In particular, if $R(x)>0$,

$$t^x = \Gamma(x+1)\sum_{n=0}^{\infty}(-)^n\binom{x}{n}L_n(t).$$

6. Let

$$w(x) = x^{-1}\pi^x = \int_0^{\pi} t^{x-1}dt.$$

Then, if $f_n=\cos(nE)$, we have

$$f_m f_n w(x) = \int_0^{\pi} t^{x-1}\cos(mt)\cos(nt)dt = C_{m,n}(x),$$

$$\begin{aligned} C_{m,n}(1) = \int_0^{\pi} \cos(mt)\cos(nt)dt &= 0 && m \neq n \\ &= (\pi/2) && m = n > 0 \\ &= \pi && m = n = 0. \end{aligned}$$

Let

$$C_n(x) = \int_0^{\pi} t^{x-1}\cos nt\, dt = \pi^x\sum_{s=0}^{\infty}(-)^s\frac{(n\pi)^{2s}}{(2s)!(x+2s)}$$

then the coefficients in an expansion

$$g(x) = \sum_{n=0}^{\infty} c_n C_n(x)$$

are given by the rule

$$\epsilon_n c_n = \lim_{x\to 0} f_n g(x),\ \epsilon_n = \pi/2,\ n>0,\ \epsilon_0 = \pi.$$

In particular

$$\begin{aligned} \pi C_m(x+y) &= C_0(x)C_m(1+y) + 2\sum_{n=1}^{\infty}C_n(x)C_{m,n}(1+y) \\ &= \sum_{n=-\infty}^{\infty}C_n(x)C_{m+n}(1+y). \end{aligned}$$

This is easily confirmed with the aid of Parseval's theorem.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

The Numerical Evaluation of a Class of Trigonometric Series

By Morgan Ward, California Institute of Technology

1. In a recent problem arising in the design of an X-ray tube, it was necessary to sum two slowly convergent trigonometric series

$$\sum_1^\infty n^{-3/2} \cos 2n\pi x, \quad \sum_1^\infty n^{-3/2} \sin 2n\pi x, \; 0 \leqq x \leqq 1,$$

to a fair degree of accuracy. It was thought that the method devised to transform these series into more rapidly converging ones might be useful to others confronted with a similar task.

2. Let us consider quite generally a trigonometric series of the form

$$F(x) = \sum_1^\infty \phi(n) e^{2n\pi i x}, \tag{2.1}$$

where $\phi(n)$ is real and such that the series converges in the interval $0 \leqq x \leqq 1$.

Let $\Delta\phi(n)$, $\Delta^2\phi(n)$ and so on, represent the successive differences $\phi(n+1) - \phi(n)$, $\Delta\phi(n+1) - \Delta\phi(n)$, of $\phi(n)$; and let us write for brevity $\Delta^r\phi(a)$ for the value of the rth difference of $\phi(n)$ when n equals a. We then have the following

Theorem. *If a is any positive integer, and if the $(s+1)$th difference of $\phi(n)$ is of invariable sign for all positive integral values of n greater than a, then*

$$F(x) = \sum_1^a \phi(n) e^{2n\pi i x} + \sum_{r=0}^{s-1} \Delta^r \phi(a+1) \left(\frac{\csc \pi x}{2}\right)^{r+1} e^{2\pi i a x + \pi i (r+1)(x+1/2)} + R, \tag{2.2}$$

where

$$| R | \leqq 2\left(\frac{\csc \pi x}{2}\right)^{s+1} | \Delta^s \phi(a+1) | . \tag{2.3}$$

If we consider the real and imaginary parts of $F(x)$ separately, we can annunciate two precisely similar theorems where the exponentials appearing in (2.2) are replaced by the corresponding cosine and sine terms.

As regards our hypothesis about the sign of the $(s+1)$th difference of $\phi(n)$, we may remark that if $\phi(t)$ may be considered as a function of the continuous variable t for all values of $t \geqq a$, and if the $(s+1)$th derivative of $\phi(t)$ exists and is of invariable sign for $t \geqq a$, then the $(s+1)$th difference of $\phi(n)$ is also of invariable sign for $n > a$. These conditions will be satisfied for example for any positive integers a and s if $\phi(n) = n^{-\beta}$, $\beta > 1$.

To demonstrate the utility of our result for practical computation, take $\phi(n) = n^{-3/2}$, $a = 5$, and $s = 2$. Then we have

$$\sum_1^\infty n^{-3/2} \cos 2n\pi x = \sum_1^5 n^{-3/2} \cos 2n\pi x - \frac{\csc \pi x}{2} 6^{-3/2} \sin 11\pi x$$

$$-\left(\frac{\csc \pi x}{2}\right)^2 \Delta 6^{-3/2} \cos 12\pi x + R,$$

where

$$|R| \leqq 2\left(\frac{\csc \pi x}{2}\right)^3 \Delta^2 6^{-3/2} \leqq 2\left(\frac{\csc \pi x}{2}\right)^3 (.0023).$$

Thus in the range $1/6 \leqq x \leqq 5/6$ where $1/2 \leqq (\csc \pi x)/2 \leqq 1$, $|R| < .005$. A comparable degree of accuracy from the series itself would require several hundred terms.

3. The above theorem may be proved as follows. Let

$$F_{(a)}(x) = \sum_{a+1}^{\infty} \phi(n) e^{2n\pi i x} \tag{3.1}$$

be the remainder of the series (2.1) after a terms, so that

$$F(x) = \sum_{n=1}^{a} \phi(n) e^{2n\pi i x} + F_{(a)}(x). \tag{3.2}$$

Then if s is any positive integer,

$$\begin{aligned} (1 - e^{2\pi i x})^{s+1} F_{(a)}(x) = &\sum_{r=0}^{s} (1 - e^{2\pi i x})^{s-r} \Delta^r \phi(a+1) e^{2\pi i (a+r+1)x} \\ &+ \sum_{n=a+1}^{\infty} \Delta^{s+1} \phi(n) e^{2\pi i (n+s+1)x}. \end{aligned} \tag{3.3}$$

For this formula is easily seen to be true when $s=0$. Assume that it is true when $s=k-1$:

$$\begin{aligned} (1 - e^{2\pi i x})^{k} F_{(a)}(x) = &\sum_{r=0}^{k-1} (1 - e^{2\pi i x})^{k-1-r} \Delta^r \phi(a+1) e^{2\pi i (a+r+1)x} \\ &+ \sum_{n=a+1}^{\infty} \Delta^{k} \phi(n) e^{2\pi i (n+k)x}. \end{aligned}$$

Multiply this expression by $1-e^{2\pi i x}$. Then

$$\begin{aligned} (1 - e^{2\pi i x})^{k+1} F_{(a)}(x) = &\sum_{r=0}^{k-1} (1 - e^{2\pi i x})^{k-r} \Delta^r \phi(a+1) e^{2\pi i (a+r+1)x} \\ &+ \Delta^k \phi(a+1) e^{2\pi i (a+k+1)x} + \sum_{n=a+2}^{\infty} \Delta^k \phi(n) e^{2\pi i (n+k)x} - \sum_{n=a+1}^{\infty} \Delta^k \phi(n) e^{2\pi i (n+k+1)x}. \end{aligned}$$

The term $\Delta^k \phi(a+1) e^{2\pi i (a+k+1)}$ can be incorporated into the first summation, on changing its upper index from $k-1$ to k. In the second summation, we replace n by $n+1$, and then combine the resulting expression with the third summation. On recalling that by definition, $\Delta^k \phi(n+1) - \Delta^k \phi(n) = \Delta^{k+1} \phi(n)$, we obtain in this manner (3.3) with $s=k$. Hence (3.3) is true when $s=0$, and if it is true for $s=k-1$, it is true for $s=k$. Therefore, by induction, it is true generally.

Now assume that $\Delta^{s+1}\phi(n)$ is of invariable sign. Then in (3.3)

$$\left|\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n)e^{2\pi i(n+s+1)x}\right| \leqq \sum_{n=a+1}^{\infty}\left|\Delta^{s+1}\phi(n)e^{2\pi i(n+s+1)x}\right|$$

$$= \pm\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n) = \pm\sum_{n=a+1}^{\infty}\Delta^{s}\phi(n+1)-\Delta^{s}\phi(n) = \left|\Delta^{s}\phi(a+1)\right|, \text{ or}$$

$$\left|\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n)e^{2\pi i(n+s+1)x}\right| \leqq \left|\Delta^{s}\phi(a+1)\right| \cdot \tag{3.4}$$

Finally, $1-e^{2\pi ix}$ may be written $2\sin\pi x e^{\pi i(x-1/2)}$. Therefore, if we divide both sides of (3.3) by $(1-e^{2\pi ix})^{s+1}$, we obtain

$$F_{(a)}(x) = \sum_{r=0}^{s}\Delta^{r}\phi(a+1)\left(\frac{\csc\pi x}{2}\right)^{r+1}e^{2\pi iax+\pi i(r+1)(x+1/2)}$$

$$+\left(\frac{\csc\pi x}{2}\right)^{s+1}\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n)e^{2\pi inx+\pi i(s+1)(x+1/2)}.$$

It follows then from (3.4) that

$$F_{(a)}(x) = \sum_{r=0}^{s-1}\Delta^{r}\phi(a+1)\left(\frac{\csc\pi x}{2}\right)^{r+1}e^{2\pi iax+\pi i(r+1)(x+1/2)} + R, \tag{3.5}$$

where

$$|R| < 2\left(\frac{\csc\pi x}{2}\right)^{s+1}\left|\Delta^{s}\phi(a+1)\right|.$$

On combining (3.2) and (3.5), we obtain the result stated in the theorem.

An Application of Stirling's Numbers

By H. J. Goldstein, New York City

1. J. Ginsburg has called attention in the February, 1928, issue of this Monthly, to the varied history of the Stirling numbers. These numbers, while perhaps as interesting and useful as the allied Bernoulli numbers, have received relatively little attention. Their properties have, however, been discussed at considerable length by N. Nielsen[1] and C. Tweedie.[2] We shall consider an elementary application, and derive, *en passant*, a relation among the numbers themselves. The latter is probably not new, but does not appear in the works cited, or in others consulted by the author.

The Stirling numbers of the first and second species, designated respectively by C_n^r and Γ_n^r $(n>0,\ r\geqq 0)$ are the coefficients in the expansions

[1] *Theorie der Gamma Funktion* (1904), pp. 66–78; and *Recherches sur les Nombres et les Polynomes de Stirling*, Annali di Matematica, vol. 10 (1904), pp. 287–318.

[2] Proc. Edinburgh Math. Soc., vol. 37 (1919), pp. 11–25.