ON THE BEHAVIOR OF NON-STATIC MODELS OF THE UNIVERSE WHEN THE COSMOLOGICAL TERM IS OMITTED

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Abstract

If the cosmological term in the equations of relativistic mechanics is set equal to zero, it has been shown by Einstein that a non-static model of the universe filled with a homogeneous distribution of incoherent matter would expand to a maximum volume and then start contracting. This, however, is a very special model of the universe filled with a highly simplified fluid, and subjected to changes which can be shown to be thermodynamically reversible; and it has recently been pointed out by one of the present authors that we can also expect a similar expansion to a maximum volume with much more general models of the universe allowing irreversible as well as reversible changes in the fluid filling the model. The present article gives a somewhat detailed analysis of the behavior of a wide class of non-static models of the universe when the cosmological term is set equal to zero, and shows that we may expect a continued succession of expansions and contractions without reference to the reversible or irreversible nature of the processes taking place in the fluid filling the model. The bearings of this finding on the problems of relativistic thermodynamics, which have already been treated by one of the present authors, are again noted.

§ 1. *Introduction.* Einstein's original equations connecting the distribution of matter and energy with the space-time metric of general relativity can be written in the form

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2}Gg_{\mu\nu} \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $G_{\mu\nu}$ is the contracted Riemann-Christoffel tensor, and $g_{\mu\nu}$ the fundamental metrical tensor. The choice of these equations as a starting point for relativistic mechanics has considerable justification. In empty space, they reduce to

$$G_{\mu\nu} = 0 \tag{2}$$

which has the support provided by the three well-known crucial tests of the general theory of relativity. In weak static gravitational fields, only one of the ten equations, that with $\mu=\nu=4$, is of importance and this can then be shown to reduce to Poisson's equation

$$4\pi\rho = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{3}$$

where ρ is the density of matter and ψ the ordinary Newtonian gravitational potential. And finally, the energy-momentum tensor $T_{\mu\nu}$ is a quantity whose divergence we wish to have equal to zero on physical grounds while the combination on the right-hand side of (1) is a quantity whose divergence is known to be identically equal to zero.

As is well-known the original equations given above were later modified by Einstein's addition of the so-called cosmological term to read

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2}Gg_{\mu\nu} + \Lambda g_{\mu\nu} \tag{4}$$

where the cosmological constant Λ is a quantity independent of the spatial and temporal coordinates, which would have to be regarded as a new fundamental constant of nature.

At the time the reason for making such a change appeared to be twofold. In the first place, by adding the cosmological term there was obtained the most general possible function, of the gravitational potentials $g_{\mu\nu}$ and their first and second differential coefficients, the divergence of which is identically equal to zero. In the second place, without the cosmological term it was impossible to construct a static model of the universe containing a finite density of matter; but by adding the cosmological term it became possible to obtain Einstein's well known static model for the universe, with reasonable values for the density of matter and radius of the model, and no disagreement with the three observational tests of the theory, provided Λ was taken as a small positive quantity.

More recently, however, the arguments in favor of changing from the original form of the equations as given by (1) have seemed less strong. In the first place, we know that Λ must in any case be a very small quantity in order to agree with the three crucial tests of the theory of relativity, and we should certainly prefer to take it equal to zero merely in the interests of simplicity and definiteness. In the second place, if we do not set Λ equal to zero, we have to inquire into the significance and magnitude of this new fundamental constant of nature, and the results of such inquiry have so far not seemed very satisfactory. Finally, it now appears evident, both from theoretical and observational points of view,[1] that non-static models of the universe are to be preferred to static models, and it is entirely possible to construct satisfactory non-static models of the universe without introducing the cosmological term, a fact that has recently been specially pointed out and emphasized by Einstein himself.[2]

For these reasons it becomes a matter of some interest to consider the consequences of omitting the cosmological term from the equations of relativistic mechanics, even though in the interests of generality we must continue to keep in mind the possibility that Λ may not be exactly equal to zero.

§2. *Purpose of the Present Article.* The purpose of the present article is to consider the general behavior of non-static models of the universe, setting

[1] On the theoretical side, as pointed out by Tolman, Proc. Nat. Acad. **16**, 320 (1930), we cannot have a static universe if matter is changing over into radiation, and as pointed out by Eddington, Monthly Notices R.A.S. **90**, 668 (1930), we cannot regard the Einstein static universe as stable. On the observational side, the red shift in the light from the extra-galactic nebulae indicates a non-static universe as first appreciated by Lemaitre, Ann. Societe Sci. Bruxelles **47**, Series A, 49 (1927).

[2] Einstein, Berl. Ber. (1931), p. 235.

the cosmological term equal to zero, and making only very general assumptions as to the nature of the fluid which fills the model, and as to the nature of the processes which occur in this fluid as the model expands or contracts.

In the article of Einstein referred to above, it was shown that on setting Λ equal to zero we must expect a non-static model of the universe, filled with incoherent matter exerting no pressure, to expand to a maximum volume and then to start contracting. This, however, was a very special model of the universe, filled with a highly simplified fluid, and subjected only to changes which can be shown to be thermodynamically reversible,[3] and it might be questioned whether we could expect a similar behavior in the case of less simple fluids and models in which irreversible processes might take place. Nevertheless, it has been pointed out in a recent article by one of us[4] that we can also expect such expansion to a maximum followed by contraction in the case of any model of the universe filled with a homogeneous distribution of fluid exerting a positive pressure, provided we set the cosmological term equal to zero. In the present article this behavior will be considered in more detail.

In the next section, §3, we shall give those mechanical equations governing the behavior of non-static universes which will be needed later. In §4, we shall then prove for any such non-static model of the universe, filled with a fluid which could exert only positive pressures, and having initially a finite volume and finite rate of expansion, that there would be a finite upper boundary beyond which the volume could not expand. Continuing in §5, we shall then show that the model would reach its maximum upper volume in a finite time and would then start contracting. And in §6, we shall show that the equations would thereafter require the contraction to continue to zero volume which would also be reached within a finite time. In §7, we shall then discuss this mathematical conclusion that the model would contract down to the exceptional point of zero volume, and show from a physical point of view that we might expect contraction to the lower limit to be followed by a renewed expansion. Finally in §8, we shall make some remarks concerning the possible application of these conclusions as to the behavior of highly idealized models, in interpreting the behavior of the actual universe.

§3. *The Mechanics of the Non-Static Universe.* An expression for the line element for a non-static model of the universe filled with a homogeneous distribution of fluid with properties which are independent of position but dependent on the time, can be derived[5] and written in the form

$$ds^2 = -\frac{e^{g(t)}}{[1 + r^2/4R^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2 \tag{5}$$

where r, θ and ϕ are the spatial coordinates, t is the time coordinate, R is a constant, and the dependence of the line element on the time is given by the exponent $g(t)$.

[3] Tolman, Phys. Rev. **38,** 1758 (1931). See 9.

[4] Tolman, Phys. Rev. **39,** 320 (1932).

[5] Tolman, Proc. Nat. Acad. **16,** 320 (1930).

For the proper pressure p_0 and proper macroscopic density ρ_{00} corresponding to this line element we can write in accordance with the principles of relativistic mechanics[6]

$$8\pi p_0 = -\frac{1}{R^2}e^{-g} - \tfrac{3}{4}\dot{g}^2 + \Lambda \tag{6a}$$

$$8\pi\rho_{00} = \frac{3}{R^2}e^{-g} + \tfrac{3}{4}\dot{g}^2 - \Lambda \tag{7a}$$

provided we retain the cosmological term, or

$$8\pi p_0 = -\frac{1}{R^2}e^{-g} - \ddot{g} - \tfrac{3}{4}\dot{g}^2 \tag{6b}$$

$$8\pi\rho_{00} = \frac{3}{R^2}e^{-g} + \tfrac{3}{4}\dot{g}^2 \tag{7b}$$

if we set the cosmological term equal to zero, as is of prime interest for the present article. The pressure and density are independent of position and dependent as shown on the exponent $g(t)$ and its time derivatives $\dot{g}$ and $\ddot{g}$.

With the above choice of coördinates particles which are at rest with respect to r, θ and ϕ will not be subject to gravitational acceleration, so that we can regard the fluid as macroscopically at rest in these coordinates. As g changes with the time, however, the proper volume

$$\delta V_0 = \frac{r^2 \sin\theta e^{3g/2}}{[1 + r^2/4R^2]^3}\delta r\delta\theta\delta\phi \tag{8}$$

associated with a small range in coordinates $\delta r\delta\theta\delta\phi$, and the total integrated proper volume of the model[7]

$$V_0 = \pi^2 R^3 e^{3g/2} \tag{8a}$$

will change with the time. Hence we can describe the changes that take place in such a universe, as g increases or decreases, as expansions or contractions in the proper volume of the elements of fluid which fill the model, and in the total proper volume of the model.

Furthermore, in accordance with the expressions for pressure and density we can easily obtain the relation[8]

$$\frac{d}{dt}(\rho_{00}\delta V_0) + p_0\frac{d}{dt}(\delta V_0) = 0 \tag{9}$$

which shows that the change in the energy content of any given small element of the fluid, as measured by a local observer using proper coordinates, will be found equal to the negative of the work performed on the surroundings.

[6] Reference 5, Eqs. 34.
[7] Tolman, Phys. Rev. **37**, 1652 (1931). Eq. (28).
[8] Tolman, Proc. Nat. Acad. **16**, 409 (1930). Eq. (4).

§4. *The Upper Boundary of Expansion.* Let us now consider such a model of the universe filled with a mixture of matter and radiation which might exert a positive pressure

$$p_0 \geqq 0 \tag{10}$$

but cannot withstand tension. And at some initial time $t=0$ let the model have a finite volume and finite rate of expansion corresponding to

$$g = g_0 \text{ and } \ddot{g} = \dot{g}_0 \tag{11}$$

where g_0 is finite and $\dot{g}_0$ is finite and positive. We shall first show, with $\Lambda = 0$, that there will be a finite upper volume beyond which the expansion cannot go, that is a finite upper boundary for the quantity g.

Combining equation (6b) with the inequality (10), we can write in general

$$\ddot{g} + \tfrac{3}{4}\dot{g}^2 + \frac{1}{R^2}e^{-g} \leqq 0. \tag{12}$$

Furthermore, since $\dot{g}$ will be positive as long as expansion continues we can multiply (12) by the positive quantity $2e^{3g/2}\dot{g}$ and write

$$2e^{3g/2}\ddot{g}\dot{g} + \frac{3}{2}e^{3g/2}\dot{g}^3 + \frac{2}{R^2}e^{g/2}\dot{g} \leqq 0 \tag{13}$$

or

$$\frac{d}{dt}(e^{3g/2}\dot{g}^2) + \frac{4}{R^2}\frac{d}{dt}(e^{g/2}) \leqq 0 \tag{14}$$

as an expression which will continue to hold as long as g continues to increase.

Integrating (14) between $t=0$ and any later time of interest $t=t$, and substituting the values for g and $\dot{g}$ at time $t=0$ as given by (11), we obtain

$$e^{3g/2}\dot{g}^2 + \frac{4}{R^2}e^{g/2} \leqq e^{3g_0/2}\dot{g}_0{}^2 + \frac{4}{R^2}e^{g_0/2} \tag{15}$$

or, taking the constant R as real for the models in which we shall be interested,

$$e^{g/2} \leqq \frac{R^2}{4}e^{3g_0/2}\dot{g}_0{}^2 + e^{g_0/2} - \frac{R^2}{4}e^{3g/2}\dot{g}^2 \tag{16}$$

as an expression which will hold as long as g continues to increase. Since g_0 and $\dot{g}_0$ are, however, finite by hypothesis this shows that there is an upper finite boundary say γ which g cannot surpass. So that we can necessarily write

$$g \leqq \gamma \tag{17}$$

where γ is finite.

§5. *Time Necessary to Reach the Maximum.* From the inequality (17) we can then evidently write

$$-\frac{1}{R^2}e^{-g} \leqq -\frac{1}{R^2}e^{-\gamma} \tag{18}$$

and combining this with (12) we obtain

$$\ddot{g} \leqq -\frac{1}{R^2}e^{-\gamma} - \tfrac{3}{4}\dot{g}^2 \tag{19}$$

or

$$\frac{d\dot{g}}{dt} \leqq -\frac{1}{R^2}e^{-\gamma}. \tag{20}$$

And integrating this between $t=0$ and any later time of interest $t=t$, we obtain

$$\dot{g} \leqq \dot{g}_0 - \frac{1}{R^2}e^{-\gamma}t \tag{21}$$

where $\dot{g}_0$ is the rate of increase in g at $t=0$.

In accordance with this expression, however, noting from (11) that $\dot{g}_0$ is positive, we see that at a finite time

$$t \leqq R^2 e^{\gamma} \dot{g}_0 \tag{22}$$

$\dot{g}$ will become equal to zero, g will pass through a maximum, and the volume will start to decrease.

§6. *Time Necessary to Reach Zero Volume.* It will also be of interest to consider the behavior of the model after passing through its maximum volume. Since $\dot{g}$ will then evidently be negative, we may this time multiply (12) by the negative quantity $2e^{3g/2}\dot{g}$, and integrating as in §4, obtain in correspondence with (15)

$$e^{3g/2}\dot{g}^2 + \frac{4}{R^2}e^{g/2} \geqq e^{3g_m/2}\dot{g}_m{}^2 + \frac{4}{R^2}e^{g_m/2} \tag{23}$$

where g_m and $\dot{g}_m$ are the values of the quantities indicated on passing through the maximum point at the time $t=t_m$. Since, however, the velocity will be zero when the model passes through its maximum volume, we may substitute

$$\dot{g}_m = 0 \tag{24}$$

and rewrite (23) in the form

$$e^{3g/2}\dot{g}^2 \geqq \frac{4}{R^2}(e^{g_m/2} - e^{g/2}) \tag{25}$$

and with $\dot{g}$ negative this gives us

$$e^{3g/4}\frac{dg}{dt} \leqq -\frac{2}{R}(e^{g_m/2} - e^{g/2})^{1/2} \tag{26}$$

provided we take the constant R not only as real but positive, which will be the case for a closed model with the positive "radius" $Re^{g/2}$.

Expression (26), however, can easily be integrated between $t=t_m$ and any later time of interest $t=t$ to give

$$-\frac{e^{g/4}}{2}(e^{g_m/2}-e^{g/2})^{1/2}+\frac{e^{g_m/2}}{2}\sin^{-1}\frac{e^{g/4}}{e^{g_m/4}}-\frac{\pi}{4}e^{g_m/2}\leqq-\frac{(t-t_m)}{2R} \tag{27}$$

or on rearranging

$$(t-t_m)\leqq R\left[e^{g/4}(e^{g_m/2}-e^{g/2})^{1/2}-e^{g_m/2}\sin^{-1}\frac{e^{g/4}}{e^{g_m/4}}+\frac{\pi}{2}e^{g_m/2}\right]. \tag{28}$$

And in accordance with this result, it is evident that $e^{g/4}$ will reach the value zero or g the value minus infinity at a finite time after the maximum

$$(t-t_m)\leqq\frac{\pi}{2}Re^{g_m/2}. \tag{29}$$

We have thus shown not only that the model, starting with a finite volume and finite rate of expansion, would reach a finite maximum volume within a finite time, but continuing beyond the maximum would go on down to zero volume within a finite time later. In addition it will be noted, by comparing equations (6a) and (6b) and examining the method of analysis which has been employed, that these results would also hold if the cosmological constant Λ were taken as a negative quantity, as well as for the case which we have treated with the cosmological term set equal to zero.

§7. *Behavior of the Model on Reaching Zero Volume.* As a result of the preceding section we have seen that our equations lead to the conclusion that the proper volume of the model would decrease to the value zero within a finite time after passing the maximum. We must now inquire into the physical significance of this result, and into the further behavior of the model after reaching this exceptional point.

In accordance with the inequality (26) we may write

$$\dot{g}\leqq-\frac{2}{Re^{3g/4}}(e^{g_m/2}-e^{g/2})^{1/2} \tag{30}$$

as an expression which holds at any time after passage of the maximum, so that on reaching zero volume with $g=-\infty$ we shall have

$$\dot{g}=-\infty. \tag{31}$$

Furthermore, in accordance with our general expression (12) we can write

$$\ddot{g}\leqq-\frac{1}{R^2}e^{-g}-\tfrac{3}{4}\dot{g}^2 \tag{32}$$

so that we shall also have

$$\ddot{g}=-\infty \tag{33}$$

on reaching zero volume. The exceptional point $g=-\infty$ is thus reached with the velocity $\dot{g}$ and the acceleration $\ddot{g}$ both equal to minus infinity.

The conditions for an analytical minimum are thus unsatisfied and the analysis fails to describe the passage through the exceptional point of zero volume. From a mathematical point of view, however, it is evident that our differential equations of motion (6b, 7b) can be satisfied if we have a renewed expansion taking place at this point, and from a physical point of view as previously emphasized by one of us[9] it is evident that contraction to zero volume could only be followed by renewed expansion. Furthermore, as noted in a similar connection by Einstein,[10] it is possible that the idealization upon which our considerations have been based should be regarded as failing in the neighborhood of zero volume[11] so that the analysis fails to give a correct description of the behavior at the lower limit of volume. It hence appears reasonable to conclude for models of the kind we are discussing that the contraction to zero volume or more generally to the lower limit of volume would result in a sudden reversal in the direction of the velocity $\dot{g}$, followed by a renewed expansion of similar character to the previous one.

§8. *Conclusion.* The main results of the foregoing analysis may now be summarized and a few remarks made concerning their significance.

It has previously been shown that the line element for any non-static model of the universe containing a uniform distribution of fluid can be written in the general form

$$ds^2 = -\frac{e^{g(t)}}{[1 + r^2/4R^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + dt^2 \qquad (34)$$

where R is a constant and the dependence of the line element on the time is given by the exponent $g(t)$. The "radius" for such a model is $Re^{g/2}$, and the changes that take place in the model, as g increases or decreases, can be described as expansions or contractions in the proper volume of the elements of the material filling the model, and in the total proper volume of the model as a whole.

Let us now consider the special case of a "closed" model with R real[12] and positive, filled with a perfect fluid which could not withstand tension, and having at some initial time a finite volume and finite rate of expansion. Applying the equations of relativistic mechanics with the cosmological term omitted, it has then been rigorously shown that such a model would expand to a finite maximum volume which would be reached within a finite time, and would then contract to zero volume which would also be reached within a finite time later. Furthermore, although the mathematical analysis fails to carry us through the exceptional point of zero volume, it has been shown plausible on physical grounds to expect that contraction to the lower limit would be followed by renewed expansion, thus leading to a continued succession of somewhat similar expansions and contractions.

[9] Tolman, Phys. Rev. **38**, 1758 (1931). See §7.

[10] Reference 2.

[11] The assumptions that the material filling the model has a perfectly homogeneous distribution, and that this material is a perfect fluid incapable of shearing stresses, may fail at very small volumes.

[12] It should be emphasized that these conclusions apply to "closed" models with R real. The theoretical possibility for "hyperbolic" models with R imaginary has recently been pointed out by Heckmann, Göttingen Nachrichten II, 126 (1931).

This result applies of course in the first instance only to the class of simplified cosmological models that we have considered. Nevertheless, in view of the fairly general nature of the assumptions that were necessary, the result may at least be taken as indicating a possibility that the actual universe or parts thereof might also exhibit such a continued succession of expansions and contractions.

Furthermore, it may again be pointed out, as was emphasized in a previous article,[13] that this result has been obtained solely by applying the principles of relativistic mechanics, without the necessity for any assumption as to the thermodynamic nature of the processes which take place in the fluid in the model as a consequence of the expansion and contraction. We are hence led to the conclusion that the continued series of expansions and contractions would recur even though the processes taking place in the fluid might be thermodynamically irreversible in character.[14]

This latter conclusion is of particular interest since the classical thermodynamics would have led us to expect that the continued occurrence of irreversible processes would result in a condition of maximum entropy where further change would be impossible. As shown in detail in the article mentioned, however, relativistic thermodynamics would permit a continued succession of irreversible expansions and contractions without the entropy ever reaching an unsurpassable maximum.

[13] Reference 4.

[14] If the processes taking place in the fluid are thermodynamically reversible we may ex pect a series of identical expansions and contractions as previously studied in detail (Ref. 9) When the processes are thermodynamically irreversible, however, we may expect a series of non-identical expansions and contractions of gradually increasing amplitude as studied in the article mentioned. (Ref. 4).