A CERTAIN CLASS OF POLYNOMIALS.*

By Morgan Ward.

Contents.

1. The remarkable properties which a sequence of polynomials

$$Y_0(x),\ Y_1(x),\ Y_2(x),\ \cdots$$

of degrees $0, 1, 2, \cdots$ in x possesses when the two functional equations

$$\frac{d\,Y_n(x)}{dx} = Y_{n-1}(x), \qquad Y_n(-x-1) = (-1)^n\, Y_n(x)$$

are satisfied have been systematically developed by N. Nielsen.† It is of some interest to consider sequences of polynomials satisfying the more general equations

$$\text{(1)} \qquad \frac{d\,Y_n(x)}{dx} = Y_{n-1}(x),$$

$$\text{(2)} \qquad Y_n(ax+b) = \tau_n\, Y_n(x), \qquad (n = 0, 1, 2, \cdots)$$

where a, b are any complex numbers. Such a sequence we call a *regular* sequence. The main properties of regular sequences are as follows:

2. *If a is not a root of unity, there is only one regular sequence; namely*

$$Y_n(x) = \frac{c}{n!}\left(x + \frac{b}{a-1}\right)^n, \qquad \tau_n = a^n \qquad (n = 0, 1, 2, \ldots).$$

If however, a is a root of unity, say a primitive πth root, there exist an infinite number of regular sequences. Let

$$(k_0 + k_1 t + k_2 t^2 + \cdots)\, e^{xt} = K_0(x) + K_1(x)\, t + K_2(x)\, t^2 + \cdots$$

* Received February 4, 1929.

† *Traité Élémentaire des Nombres de Bernoulli*, Paris 1923.

be the generating function of any such sequence. Then if $r\pi \leqq n < (r+1)\pi$, k_n *may be expressed as a linear function of* $k_0, k_\pi, k_{2\pi}, \cdots, k_{r\pi}$

$$k_n = \sum_{s=0}^{r} k_{s\pi} H'_{n-s\pi},$$

where $H'_0, H'_1, H'_2, \cdots$ *depend only on* a *and* b *and are independent of the particular regular sequence* $[K_n(x)]$ *we have selected. Moreover, if* a *is a given* π*th root of unity, all possible solutions* (1) *and* (2) *are obtained by giving* $k_0, k_\pi, k_{2\pi}, \cdots$ *the proper values.*

3. We now proceed to the proof of these results. A sequence of polynomials

$$H_0(x),\ H_1(x),\ H_2(x), \cdots$$

of degrees $0, 1, 2, \cdots$ in x which satisfies (1) is said to be *harmonic.** The following properties of harmonic sequences are easily proved.*

(i) If $[H_n(x)]$ is a harmonic sequence, there exists a sequence of constants

$$[h_n]:\quad h_0, h_1, h_2, \cdots, h_n, \cdots,$$

such that for all values of n,

$$H_n(x) = \frac{h_0 x^n}{n!} + \frac{h_1 x^{n-1}}{(n-1)!} + \frac{h_2 x^{n-2}}{(n-2)!} + \cdots + h_n, \qquad H_n(0) = h_n.$$

We denote such a sequence by $[H_n(x), h_n]$.

(ii) If†

$$H(x, t) = H_0(x) + H_1(x)t + H_2(x)t^2 + \cdots,$$
$$h(t) = h_0 + h_1 t + h_2 t^2 + \cdots$$

are the generating functions of the sequences $[H_n(x)]$, $[h_n]$

$$H(x, t) = e^{xt} h(t).$$

(iii) If b is any constant,

$$H_n(x+b) = H_n(b) + \frac{x H_{n-1}(b)}{1!} + \frac{x^2 H_{n-2}(b)}{2!} + \cdots + \frac{x^n H_0(b)}{n!}.$$

(iv) Let $[K_n(x), k_n]$ be a second harmonic sequence. Then there exists a unique ordinary sequence $[\alpha_n]$ such that for all values of n

$$K_n(x) = \alpha_0 H_n(x) + \alpha_1 H_{n-1}(x) + \cdots + \alpha_n H_0(x).$$

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* Nielsen, l. c., chap. III, section XI.

† This property of harmonic sequences is substantially due to Appell: Annales de l'École Normale, (2) 10 (1880), 119–120.

4. We pass now to regular sequences. If in (2), $a = 0$ or 1, the sequences are trivial, so that we shall exclude these cases in all that follows. It is clear that for any sequence $[L_n(x)]$ satisfying (2)

$$\tau_n = a^n, \quad L_n(b) = a^n L_n(0) \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

Now suppose $[H_n(x)]$ is a harmonic sequence for which

$$H_n(b) = a^n H_n(0) \qquad (n = 0, 1, 2, \cdots).$$

Then by (iii) and our hypothesis

$$H_n(ax+b) = \sum_{s=0}^{n} H_{n-s}(b)\, a^s \frac{x^s}{s!} = \sum_{s=0}^{n} a^{n-s} H_{n-s}(0)\, a^s \frac{x^s}{s!},$$
$$H_n(ax+b) = a^n H_n(x)$$

by (i). Hence we have proved

THEOREM 1. *The necessary and sufficient condition that a harmonic sequence* $[H_n(x), h_n]$ *be regular is that*

$$H_n(b) = a^n H_n(0) \qquad (n = 0, 1, 2, \cdots). \tag{4}$$

5. If we expand the left side of (4) by (i), we obtain

$$\sum_{r=0}^{n} \frac{h_{n-r} b^r}{r!} = a^n h_n \qquad (n = 0, 1, 2, \cdots), \tag{5}$$

so that

$$\begin{array}{llllll}
h_0 & & & & & = h_0, \\
\frac{b h_0}{1!} + & h_1 & & & & = a h_1, \\
\frac{b^2 h_0}{2!} + & \frac{b h_1}{1!} & + \quad h_2 & & & = a^2 h_2, \\
\frac{b^3 h_0}{3!} + & \frac{b^2 h_1}{2!} & + \quad \frac{b h_2}{1!} & + \quad h_3 & & = a^3 h_3, \\
\cdots & & & & & \\
\cdots & & & & & \\
\end{array}$$
$$\frac{b^r h_0}{r!} + \frac{b^{r-1} h_1}{(r-1)!} + \frac{b^{r-2} h_2}{(r-2)!} + \frac{b^{r-3} h_3}{(r-3)!} + \cdots + \frac{b h_r}{1!} + h_r = a^r h_r,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

If we assume $a^k \neq 1$ $(1 \leqq k \leqq r)$ we can solve the first $r+1$ of these equations for h_r in terms of h_0, a, b by determinants. We thus obtain after a few simplifications

$$h_r = \frac{h_0 b^r \Delta_r(a)}{(a-1)(a^2-1)\cdots(a^r-1)}, \tag{6}$$

where $\Delta_r(a)$ is given by

$$\Delta_r(a) = \begin{vmatrix} \frac{1}{1!} & 1-a & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1-a^2 & 0 & \cdots & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1-a^3 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \frac{1}{1!} & 1-a^{r-1} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \frac{1}{(r-3)!} & \cdots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

Moreover this value of h_r is unique. We find by solving for h_0, h_1 and h_2, provided $a^2 \neq 1$

$$h_0 = \frac{h_0 \lambda^0}{0!}, \qquad h_1 = \frac{h_0 \lambda^1}{1!}, \qquad h_2 = \frac{h_0 \lambda^2}{2!}$$

where

$$\lambda = \frac{b}{a-1}. \tag{7}$$

Hence let us assume

$$h_r = \frac{h_0 \lambda^r}{r!}, \qquad a^{r+1} \neq 1 \qquad (0 \leqq r \leqq k). \tag{8}$$

From (5) and our hypothesis

$$(a^{k+1}-1) h_{k+1} = \sum_{r=0}^{k} \frac{h_r b^{k+1-r}}{(k+1-r)!} = h_0 b^{k+1} \sum_{r=0}^{k} \frac{1}{r!\,(k+1-r)!\,(a-1)^r}.$$

Then

$$\begin{aligned}(k+1)!\,(a-1)^{k+1}(a^{k+1}-1)\,h_{k+1} &= h_0 b^{k+1} \sum_{r=0}^{k} \binom{k+1}{r} (a-1)^{k+1-r} \\ &= h_0 b^{k+1} (a^{k+1}-1),\end{aligned}$$

so that by (7) and (8)

$$h_{k+1} = \frac{h_0 \lambda^{k+1}}{(k+1)!}.$$

Thus by induction (6) holds for all values of r, provided a is not a root of unity. On comparing (6) and (8) we deduce

THEOREM 2. *The determinant* $\Delta_r(a)$ *has the value*

$$\Delta_r(a) = (1+a)(1+a+a^2) \cdots (1+a+a^2+\cdots+a^r)/r!$$

Moreover $\Delta_r(1) = 1$, $\Delta_r(0) = 1/r$, *and* $\Delta_r(a)$ *vanishes if* a *is any primitive* π*th root of unity* $(2 \leqq \pi \leqq r)$.

6. From (8) and (i) it follows that

$$(9) \qquad H_n(x) = \sum_{r=0}^{n} \frac{h_0}{r!\,(n-r)!} \lambda^{n-r} x^r = \frac{h_0}{n!}(x+\lambda)^n.$$

If (9) holds,

$$\frac{dH_n(x)}{dx} = H_{n-1}(x), \quad H_n(ax+b) = \frac{h_0}{n!}\left(ax+b+\frac{b}{a-1}\right)^n = a^n H_n(x),$$

so that (9) is a regular sequence whether or not a is a root of unity. On collecting these results, we have

THEOREM 3. *A sequence* $[H_n(x)]$ *satisfying the two functional equations*

$$(1) \qquad \frac{dY_n(x)}{dx} = Y_{n-1}(x),$$

$$(2) \qquad Y_n(ax+b) = a^n Y_n(x), \qquad (a \neq 0, 1),$$

is always given by

$$(8) \qquad h_n = \frac{h_0 \lambda^n}{n!},$$

$$(9) \qquad H_n(x) = \frac{h_0 (x+\lambda)^n}{n!},$$

where h_0 *is an arbitrary constant, and*

$$(7) \qquad \lambda = \frac{b}{a-1}.$$

Moreover if a *is not a root of unity, this solution of* (1) *and* (2) *is unique.*

7. The first result stated in section 2 is now proved. The regular sequences for which a is a root of unity are of much greater interest. It will be convenient in studying them to define $H_n(x)$ to mean the polynomial $\frac{(x+\lambda)^n}{n!}$ which gives the simplest regular sequence and to define a *cyclic sequence of order* π to mean any solution of (1) and (2) for which a is a primitive πth root of unity. Then

$$(10) \qquad a^r = 1$$

when and only when $r \equiv 0 \bmod \pi$.

THEOREM 4. *If* $[K_n(x), k_n]$ *is a cyclic sequence of order* π *then* $K_n(x)$ *may be uniquely represented in the form*

$$(11) \qquad K_n(x) = \alpha_0 H_n(x) + \alpha_\pi H_{n\pi}(x) + \cdots + \alpha_{r\pi} H_{n-r\pi}(x)$$

where $\alpha_0, \alpha_\pi, \cdots, \alpha_{r\pi}$ *are constants, and* $r\pi \leqq n < (r+1)\pi$. *Moreover every sequence of the form* (11) *is a cyclic sequence of order* π.

For since $[K_n(x)]$ and $[H_n(x)]$ are both harmonic sequences, there exists by (iv) a unique ordinary sequence $[\alpha_n]$ such that

$$K_n(x) = \alpha_0 H_n(x) + \alpha_1 H_{n-1}(x) + \cdots + \alpha_n H_0(x).$$

Hence the first part of the theorem will be proved if we can show $\alpha_r = 0$, $r \not\equiv 0 \bmod \pi$. Now

$$K_n(ax+b) = \sum_{r=0}^{n} H_{n-r}(ax+b) = \sum_{r=0}^{n} \alpha_r a^{n-r} H_{n-r}(x),$$

$$K_n(ax+b) = a^n K_n(x) = \sum_{r=0}^{n} \alpha_r a^n H_{n-r}(x).$$

Write $x + \lambda = y$ so that $H_{n-r}(x) = \dfrac{y^{n-r}}{(n-r)!}$. Then we have identically in y

$$\sum_{r=0}^{n} \frac{\alpha_r a^{n-r}}{(n-r)!} y^{n-r} = \sum_{r=0}^{n} \frac{\alpha_r a^n y^{n-r}}{(n-r)!}$$

so that by equating coefficients of corresponding powers of y,

$$\alpha_r (a^r - 1) = 0 \qquad (r = 0, 1, 2, \cdots, n).$$

(11) now follows from (10). The last part of the theorem is obvious from (9) and (10).

If we write in (11) first $x = -\lambda$ and then $x = 0$ we obtain

THEOREM 5. *For any cyclic sequence* $[K_n(x)]$ *of order* π,

$$\begin{aligned} K_n(-\lambda) &= 0, \qquad n \not\equiv 0 \bmod \pi, \\ &= \alpha_n, \qquad n \equiv 0 \bmod \pi, \end{aligned}$$

$$(12) \quad a^{-n} K_n(b) = K_n(0) = k_n = \frac{\alpha_0 \lambda^n}{n!} + \frac{\alpha_\pi \lambda^{n-\pi}}{(n-\pi)!} + \cdots + \frac{\alpha_{r\pi} \lambda^{n-r\pi}}{(n-r\pi)!},$$

where $r\pi \leqq n < (r+1)\pi$ *and* $n = 0, 1, 2, \cdots$.

We have from (12)

$$\begin{aligned} k_0 &= \alpha_0, \\ k_\pi &= \frac{\alpha_0 \lambda^\pi}{\pi!} + \alpha_\pi, \\ k_{2\pi} &= \frac{\alpha_0 \lambda^{2\pi}}{(2\pi)!} + \frac{\alpha_\pi \lambda^\pi}{\pi!} + \alpha_{2\pi}, \\ k_{3\pi} &= \frac{\alpha_0 \lambda^{3\pi}}{(3\pi)!} + \frac{\alpha_\pi}{(2\pi)!} \lambda^{2\pi} + \frac{\alpha_{2\pi}}{\pi!} \lambda^\pi + \alpha_{3\pi}, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Hence $\alpha_0, \alpha_\pi, \alpha_{2\pi} \cdots$ are uniquely determined in terms of $k_0, k_\pi, k_{2\pi}, \cdots$. Since (12) gives k_n for all values of n in terms of $\alpha_0, \alpha_\pi, \alpha_{2\pi}, \cdots$ we have proved

THEOREM 6. *If $[K_n(x), k_n]$ is any cyclic sequence of order π and if the values of $k_0, k_\pi, k_{2\pi}, \cdots$ are given, then all the other k_n are uniquely determined.*

8. We shall now give the explicit expression for k_n as a function of $k_0, k_\pi, k_{2\pi}, \cdots$ proving the last result in section 2. It is convenient to borrow a definition from the calculus of generating functions.

Given any sequence of constants

$$[c_n]: \qquad c_0, c_1, c_2, \cdots \qquad\qquad (c_0 \neq 0),$$

the sequence

$$[c'_n]: \qquad c'_0, c'_1, c'_2, \cdots$$

determined by the equations:

$$c'_0 c_0 = 1$$
$$c_0 c'_n + c_1 c'_{n-1} + c_2 c'_{n-2} + \cdots + c_n c'_0 = 0 \quad (n = 1, 2, 3, \cdots)$$

is called the *inverse* of the sequence $[c_n]$. It is clear that the generating functions $c(t)$, $c'(t)$ of the two sequences are reciprocals of each other. Assume that $k_0 \neq 0$, and consider the expression

$$A(z^\pi) = \frac{k_0 + k_\pi z^\pi + k_{2\pi} z^{2\pi} + \cdots}{1 + \frac{\lambda^\pi}{\pi!} z^\pi + \frac{\lambda^{2\pi}}{(2\pi)!} z^{2\pi} + \cdots} = u_0 + u_\pi z^\pi + u_{2\pi} z^{2\pi} + \cdots.$$

On clearing of fractions, we have the following set of equations to determine $u_0, u_\pi, u_{2\pi}, \cdots$,

$$k_0 = u_0,$$
$$k_\pi = \frac{u_0 \lambda^\pi}{\pi!} + u_\pi,$$
$$k_{2\pi} = \frac{u_0 \lambda^{2\pi}}{(2\pi)!} + \frac{u_\pi \lambda^\pi}{\pi!} + u_{2\pi},$$
$$\cdots\cdots\cdots\cdots$$

Since these equations become identical with the equations (12) on replacing u by α we have proved

THEOREM 7. *The generating function of the sequence $\alpha_0, \alpha_\pi, \alpha_{2\pi}, \cdots$ in Theorem 5 is given by*

$$A(z^\pi) = \frac{K(z^\pi)}{H(z^\pi)}$$

where

$$K(z^\pi) = \sum_{n=0}^{\infty} k_{n\pi} z^{n\pi},$$

$$H(z^\pi) = \sum_{n=0}^{\infty} \frac{1}{(n\pi)!} \lambda^{n\pi} z^{n\pi} = \frac{1}{\pi} \{e^{a\lambda z} + e^{a^2\lambda z} + \cdots + e^{a^\pi \lambda z}\}.$$

The reciprocal of $H(z^\pi)$ generates the sequence inverse to $\left[\frac{\lambda^{n\pi}}{(n\pi)!}\right]$ since by (8)

$$h_{n\pi} = \frac{\lambda^{n\pi}}{(n\pi)!}$$

we write

$$\frac{1}{H(z^\pi)} = H'(z^\pi) = h'_0 + h'_\pi z^\pi + h'_{2\pi} z^{2\pi} + \cdots. \tag{13}$$

If $\pi = 2$, $a = b = -1$, the case Nielsen has studied*,

$$H'(z^\pi) = \operatorname{sech} \frac{z}{2} = E_0 + \frac{E_2 z^2}{2^2 2!} + \frac{E_4 z^4}{2^4 4!} + \cdots$$

where E_0, E_2, E_4, $\cdots$ are Euler's numbers.

Moreover it is clear that

$$\alpha_{n\pi} = k_{n\pi} h'_0 + k_{(n-1)\pi} h'_\pi + \cdots + k_0 h'_{n\pi}. \tag{14}$$

Let $[H''_n(x), h''_n]$ denote the cyclic sequence defined by

$$H''_n(x) = \frac{h'_0(x+\lambda)^n}{n!} + \frac{h'_\pi(x+\lambda)^{n-\pi}}{(n-\pi)!} + \frac{h'_{2\pi}(x+\lambda)^{n-2\pi}}{(n-2\pi)!} + \cdots$$
$$\cdots + \frac{h'_{r\pi}(x+\lambda)^{n-r\pi}}{(n-r\pi)!},$$

where as usual $r\pi \leqq n \leqq (r+1)\pi$. Then by (12), if $[K_n(x), k_n]$ is any cyclic sequence

$$k_n = \sum_{s=0}^{r} \frac{\alpha_{s\pi} \lambda^{n-s\pi}}{(n-s\pi)!} = \sum_{t=0}^{r} \sum_{s=t}^{r} \frac{k_{t\pi} h'_{(s-t)\pi} \lambda^{n-s\pi}}{(n-s\pi)!}$$

on substituting from (14) for $\alpha_{s\pi}$ and changing the order of summation. Now

$$\sum_{s=t}^{s=r} \frac{h'_{(s-t)\pi} \lambda^{n-s\pi}}{(n-s\pi)!} = \sum_{u=0}^{u=r-t} \frac{h'_{u\pi} \lambda^{n-t\pi-u\pi}}{(n-t\pi-u\pi)!} = H''_{n-t\pi}(0) = h''_{n-t\pi}.$$

Furthermore if

$$H'(t^\pi) e^{tx} = H'_0(x) + H'_1(x) t + H'_2(x) t^2 + \cdots.$$

* Nielsen, l. c., chapter VI.

Then

$$H'_n(x) = \frac{h'_0\, x^n}{n!} + \frac{h'_\pi\, x^{n-\pi}}{(n-\pi)!} + \cdots + \frac{h'_{r\pi}\, x^{n-r\pi}}{(n-r\pi)!} = H''_n(x-\lambda).$$

Hence we have proved

THEOREM 8. *If* $[K_n(x), k_n]$ *is any cyclic sequence of order* π

$$k_n = \sum_{s=0}^{r} k_{s\pi}\, H'_{n-s\pi}(\lambda),$$

where $r\pi \leqq n < (r+1)\pi$, $\lambda = \dfrac{b}{a-1}$ *and the generating function of the polynomials* $H'_n(x)$ *is*

$$\pi e^{tx}[e^{at} + e^{a^2 t} + \cdots + e^{a^\pi t}]^{-1}.$$

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