Note on an arithmetical property of recurring series.

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1. In 1921, Siegel[1]) proved by the use of Thue's theorem a result equivalent to the following:

"If the sequence

$$(U) \qquad U_0, U_1, U_2, \ldots$$

is a rational solution of the difference equation

$$(1.1) \qquad \Omega_{n+3} = P\Omega_{n+2} - Q\Omega_{n+1} + \Omega_n, \quad P, Q \text{ rational integers,}$$

then only a finite number of terms of the sequence can vanish unless the polynomial

$$(1.2) \qquad F(x) = x^3 - Px^2 + Qx \pm 1$$

associated with (1.1) is of one or the other of the forms

$$(x \pm 1)(x^2 + 1) \qquad \text{or} \qquad (x \pm 1)(x^2 \pm x + 1)\text{''}.$$

I wish to show here that as a simple consequence of the fundamental results of Delaunay[2]) and Nagell[3]) concerning the solution of the cubic diophantine equation

$$(1.3) \qquad \Phi(u, v) = Au^3 + Bu^2v + Cuv^2 + Dv^3 = 1,$$
$$A, B, C, D \text{ rational integers,}$$

that in general at most *three* terms of the sequence (U) can vanish provided that the discriminant of the associated polynomial is negative[4]).

2. For let us assume that the polynomial $F(x)$ is irreducible in the field of rationals, has a negative discriminant, and that the sequence (U) contains $N \geqq 1$ vanishing terms. Without affecting N, we may assume that the constant Term of $F(x)$ is $+1$, and that the first non-vanishing term of (U) is U_0, and that U_1 and U_2 are co-prime integers.

If $(X), (Y), (Z)$ denote those particular solutions of (1.1) with the initial values $1, 0, 0; 0, 1, 0; 0, 0, 1$ respectively, then it is easily shown that $U_n = U_0X_n + U_1Y_n + U_2Z_n, \quad \alpha^n = X_n + Y_n\alpha + Z_n\alpha^2, \quad n = 0, 1, \ldots,$

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[1]) Tohoku Journal 20 (1921), S. 26—31.

[2]) Compt. Rend. 171 (1920), S. 136.

[3]) Math. Zeitschr. 28 (1928), S. 10—29.

[4]) If the discriminant of $F(x)$ is positive, so that all the roots of $F(x) = 0$ are real, the finiteness of the number of zeros in the sequence (U) is trivial, and extends to the case when P, Q, U_0, U_1, U_2 are real numbers and the constant term of $F(x)$ is not unity.

where α is any root of $F(x) = 0$. Since $U_0 = 0$, $(U_1, U_2) = 1$, $U_n = 0$ when and only when $Y_n = U_2 T_n$, $Z_n = -U_1 T_n$, T_n an integer. Thus $U_n = 0$ when and only when the norm of the algebraic integer $X_n + T_n(U_2\alpha - U_1\alpha^2)$ is unity; that is when and only when

$$(2.1) \qquad A X_n^3 + B X_n^2 T_n + C X_n T_n^2 + D T_n^3 = 1$$

where

$$A = 1, \qquad C = Q U_2^2 + (3 - PQ) U_1 U_2 + (Q^2 - 2P) U_1^2,$$
$$B = P U_2 + (2Q - P^2) U_1, \quad D = U_2^3 - P U_2^2 U_1 + Q U_2 U_1^2 - U_1^3.$$

Hence $u = X_n^*$, $v = T_n$ is a solution of the diophantine equation (1.2).

Owing to our hypotheses upon $F(x)$, the form $\Phi(u, v)$ is irreducible and has a negative discriminant. Therefore, by Nagell's main theorem[5]), the diophantine equation has at most three integral solutions unless the form $\Phi(u, v)$ is equivalent to $u^3 + uv^2 + v^3$ or $u^3 - u^2 v + uv^2 + v^3$, when it has exactly four solutions, or to $u^3 - u^2 v + v^3$ when it has exactly five solutions.

Since $F(x)$ is irreducible, we cannot have $X_n = X_{n'}$, $T_n = T_{n'}$ unless $n = n'$. Hence the sequence (U) has in general at most three vanishing terms, and never more than five if the discriminant of $F(x)$ is negative.

3. It is possible to obtain a result analagous to Siegel's for the quartic difference equation

$$(3.1) \quad \Omega_{n+4} = P\Omega_{n+3} - Q\Omega_{n+2} + R\Omega_{n+1} \pm \Omega_n, \quad P, Q, R \text{ rational integers}$$

by a similar use of Thue's theorem; namely,

"If the sequence

$$(V) \qquad V_0, V_1, V_2, \ldots$$

is a rational solution of the difference equation (3.1), and if a is a fixed positive integer, then there are only a finite number of *pairs* of terms

$$V_{n_1}, V_{n_1+a};\ V_{n_2}, V_{n_2+a};\ V_{n_3}, V_{n_3+a}; \ldots \qquad n_1 < n_2 < n_3 < \ldots$$

of the sequence (V) can vanish, provided that the associated polynomial

$$(3.2) \qquad G(x) = x^4 - Px^3 + Qx^2 - Rx \pm 1$$

is irreducible, and that its roots cannot be obtained by solving a chain of quadratic equations".

We may assume that V_0, V_a is the first pair of terms of (V) to vanish, and that V_1, V_2, V_3 are co-prime rational integers.

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[5]) Math. Zeitschr. 28 (1928), S. 10.