] Bellman, R., "Wigert's Approximate Functional Equation and the Riemann Zeta-Function," *Duke Math. J.*, **16**, 547–552 (1949).

[2] Hardy, G. H., "On Dirichlet's Divisor Problem," *Proc. Lond. Math. Soc.*, **15**, 1–20 (1916).

[3] Hardy, G. H., "Some Multiple Integrals," *Quart. J. Math.*, **39**, 357–375 (1908).

[4] Maass, H., "Uber eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichetschen Reihen durch Funktionalgleichungen," *Math. Ann.*, **121**, 141–183 (1949).

[5] Siegel, C. L., "Uber die analytische Theorie der quadratischen Formen," *Ann. Math.*, **136**, 527–606 (1935).

ARITHMETICAL PROPERTIES OF POLYNOMIALS ASSOCIATED WITH THE LEMNISCATE ELLIPTIC FUNCTIONS

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Communicated by H. S. Vandiver, April 18, 1950

1. I have studied elsewhere the arithmetical properties of certain polynomials associated with the real multiplication of elliptic functions.[1] Such polynomials include as a special case the function $U_n = (a^b - b)_n/(a - b)$ first systematically studied by Lucas[2] and Sylvester[3] when expressed as a polynomial in $P = a + b$ and $Q = ab$.

I have recently investigated the polynomials associated with the simplest type of complex multiplication of elliptic functions; namely, the so-called lemniscate case for which the period ratio τ has the value $i = \sqrt{-1}$ and the Weierstrass invariant g_3 is zero.

In the account which follows, the small greek letters α, ϵ, λ, μ, ν and π will be used for elements of the ring G of Gaussian integers. $\bar{\alpha}$ and $N\alpha$ denote the conjugate and norm of α in G. α is said to be odd, oddly even or totally even according as $N\alpha$ is congruent to one, two or zero modulo 4. The letter ϵ is reserved for denoting any one of the four units ± 1, $\pm i$ of the ring G.

2. Let u be a complex variable, and $\wp(u)$ the Weierstrass $\wp$-function formed with the invariants $g_2 = 4w$, $g_3 = 0$. Let $E\mu = E\mu(u)$ equal 1,

$\sqrt{\mathcal{P}(u)}$ or $\mathcal{P}'(u)$ according as μ is odd, oddly even or totally even. Finally let

$$\Psi_\mu = \Psi_\mu(u) = \sigma(\mu u)/\sigma(u)^{N\mu} \tag{1}$$

where $\sigma(u)$ is the Weierstrass sigma function. Then $\Psi_\mu \div E_\mu$ is an even elliptic function with the same periods as $\mathcal{P}(u)$. More specifically,

$$\Psi_\mu(u) = E_\mu(u)P_\mu(z, w) \tag{2}$$

where

$$P_\mu = P_\mu(z, w) = \sum_{r=0}^{q} \pi_r z^{q-2r} w^r \tag{3}$$

is a polynomial in $z = \mathcal{P}(u)$ and $w = \frac{g_2}{4}$ whose coefficients π_r are Gaussian integers with $\pi_0 = \mu$. The degree q of P_μ in z depends in a simple way on $N\mu$. The arithmetical properties of these polynomials were the object of the investigation; (3) is the elliptic function analog of the cyclotomic polynomial $\frac{z^n - 1}{z - 1}$ associated with Lucas' U_n.

3. The arithmetical properties of the polynomials P_μ closely parallel the properties of Lucas' U_n. The main new feature of interest (not occurring in the real multiplication case) is a genuine double numerical periodicity when the free variables z and w are given fixed values in G, and the residues of the resulting sequence in G are considered for moduli in G. Indeed Lucas claimed in his fundamental paper and elsewhere to have discovered doubly periodic numerical functions connected with the elliptic functions, but he apparently published nothing on this subject.[4]

The Lucas polynomial U_n may be defined as the solution of a simple difference equation with prescribed initial values. The function Ψ_μ may be similarly defined as a solution of the difference equation

$$\Omega_{\mu+\nu}\Omega_{\mu-\nu} = \Omega_{\mu+1}\Omega_{\mu-1}^2 - \Omega_{\nu+1}\Omega_{\nu-1}^2 \tag{4}$$

with prescribed initial values; in particular, $\Psi_0 = 0$ and $\Psi_\epsilon = \epsilon$. (A table of the corresponding initial values of P_μ for small $N\mu$ is given at the close of the paper.)

Consequently, just as in the real multiplication case,[5] the polynomials P_μ may be defined purely algebraically as modified solutions of (4). On using this algebraic definition in conjunction with the function-theoretic definitions (1) and (2), the following results were obtained.

(i) *If z, w are indeterminates, the correspondence $\nu \to P_\mu(z, w)$ is a mapping of the ring G into the polynomial ring $G(z, w)$ which preserves*

division; that is ν divides μ in G implies that P_ν divides P_μ in $G(z, w)$. Furthermore,

$$P_\epsilon = \epsilon, \qquad P_{\epsilon\mu} = \epsilon P_\mu, \qquad P_{\bar{\mu}} = \bar{P}_\mu.$$

Therefore if μ is a rational integer, all the coefficients π_τ of P_μ are rational integers, and P_μ reduces to the polynomial of the real multiplication case studied in reference 1.

Let z_0, w_0 be fixed rational integers. Then $h_\nu = P_\nu(z_0, w_0)$ is a Gaussian integer and the correspondence $\nu \to h_\nu$ is a mapping of G into itself preserving division. Let π from now on denote a fixed Gaussian prime. An integer λ is called a zero of h_ν modulo π if $h_\lambda \equiv 0 \pmod{\pi}$ and a rank of apparition of π in $\{h_\nu\}$ if $h_\lambda \equiv 0 \pmod{\pi}$ but $h_\mu \not\equiv 0 \pmod{\pi}$ for μ any proper divisor of λ.

(ii) *If π is odd, the zeros of the prime π in $\{h_\nu\}$ form an ideal m which is never the zero ideal.* Furthermore if λ is any rank of apparition of π in $\{h_\nu\}$, m is the principal ideal determined by λ.

(iii) *If π is an odd complex Gaussian prime, then*[6]

$$P_\mu(z, w) \equiv P_\mu(0, w) \pmod{\pi}.$$

(iv) *The sequence $\{h_\nu\}$ becomes numerically periodic modulo π. The moduli of its periods is contained in the ideal m of its zeros modulo π.*

(v) *Given a specific term h_λ of $\{h_\nu\}$, the only odd primes π which can have rank of apparition λ in $\{h_\nu\}$ are either divisors of λ, or primes for which the polynomial $P_\lambda(z, w)$ splits completely into linear factors or completely into quadratic factors in the residue class ring $G(z, w)/(\pi)$.* Such primes lie in arithmetical progressions whose common constant difference is a function of λ alone.[7]

(v) generalizes the well-known result of Lucas and Sylvester that if P and Q are rational integers, all primitive prime divisors of U_l are either divisors of l or of the form $kl \pm 1$.

The first few polynomials P_μ are as follows: $P_0 = 0$, $P_1 = 1$, $P_i = i$, $P_{1+i} = 1 + i$, $P_2 = 2$, $P_3 = 3z^4 - 6wz^2 - w^2$, $P_{1+2i} = (1 + 2i)z^2 - w$, $P_{3+i} = (3 + i)z^4 - 2(1 + 3i)wz^2 + (3 + i)w^2$. All the remaining P_μ can be calculated from the recursion (4) and the relations $P_{\bar{\mu}} = \bar{P}_\mu$, $P_{\epsilon\mu} = \epsilon P_\mu$.

Qualitatively similar results hold for the polynomials associated with any complex multiplication of $\wp(u)$.[8]

A more complete account of these and other results with proofs will be published elsewhere.

[1] *Am. J. Math.*, **70**, 31–74 (1948). Various algebraic properties of these polynomials are developed in Halphen's treatise on elliptic functions.

[2] *Ibid.*, **1**, 184–240, 289–321 (1878).

[3] *Ibid.*, **2**, 357–380 (1879).

[4] In particular, Lucas stated to C. A. Laisant that there was a remarkable connection between his doubly periodic numerical functions and Fermat's last theorem. See Bell, E. T., *Bull. Am. Math. Soc.*, **29**, 401–406 (1923). The crux of the matter is to understand what Lucas meant by "double periodicity." Since the modules of the ring of Integers are all principal ideals, no numerical function of the rational integer n can be doubly periodic. The simplest case in which double periodicity in the usually understood sense can occur is for numerical functions over the ring of Gaussian integers.

[5] See Chapter V of reference 1.

[6] Due to Eisenstein for the Jacobian lemniscate polynomials and used by him to prove the biquadratic reciprocity law. See his *Math. Abb.*, third paper or *J. Math.* (*Crelle*), **30**, 184–187 (1846).

[7] This result follows from Abel's theorem that the Galois group of the equation $P_\mu(z, w) = 0$ in z is commutative and of order q.

[8] The equi-harmonic case when the period ratio τ is a complex cube root of unity and the invariant g_2 vanishes is being studied in detail by Lincoln K. Durst.