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This thesis is dedicated to my parents, Yuchen and Liquan, who have raised me to be the person I am today. Also this thesis is dedicated to my wife Weiwei, who has been given me great source of motivation and inspiration.

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ABSTRACT

Yang, Shan. Ph.D., Purdue University, May 2010. Credit Risk Modeling Under Incomplete Information . Major Professor: Jin Ma.

In this work we study a class of credit default models with imperfect information. We combine the ideas of both structural and reduced form models, within a partial observation framework in which the information could even be delayed. Assuming that default is triggered by the touch-down of the firm total asset process to a prescribed and possibly random barrier, our main purpose is to obtain the default probability, as a continuous function of a hidden Markovian factor process, conditioning on the observed continuous and jump information. We show that a "separation principle" of nonlinear filtering is still valid in such a setting, and the default intensity can be estimated through the filtered factor process, which is the solution of a Riccati-type of Stochastic SDE driven by the underlying Brownian motion and counting process. Some Bayesian inference theory is also applied to obtain our solutions.

1. INTRODUCTION

The literature of credit risk can be traced back to the original works of Black and Scholes [6] and Merton [36]. There have been mainly two types of models: the structural model and the reduced form model (a.k.a the intensity model). Since the seminal works by Merton [36] and Black-Cox [5], the structural model has been considered as an intrinsic way to understand the default, which is triggered by the event that the total asset of the firm hits a prescribed, possibly random (lower) barrier. The default probability is therefore thought of as the "survival probability" of the total asset cash flow, a quite standard notion in survival analysis (we refer to [32], [33] for a general exposition on the structure model). It has been commonly recognized, however, that a drawback of the structural model is that the accounting information or the future cash flow is not transparent to the market at all time. Nevertheless, the recent experiences in the banking industry strongly suggest that the quality of the firm's cash flow should still be the most direct and important factor to indicate default, and from this perspective the structural model does have its advantage.

The reduced form model, on the other hand, is based more directly on the market price of the defaultable instruments. In other words, unlike the structural model, the reduced form model bootstraps the default probability from the market prices of the defaultable products, and hence it reflects the market reaction to the health of the firm capital structure. We refer to the works of Jarrow-Turnbull [21] [22], Duffie-Lando [14], and Lando [31], for more detailed account of the reduced model. (See also Ammann [1], Bielecki-Rutkovski [4], Duffie-Singleton [15] for various perspectives on this model.) It has been noted that one way to differentiate the structural model and reduced form model is to consider the amount of information available (or in probabilistic terms, the size of the filtration). For example, Jarrow-Protter [20] indicated that the default time defined in a structural model is usually predictable, with respect to the filtration based on the firm's accounting history, but in an intensity model, the default time is usually an exogenous *totally inaccessible* stopping time with respect to the same filtration. Furthermore, one can go from one model to the other by either enlarge or shrink the filtration.

This naturally leads to the issue of finding the default probability under incomplete information. The earliest work in this direction was done by Duffie and Lando [13], in which the total asset is only discretely observable with accounting noise, and the default intensity is derived using the nonlinear filtering technique. The problem was later extended to the case of continuous time observation by Kusuoka [27]. The filtering-type of results for the default intensity can be found in the works by Kusuoka [27], Duffie [13], Collin-Dufresne et al. [10]. Other works involving intensity models using filtering techniques include: Guo et al. [18] (regime switch model with delayed information), Capponi and Cvitanic [7] (factor driven regime switching with partially observable factor), Frey and Runggaldier [16] (default contagion problem), to mentions a few.

In this work we consider a structural model but with partial and delayed information. An intensity based argument is then naturally brought into play to incorporate the imperfectness of the information. This then leads to a model that combines the main features of both structural and reduced form models. Our framework extends some existing discrete observation/factor models, e.g., the delayed information of [18] and the hidden Markov chain factor model in [7] to the continuous time setting. Furthermore, similar to [16] we shall allow Poisson jump type observations. Therefore our model in a sense provides a unified framework for many existing credit default models, for which the evaluation/estimation of the default probability is the main concern.

Our model can be more precisely described as follows. In light of [18], we begin by assuming that the accounting information of the firm is released finite number of times every year. Besides the accounting report, we assume that the true value of firm's assets can only be observed continuously via its stocks values, but can also be revealed by some unpredictable events. Therefore the we shall assume that the observation process consists of two components: one takes the form of a geometric Brownian motion, and the other is a pure jump process, driven by an independent Poisson random measure. Borrowing the idea of "frailty", we assume that there is an unobservable (common) factor process that affects all the dynamics involved. We note that the factor models proposed here has similar features as the typical return-earnings models by Kormendi and Lipe [26], and partial information models by Lakner [28].

The main task of this work is to evaluate the conditional default (or survival) probability, given all the available information, from both the market and firm's accounting reports. The guiding idea is the well-known "separation principle" in nonlinear filtering. Namely, we shall first estimate the factor process θ by a standard nonlinear filtering method, but with a combined continuous and counting process observations. We then derive the dynamics of the forward and instantaneous intensity of the default probability, which turns out to be a Riccati-type of stochastic differential equations driven by the observable random noises. In particular, we prove that an intensity model can be derived from the given structural mode, or more precisely it is a specific form of structural model when the information is not complete. This way we establish a quantitative connection between firm accounting information and market information. We should point out that some of our arguments are based on the standard nonlinear filtering theory (cf. e.g., [24] and [35]), as well as some recent works involving counting process observations [38] and stochastic volatility [11].

The rest of the thesis is organized as follows. In chapter 2 we make a general introduction to the classical structural model and reduced form model, where the survival probabilities and the pricing of the defaultable claims are derived under both models. Chapter 3 is the main part of our work, we introduce our imperfect information on both of the two models basis. The survival probability and the intensities in our model will be defined and derived as a term structure differential equation from the capital structure assumption by extending Bayesian inference theory. We will also derive the estimation of the hidden factor and the dynamics of the SDE's under some general assumption.

2. CREDIT RISK MODELS REVISITED

In this chapter we review some standard results in credit default theory. These results were proved in various literature, we collected them for ready references.

2.1 Structural Model

Structural model is the earliest type of credit models, based on the assumption that the default of the firm is triggered by the company total asset dropping down below certain level. In 1974, Merton constructed the first structural model [36] using both corporate financial structure and Black-Scholes' framework. Assuming a single debt (or corporate bond) capital structure, he considered the case when default happens only at the debt maturity. Based on the similar capital structures, Merton' s model have been extended extensively, from various perspectives, to cases including multiple default times, multiple debts, and those cases with complicate capital structures. We refer to Black-Cox [5], Geske [17], Leland [32], among many others. The following is a brief description of the Merton model, for ready references.

2.1.1 Merton model

Through out this section, we assume that all the randomness are defined in a filtered probability space $(\Omega, \mathcal{F}, P; \mathbb{F})$, where $\mathbb{F} = \{\mathcal{F}_t\}$ is the natural filtration generated by a standard Brownian motion W, augmented by all the P-null sets in $\mathcal{F}$. Thus the filtration $\mathbb{F}$ satisfies the *usual hypotheses.*

We denote the total asset process by $X = \{X_t : 0 \leq t \leq T\}$, which is defined as the $\mathbb{F}$ adapted solution of the stochastic differential equation:

$$X_t = x_0 + \int_0^t X_u(r-d)du + \int_0^t X_u \sigma dW_u, \qquad t \in [0,T], \tag{2.1}$$

where r, d and σ are finite and constants. It is assumed that there is only one corporate debt, initiated at time 0 and matured at time T, defined as the finite, deterministic debt process:

$$\nu(t) = \begin{cases} L_0 \cdot e^{\int_0^t r du} & 0 \leq u < T \\ L & u = T, \end{cases} \tag{2.2}$$

where $\nu(0) = L_0 < x_0$ and $\nu(T) = L < \infty$. The default time is defined as a random but single valued time:

$$\tau = T \cdot 1_{\{X_T < \nu(T)\}}. \tag{2.3}$$

Finally, we denote the conditional survival probability at any time $t \in [0, T]$ by $D(t, T)$, it is then obvious that it can be written as:

$$D(t,T) = P\{\tau > T | \mathcal{F}_t\} = P\{X_T \geq \nu(T) | \mathcal{F}_t\} = E\{1_{\{X_T \geq \nu(T)\}} | \mathcal{F}_t\}. \tag{2.4}$$

It is clear by the definition that the corporate debt worths $\nu(T) \wedge X_T$ at terminal time T. In this spirit, a Black-Scholes type solution can be derived for both survival probability and corporate debt price. The following proposition shows the details.

Proposition 2.1.1 *Let X be the stochastic process defined in (2.1), and let $X_t = x_t$ at present time t, then the survival probability $D(t, T)$ defined in (2.4) is of the following forms:*

$$D(t,T) = N(d_2)$$

and the bond price, which we denote by $B(t, T)$ at time t, is of the form

$$B(t,T) = e^{-r(T-t)} L N(d_2) + e^{-d(T-t)} x_t N(-d_1),$$

where

$$d_1 = \frac{\log[x_t/L] + (r - d + \frac{1}{2}\sigma^2)(T-t)}{\sigma(T-t)} \qquad \textit{and}$$

$$d_2 = d_1 - \sigma(T-t).$$

Proof The proof for the survival probability $D(t,T)$ is by definition and straight forward:

$$\begin{aligned} D(t,T) &= P\{X_T > \nu(T) \mid \mathcal{F}_t\} \\ &= P\{X_t e^{(r-d-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} > \nu(T)\} \\ &= P\{\frac{W_T - W_t}{\sqrt{T-t}} > \frac{\log[L/x_t] - (r-d-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\} \\ &= N(d_2). \end{aligned}$$

Similarly, the debt value at maturity equates $\min[X_T, \nu(T)]$. Thus,

$$\begin{aligned} B(t,T) &= E\{e^{-r(T-t)}\min[X_T, \nu(T)] \mid x_t\} \\ &= E\{e^{-r(T-t)}\nu(T) - \max[\nu(T) - X_T, 0] \mid x_t\} \\ &= E\{e^{-r(T-t)}L \mid \mathcal{F}_t\} - E\{e^{-r(T-t)}\max[L - X_T, 0] \mid x_t\} \\ &= e^{-r(T-t)}LN(d_2) + e^{-d(T-t)}x_t N(-d_1). \end{aligned}$$

The last equality holds by the application of Black-Schole put option pricing formula. ■

Remark 2.1.2 The Merton model was the first model for corporate bond pricing. It is constructed under the assumption that there is only one debt (bond) with a determined maturity time T. This assumption restricted the modeling to be a model with one single time period and one default event. Thus Merton model can be barely applicable to the current financial market situation, however the idea of modeling defaults from capital structure is intuitive and essential for credit risk modeling.

2.1.2 The Black and Cox Model

Black and Cox model [5] is the first extension of Merton's model, with respective to multiple default times. While inheriting the same assumptions on total asset process X and single debt $\nu(t)$ structure as Merton model, this model emphasis that, default can happen at any time between the issue and maturity of a debt. The default is defined to happen, if at any time t, the total asset X_t is below the total debt $\nu(t)$. Mathematically, by this change, the modeling of survival becomes a first passage time problem.

Same as Merton model, let us assume that the firm issued a single debt(or corporate bond) with maturity T at time 0. Assume the total asset of the firm X has the same definition as in (2.1), and the debt process ν is defined the same as (2.2).

The Black-Cox default time of the firm can be defined as:

$$\tau = \min\Big(\inf_{0\leq u<T}\{u:\ X_u < \nu(u)\},\quad T\cdot 1_{\{X_T<\nu(T)\}} + \infty\cdot 1_{\{X_T\geq\nu(T)\}}\Big).$$

In addition, we define the recovery process of the bond as the bond pay off ratio to the total asset value at default time. And assume it has a constant value β_1 if default before T and β_2 if default at time T. Under the risk neutral probability measure P, the defaultable debt(bond) price process can be derived as:

$$\begin{aligned} B(t,T) &= E\{e^{-r(T-t)}L\cdot 1_{\{\tau>T,X_T\geq L\}}|\mathcal{F}_t\} + E\{e^{-r(T-t)}\beta_2 X_T 1_{\{\tau=T,X_T<L\}}|\mathcal{F}_t\} \\ &\quad + E\{e^{-r(T-t)}\beta_1 X_\tau\cdot 1_{\{t<\tau<T\}}|\mathcal{F}_t\}. \end{aligned} \tag{2.5}$$

The following lemma introduce the solution for the first passage time of a geometric Brownian motion, which is the main tool to solve for the debt price:

Lemma 2.1.3 *Suppose X_t is a geometric Brownian motion process solving the SDE (2.1), where we let $d=0$. And assume $X_t = y_1 > \nu$ and $\tau > t$, then the probability that $X_T > y_2$ and $\inf_{t<u<T}\{X_u\} > \nu$ is defined as:*

$$\begin{aligned} &\Psi\big(\mu,\sigma,y_1,y_2,\nu,T-t\big) \\ = &\ P\{\inf_{t\leq u\leq T}\{X_u\} > \nu \quad \textit{and} \quad X_T > y_2 \mid X_t = y_1\} \\ = &\ N\big[d_2(\mu,\sigma,y_1,y_2,\nu,T-t)\big] \\ &- \exp\Big[\frac{(2\mu-\sigma^2)(\log[y_2/y_1])}{\sigma^2}\Big]\cdot N\big[d_1(\mu,\sigma,y_1,y_2,\nu,T-t)\big], \end{aligned}$$

where the $N(\cdot)$ is the standard normal distribution formula and $d_1(\mu,\sigma,y_1,y_2,\nu,\Delta t)$ and $d_2(\mu,\sigma,y_1,y_2,\nu,\Delta t)$ are of the forms:

$$d_1(\mu,\sigma,y_1,y_2,\nu,T-t) = \frac{(\mu-\frac{1}{2}\sigma^2)(T-t) + 2\log[y_2] - \log[y_1] - \log[\nu]}{\sigma\sqrt{T-t}},$$

and

$$d_2(\mu,\sigma,y_1,y_2,\nu,T-t)=d_1(\mu,\sigma,y_1,y_2,\nu,T-t)-\frac{2\log[y_2/y_1]}{\sigma\sqrt{T-t}}.$$

Proof The proof of this lemma could be found in Harrison [19]. Here we give the sketch of the proof steps. Given a Brownian motion with constant drift: $W_t^* = bt+\sigma W_t$ with initial value $W_0^*=0$ and a constant barrier z with the condition $z<0$, the survival probability of W_t^* to z on the interval $(0,t)$ is defined as: $\Phi(b,\sigma,z,y,t) = P\big(inf_{0<u<t}W_u^* > z, W_t^* \geq y\big)$. Then we have an explicit solution formula for $\Phi(b,\sigma,z,y,t)$ and $y>\nu$, which is of the well known form:

$$\Phi(b,\sigma,z,y,t)=N\big(\frac{bt-z}{\sigma\sqrt{t}}\big)-e^{\frac{2by}{\sigma^2}}N\big(\frac{bt+2y-z}{\sigma\sqrt{t}}\big) \tag{2.6}$$

Note the nature log of the process X_t is a Brownian Motion process with constant drift, thus the plugging the nature log formula of X_u into (2.6), one can prove the equation in lemma holds. ∎

Proposition 2.1.4 *Let $\Psi(\mu,\sigma,y_1,y_2,\nu,T-t)$ be the survival probability function defined in Lemma 2.1.3, then the Black-Cox model bond price defined by (2.5) can be derived as:*

$$B(t,T)=e^{-r(T-t)}\big\{B_1(t,T)+B_2(t,T)+B_3(t,T)\big\},$$

where

$$B_1(t,T)=(1-\beta_2)L\Psi\big(0,\sigma,(\frac{x_t}{L_0})e^{-rt},(\frac{L}{L_0})e^{-rt},1,T-t\big),$$

$$B_2(t,T)=(\beta_2-\beta_1)e^{rT}L_0\Psi(0,\sigma,(\frac{x_t}{L_0})e^{-rt},e^{-r(T-t)},1,T-t), \qquad \textit{and}$$

$$\begin{aligned}B_3(t,T) &= -\int_0^{L_0e^{rT}}\beta_1\cdot\Psi(0,\sigma,(\frac{x_t}{L_0})e^{-rt},(\frac{y_2}{L_0})e^{-rt},1,T-t)\big)dy_2\\ &\quad -\int_{L_0e^{rT}}^{L}\beta_2\cdot\Psi\big(0,\sigma,(\frac{x_t}{L_0})e^{-rt},(\frac{y_2}{L_0})e^{-rt},1,T-t\big)dy_2.\end{aligned}$$

Proof Note here,

$$\begin{aligned}
\{\tau > T, X_T \geq L\} &= \{X_T > L \quad and \quad \inf_{t\leq u<T} X_u > L_0 e^{ru}\}, \\
\{\tau \geq T, X_T = L\} &= \{X_T \leq L \quad and \quad \inf_{t\leq u<T} X_u > L_0 e^{ru}\} \quad and \\
\{t < \tau < T\} &= \{\inf_{t\leq u<T} X_u \leq L_0 e^{ru}\}.
\end{aligned}$$

Let $\tilde{X}_t = X_t/\nu(t) = (\frac{x_0}{L_0})e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$. Then $\tilde{X}_t$ has zero drift, and the equivalent default barrier of $\tilde{X}_t$ becomes $\nu(t) = 1$, $\forall t \in [0, T)$ and $\nu(T) = \big(\frac{L}{L_0}\big)e^{-rT}$, $t = T$. Thus by applying Lemma 2.1.3 and integration by parts, we have

$$\begin{aligned}
& E\{e^{-r(T-t)} \cdot L \cdot 1_{\{\tau>T, X_T\geq L\}}|\mathcal{F}_t\} \\
= \;& e^{-r(T-t)} \cdot L \cdot \Psi\big(0, \sigma, (\frac{x_t}{L_0})e^{-rt}, (\frac{L}{L_0})e^{-rt}, 1, T-t\big),
\end{aligned}$$

$$\begin{aligned}
& E\{e^{-r(T-t)} \cdot \beta_2 \cdot X_T 1_{\{\tau=T, X_T<L\}}|\mathcal{F}_t\} \\
= \;& \int_{L_0 e^{rT}}^{L} e^{-r(T-t)}\beta_2 y_2\big[-\frac{\partial}{\partial y_2}\Psi\big(0, \sigma, (\frac{x_t}{L_0})e^{-rt}, (\frac{y_2}{L_0})e^{-rt}, 1, T-t\big)\big]dy_2 \\
= \;& e^{-r(T-t)}\big\{\beta_2 L_0 e^{rT}\Psi\big(0, \sigma, (\frac{x_t}{L_0})e^{-rt}, e^{-r(T-t)}, 1, T-t\big) \\
= \;& -\beta_2 \cdot L \cdot \Psi\big(0, \sigma, (\frac{x_t}{L_0})e^{-rt}, (\frac{L}{L_0})e^{-rt}, 1, T-t\big) \\
= \;& -\int_{L_0 e^{rT}}^{L} \beta_2 \cdot \Psi\big(0, \sigma, (\frac{x_t}{L_0})e^{-rt}, (\frac{y_2}{L_0})e^{-rt}, 1, T-t\big)dy_2\big\}
\end{aligned}$$

and

$$\begin{aligned}
& E\{e^{r(T-t)} \cdot \beta_1 \cdot X_\tau \cdot 1_{\{t<\tau<T\}}|\mathcal{F}_t\} \\
= \;& \int_0^{L_0 e^{rT}} e^{-r(T-t)}\beta_1 y_2\big(-\frac{\partial}{\partial y_2}\Psi(0, \sigma, \big(\frac{x_t}{L_0}\big)e^{-rt}, \big(\frac{y_2}{L_0}\big)e^{-rt}, 1, T-t)\big)dy_2 \\
= \;& e^{-r(T-t)}\big\{-\beta_1 L_0 e^{rT}\Psi(0, \sigma, \big(\frac{x_t}{L_0}\big)e^{-rt}, e^{-r(T-t)}, 1, T-t) \\
& -\int_0^{L_0 e^{rT}} \beta_1 \cdot \Psi(0, \sigma, \big(\frac{x_t}{L_0}\big)e^{-rt}, \big(\frac{y_2}{L_0}\big)e^{-rt}, 1, T-t)\big)dy_2\big\}.
\end{aligned}$$

Thus add the three equations together we can prove the relations in the Proposition.

■

The Black and Cox model was the first one to model the default of a corporate bond as a first passage time problem. Other extensions of structural model, we refer to [32], [33], [39] and so on, are based on the same idea of default assumption.

2.2 The Intensity Based Models

A main difference of an intensity model from a structural model is that, by modeling the default as an intensity based jump process, the default is totally inaccessible. The modeling of default, is reduced from modeling capital structure to modeling the intensity itself. In the intensity model the survival process is modeled in a similar way as the discount process, which makes it easier to be applied to the pricing of credit defaultable products. Comparing with structural model, the intensity model is more convenient and more broadly used by academic researchers and industry practitioners. The intensity model of credit products were studied by, among others, Jarrow [21], Jarrow-Turbull [22], Duffie-Huang [12], Lando [29], Lando [30], Duffie-Singleton [14], Belanger et al. [3] and so on. In this section, we will first introduce the classic construction of the intensity, hazard process and survival process in a general form similarly as [3]. Then we introduce the application of these processes in pricing of credit default products.

2.2.1 Intensity and hazard process

Although default is modeled as an intensity based jump process, it can still be triggered by the touch down of some process to certain random barrier. That is, the default of the company is assumed to be at the time when a non-decreasing stochastic hazard process is below a random barrier. Without loss of generality, we assume a maximum time through out this section is T. Mathematically, we define the probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, where $\mathbb{F} = \{\mathcal{F}_t\}$ is a natural filtration generated by a random process $\lambda(t)$ and augmented by $\mathbb{P}$-null sets. We denote the hazard process by $\Gamma = \{\Gamma_t : 0 \leq t \leq T\}$ and the intensity process by $\lambda = \{\lambda(t) : 0 \leq t \leq T\}$.

Definition 2.2.1 *(i) The intensity process λ is an $\mathbb{F}$-adapted process satisfying*

$$0 < \lambda_t < \infty, \quad \forall t \in [0, T] \qquad \textit{and} \qquad \int_0^T \lambda_t dt < \infty, \quad P - a.s.$$

(ii) The hazard process is defined by $\Gamma_t = \int_0^t \lambda_u du < \infty$*, an* $\mathbb{F}$*-adapted, non-decreasing process.*

(iii) Let Ξ *be a exponential random variable, independent of* $\mathbb{F}$*. The default time is defined by*

$$\tau = \inf\{t : \Gamma_t \geq \Xi\}.$$

■

Given a default time τ, we define the default indicator process $H(t) \triangleq \mathbf{1}_{\{\tau \leq t\}}$. The default filtration is defined by $\mathbb{H} = \{\mathcal{H}_t\}$, the augmented filtration generated by the single jump process $H(t)$. Define the filtration $\mathbb{F}^H$ as: $\mathbb{F}^H = \mathbb{F} \vee \mathbb{H}$.

The conditional cumulated default probabilities can be defined from the default time τ:

Definition 2.2.2 *Define the non-increasing default process* $F(t)$ *of the form:*

$$F(t) = P\{\tau \leq t | \mathcal{F}_t\}. \tag{2.7}$$

Correspondingly, the non-decreasing survival process is defined as:

$$D(t) = 1 - F(t) = P\{\tau > t | \mathcal{F}_t\}. \tag{2.8}$$

We note that Definition 2.2.1 implies that $\{\tau < t\} = \{\Gamma_t \leq \Xi\}$. Noting that Ξ is independent of $\mathbb{F}$, we deduce from (2.7) that, for any $t \in [0, T]$,

$$\begin{cases} F(t) = P\{\tau \leq t | \mathcal{F}_t\} = P\{\Gamma_t \leq \Xi | \mathcal{F}_t\} = 1 - \exp(-\Gamma_t), \\ D(t) = P\{\tau > t | \mathcal{F}_t\} = P\{\Gamma_t > \Xi | \mathcal{F}_t\} = \exp(-\Gamma_t). \end{cases} \tag{2.9}$$

The following Bayesian Formula will be frequently used.

Lemma 2.2.3 *Let* X *be an* $\mathbb{F}^H$*-adapted process. Then the following identity holds for all* $t \in [0, T]$*:*

$$E\{(1 - H(T))X_T | \mathcal{F}_t \vee \mathcal{H}_t\} = (1 - H(t)) \frac{E\{(1 - H(T))X_T | \mathcal{F}_t\}}{E\{1 - H(t) | \mathcal{F}_t\}}. \tag{2.10}$$

Proof We borrow the idea of Jeanblanc-Rutkowski [23]. Define, for fixed $t \in [0, T]$

$$\mathcal{X}_t = E\{(1 - H(T))X_T | \mathcal{F}_t \vee \mathcal{H}_t\},$$

then it holds that

$$\begin{aligned} \mathcal{X}_t &= E\{(1 - H(T))X_T | \mathcal{F}_t \vee \mathcal{H}_t\} = E\{(1 - H(t))(1 - H(T))X_T | \mathcal{F}_t \vee \mathcal{H}_t\} \\ &= (1 - H(t))\mathcal{X}_t. \end{aligned}$$

We claim that there exists an $\mathcal{F}_t$-measurable random variable $\tilde{\mathcal{X}}_t$, such that

$$(1 - H(t))\tilde{\mathcal{X}}_t = (1 - H(t))\mathcal{X}_t.$$

Indeed, note that $\mathcal{H}_t = \sigma(1_{\{\tau > s\}} : 0 \le s \le t)$, and for any $s \le t$, $\{\tau > s\} \supseteq \{\tau > t\}$. Thus one can check that $\forall H \in \mathcal{H}_t$, either $H \cap \{\tau > t\} = \emptyset$ or $H \cap \{\tau > t\} = \{\tau > t\}$. Thus

$$\mathcal{F}_t \vee \mathcal{H}_t = \{A \in \mathcal{F}_t \vee \mathcal{H}_t : \exists B \in \mathcal{F}_t, A \cap \{\tau > t\} = B \cap \{\tau > t\}\}.$$

Consequently we have

$$\begin{aligned} \tilde{\mathcal{X}}_t E\{(1 - H(t)) | \mathcal{F}_t\} &= E\{(1 - H(t))\tilde{\mathcal{X}}_t | \mathcal{F}_t\} \\ &= E\{\mathcal{X}_t | \mathcal{F}_t\} \\ &= E\{E\{(1 - H(T))X_T | \mathcal{F}_t \vee \mathcal{H}_t\} | \mathcal{F}_t\} \\ &= E\{(1 - H(T))X_T | \mathcal{F}_t\}, \end{aligned}$$

which leads to the desired identity (2.10). ■

We now define two $\mathbb{F}^H$-adapted, non-decreasing *survival processes*:

$$\begin{aligned} \tilde{D}(t, T) &= E\{1 - H(T) | \mathcal{F}_t \vee \mathcal{H}_t\}, \\ D(t, T) &= E\{1 - H(T) | \mathcal{F}_t\}. \end{aligned} \tag{2.11}$$

Then Lamma 2.2.3 indicates:

$$\tilde{D}(t, T) = (1 - H(t))\tilde{D}(t, T)$$

and

$$\begin{aligned}
\tilde{D}(t,T) &= (1-H(t))\cdot E\{1-H(T)|\mathcal{F}_t \vee \mathcal{H}_t\} \\
&= (1-H(t))\frac{E\{1-H(T)|\mathcal{F}_t\}}{E\{1-H(t)|\mathcal{F}_t\}} \\
&= (1-H(t))D(t,T)/D(t,t) \\
&= (1-H(t))E\{\exp(-\Gamma_T)|\mathcal{F}_t\}/E\{\exp(-\Gamma_t)|\mathcal{F}_t\} \\
&= (1-H(t))E\{\exp(-\int_0^T \lambda(u)du)|\mathcal{F}_t\}/E\{\exp(-\int_0^t \lambda(u)du)|\mathcal{F}_t\} \\
&= (1-H(t))E\{\exp(-\int_t^T \lambda(u)du)|\mathcal{F}_t\}.
\end{aligned}$$

We note that the definitions above started from the default hazard process Γ_t and the survival process $D(t,T)$ is constructed as a probability deduced by Γ_t. Equivalently, one can also start from the survival process $D(t,T)$ and derive the instantaneous default intensity from it.

Definition 2.2.4 *Assume the survival probability $D(t,T)$ defined in (2.11) is continuous on T. The default intensity at time t is defined as a $\mathcal{F}_t$ adapted non-negative process as:*

$$\lambda(t) = \lim_{h\downarrow 0}\frac{1}{h}P\{t<\tau\le t+h|\mathcal{F}_t \vee \mathcal{H}_t\}.$$

The $\mathcal{F}_t$ adapted forward default intensity $f(t,T)$ is defined as:

$$f(t,T) = \lim_{h\downarrow 0}\frac{1}{h}P\{T<\tau\le T+h|\mathcal{F}_t \vee \mathcal{H}_T\}.$$

From above definition, by assuming that $D(t,t)$ and $D(t,T)$ are càdlàg and non-decreasing, we can directly derive:

$$\begin{aligned}
\lambda(t) &= \lim_{h\downarrow 0}\frac{1}{h}E\{(1-H(t))H(t+h)|\mathcal{F}_t \vee \mathcal{H}_t\} \qquad (2.12) \\
&= \lim_{h\downarrow 0}\frac{1}{h}E\{(1-H(t))-(1-H(t+h))|\mathcal{F}_t\}/E\{(1-H(t))|\mathcal{F}_t\} \\
&= \lim_{h\downarrow 0}\frac{1}{D(t,t)\cdot h}(D(t,t)-D(t,t+h)) \\
&= \frac{-1}{D(t,t)}\frac{\partial}{\partial T}D(t,t)
\end{aligned}$$

and

$$\begin{aligned}
f(t,T) &= \lim_{h\downarrow 0}\frac{1}{h}E\{(1-H(T))H(T+h)|\mathcal{F}_t\vee\mathcal{H}_T\} \qquad (2.13)\\
&= \lim_{h\downarrow 0}\frac{1}{h}E\{(1-H(T))-(1-H(T+h))|\mathcal{F}_t\}/E\{(1-H(T))|\mathcal{F}_t\}\\
&= \lim_{h\downarrow 0}\frac{1}{D(t,t)\cdot h}(D(t,T)-D(t,T+h))\\
&= \frac{-1}{D(t,T)}\frac{\partial}{\partial T}D(t,T).
\end{aligned}$$

From (2.12) and (2.13), we get the following equations by direct integration:

$$D(t) = \ D(t,t) = E\{\exp(-\int_0^t \lambda(u)du)|\mathcal{F}_t\}$$

and

$$D(t,T) = E\{\exp(-\int_0^T \lambda(u)du)|\mathcal{F}_t\} = \exp(-\int_0^T f(t,u))du,$$

where $f(t,s) = f(s,s) = \lambda(s)$ for $s < t$.

2.2.2 Defaultable Claims Valuation

In the extended probability space $(\Omega, \mathcal{F}, \mathbb{P};\ \mathbb{F}^H)$, denote a defaultable claim by (X, C, Z, T, τ), which stands for a financial contract over the duration $[0, T]$, with an incoming cash flow $\{C_{t_i} - C_{t_{i-1}}\}_{i=1}^n$, at determined times $0 = t_0 < t_1 < \cdots < t_n = T$, up to the default time τ, a face value of X, and a default payoff Z $(Z < 0)$. The typical examples of a defaultable claim include the *corporate bond*, for which $X = 1$, and C is the dividends; and *a credit defaultable swap* (CDS), for which $X = 0$, and C is the *Cumulated Spread*.

Define V_t as the value of a defaultable claim (X, C, Z, T, τ) at time t, then we have:

$$\begin{aligned}
V_t &= E\big\{B(t,T)(1-H(T))X + \int_t^T B(t,u)(1-H(u))dC_u \qquad (2.14)\\
&\quad + \int_t^T B(t,u)Z_u dH_u \mid \mathcal{F}_t\vee\mathcal{H}_t\big\},
\end{aligned}$$

where $B(t,T) = \exp(-\int_t^T r(u)du)$, which denotes the bond price, is $\mathbb{F}$ adapted.

Proposition 2.2.5 *Assume the defaultable claim (X, C, Z, T, τ) satisfies the conditions:*

$E\{\int_0^T B(0,u)\exp(-\int_0^T \lambda(s)ds)dC_u\} < \infty$ *and* $E\{\sup_{u\in(t,T]} Z_u\} < \infty$.

Denote $\beta(t,T) = B(t,T)\exp(-\int_t^T \lambda(u)du)$, then the value V_t of the claim could be derived as:

$$V_t = (1 - H(t))E\big\{\beta(t,T)X + \int_t^T \beta(t,u)dC_u + \int_t^T \beta(t,u)Z_u\lambda(u)du \mid \mathcal{F}_t\big\}. \quad (2.15)$$

Proof Break down V_t formula (2.14) into three parts: $V_t = V_t^1 + V_t^2 + V_t^3$, and

$$V_t^1 = E\big\{B(t,T)X(1 - H(T)) \mid \mathcal{F}_t \vee \mathcal{H}_t\big\};$$

$$V_t^2 = E\{\int_t^T B(t,u)(1 - H(u))dC_u \mid \mathcal{F}_t \vee \mathcal{H}_t\};$$

$$V_t^3 = E\{\int_t^T B(t,u)Z_u dH_u \mid \mathcal{F}_t \vee \mathcal{H}_t\big\}.$$

Then we show V_t^1, V_t^2 and V_t^3 equates the terms on the right hand side of (2.15). First, applying lemma (2.2.3), we have,

$$\begin{aligned} V_t^1 &= (1 - H(t))\frac{E\{B(t,T)(1 - H(T))X \mid \mathcal{F}_t\}}{E\{1 - H(t) \mid \mathcal{F}_t\}} \quad (2.16) \\ &= (1 - H(t))\frac{E\{B(t,T)X\exp(-\int_0^T \lambda(u)du) \mid \mathcal{F}_t\}}{E\{\int_0^t \lambda(u)du \mid \mathcal{F}_t\}} \\ &= (1 - H(t))E\{B(t,T)X\exp(-\int_t^T \lambda(u)du) \mid \mathcal{F}_t\} \\ &= (1 - H(t))E\{\beta(t,u)Xdu \mid \mathcal{F}_t\}. \end{aligned}$$

Similarly, from the assumption that $\int_0^T \beta(t,u)dC_u < \infty$ and C_t is deterministic,

$$\begin{aligned} V_t^2 &= E\{\int_t^T B(t,u)(1-H(u))dC_u \mid \mathcal{F}_t \vee \mathcal{H}_t\} \quad (2.17)\\ &= \int_t^T E\{B(t,u)(1-H(u)) \mid \mathcal{F}_t \vee \mathcal{H}_t\}dC_u \\ &= \int_t^T E\{\beta(t,u) \mid \mathcal{F}_t\}dC_u \\ &= \int_t^T E\{\beta(t,u)dC_u \mid \mathcal{F}_t\} \end{aligned}$$

Note that H_u is a single jump process and $\{Z_u\}$ is deterministic,

$$\begin{aligned} V_t^3 &= E\Big\{\int_t^T B(t,u)Z_u dH_u \mid \mathcal{F}_t \vee \mathcal{H}_t\Big\} \quad (2.18)\\ &= E\Big\{\Sigma_{u\in(t,T]} B(t,u)Z_u[H_u - H_{u-}] \mid \mathcal{F}_t \vee \mathcal{H}_t\Big\} \\ &= E\Big\{-\int_t^T B(t,u)Z_u d\tilde{D}(t,u) \mid \mathcal{F}_t\Big\} \\ &= E\Big\{\int_t^T \beta(t,u)Z_u\lambda(u)du \mid \mathcal{F}_t\Big\}. \end{aligned}$$

(2.19)

Sum (2.16), (2.17) and (2.18) together, we proved that equation (2.14) holds. ■

It is now clear that the pricing of defaultable products relies on the survival process $D(t,T)$. It is obvious that from the pricing formula, the intensity is easier to be applied in pricing processes. However, the intensity based model doesn't have the corporate capital structure or financial leverage involved in the pricing process. Further more, the more derivative products, such as credit default swap option, requires the dynamic of $D(t,T)$. This drives our goal to exploring the dynamics of survival process and even more comprehensive model.

3. DEFAULT UNDER INCOMPLETE INFORMATION

We now introduce our model of default under incomplete information (or imperfect information). This model is based on the assumption that the complete accounting information of the company is not available all time. In practice, the delayed information is released periodically and at the same time, where as the equity information available all the time. The delayed accounting information and the continuously observed equity information are the main factors of the model. Hence this chapter is structured in a similar way: First section we introduce the model and the foundation of the problems. Examples of different delayed information models will be introduced and compared with our model in the second section. Then we solve our problem and derive the term structure equations of the survival process and default intensities.

3.1 The Model and Problem Formulation

Throughout this chapter we assume that all the randomness are defined on a given filtered probability space $(\Omega, \mathcal{F}, P; \mathbb{F})$, where $\mathbb{F} = \{\mathcal{F}_t\}$ is a filtration satisfying the *usual hypotheses*. We assume that the probability space is rich enough to carry a multi-dimensional standard Brownian motion (W, W^θ, W^S), and an independent Poisson random measure μ defined on $[0, \infty) \times \mathbb{R}$, with *Levy* measure ν. Since the dimensions of W, W^θ, and W^S are not essential in our discussion, we shall henceforth assume that they are all one dimensional. In what follows we shall denote the factor process by θ, and the observable stock prices of the firm by S. The meanings of W^θ and W^S are therefore obvious.

In light of the delayed information model [18], we assume that the following structure for the information collection. Assume that the firm releases its accounting information at fixed times $0 = t_0 < t_1 < t_2 < \cdots < t_N = \bar{T}$. At each time t_k, we have full exposure of accounting information up to time t_k, we assume that for any $t \in [t_k, t_{k+1})$, there will be no further information available. Mathematically, let us define, for any time $t \in [0, T]$, the delayed time:

$$[t] = \sum_{i=0}^{N-1} t_k \mathbf{1}_{[t_k, t_{k+1})}(t), \qquad t \in [0, T]. \tag{3.1}$$

Also, for the given filtration $\mathcal{F}_t$, we define the delayed filtration at time t by $\mathcal{F}_t^d$.

Next, we define $\mathbb{F}^W \stackrel{\triangle}{=} \{\mathcal{F}_t^W\}_{t\ge0}$, $\mathbb{F}^\theta \stackrel{\triangle}{=} \{\mathcal{F}_t^\theta\}_{t\ge0}$, and $\mathbb{F}^S \stackrel{\triangle}{=} \{\mathcal{F}_t^S\}_{t\ge0}$ be the natural filtrations generated by the Brownian motions W, W^θ, and W^S, respectively, all augmented by the $\mathbb{P}$-null sets in $\mathcal{F}$ so that they all satisfy the usual hypotheses. We then define, For any fixed time t, a filtration $\mathbb{F}^{W,\theta,t} \stackrel{\triangle}{=} \{\mathcal{F}_u^{W,\theta,t}\}_{u\ge t}$, where

$$\mathcal{F}_u^{W,\theta,t} = \mathcal{F}_u^W \vee \mathcal{F}_t^\theta, \qquad u \ge [t]. \tag{3.2}$$

It should be noted that in our model the "present time" t is a special parameter which determines not only the amount of the information collected, but also the parameters of the future dynamics to be estimated. This is based on the premise that with more and more past information of the parameters will be more and more accurately identified. These points will be made clear in our discussions below.

3.1.1 Dynamics of total asset and factor.

From this point on we shall fix a starting time $t \in [0, T]$, which can be thought of as the *present time*. We first describe the unobservable factor process $\theta = \{\theta(s) : s \ge 0\}$. We assume, which takes the form of a continuous Markov process, satisfying the following SDE starting from t:

$$\theta(t) = \theta_0 + \int_0^t \mu^\theta(s, \theta(s))ds + \int_0^t \sigma^\theta(s, \theta(s))dW_s^\theta, \quad t \in [0, T], \tag{3.3}$$

where μ^θ and σ^θ are two deterministic functions, to be specified later. Clearly, θ is $\mathbb{F}^\theta$-adapted, and is independent of $\mathbb{F}^W$ and $\mathbb{F}^S$. To describe the dynamics of total asset, denoted by $X = \{X_t : 0 \le t \le T\}$ in the sequel, we first note that for each $t \in [0, T]$, if $t \in [t_k, t_{k+1})$, then value $X_{[t]} = x_k$ is a given deterministic number, representing the firm's value via its accounting report. The dynamics of X will be "reset" at t_k, with a set of coefficients, depending on the factor θ, or its estimated value, at time t. To be more precise, we have the following definition.

Definition 3.1.1 *For each time $t \in [0, T]$, assuming $t \in [t_k, t_{k+1})$, the total asset X is the $\mathbb{F}^{W,\theta,t}$-adapted solution to the following SDE:*

$$X_u^t = x_{[t]} + \int_{[t]}^u X_s^t \mu(\theta(t), t, [s])ds + \int_{[t]}^u X_s^t \sigma(\theta(t), t, [s])dW_s, \quad [t] \leq u \leq T, \quad (3.4)$$

where $\mu(\theta, t, t_k) = \mu_k(\theta, t)$ and $\sigma(\theta, t, t_k) = \sigma_k(\theta, t)$, $k = 1, 2, \cdots, N$ are two sequences of (deterministic) functions of θ and t.

Remark 3.1.2 (i) We should note that for each fixed time t, say, $u \in [t_k, t_{k+1})$, the coefficients $\mu(\theta(t), t, t_k)$ and $\sigma(\theta(t), t, t_k)$ in (3.4) are constants for $u \in [t_k, t_{k+1})$. In other words, X_u^t is in fact $\mathbb{F}^{W,\theta,t}$-adapted geometric Brownian motion for $u \geq [t]$ with parameter sets $(\mu(\theta(t), t, t_k), \sigma(\theta(t), t, t_k))$ when $u \in [t_k, t_{k+1})$. The dependence of μ and σ on $(\theta(t), t)$ reflects the possibility that the piecewise constant coefficients could be adjusted at each time t, for instance, by a parameter identification process based on the observations up to time t.

(ii) The Merton Structural model is a special case of (3.4) in which μ and σ are simply constants. Another slightly more general model is that X_u is the solution to an SDE with piecewise constant parameters. For example, in the defaultable swap option modeling, one often assume that the firm's asset follows an SDE with periodic constat drift and volatility pairs $\{(\mu_k, \sigma_k)\}_{k \geq 0}$:

$$X_T = X_0 + \int_0^T X_u \mu(u)du + \int_0^T X_u \sigma(u)dW_u, \qquad T \in [0, \bar{T}], \qquad (3.5)$$

where $\mu(u) = \mu_k$ and $\sigma(u) = \sigma_k$ if $u \in [t_k, t_{k+1}), k = 1, 2, ...$

If we assume that at each time t, $\mu_k = \mu(\theta(t), t, t_k)$ and $\sigma_k = \sigma(\theta(t), t, t_k)$, then $X = X_u^t$ would be the solution of SDE (3.4). ■

Having defined the asset dynamic, the default time can be defined in the following traditional way: assume that present time $t = 0$, and given a default barrier $\nu \triangleq \nu(\theta(.), ., .)$, which may depend on both current time t and the factor $\theta(t)$:

$$\tau_0 = \inf\{u > 0 : X_u^0 \leq \nu(\theta(0), 0, 0) \mid X_0 > \nu(\theta(0), 0, 0)\},$$

Note that if one starts from the present time t, then the default after time t is of the form:

$$\tau_t = \inf\{u > [t] : X_u^t \leq \nu(\theta(t), t, [u]) \mid X_{[t]} = x > \nu(\theta([t]), [t], [t])\}. \tag{3.6}$$

One should note that in the above $\theta(\cdot)$ is in general not observable. An observation processes carrying the accounting information is thus in order.

3.1.2 Observation processes

In what follows we assume that, besides the accounting information announced at each $[t]$, one can observe the firm's stock price and some sudden events related to the firm's performance (e.g., the jumps of the firm's CDS spread). To be more precise, we assume that the stock price $S = \{S_t\}$ satisfies the following SDE:

$$S_t = s_0 + \int_0^t S_u \mu^S(\theta(u))du + \int_0^t S_u \sigma^S(u) dW_u^S, \qquad t \geq 0. \tag{3.7}$$

We shall assume that the volatility $\sigma^S(t) = \sigma(t)$ is deterministic, and is observable from the market. But this can be easily extended to the case where σ depends on the starting time t, in its future dynamics as we see in (3.4).

Another important observation process which plays an important role in our discussion is of a pure jump form. To be more precise, we assume for a (deterministic) function $\nu : E \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, the standard Poisson measure has a measure of the form $\nu(x, t)$ and the Poisson process based on this measure.

Thus we can define the compensated Poisson process $\tilde{Y}_t$ as:

$$Y_t \ = Y_0 + \int_0^t \int_E k(x, \theta(t)) \tilde{N}(dx, dt), \tag{3.8}$$

where $\tilde{N}(dx,dt) = N(dx,dt) - v(dx,dt)$ is the compensated standard Poisson measure.

Clearly, if we let $\{\mathcal{F}_t^Y\}$ be the augmented natural filtration generated the pure jump process $\{Y_t\}_{t\geq 0}$, then the compensated process $\tilde{Y}_t$ is an $\mathbb{F}^Y$-martingale.

To make sure that all SDEs that we saw above will well-posed, let us make the following *Standing Assumptions*:

(A1) There exists a constant $L > 0$ such that

$$\begin{cases} |\mu^\theta(t,x) - \mu^\theta(t,y)| + |\sigma^\theta(t,x) - \sigma^\theta(t,y)| \leq L|x-y|, \\ |\mu^\theta(t,x)| + |\sigma^\theta(t,x)| \leq L(1+|x|). \end{cases} \qquad \forall x,y \in \mathbb{R}; \quad (3.9)$$

(A2) There exists constants $L > 0$ and $C > 0$ such that for all $x, y \in \mathbb{R}$:

$$\begin{cases} |\mu^S(x) - \mu^S(y)| + |\sigma^S(x) - \sigma^S(y)| \leq L|x-y|, \\ \mu^S(x)| \leq L(1+|x|), \end{cases} \qquad \forall x,y \in \mathbb{R} \quad (3.10)$$

and

$$|\sigma^S(t)] \geq C > 0, \qquad t \in [0,T]. \qquad (3.11)$$

(A3) There exists constants $0 < c < C$ such that the intensity function $k(\cdot,\cdot)$ in (3.8) satisfies

$$0 < c \leq |k(x,y)| \leq C, \qquad \forall x \in \mathbb{R},\ y \in \mathbb{R}, \qquad (3.12)$$

We remark that under assumption **(A1)** the SDE (3.3) has a unique strong solution θ which is square integrable. Thus the (3.10) implies that

$$E\{\int_0^T |\mu^S(\theta(t))|^2\} < \infty.$$

Consequently, the SDE (3.7) has a strong solution, and our observation equations are all well-posed.

Having defined the observable processes, we now define the filtration of observation as follows. First, denote $\mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^Y$, $t \in [0,T]$, to be the total observable filtration. Then the market information filtration is

$$\mathcal{M}_t \stackrel{\triangle}{=} \mathcal{F}_{[t]}^W \vee \mathcal{F}_{[t]}^\theta \vee \mathcal{G}_t, \qquad t \in [0,T]. \qquad (3.13)$$

Finally, let τ be the default time, we define $\mathcal{H}_t$ to be the filtration generated by the default indicator process $H(t) = \mathbf{1}_{\{\tau_t \le t\}}$. It is then easy to see that the following relations among the filtration should be true:

$$\mathcal{F}^W_{[t]} \subseteq \mathcal{M}_t \subseteq \mathcal{M}_t \vee \mathcal{F}^\theta_t \subseteq \mathcal{M}_t \vee \mathcal{F}^\theta_t \vee \mathcal{H}_t. \tag{3.14}$$

3.1.3 The conditional default probabilities and their intensities

We are now ready to give a mathematical description of our main objectives. We begin by giving the definitions of the conditional survival probabilities that we are interested in. Assuming first that we can completely observe the factor process $\theta = \{\theta(t) : t \ge 0\}$, then for each given time $t \in [0, T]$, we are interested in the following quatities:

- The instantaneous conditional default probability at t:

$$F(t) = P\{\tau_t \le t | \mathcal{M}_t\} = P\Big\{ \inf_{[t]<u\le t} X^t_u \le \nu(\theta(t), t, [u]) \Big| \mathcal{M}_t \Big\}. \tag{3.15}$$

- The instantaneous conditional survival probability at t:

$$D(t) = P\{\tau_t > t | \mathcal{M}_t\} = P\Big\{ \inf_{[t]<u\le t} X^t_u > \nu(\theta(t), t, [u]) \Big| \mathcal{M}_t \Big\}. \tag{3.16}$$

- The forward conditional survival probability from time t to T:

$$D(t,T) = P\{\tau_t > T | \mathcal{M}_t\} = P\Big\{ \inf_{[t]<u\le T} X^t_u > \nu(\theta(t), t, [u]) \Big| \mathcal{M}_t \Big\}. \tag{3.17}$$

We often consider the following "future survival probability" defined as

$$\tilde{D}(t,T) = P\{\tau_t > T | \mathcal{M}_t \vee \mathcal{H}_t\}. \tag{3.18}$$

We remark that the instantaneous default probability indicates that, given the last accounting report, and before the next one coming out, the probability that the firm is already default at present time t, given all the information that one can get from the market. The meaning of D is similar. It is clear that F is non-decreasing and D is non-increasing, thus we can assume that they both have càdlàg paths. Furthermore, the difference between $D(t,T)$ and $\tilde{D}(t,T)$ is that $\tilde{D}(t,T)$ is the probability that the firm will survive at least to T, knowing that it is "alive" at the present time. But $D(t,T)$ does not exclude the event that the firm is already bankrupt at t.

Our main task is to find the following (conditional) default probabilities:

$$\begin{aligned} G(\theta(t),t,T) &\triangleq P\Big\{\inf_{[t]<u<T} X_u^t > \nu(\theta(t),t,[u])\Big|\mathcal{F}_t^\theta \vee \mathcal{M}_t\Big\} \qquad (3.19) \\ G(\theta(t),t,T,y) &\triangleq P\Big\{\inf_{[t]<u<T} X_u^t > \nu(\theta(t),t,[u]), X_T^t > y\Big|\mathcal{F}_t^\theta \vee \mathcal{M}_t\Big\},\ \forall y > \nu. \end{aligned}$$

It is readily seen that

$$D(t,T) = P\{\tau > T|\mathcal{M}_t\} = P\Big\{\inf_{[t]<u\le T} X_u^t > \nu(\theta(t),t,[u])\Big|\mathcal{M}_t\Big\} = E\{G(\theta(t),t,T)|\mathcal{M}_t\}.$$

It is now clear that if we could find the explicit form of the function G, then the "separation principle" of nonlinear filtering then indicates that we need only find the conditional density of $\theta(t)$ given the observation processes S and Y up to time t. But this is a nonlinear filtering problem which combines both continuous and pure jump types of observations. We shall carry out these two tasks separately in the following sections.

3.2 The Default Term Structure

In what follows we show that as a hybrid model our model inherits many properties of a reduced form model introduced in Section 2.2.1. We first extend the Bayesian property of Lemma 2.2.3 in Section 2.2.1 to our hybrid model where the random variable in the expectation is measurable with respect to a larger filtration. Then we show, based on this extended Bayesian Lemma, conditional survival process and default intensity can be defined and derived in the same way as in the regular intensity based model.

3.2.1 Default Intensity under the delayed information

Recall from last section, survival process is defined as a first passage time formula of the $\mathbb{F}^{W,\theta,t}$ adapted process X_u^t conditional under the smaller delayed filtration $\mathcal{M}_t$. In order to extend the Bayesian Lemma to fit our model, we introduce the following lemma:

Lemma 3.2.1 *Suppose A has finite expectation and is $\mathcal{H}$-measurable and $\mathcal{D}$ is independent of $\mathcal{G} \vee \mathcal{H}$. Then*

$$E[A|\mathcal{G} \vee \mathcal{D}] = E[A|\mathcal{G}]$$

Proof It suffices to show that, $\forall$ G $\in \mathcal{G}$ and $\forall D \in \mathcal{D}$,

$$\int_{G\cap D} E[A|\mathcal{G}]dP = \int_{G\cap D} AdP.$$

That is:

$$\int_{G\cap D} E[A|\mathcal{G}]dP = E[1_D 1_G E[A|\mathcal{G}]] = E[1_D E[1_G A|\mathcal{G}]] = E[1_D 1_G A] = \int_{D\cap G} AdP.$$

■

Now we are ready to give the Bayesian Lemma for our model:

Lemma 3.2.2 *Suppose two general filtrations $\mathbb{F}$ and $\mathbb{M}$ satisfies the condition: $\mathbb{M} \subset \mathbb{F}$ and they are independent with the survival filtration $\mathbb{H}$. Then for any $\mathbb{F} \vee \mathbb{H}$ adapted random process X, the following form of Bayesian equation holds:*

$$E\{(1 - H(T))X|\mathcal{M}_t \vee \mathcal{H}_t\} = (1 - H(t))\frac{E\{(1 - H(T))X|\mathcal{M}_t\}}{E\{1 - H(t)|\mathcal{M}_t\}}.$$

Proof In Lemma 2.2.3 we have proved that, if X is $\mathbb{F} \vee \mathbb{H}$ adapted and denote $\mathcal{X}_t = E\{(1 - H(T))X|\mathcal{F}_t \vee \mathcal{H}_t\}$, then $\mathcal{X}_t = (1 - H(t))\mathcal{X}_t$ and $\exists$ an $\mathbb{F}$ adapted process $\tilde{\mathcal{X}}_t$, such that $(1 - H(t))\tilde{\mathcal{X}}_t = (1 - H(t))\mathcal{X}_t$.
Next we show, let $\mathcal{X}_t^* = E\{\tilde{\mathcal{X}}_t|\mathcal{M}_t\}$, then $\mathcal{X}_t^*$ is $\mathbb{M}$ adapted and

$$(1 - H(t))\mathcal{X}_t^* = E\{\mathcal{X}_t|\mathcal{M}_t \vee \mathcal{H}_t\} = E\{(1 - H(T))X|\mathcal{M}_t \vee \mathcal{H}_t\}.$$

Note from $\mathcal{M}_t \subset \mathcal{F}_t$ and $1 - H(t)$ is $\mathcal{M}_t \vee \mathcal{H}_t$ adapted,

$$
\begin{aligned}
& E\{(1-H(t))\mathcal{X}_t|\mathcal{M}_t \vee \mathcal{H}_t\} \\
= \; & E\{(1-H(t))\tilde{\mathcal{X}}_t|\mathcal{M}_t \vee \mathcal{H}_t\} \\
= \; & (1-H(t))E\{\tilde{\mathcal{X}}_t|\mathcal{M}_t \vee \mathcal{H}_t\} \\
= \; & (1-H(t))E\{\tilde{\mathcal{X}}_t|\mathcal{M}_t\} \\
= \; & (1-H(t))\mathcal{X}_t^*.
\end{aligned}
$$

The second last equality holds if $\tilde{\mathcal{X}}_t$ is independent with $\mathcal{H}_t$. This is by Lemma 3.2.1. The rest of the proof is straight forward:

$$
\begin{aligned}
& \mathcal{X}_t^* E\{(1-H(t))|\mathcal{M}_t\} \\
= \; & E\{(1-H(t))\mathcal{X}_t^*|\mathcal{M}_t\} \\
= \; & E\{E\{(1-H(t))\mathcal{X}_t^*|\mathcal{M}_t \vee \mathcal{H}_t\}|\mathcal{M}_t\} \\
= \; & E\{E\{\mathcal{X}_t|\mathcal{F}_t \vee \mathcal{H}_t\}|\mathcal{M}_t\} \\
= \; & E\{(1-H(T))X|\mathcal{M}_t\}.
\end{aligned}
$$

■

Now with above lemma, we can define the default intensity in the traditional way:

Definition 3.2.3 *The default intensity at time t is defined as a $\mathbb{M}$ adapted process as the following:*

$$\lambda(t) = \lim_{h \downarrow 0} \frac{1}{h} P\{t < \tau \le t + h|\mathcal{M}_t \vee \mathcal{H}_t\}$$

The forward default intensity $f(t,T)$ is defined as:

$$f(t,T) = \lim_{h \downarrow 0} \frac{1}{h} P\{T < \tau \le T + h|\mathcal{M}_t \vee \mathcal{H}_T\}$$

Now let us suppose the survival probabilities defined by (3.17) and (3.18) are both continuous and differentiable on the future time T. (We will prove in later section, that by our assumption of default, this is true.) Our model definition of intensity and and forward intensity makes it the same as the classic definitions. Then directly from above definition and Lemma 3.2.2 to the traditional process (2.12) and (2.13), we have the spot intensity is of the form:

$$\lambda(t) = -\frac{1}{D_t}\frac{\partial}{\partial T}D(t,t), \tag{3.20}$$

where as the forward default intensity $f(t,T)$ can be written as:

$$f(t,T) = -\frac{1}{D(t,T)}\frac{\partial}{\partial T}D(t,T). \tag{3.21}$$

By definition, we know:

$$\lambda(t) = f(t,t).$$

Similarly, the survival processes are of the condtional integration forms:

$$\begin{aligned} D(t,T) &= H(t)E\Big\{exp\big(-\int_{[t]}^{T}\lambda(u)du\big)|\mathcal{M}_t\Big\} \\ &= H(t)exp\big(-\int_{[t]}^{T}f(t,u)du\big), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \tilde{D}(t,T) &= H(t)E\Big\{exp\big(-\int_{t}^{T}\lambda(u)du\big)|\mathcal{M}_t\Big\} \\ &= H(t)exp\big(-\int_{t}^{T}f(t,u)du\big). \end{aligned} \tag{3.23}$$

And $\forall s < t,\ f(t,s) = f(s,s) = \lambda(s)$.

Definition 3.2.4 *For any t and T such that $t < T < \bar{T}$, the forward hazard process $\Lambda(t,T)$ is defined as:*

$$\Lambda(t,T) = \int_{[t]}^{T} f(t,u)du.$$

Fix time t, then $\Lambda(t,\cdot)$ has a differentiation form:

$$d\Lambda(t,\cdot) = -\frac{dD(t,\cdot)}{D(t,\cdot)}.$$

Thus $D(t,T) = e^{-\Lambda(t,T)}$ and similarly we can define $\Lambda(t) = \Lambda(t,t)$.

It is now clear that, given our model setting of default process indicated by the total asset process X, the general or the extended properties of the survival processes in traditional intensity based models also hold in our model. Recall the valuation process for a defaultable claim in section 2.2.2, if the survival processes satisfies and general properties, the pricing formula could be applied to our model.

3.2.2 Survival under delayed information

Given the process $\{X_u^t\}$ is adapted to $\mathbb{F}^{W,\theta,t}$, let us fix the present time t and the random factor $\theta(t)$ at t, then X_u^t is a geometric Brownian motion process with piecewise constant coefficients. And the survival probability of the X at time t and under $\mathcal{F}_u^{W,\theta,t}$ can be derived as a first passage time problem.

Proposition 3.2.5 *Let X_u^t be the solution of the differential equation (3.4), assume the present time $t \in [t_s, t_{s+1})$, maturity $T \in [t_k, t_{k+1})$ and $t_k > t_s$. Then the conditional survival probabilities $G(\theta(t), t, T)$ and $G(\theta(t), t, T, y)$ defined in (3.19) can be solved by the following:*

$$\begin{aligned}
& G(\theta(t), t, T) \qquad (3.24)\\
= & \int_{\nu(\theta(t),t,t_k)}^{\infty} \cdots \int_{\nu(\theta(t),t,t_s)}^{\infty} \psi\big(\mu\big(\theta(t), t, t_k\big), \sigma\big(\theta(t), t, t_k\big), y_k, y, \nu(\theta(t), t, t_k), T - t_k\big) \\
& \times \psi\big(\mu\big(\theta(t), t, t_{k-1}\big), \sigma\big(\theta(t), t, t_{k-1}\big), y_{k-1}, y_k, \nu(\theta(t), t, t_{k-1}), t_k - t_{k-1}\big) \cdots \\
& \times \psi\big(\mu\big(\theta(t), t, t_s\big), \sigma\big(\theta(t), t_s\big), X_{t_s}, y_{s+1}, \nu(\theta(t), t, t_s), t_{s+1} - t_s\big) dy_{s+1} dy_{s+2} \cdots dy_k dy.
\end{aligned}$$

where the function $\psi\big(\mu, \sigma, y_1, y_2, \nu, \Delta t\big)$ is partial derivative of $\Psi\big(\mu, \sigma, y_1, y_2, \nu, \Delta t\big)$ on y_2 and $\Psi\big(\mu, \sigma, y_1, y_2, \nu, \Delta t\big)$ is the survival probability formula defined in Lemma 2.1.3.

$$\begin{aligned}
& \psi\big(\mu, \sigma, y_1, y_2, \nu, \Delta t\big) \qquad (3.25)\\
= & \Big(\frac{2\mu - \sigma^2}{y_2 \sigma^2}\Big) \cdot \exp\Big(\frac{(2\mu - \sigma^2)\log[y_2/y_1]}{\sigma^2}\Big) \cdot N\big(d_1(\mu, \sigma, y_1, y_2, \nu, \Delta t)\big) \\
& + \frac{2}{y_2 \sigma \sqrt{\Delta t}} \cdot \exp\Big(\frac{(2\mu - \sigma^2)\log[y_2/y_1]}{\sigma^2}\Big) \cdot n\big(d_1(\mu, \sigma, y_1, y_2, \nu, \Delta t)\big),
\end{aligned}$$

for $d_1(\mu, \sigma, y_1, y_2, \nu, \Delta t)$ and $d_2(\mu, \sigma, y_1, y_2, \nu, \Delta t)$ defined in Lemma 2.1.3.

Proof At a fixed time t, solving the SDE (3.4) for $\{X_u^t\}$ on any time interval (t_{s^*}, t_{s^*+1}) and $s < s^* < t$, we have:

$$X_u^t = X_{t_{s^*}} exp\big\{ \big(\mu(\theta(t), t, t_{s^*}) - \frac{1}{2}\sigma^2(\theta(t), t, t_{s^*})\big)(u - t_{s^*}) + \sigma(\theta(t), t, t_{s^*}) W_u \big\};$$
$$\forall u \in [t_{s^*}, t_{s^*+1}).$$

The default in a single time interval (t_{s^*}, t_{s^*+1}), where the coefficients of X is constant, is the same as Black-Cox model default probability. The survival process $G(\theta(t), t, T, y)$ is of the form:

$$\begin{aligned} & G(\theta(t), t, t_{s^*+1}, y) \\ = \; & P\big\{ \inf_{t_{s^*} < u < t_{s^*+1}} X_u^t \geq \nu(\theta(t), t, t_{s^*}),\; X_{t_{s^*+1}}^t > y | \mathcal{F}_{t_{s^*}}^W \vee \mathcal{F}_t^\theta \vee \mathcal{F}_t^S \vee \mathcal{F}_t^Y \big\} \\ = \; & \Psi\Big(\mu\big(\theta(t), t, t_{s^*}\big), \sigma\big(\theta(t), t, t_{s^*}\big), X_{t_{s^*}}, y, \nu(\theta(t), t, t_{s^*}), t_{s^*+1} - t_{s^*}\Big), \end{aligned}$$

where the last equality holds for the reason that $\mathbb{F}^W$ is independent with $\mathbb{F}^\theta$, $\mathbb{F}^S$ and $\mathbb{F}^Y$.

Now let us consider the survival probability on multiple time periods. That is the case when $[t] = t_s$ and $[T] = t_k$, where $k - s > 1$.

$$\begin{aligned} & G(\theta(t), t, T, \bar{y}) \qquad (3.26) \\ = \; & P\{\tau > T,\; X_T^t > \bar{y} \mid \tau > [T], \mathcal{F}_t^\theta \vee \mathcal{M}_t\} \cdot P\{\tau > [T] \mid \mathcal{F}_t^\theta \vee \mathcal{M}_t\} \\ = \; & \int_{\nu(\theta(t),t,t_k)}^{\infty} P\big(\tau > T,\; X_T^t > \bar{y} \mid X_{[T]}^t = y, \tau > [T], \mathcal{F}_t^\theta \vee \mathcal{M}_t\big) \\ & \times P\big(X_{[T]}^t \in dy | \tau > [T], \mathcal{F}_t^\theta \vee \mathcal{M}_t\big) \cdot P\big(\tau > [T] | \mathcal{F}_t^\theta \vee \mathcal{M}_t\big) \\ = \; & \int_{\nu(\theta(t),t,t_k)}^{\infty} P\big(\tau > T, X_T^t > \bar{y} | X_{[T]}^t = y, \tau > [T], \mathcal{F}_t^\theta \vee \mathcal{M}_t\big) \\ & \times P\big(X_{[T]}^t \in dy, \tau > [T] | \mathcal{F}_t^\theta \vee \mathcal{M}_t\big) \\ = \; & \int_{\nu(\theta(t),t,t_k)}^{\infty} \Psi\Big(\mu\big(\theta(t), t, t_k\big), \sigma\big(\theta(t), t, t_k\big), y, \bar{y}, \nu\big(\theta(t), t, t_k\big), T - [T]\Big) \\ & \times \big[- \frac{\partial}{\partial y} G(\theta(t), t, [T], y) \big] dy. \end{aligned}$$

By taking partial derivative of $\bar{y}$ on both sizes of above equation, we have:

$$
\begin{aligned}
&\frac{\partial}{\partial \bar{y}} G(\theta(t), t, T, \bar{y}) \qquad (3.27)\\
= &\int_{\nu(\theta(t),t,t_k)}^{\infty} \psi\Big(\mu\big(\theta(t), t, t_k\big), \sigma\big(\theta(t), t, t_k\big), y, \bar{y}, \nu(\theta(t), t, t_k), T - [T]\Big)\\
&\times\big[\frac{\partial}{\partial y} G(\theta(t), t, [T], y)\big] dy.
\end{aligned}
$$

Expand the integration recursively, formula (3.27) can be derived as:

$$
\begin{aligned}
&\frac{\partial}{\partial \bar{y}} G(\theta(t), t, T, \bar{y})\\
= &\int_{\nu(\theta(t),t,t_k)}^{\infty} \cdots \int_{\nu(\theta(t),t,t_s)}^{\infty} \psi\big(\mu\big(\theta(t), t, t_k\big), \sigma\big(\theta(t), t, t_k\big), y_k, \bar{y}, \nu(\theta(t), t, t_k), T - t_k\big)\\
&\times\psi\big(\mu\big(\theta(t), t, t_{k-1}\big), \sigma\big(\theta(t), t, t_{k-1}\big), y_{k-1}, y_k, \nu(\theta(t), t, t_{k-1}), t_k - t_{k-1}\big)\cdots\\
&\times\psi\big(\mu\big(\theta(t), t, t_s\big), \sigma\big(\theta(t), t_s\big), X_{t_s}, y_{s+1}, \nu(\theta(t), t, t_s), t_{s+1} - t_s\big) dy_k \cdots \cdot dy_{s+1}.
\end{aligned}
$$

Integrating both sizes on $\bar{y}$, we can prove the formula in the Proposition. ■

Note, by definition, we have $G(\theta(t), t, T) = G\big(\theta(t), t, T, \nu(\theta(t), t, [T])\big)$ and the density function $g(\theta(t), t, T) = \frac{\partial}{\partial T} G(\theta(t), t, T)$. Thus we have

$$
\begin{aligned}
&g(\theta(t), t, T) \qquad (3.28)\\
= &\int_{\nu(\theta(t),t,t_k)}^{\infty} \cdots \int_{\nu(\theta(t),t,t_s)}^{\infty} \psi_T\big(\mu\big(\theta(t), t, t_k\big), \sigma\big(\theta(t), t, t_k\big), y_k, y, \nu(\theta(t), t, t_k), T - t_k\big)\\
&\times\psi\big(\mu\big(\theta(t), t, t_{k-1}\big), \sigma\big(\theta(t), t, t_{k-1}\big), y_{k-1}, y_k, \nu(\theta(t), t, t_{k-1}), t_k - t_{k-1}\big)\cdots\\
&\times\psi\big(\mu\big(\theta(t), t, t_s\big), \sigma\big(\theta(t), t_s\big), X_{t_s}, y_{s+1}, \nu(\theta(t), t, t_s), t_{s+1} - t_s\big) dy dy_k \cdots dy_{s+1}.
\end{aligned}
$$

It should be noted that, by our model setting, both the survival probability formula and the survival density formula are continuous functions of T. And the survival functions are differentiable on θ if $\mu(\theta(t), t, u)$, $\sigma(\theta(t), t, u)$ and $\nu(\theta(t), t, u)$ are differentiable on θ. The survival formulae are $\mathbb{F}^\theta \vee \mathbb{M}$ adapted. Given our goal of estimating default under available information filtration, our main task task is left with calibrating the conditional expectations of the survival functions under filtration $\mathbb{M}$, which will be accomplished in later sections.

3.3 Examples of delayed and imperfect information models

We finished the set up of our model and our main problem. At this point, we would like to show and compare our work with the current existing models. For the consistence purpose, we will use out model notations in the examples.

3.3.1 Gausian Copula model

Gausian Copula model, which is originated by Li [34], is the benchmark of credit default obligation (CDO) pricing model. In this model, survival probability of a single CDS in certain CDO portfolio is assumed to be standard Gausian. When dealing with the correlated default of the CDS portfolio, the Gaussian random variables in each of the CDS's are assumed to be correlated. Equivalently, it is to assume the minimum of the total asset X during certain time interval to be a pure Brownian motion process without delayed information in our model. As we are interested in the default of a single entity, our introduction below will focus on single default in Gausian Copula model.

Assume at present time $t \in (t_s, t_{s+1})$, we are interested in the survival probability at time intervals $[t, t_{s^*}]$, where $s^* = t_s,\ t_{s+1}, ...t_k$. Assume at each interval $[t, t_{s^*}]$, the random process $\inf_{[t]<u<T} X_u^{i,t} = X^* = \rho_{s^*} Z + \sqrt{1-\rho_{s^*}^2} Z^{X_i}$, where Z and Z^{X2} are independent standard normal random variables. We simplify and assume the default barrier $\nu_i(\theta(t), t, [u]) = \theta_i(t)$.

Proposition 3.3.1 *Based on above setting of defaults, the i^{th} CDS survival probability $D_i(t,T)$ is of the form:*

$$D_i(t,T) = G_i(\theta_i(t), t, T) = 1 - N(\theta_i(t)), \qquad \forall T > t.$$

Proof This is directly from the assumption of Normal distribution. ■

Based on above modeling of single survival probabilities, the CDO tranche survival probability can be calculated by the following process:

1^{st} Recall the pricing formula of a defaultable claim in section 2.2.2, the price of a CDS contract could be derived as a function of the periodic survival probabilities.

2^{nd} Given the survival probabilities $D_i(t,T)$ are of the same formula, reverse the function to calculate $D_i(t,T)$ from CDS prices. And Calculate each $\theta_i(t)$ from the survival probability.

3^{rd} And the CDO contract survival probability is calculated as

$$P_{survival}(\vec{\theta}(t),t,T) = \int_z \Pi_{i=1,2,..N} P\{\rho z + \sqrt{1-\rho^2} Z^{X_i} > \theta_i(t) | Z = z\} n(z) dz.$$

In Gausian Copula model, the single default assumption is simplified for the purpose the calculation of portfolio default. However we can still see, the starting point of this model is still from the structural model way. It also worth mention that the extension works on Gausian models are mostly based on expanding the assumption of minimum total asset.

3.3.2 GJZ delayed information model

The Guo-Jarrow-Zeng (GJZ) model [18] is based on the same delayed information setting as our model. Their model is different on the assumption of the total asset, where $\{X_u^t\}$ is geometric Brownian motion with regime switching coefficients $\mu(\theta(t))$ and $\sigma(\theta(t))$. In addition, θ is assumed to represent the state factor of the regime switch and is adapted to $\mathbb{F}$. Combining with the transition densities of the coefficients, the survival probability could be derived in a similar way as in formula (3.24).

Mathematically, we assume X_u^t to be the solution of the following differential equation:

$$dX_u^t = x_{[t]} + \int_{[t]}^{u} X_s^t \mu(\theta(s)) ds + \int_{[t]}^{u} X_s^t \sigma(\theta(s)) dW_s, \quad t \in [0,T], \quad u \in [[t],T],$$

where $\mu(\theta(t))$ and $\sigma(\theta(t))$ are regime switch process taking value from two finite sets $\{\mu_1,\ \mu_2\ ,...\mu_M\}$ and $\{\sigma_1,\ \sigma_2\ ,...\sigma_M\}$. $\theta(t)$ is the random process that indicates the switch. This setting is a good fit of the credit rating model as the regime switch is indicated by the change of the firm quality.

3.3.3 Default under incomplete information models

Now we introduce two incomplete information models in addition to the delayed information model, the Capponi-Cvitanic (CC) model [7] and Frey-Runggaldier (FR) model [16].

In C-C model, the total asset is a geometric Bronian motion driven by a hidden random factor θ with discrete observations of a continuous stochastic process S_t, which also contains θ in its drift.

Assume the present time $t \in [t_s, t_{s+1})$ and X_u^t is the solution of the following SDE:

$$X_u^t = X_{[t]} + \int_{[t]}^{u} X_s^t \mu(\theta([s]))ds + \int_{[t]}^{u} X_s^t \sigma(\theta([s]))dW_s, \quad t \in [0, T], \quad u \in [[t], T],$$

where $[s]$ is defined the same as our delayed setting and $\theta([s])$ is a Markov process on finite states.

Assume the $\mathcal{F}_t^S$ adapted observable process S_t is continuous and it solves the SDE:

$$S_{t_k} = X_{t_k} + \int_{t_{k-1}}^{t_k} h(\theta(t_{k-1}))dt + \int_{t_{k-1}}^{t_k} \sigma dW_t, \quad k = s+1, s+2, ...,$$

The process $\{S_t\}$ can only be observed at the end of each time periods after time t_s.

The $\mathcal{F}_t^\theta$ adapted hidden factor θ also has a dynamic form:

$$\theta(t_k) = \Gamma(\theta(t_{k-1}), x_{t_k}, \omega_k).$$

So when $t \in [t_s, t_{s+1})$, the default probability at time t_{s+1} becomes the following conditional expectation:

$$\begin{aligned} D(t_{s+1}) &= E\{G(\theta(t_s), t_s, t_{s+1}) \mid S_{t_{s+1}}\} \\ &= E\{\Psi(\mu(\theta(t_s)), \sigma(\theta(t_s)), x_{t_s}, \nu, \nu, t_{s+1} - t_s) \mid S_{t_{s+1}}\}. \end{aligned}$$

Frey and Runggaldier (F-R) model applied nonlinear filtering method to the default contagion case. In their model,the factor process $\theta(t)$ is defined as a multi-dimensional vector which enjoys the differential equation:

$$\begin{aligned}\theta(t) &= \theta([t]) + \int_{[t]}^{t} b(\theta(u-), \vec{Y}_{u-})du + \int_{[t]}^{t} \sigma(\theta(u-), \vec{Y}_{u-})dW_u \\ &\quad + \int_{[t]}^{t}\int_{E} K(\theta(u-), \vec{Y}_{u-})d\mathcal{N}(du, ds).\end{aligned}$$

$\vec{Y}_t$ is an $\mathcal{M}_t$ adapted observable vector of jump processes:

$$Y_{t,j} = Y_{[t],j} + \int_{[t]}^{t}\int_{E} K^Y(\theta(u-), \vec{Y}_{u-})d\mathcal{N}(du, ds).$$

The observable process S_t is the solution of the SDE:

$$S_t \quad = s_0 + \int_0^t h(\theta(u), S_u)du + \sigma dW_u, \qquad t \in [0, \bar{T}],$$

This is a stochastic intensity model with nonlinear filtering. The survival process is defined as a function of random factor $\theta(t)$. The two observation processes, including the continuous diffusion process $\{S_t\}$ and the poisson jump measure $\mathcal{N}(t, E)$ are similar as in our model. However it is different from ours on the point that, the observable jump process indicates the jump of the hidden factor directly. Hence the jump is not applied to the nonlinear estimation and the estimation is made only between the jumps.

3.4 The Nonlinear Bayesian Inference via filtering

Through out this section, we focus on the solution to the survival probability $D(t,T)$ defined by (3.24), which is the expectation of the nonlinear θ function $G(\theta, t, T)$ conditioned by $\mathcal{M}_t$. This is a typical nonlinear filtering problem with continuously observable continuous process and jump process. As we will show in the follwing, the Bayesian inference method will be extended to solve this particular problem. The solutions, the estimation of the forward and spot intensity, will be given as solutions of certain stochastic differential equations.

3.4.1 Default intensity Driven by stochastic factor

Assume assumptions **(A1)** through **(A3)** hold. We redefine our Probability measure P conditional expectation of G as a functional:

$$\pi(G,t,T) = D(t,T) = E^P\{G(\theta(t),t,T)|\mathcal{M}_t\}. \tag{3.29}$$

Similarly, the conditional expectation of $g(\theta(t),t,T)$ defined in (3.28) as the functional:

$$\pi(g,t,T) = E^P\{g(\theta(t),t,T)|\mathcal{M}_t\}. \tag{3.30}$$

Note by definition of $g(\theta(t),t,T)$ and the finite integrable of $G(\theta(t),t,T)$, Fubini's theorem implies:

$$\frac{\partial}{\partial T}\pi(G,t,T) = \pi(g,t,T).$$

Then by the formulae (3.20) and (3.21), we have:

$$\lambda(t) = -\frac{\pi(g,t,t)}{\pi(G,t,t)} \qquad \textit{and} \qquad f(t,T) = -\frac{\pi(g,t,T)}{\pi(G,t,T)}. \tag{3.31}$$

From above formulae, the spot and forward intensities are formalized as a standard nonlinear filtering problem. The proof of this type of problem is shown in the next subsection.

3.4.2 Change of probability measure

We extend the traditional Bayesian inference method to solve our filtering problem. At the first step, we change the probability measure P to a probability Q, under which the observable processes are martingales. As our observation contains both continuous and jump observable processes, we introduce the following version of Girsanov theorem:

Lemma 3.4.1 *Define $l(\theta(u),u)$ and $\gamma(\theta(u))$ of the forms:*

$$l(\theta(u),u) = \frac{\mu^S(\theta(u))}{\sigma^S(u)},$$

$$\gamma(\theta(u)) = \int_E k(x, \theta(u))\nu(dx, du), \ \ and$$

$$\gamma^*(\theta(u)) = \frac{\int_E k(x, \theta(u))\nu(dx, du)}{\int_E \nu(dx, du)}.$$

*Assume assumptions **(A1)** through **(A3)** in section 3.1.2 hold, then there exists a probability measure Q, such that* $\frac{dQ}{dP} = L_T^{-1}$, *and*

$$\begin{aligned} L_t^{-1} &= exp\{ \int_0^t \int_E [\log(k(x, \theta(u))) - \log(\gamma^*(\theta(u)))] N(dx, du) \\ &- \int_0^t \int_E [\frac{k(x, \theta(u))}{\gamma^*(\theta(u))} - 1]\nu(dx, du) - \int_0^t l(\theta(u), u) dW_u^S - \frac{1}{2}\int_0^t l(\theta(u), u)^2 du \} \end{aligned}$$

is an $\mathcal{F}_t$*-martingale under P, and Q is an equivalent probability measure to P. And by the measure Q,*

- *The process* $W_t^{S*} = W_t^S + \int_0^t l(\theta(u), u)du$ *is a Brownian motion under probability Q.*
- *The Poisson process* $\{\tilde{Y}\}_t$ *is of the form:*

$$\tilde{Y}_t = \int_0^t \gamma(\theta(u)) d\tilde{Y}_t^*,$$

under probability measure Q, where $Y_t^* = \int_0^t \frac{1}{\int_E \nu(dx, u)} \int_E N(dx, du)$, *is a standard Poisson process with unit intensity under Q.*

Proof First, we show that L_t^{-1} is an $\mathcal{F}_t$ square integrable martingale and the probability measure Q is an equivalent probability measure to P. By assumption **(A2)** and **(A3)**, we know that $\mu^S(\theta(t))$ is in L^2, σ^S is bounded below and $\gamma(\theta(t))$ is bounded below and above, which implies:

$$\begin{aligned} &E\big[exp\{\frac{1}{2}\int_0^{\bar{T}} l^2(\theta(u), u)du\}\big] < \infty \qquad and \\ &E\big[exp\{\int_0^{\bar{T}} [\gamma(\theta(u)) - 1]du\}\big] < \infty. \end{aligned} \tag{3.32}$$

Let $Z_t = L_t^{-1}$. Express Z_t as $Z_t = Z_t^1 Z_t^2$, where

$$Z_t^1 = \exp\{- \int_0^t l(\theta(u), u)dW_u^S - \frac{1}{2}\int_0^t l(\theta(u), u)^2 du\}$$

and

$$\begin{aligned} Z_t^2 \quad &= \exp\{\int_0^t \int_E [\log(k(x,\theta(u))) - \log(\gamma^*(\theta(u)))] N(dx,du) \\ &- \int_0^t \int_E [\frac{k(x,\theta(u))}{\gamma^*(\theta(u))} - 1]\nu(dx,du)\}. \end{aligned}$$

It is well known that Ito's lemma implies:

$$dZ_t^1 = Z_t^1 l(\theta(t),t) dW_t.$$

On the other hand, apply Ito's lemma form jump process, we have

$$dZ_t^2 \quad = \quad Z_{t-}^2 \int_E (\frac{k(x,\theta(u))}{\gamma^*(\theta(u))} - 1)\tilde{N}(dx,dt),$$

where $\tilde{N}(x,t)$ is the compensated process of the Poisson measure $N(x,t)$.

Thus,

$$\begin{aligned} Z_t &= Z_t^1 Z_t^2 \\ &= 1 + \int_0^t Z_s^1 dZ_s^2 + \int_0^t Z_{s-}^2 dZ_s^1 + [Z^1, Z^2]_t \\ &= 1 + \int_0^t Z_s^1 dZ_s^2 + \int_0^t Z_{s-}^2 dZ_s^1 + \Sigma_{0<s<t} \Delta Z_s^1 \Delta Z_s^2 \\ &= 1 + \int_0^t Z_s^1 dZ_s^2 + \int_0^t Z_s^2 dZ_s^1. \end{aligned}$$

Take the differentiation form of above Z_t, we have

$$dZ_t = \int_E (\frac{k(x,\theta(u))}{\gamma^*(\theta(u))} - 1) Z_{t-} d\tilde{N}(dx,dt) + l(\theta(t),t) Z_t dW_t. \tag{3.33}$$

Then we can conclude that L_t^{-1} is the solution to the following SDE:

$$L_t^{-1} = 1 + \int_0^t \int_E L_{u-}^{-1} [\frac{k(x,\theta(u))}{\gamma^*(\theta(u))} - 1] d\tilde{N}(x,u) + \int_0^t L_u^{-1} l(\theta(u),u) dW_u^S. \tag{3.34}$$

Inversely, L_t is a square integrable Q-martingale. It is a equivalent martingale measure to P.

Second, we show the corresponding changes of the random processes under the transformation. The following relation on probability transform is from Philip Protter [37], which is a well known result of Girsanov transformation dealing with both continuous and jump processes.

Let the square integrable process Z_t be the Radon-Nikodym Derivative of P and Q, that is, $\frac{dQ}{dP} = P$, where $\{Z_t\}_{t\geq 0}$ is a Q-martingale. Then for any P martingale M_t, $M_t^* = M_t \cdot Z_t$ is a Q martingale, and under Q,

$$M^*(t) = M_t - \int_0^t \frac{1}{Z_u} d[M, Z]_u. \tag{3.35}$$

Now we apply above formula to complete the proof of our theorem. For the change of the Brownian motion, it suffices to show that:

$$[Z, W]_t = \int_0^t Z_u l(\theta(u), u) du.$$

Plug the Z_t function (3.33) into the definition form of quadratic variation, we have:

$$[Z, W]_t = \Sigma_{0<u<t} \Delta Z_u \Delta W_u = \Sigma_{0<u<t} l(\theta(u), u) Z_u [\Delta W_u]^2 = \int_0^t l(\theta(u), u) Z_u du.$$

Thus formula 3.35 implies that

$$W_t^{S*} = W_t^S + \int_0^t l(\theta(u), u) du$$

is a Martingale under the measure Q. By Levy's theorem, it is a Brownian Motion.

Again, to the compensated Poisson process $\tilde{Y}_t$, applying formula (3.35), we have:

$$\tilde{Y}_t' = \tilde{Y}_t - \int_0^t \frac{1}{Z_t} d[\tilde{Y}, Z]_t$$

is a Q-martingale. And

$$[\tilde{Y}, Z]_t = \Sigma_{0<u<t} \Delta \tilde{Y}_u \Delta Z_u = \int_0^t \int_E Z_{u-} \left(\frac{k(x, \theta(u))}{\gamma^*(\theta(u))} - 1\right) k(x, \theta(u)) N(dx, du).$$

Now expand the right hand side of the martingale $\tilde{Y}_t'$:

$$\begin{aligned}
\tilde{Y}_t' = \quad & \int_0^t \int_E k(x(\theta(u)), u) \tilde{N}(dx(\theta(u)), du) \\
& - \int_0^t \int_E \frac{Z_{u-}(\frac{k(x,\theta(u))}{\gamma^*(\theta(u))} - 1) k(x, \theta(u))}{Z_{u-}(\frac{k(x,\theta(u))}{\gamma^*(\theta(u))} - 1) + Z_{u-}} N(dx, du) \\
= \quad & \int_0^t \int_E k(x(\theta(u)), u) N(dx, du) - \int_0^t \int_E k(x, \theta(u)) \nu(dx, du) \\
& - \int_0^t \int_E (k(x, \theta(u)) - \gamma^*(\theta(u))) N(dx, du) \\
= \quad & \int_0^t \gamma(\theta(u)) \int_E \frac{1}{\int_E \nu(dx, u)} \tilde{N}(dx, du).
\end{aligned}$$

Let $Y_t^* = \int_E \frac{1}{\int_E \nu(dx,u)} N(dx,du)$ and $\tilde{Y}_t^* = Y_t^* - t$, then $\tilde{Y}_t' = \int_0^t \gamma(\theta(u)) d\tilde{Y}_u^*$. By Ito's lemma we know $\tilde{Y}_t'$ is a martingale, thus $\tilde{Y}_t^*$ is a standard Poisson process.

That indicates: Y_t becomes a counting process with intensity:

$$\gamma(t,\theta(t)) = \int_E k(x,\theta(t))\nu(dx,t)$$

after the transformation. ■

Remark 3.4.2 We denote the Radon Nikodym derivative L_t^{-1} in a inverse way for the purpose of future work on Bayesian inference. One can check in the similarly way that $\frac{dP}{dQ} = L_T$ is also the Radon Nikodym derivative between P and Q (A Fubini theorem for change of integral order will be applied in this process). And L_t is a Q martingale, which solves of the following equation:

$$L_t \;= 1 + \int_0^t L_{u-}[\gamma(\theta(u)) - 1] d\tilde{Y}_t^* + \int_0^t L_u l(\theta(u),u) dW_u^{S*}. \tag{3.36}$$

Under probability measure Q, the solution L_t for 3.36 is:

$$\begin{aligned} L_t &= \exp\{\int_0^t \log(\gamma(\theta(u)) dY_u^* - \int_0^t [\gamma(\theta(u)) - 1] du \qquad (3.37) \\ &+ \int_0^t l(\theta(u),u) dW_u^{S*} - \frac{1}{2}\int_0^t l^2(\theta(u),u) du\}. \end{aligned}$$

Here W_t^{S*} and Y_t^* have the same properties as in Lemma 3.4.1. One can easily check the equality of 3.37 by taking the relations in lemma.

3.4.3 Bayesian Inference via filtering

Bayes formula, which is also known as Kallianpur-Striebel formula, states the relations between the probability transformations under imperfect information filtration. With this relation, we can derive our estimation formula under Q and transfer it back to P. In what follows, we introduce the well known theorem without proof.

Lemma 3.4.3 *Let Q and P be two equivalent probability measures, where L_t is the Radon-Nikodym derivative between Q and P, such that*

$$\frac{\partial Q}{\partial P} = L_T.$$

For any measurable function $f(\theta(t),t,T)$ of R-valued and $\mathbb{F}^\theta$ adapted process $\theta(t)$ on a finite time interval $[0,\bar{T}]$,

$$\pi(f,t,T) = E^{\tilde{P}}\{f(\theta(t),t,T)|\mathcal{M}_t\} = \frac{E^Q\{f(\theta(t),t,T)L_t|\mathcal{M}_t\}}{E^Q\{L_t|\mathcal{M}_t\}}. \tag{3.38}$$

Now our conditional expectation of $G(\theta(t),t,T)$ and $g(\theta(t),t,T)$ can be written in the forms:

$$\phi(G,t,T) = E^Q\{G(\theta(t),t,T)L_t|\mathcal{M}_t\}, \qquad \textit{and} \tag{3.39}$$

$$\phi(g,t,T) = E^Q\{g(\theta(t),t,T)L_t|\mathcal{M}_t\}. \tag{3.40}$$

Thus by Lemma 3.4.3,

$$\pi(g,t,T) = \frac{\phi(g,t,T)}{\phi(1,t,T)} \quad \textit{and} \quad \pi(G,t,T) = \frac{\phi(G,t,T)}{\phi(1,t,T)}. \tag{3.41}$$

Thus the formula 3.31 becomes:

$$f(t,T) = -\frac{\phi(g,t,T)}{\phi(G,t,T)} \qquad \textit{and} \qquad \lambda(t) = -\frac{\phi(g,t,t)}{\phi(G,t,t)}.$$

The differential equation of $\pi(g,t,T)$ is called normalized filtering equation where as $\phi(g,t,T)$ is called the unnormalized equation. Now the estimation of $\pi(g,t,T)$ under probability measure P becomes the estimation under probability measure Q. We would like to explore the dynamic forms of the $\phi(.,t,T)$ formulas, and before introducing the main theorems on filtering equations, we prove a series of lemmas in the following.

First, under the conditional expectation, we show Fubini's theorem holds:

Lemma 3.4.4 *Let $\{M_t\}_{t\geq 0}$ be a $\mathbb{M}$ semi-martingale, then for any càdlàg square integrable, $\mathbb{F}$ adapted process F_t, such that:*

$$E[|\int_0^{\bar{T}} F_u dM_u|] \leq K \cdot E[\int_0^{\bar{T}} |F_u|^2 du]^{\frac{1}{2}} < \infty,$$

then for each $t > 0$, the following equation holds:

$$E[\int_0^t F_u dM_u|\mathcal{M}_t] = \int_0^t E[F_u|\mathcal{M}_t]dM_u.$$

Proof Let $\mathcal{S}$ denote the collection of all $\mathcal{F}_t$ predictable simple functions, $\mathcal{S}$ is dense in the collection of $\mathcal{F}_t$ adapted càdlàg functions. That is, for every càdlàg square integrable process F_t, we have a series of simple functions $\{F_t^n\}_{n\geq 0}$ such that, $F_t^n = F_0 + \Sigma_{i=0}^n F_{U_i}^n \cdot 1_{[U_i,U_{i+1}]}$ for the stopping times $0 = U_0 < U_1 < ... < U_n = t$ and $\{F_t^n\}$ converges to F_t uniformly on compact sets. Now consider an arbitrary simple function F_t^n:

$$\begin{aligned} E[\int_0^t F_u^n dM_u|\mathcal{M}_t] = & \quad E[\Sigma_{i=0}^n F_{U_i}^n \Delta M_{U_i}|\mathcal{M}_t] = \Sigma_{i=0}^n E[F_{U_i}^n|\mathcal{M}_t]\Delta M_{U_i} \\ = & \quad \int_0^t E[F_u^n|\mathcal{M}_t]dM_u. \end{aligned}$$

On the other hand, if F_t^n converges to F_t uniformly, then Fubini's theorem implies the following:

$$\begin{aligned} & E\{\big|E[\int_0^t F_u dM_u|\mathcal{M}_t] - E[\int_0^t F_u^n dM_u|\mathcal{M}_t]\big|\} \qquad (3.42) \\ = \quad & E\big|\int_0^t (F_u - F_u^n)dM_u\big| \\ \leq \quad & K \cdot E[\int_0^t |F_u - F_u^n|^2 du]^{\frac{1}{2}} \\ \leq \quad & K \cdot \big\{\int_0^t \{E|(F_u - F_u^n)|\}^2 du\big\}^{\frac{1}{2}} \\ \leq \quad & K \cdot \epsilon_n \cdot \sqrt{t}. \end{aligned}$$

And

$$\begin{aligned} & E\{\big|\int_0^t E[F_u|\mathcal{M}_t]dM_u - \int_0^t E[F_u^n|\mathcal{M}_t]dM_u\big|\} \qquad (3.43) \\ = \quad & E\{\big|\int_0^t E[(F_u - F_u^n)|\mathcal{M}_t]dM_u\big|\} \\ \leq \quad & K \cdot E\big\{\int_0^t \{E[(F_u - F_u^n)|\mathcal{M}_t]\}^2 du\big\}^{\frac{1}{2}} \\ \leq \quad & K \cdot \big\{\int_0^t \{E|(F_u - F_u^n)|\}^2 ds\big\}^{\frac{1}{2}} \\ \leq \quad & K \cdot \epsilon_n \cdot \sqrt{t}. \end{aligned}$$

Thus (3.42) and (3.43) implies,

$$E[\int_0^t F_u dM_u|\mathcal{M}_t] = \int_0^t E[F_u|\mathcal{M}_t]dM_u.$$

For every F_u satisfies the condition in the lemma. ■

Next we introduce the following martingale lemma,

Lemma 3.4.5 *Let $f(\theta(t), t, T)$ be any measurable function from $\mathbb{R} \times [0, \bar{T}] \times [0, \bar{T}]$ to $\mathbb{R}$ be a $\mathbb{C}^2$ function on θ, t and T with generator $\mathcal{A}^\theta$. Then for any time t and T such that $t < T < \bar{T}$, the functions:*

$$M_f^{\mathcal{A}^\theta}(t,T) = f(\theta(t), t, T) - f(\theta([t]), [t], T) - \int_{[t]}^{t} \mathcal{A}^\theta f(\theta(u), u, T)du, \tag{3.44}$$

and

$$M_f^{\mathcal{A}^\theta}(t) = f(\theta(t), t, t) - f(\theta([t]), [t], t) - \int_{[t]}^{t} \mathcal{A}^\theta f(\theta(u), u, t)du \tag{3.45}$$

are $\mathbb{F}^\theta$ martingales on $\big[[t], \bar{T}\big)$.

Proof Fix time T, apply Ito's lemma to the $\mathbb{C}^2$ function f on variable $\theta(t)$, we have

$$\begin{aligned} & f(\theta(t), t, T) \\ = \; & f(\theta([t]), [t], T) + \int_{[t]}^{t} \mathcal{A}^\theta f(\theta(u), u, T)du + \int_{[t]}^{t} \sigma^\theta(\theta(u), u)dW_u^\theta. \end{aligned}$$

By definition, $\sigma^\theta(\theta(u), u)$ is $\{\mathcal{F}_t^\theta\}$ measurable and σ^θ is integratable indicates,

$$\int_{[t]}^{t} \sigma^\theta(\theta(u), u)dW_u^\theta$$

is a $\mathcal{F}_t^\theta$-Martingale, so is $M_g^{\mathcal{A}^\theta}(t,T)$. Let $T = t$, we have the same property on $M_g^{\mathcal{A}^\theta}(t)$.
■

Proposition 3.4.6 *Suppose the process $\theta(t)$ is independent of $\mathbb{M}$. Then from Lemma 3.4.5, we know that $M_f^{\mathcal{A}}(t,T)$ is a $\mathcal{F}_t^\theta$ martingale, which is independent with $\mathbb{M}$. For any $\mathcal{F}_t^\theta$ adapted process $U(t)$, such that $E^Q\big[\int_0^{\bar{T}} U(u)dM_f^{\mathcal{A}^\theta}(u,T)\big]^2 < \infty$,*

$$E^Q[\int_{[t]}^{t} U(u)dM_f^{\mathcal{A}^\theta}(u,T) \mid \mathcal{M}_t] = 0.$$

Proof This is the application of Lemma 3.2.1. We know from definition, $\mathcal{M}_t = \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]} \vee \mathcal{F}^S_t \vee \mathcal{F}^Y_t$, and $\mathcal{F}^S_t$ and $\mathcal{F}^Y_t$ are independent with $\mathcal{F}^\theta_t$, then

$$\begin{aligned}
& E^Q[\int_0^t U(u) dM_f^{\mathcal{A}^\theta}(u,T) | \mathcal{M}_t] \\
= \; & E^Q[\int_0^t U(u) dM_f^{\mathcal{A}^\theta}(u,T) \mid \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]} \vee \mathcal{F}^S_t \vee \mathcal{F}^Y_t] \\
= \; & E^Q[\int_0^t U(u) dM_f^{\mathcal{A}^\theta}(u,T) \mid \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]}] \\
= \; & 0.
\end{aligned}$$

■

Now we are ready to introduce the main filtering theorem.

Theorem 3.4.7 *Under the probability measure Q, then for each time $t \in [0,T]$, the conditional expectations $\phi(g,t)$ and $\phi(G,t)$ in (3.41) are the solutions to the following SDEs:*

$$\begin{aligned}
\phi(g,t,T) &= \phi(g,[t],T) + \int_{[t]}^t \phi(\mathcal{A}^\theta g, u, T) du + \int_{[t]}^t \phi[(\gamma-1)g, u-, T] d\tilde{Y}_u^* \\
&\quad + \int_{[t]}^t \phi(lg, u, T) dW_u^{S*} \qquad (3.46) \\
&= \phi(g,[t],T) + \int_{[t]}^t [\phi(\mathcal{A}^\theta g, u, T) + \phi(g,u,T) - \phi(\gamma g, u, T)] du \\
&\quad + \int_{[t]}^t \phi[(\gamma-1)g, u-, T] dY_u^* + \int_{[t]}^t \phi(lg, u, T) dW_u^{S*},
\end{aligned}$$

and

$$\begin{aligned}
\phi(G,t,T) &= \phi(G,[t],T) + \int_{[t]}^t \phi(\mathcal{A}^\theta G, u, T) du + \int_{[t]}^t \phi[(\gamma-1)G, u, T] d\tilde{Y}_u^* \\
&\quad + \int_{[t]}^t \phi(lG, u, T) dW_u^{S*} \qquad (3.47) \\
&= \phi(G,[t],T) + \int_{[t]}^t [\phi(\mathcal{A}^\theta G, u, T) + \phi(G,u,T) - \phi(\gamma G, u, T)] du \\
&\quad + \int_{[t]}^t \phi[(\gamma-1)G, u, T] dY_u^* + \int_{[t]}^t \phi(lG, u, T) dW_u^{S*},
\end{aligned}$$

where the generater $\mathcal{A}^\theta$ is of the form:

$$\mathcal{A}^\theta = \frac{\partial}{\partial t} + \mu^\theta \frac{\partial}{\partial \theta} + \frac{1}{2}\sigma^{\theta 2}\frac{\partial^2}{\partial \theta^2}. \tag{3.48}$$

Proof First we would like to introduce Ito's formula for a measurable function of two random processes, which will be prevailing in the rest of our work.

Let $\varphi(U_1, U_2)$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, be a C^2 real function on both variables. For any semi-martingales U_1, U_2, the following equation holds:

$$\begin{aligned}
& \varphi(U_1(t), U_2(t)) \qquad\qquad (3.49)\\
= \ & \varphi(U_1(0), U_2(0)) + \int_0^t \varphi_{u_1}(U_1(s-), U_2(s-))dU_1(s) \\
& + \int_0^t \varphi_{u_2}(U_1(s-), U_2(s-))dU_2(s) + \int_0^t \varphi_{u_1 u_2}(U_1(s), U_2(s))d[U_1, U_2]^c \\
& + \frac{1}{2}\{ \int_0^t \varphi_{u_1 u_1}(U_1(s), U_2(s))d[U_1]^c + \int_0^t \varphi_{u_2 u_2}(U_1(s), U_2(s))d[U_2]^c \} \\
& + \Sigma_{s \le t}\big\{\varphi(U_1(s), U_2(s)) - \varphi(U_1(s-), U_2(s-)) - \varphi_{u_1}\Delta U_1(s) - \varphi_{u_2}\Delta U_2(s)\big\}.
\end{aligned}$$

Let $\varphi(g(\theta(t), t, T), L(t)) = g(\theta(t), t, T) \times L(t)$. By applying formula 3.49, one has:

$$\begin{aligned}
& g(\theta(t), t, T)L(t) \qquad\qquad (3.50)\\
= \ & g(\theta([t]), [t], T)L([t]) + \int_{[t]}^t \mathcal{A}^\theta g(\theta(u), u, T)L(u)du + \int_{[t]}^t g(\theta(u-), u-, T)dL(u) \\
& + \int_{[t]}^t L(u-)dM_g^{\mathcal{A}^\theta}(u, T) \\
= \ & g(\theta([t]), [t], T)L([t]) \\
& + \int_{[t]}^t \big[\mathcal{A}^\theta g(\theta(u), u, T) + (1 - \gamma(\theta(u)))g(\theta(u), u, T)\big]L(u)du \\
& + \int_{[t]}^t \big[\gamma(\theta(u-)) - 1\big]g(\theta(u-), u-, T)L(u-)dY_u^* \\
& + \int_{[t]}^t g(\theta(u), u, T)l(u)L(u)dW_u^{S*} + \int_{[t]}^t L(u-)dM_g^{\mathcal{A}^\theta}(u, T),
\end{aligned}$$

where $M_g^{\mathcal{A}^\theta}(t, T) = g(\theta(t), t, T) - g(\theta([t]), [t], T) - \int_{[t]}^t \mathcal{A}^\theta g(\theta(u), u, T)du$ is defined in Lemma 3.4.5.

Now by taking $\mathcal{M}_t$ conditional expectation of (3.50) under Q, we want to show that the equations (3.46) and (3.47) hold.

First of all, notice that Jensen's inequality implies:

$$E^Q[|\int_0^t F_u dW_u|] \leq \{E^Q[\int_0^t F_u dW_u]^2\}^{\frac{1}{2}} = \{E^Q[\int_0^t |F_u|^2 du]\}^{\frac{1}{2}}$$

and

$$\begin{aligned} E^Q\{|\int_0^t F_u d\tilde{Y}_u^*|\} &\leq \{E^Q|\int_0^t F_u d\tilde{Y}_u^*|^2\}^{\frac{1}{2}} = \left\{E^Q \int_0^t |F_u|^2 d[\tilde{Y}^*, \tilde{Y}^*]_u\right\}^{\frac{1}{2}} \\ &= \{E^Q[\int_0^t |F_u|^2 du]\}^{\frac{1}{2}}. \end{aligned}$$

Thus by Lemma 3.4.4, we can move the conditional expectation inside the integration for the first three integrations in 3.50:

$$\begin{aligned} & \phi(g,t,T) \qquad (3.51) \\ = \; & E\{g(\theta(t),t,T) \mid \mathcal{M}_t\} \\ = \; & \phi(g,[t],T) + E^Q\{\int_{[t]}^t [\mathcal{A}^\theta g(\theta(u),u,T) + (1-\gamma(\theta(u)))g(\theta(u),u,T)]L(u)du \\ & + \int_{[t]}^t [\gamma(\theta(u-))-1]g(\theta(u-),u,T)L(u-)dY_u^* \\ & + \int_{[t]}^t g(\theta(u),u,T)l(u)L(u)dW_u^{S*} + \int_{[t]}^t L(u-)dM_g^{\mathcal{A}^\theta}(u,T) \mid \mathcal{M}_t\big\} \\ = \; & \phi(g,[t],T) + \int_{[t]}^t E^Q\{[\mathcal{A}^\theta g(\theta(u),u,T) + (1-\gamma(\theta(u)))g(\theta(u),u,T)]L(u)|\mathcal{M}_t\}du \\ & + \int_{[t]}^t E^Q\{[\gamma(\theta(u-))-1]g(\theta(u-),u,T)L(u-)|\mathcal{M}_t\}dY_u^* \\ & + \int_{[t]}^t E^Q\{g(\theta(u),u,T)l(u)L(u)|\mathcal{M}_t\}dW_u^{S*} \\ & + E^Q\{\int_{[t]}^t L(u-)dM_g^{\mathcal{A}^\theta}(u,T) \mid \mathcal{M}_t\}. \end{aligned}$$

Second, we show, in general, if a measurable function $f(\theta(t),t,T)$ is on $\mathbb{R}$ and differentiable on θ, t and T, then $\forall u \in [[t],t)$,

$$E\{f(\theta(u),u,T) \mid \mathcal{M}_t\} = E\{f(\theta(u),u,T) \mid \mathcal{M}_u\}.$$

By definition $\mathcal{M}_t = \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]} \vee \mathcal{F}^S_t \vee \mathcal{F}^Y_t$ and $\mathcal{M}_u = \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]} \vee \mathcal{F}^S_u \vee \mathcal{F}^Y_u$ for $\forall u \in [[t], t]$, and we know that $\mathcal{F}^S_t \vee \mathcal{F}^Y_t$ is independent with $\mathcal{F}^\theta_t$. Apply Lemma 3.2.1 twice, we have

$$
\begin{aligned}
E\{f(\theta(u), u, T) \mid \mathcal{M}_t\} &= E\{f(\theta(u), u, T) \mid \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]} \vee \mathcal{F}^S_t \vee \mathcal{F}^Y_t\} \\
&= E\{f(\theta(u), u, T) \mid \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]}\} \\
&= E\{f(\theta(u), u, T) \mid \mathcal{F}^W_{[t]} \vee \mathcal{F}^\theta_{[t]} \vee \mathcal{F}^S_u \vee \mathcal{F}^Y_u\} \\
&= E\{f(\theta(u), u, T) \mid \mathcal{M}_u\}.
\end{aligned}
$$

Third of all, from the result of Proposition 3.4.6, we know that the conditional expectation of the last term in the integration form (3.50) under the filtration $\mathcal{M}_t$ becomes zero.

To sum up, function (3.51) becomes:

$$
\begin{aligned}
&\phi(g, t, T) \\
= \; & g(\theta([t]), [t], T) + \int_{[t]}^t E^Q\big\{\big[\mathcal{A}^\theta g(\theta(u), u, T) + (1 - \gamma(\theta(u)))g(\theta(u), u, T)\big]L(u) \mid \mathcal{M}_u\big\}du \\
&+ \int_{[t]}^t E^Q\big\{\big[\gamma(\theta(u-)) - 1\big]g(\theta(u-), u, T)L(u-) \mid \mathcal{M}_u\big\}dY^*_u \\
&+ \int_{[t]}^t E^Q\big\{g(\theta(u), u, T)l(u)L(u) \mid \mathcal{M}_u\big\}dW^{S*}_u.
\end{aligned}
$$

That is, by definition, to say that the equations (3.46) and (3.47) in the theorem hold.

■

This theorem is the essential estimation for our target estimation functions as it is shown in Lemma 3.4.3, we can derive the $\pi(., t, T)$ functions from $\phi(., t, T)$.

For the reasons described above, we move forward to the estimation functions under measure P from measure Q. This is mainly the application of Bayes theorem. Before proving the main theorem, we introduce the useful lemma:

Lemma 3.4.8 *For the $\mathbb{F}^S$ adapted process S_t defined in 3.7, there exists an $\mathbb{M}$ adapted Brownian motion $\bar{W}^S_t$, such that S_t is the solution of the SDE:*

$$S_t = s_0 + \int_0^t S_u \cdot \pi(\mu^S, u)du + \int_0^t S_u\sigma^S(u)d\bar{W}^S_u\}, \qquad t \in [0, \bar{T}] \tag{3.52}$$

Similarly the Poisson process Y_t defined in (3.8) can also be a $\mathbb{M}$ adapted Poisson process with intensity $\pi(\gamma, t, T)$.

Proof The proof of a general version of the existence of $\bar{W}_t^S$ could be found in Theorem 7.17 of Liptser and Shiryaev [35]. In our own case, we show the proof by the construction of $\bar{W}_t$ as the following:

$$\bar{W}_t^S = \int_0^t \frac{\mu^S(\theta(u)) - \pi(\mu^S, u)}{\sigma^S(u)} du + W_t^S,$$

Levy theorem implies that $\bar{W}_t^S$ is a $\mathcal{M}_t$ Brownian motion. Thus plus above relation into SDE (3.7) one gets the SDE of S_t the same as in Lamma 3.4.8.

Similarly, denote $\bar{Y}_t = Y_t - \int_0^t \pi(\gamma, u, T)du$, then $\bar{Y}_t = \tilde{Y}_t + \big(\int_0^t \gamma(\theta(u))du - \int_0^t \pi(\gamma, u, T)du\big)$. Thus $\bar{Y}_t$ is a $\mathcal{M}_t$ adapted compensated Poisson process. ∎

The following shows the main results of our work, the term structure equation of forward default intensity $f(t, T)$ under partial information setting.

Theorem 3.4.9 *Let $\pi(g, t, T)$ and $\pi(G, t, T)$ be the conditional expectations defined by (3.29) and (3.30). Then under probability measure P, there exist $\mathcal{M}_t$ adapted Brownian motion $\bar{W}_t^S$ and $\mathcal{M}_t$ adapted Poisson process Y_t with intensity $\pi(\gamma, t, T$, such that, the following stochastic differential equations hold:*

$$\begin{aligned}\pi(g,t,T) &= \pi(g,[t],T) + \int_{[t]}^t [\pi(\mathcal{A}^\theta g,u,T) - \pi(\gamma g,u,T) + \pi(g,u,T)\pi(\gamma,u,T)]du \\ &\quad + \int_{[t]}^t [\pi(lg,u,T) - \pi(g,u,T)\pi(l,u,T)]d\bar{W}_u^S \qquad (3.53) \\ &\quad + \int_{[t]}^t \big[\frac{\pi(\gamma g,u-,T)}{\pi(\gamma,u-,T)} - \pi(g,u-,T)\big]dY_u,\end{aligned}$$

and

$$\begin{aligned}\pi(G,t,T) &= \pi(G,[t],T) \qquad (3.54) \\ &\quad + \int_{[t]}^t [\pi(\mathcal{A}^\theta G,u,T) - \pi(\gamma G,u,T) + \pi(G,u,T)\pi(\gamma,u,T)]du \\ &\quad + \int_{[t]}^t [\pi(lG,u,T) - \pi(G,u,T)\pi(l,u,T)]d\bar{W}_u^S \\ &\quad + \int_{[t]}^t \big[\frac{\pi(\gamma G,u-,T)}{\pi(\gamma,u-,T)} - \pi(G,u-,T)\big]dY_u.\end{aligned}$$

Proof The existence of $\{\bar{W}_t\}$ is proved in Lemma 3.4.8. Note recall the result of Lemma 3.35, we have the following relation

$$W_t^{S*} = \bar{W}_t^S + \int_0^t \frac{\pi(\mu^S, u)}{\sigma^S(u)} du = \bar{W}_t^S + \int_0^t \pi(l, u, T) du. \tag{3.55}$$

On the other hand, set the formula (3.49) as $\varphi(U_1, U_2) = \dfrac{U_1(t)}{U_2(t)}$, then apply the formula to $\pi(g, t, T) = \dfrac{\phi(g, t, T)}{\phi(1, t, T)}$ and $\pi(G, t, T) = \dfrac{\phi(G, t, T)}{\phi(1, t, T)}$.

From Theorem 3.4.7, we have $\phi(g, t, T)$ and $\phi(G, t, T)$ are the solutions of the SDE given in (3.46) and (3.47). $\phi(1, t, T)$ is a special solution of (3.46) when $g(\theta(t), t, T) = 1$.

Plug (3.55) into the expansion of above fractions, we have:

$$\begin{aligned}
& d\pi(g, t, T) \\
= \; & \frac{d\phi(g, t, T)}{\phi(1, t, T)} - \frac{\phi(g, t, T)}{\phi^2(1, t, T)} d\phi(1, t, T) - \frac{1}{\phi^2(1, t, T)} d[\phi(g, , T), \phi(1, , T)]_t^c \\
& + \frac{\phi(g, t, T)}{\phi^3(1, t, T)} d[\phi(1, , T)]_t^c + \frac{\phi(g, t, T)}{\phi(1, t, T)} - \frac{\phi(g, t-, T)}{\phi(1, t-, T)} - \frac{1}{\phi(1, t, T)} \Delta\phi(g, t, T) \\
& + \frac{\phi(g, t, T)}{\phi^2(1, t, T)} \Delta\phi(1, t, T) \\
= \; & \big[\pi(\mathcal{A}^\theta g, t, T) - \pi(\gamma g, t, T) + \pi(g, t, T)\pi(\gamma, t, T)\big] dt \\
& + \big[\pi(lg, t, T) - \pi(g, t, T)\pi(l, t, T)\big] d\bar{W}_t^S + \big[\frac{\pi(\gamma g, t-, T)}{\pi(\gamma, t-, T)} - \pi(g, t-, T)\big] dY_t.
\end{aligned}$$

Thus we can see that $\pi(g, t, T)$ is the solution to equation (3.53). Similarly arguments shows that $\pi(G, t, T)$ to be the solution to (3.54). ■

Ito's formula implies the following propositions:

Proposition 3.4.10 *For a fixed time T, denote $\nu(g, t, T) = \dfrac{\pi(g, t, T)}{\pi(G, t, T)}$, $\forall t \in [0, T]$, then under the probability measure P, the forward survival intensity, which is defined by $f(t, T) = -\nu(g, t, T)$ enjoys the following differential equation:*

$$\begin{aligned}
f(t, T) \; = \; & f([t], T) + \int_{[t]}^t \big[A(u, T) f(u, T) + B(u, T)\big] du \\
& + \int_{[t]}^t \big[C(u, T) f(u, T) + D(u, T)\big] d\bar{W}_u^S \\
& - \int_{[t]}^t \big[f(u-, T) + \frac{\nu(\gamma g, u-, T)}{\nu(\gamma G, u-, T)}\big] dY_u,
\end{aligned} \tag{3.56}$$

where $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$ and $E(t,T)$ represent the following:

$$\begin{aligned}
A(t,T) &= -\nu(\mathcal{A}^\theta G,t,T)+\nu(\gamma G,t,T)+\nu^2(lG,t,T)-\nu(lG,t,T)\pi(l,t,T),\\
B(t,T) &= -\nu(\mathcal{A}^\theta g,t,T)+\nu(\gamma g,t,T)+\nu(lg,t,T)\nu(lG,t,T)-\nu(lg,t,T)\pi(l,t,T),\\
C(t,T) &= -\nu(lG,t,T),\\
D(t,T) &= -\nu(lg,t,T).
\end{aligned}$$

The spot survival intensity $\lambda(t)$ satisfies the term structure equation:

$$\begin{aligned}
\lambda(t) &= \int_{[t]}^{t}\left[\lambda^2(u)+A(u)\lambda(u)+B(u)\right]du+\int_{[t]}^{t}[C(u)\lambda(u)+D(u)]dW_u^S \quad (3.57)\\
&\quad -\int_{[t]}^{t}[\lambda(u-)+\frac{\nu(\gamma g,u-,t)}{\nu(\gamma G,u-,t)}]dY_u.
\end{aligned}$$

Similarly $A(t)$, $B(t)$,$C(t)$,$D(t)$ and $E(t)$ represent the following:

$$\begin{aligned}
A(t) &= -\nu(\mathcal{A}^\theta G,t,t)+\nu(\gamma G,t,t)+\nu^2(lG,t,t)-\nu(lG,t,t)\pi(l,t,t), && (3.58)\\
B(t) &= -\nu(\mathcal{A}^\theta g,t,t)-\nu(g_T,t,t)+\nu(\gamma g,t,t)+\nu(lg,t,t)\nu(lG,t,t) && (3.59)\\
&\quad -\nu(lg,t,t)\pi(l,t,t), && (3.60)\\
C(t) &= -\nu(lG,t,t), && (3.61)\\
D(t) &= -\nu(lg,t,t). && (3.62)
\end{aligned}$$

Proof We repeat the processes in the proof of Theorem 3.4.9, that is, setting $\varphi(U_1,U_2)=\dfrac{U_1(t)}{U_2(t)}$ in formula (3.49).

Ito's formula to $\nu(g,t,T)$, we have, for any fixed $T>t$:

$$\begin{aligned}
& d\nu(g,u,T) && (3.63)\\
=\ & \Big[\nu(\mathcal{A}^\theta g,u,T)-\nu(\gamma g,u,T)-\nu(g,u,t)\nu(\mathcal{A}^\theta G,u,t)+\nu(g,u,t)\nu(\gamma G,u,t)\\
& +\nu(g,u,t)\nu^2(lG,u,t)-\nu(g,u,t)\nu(lG,u,t)\pi(l,u,T)+\nu(lg,u,t)\pi(l,u,t)\\
& -\nu(lg,u,t)\nu(lG,u,t)\Big]du+\big[\nu(lg,u,t)-\nu(g,u,t)\nu(lG,u,t)\big]dW_u^S\\
& +\big[\frac{\nu(\gamma g,u-,t)}{\nu(\gamma G,u-,t)}-\nu(g,u-,t)\big]dY_u.
\end{aligned}$$

Plug $f(t,T)=-\nu(g,t,T)$ into the above equation, we have equation (3.56).

Recall the relation between forward rate and short rate. Here we can do the similar argument on forward intensity and spot intensity. By definition, we have $\lambda(t) = f(t,t), \forall t \in [0,T]$. Then the expansion holds:

$$d\lambda(t) = df(t,T)|_{T=t} + \frac{\partial f(t,t)}{\partial T}dt. \tag{3.64}$$

Fubini's theorem implies:

$$\frac{\partial}{\partial T}\pi(g,t,T) = \frac{\partial}{\partial T}E\big\{g(\theta(t),t,T)|\mathcal{M}_t\big\} = E\big\{\frac{\partial}{\partial T}g(\theta(t),t,T)|\mathcal{M}_t\big\} = \pi(g_T,t,t).$$

So we have:

$$\begin{aligned}\frac{\partial}{\partial T}f(t,t) &= \frac{\partial}{\partial T}\Big[\frac{\pi(g,t,t)}{\pi(G,t,t)}\Big] \\ &= \frac{\pi(g_T,t,t)\pi(G,t,t) - \pi^2(g,t,t)}{\pi^2(G,t,t)} \\ &= \nu(g_T,t,t) - \nu^2(g,t,t).\end{aligned} \tag{3.65}$$

Thus (3.56), (3.64) and (3.65) implies the exact form of $\lambda(t)$ as in the Theorem. ■

The above proposition is the key result of our work. As we are trying to find the term structure of default intensities from liquid market, this filtering method help us to understand how the default term structure would be indicated from the equity and debt markets, which are considered as the main factors of the firm structure.

Remark 3.4.11 In case that $T = t$, the stochastic differential equations for $\pi(g,t,t)$ and $\pi(G,t,t)$ are:

$$\begin{aligned}\pi(g,t,t) &= \int_{[t]}^{t}[\pi(g_T,u,u) + \pi(\mathcal{A}^\theta g,u,u) - \pi(\gamma g,u,u) + \pi(g,u,u)\pi(\gamma,u,u)]du \\ &+ \int_{[t]}^{t}[\pi(lg,u,u) - \pi(g,u,u)\pi(l,u,u)]dW_u^S \\ &+ \int_{[t]}^{t}\big[\frac{\pi(\gamma g,u-,u-)}{\pi(\gamma,u-,u-)} - \pi(g,u-,u-)\big]dY_u,\end{aligned}$$

and

$$\begin{aligned}\pi(G,t,t) &= \int_{[t]}^{t}[\pi(g,u,u)+\pi(\mathcal{A}^{\theta}G,u,u)-\pi(\gamma G,u,u)+\pi(G,u,u)\pi(\gamma,u,u)]du \\ &+\int_{[t]}^{t}[\pi(lG,u,u)-\pi(G,u,u)\pi(l,u,u)]dW_u^S \\ &+\int_{[t]}^{t}\big[\frac{\pi(\gamma G,u-,u-)}{\pi(\gamma,u-,u-)}-\pi(G,u-,u-)\big]dY_u.\end{aligned}$$

The estimations we derived in this section all have the common character of linking liquid market information to estimate credit market terms. This is the major difference between our model and other ones.

3.5 Estimations

We derived the term structure equations for the instantaneous default intensity $\lambda(t)$ and forward default intensity $f(t,T)$. Solving for the explicit solutions directly from these two differential equations relies on the estimations of the parameters in the term structure SDE. In this section we will focus on the estimation of the θ functions.

Two estimation methods of will be introduced in the rest of this section.

The first one is direct estimation of θ itself. Once we know the estimated dynamic form of θ, we can plug it back to the θ functions.

Secondly, in the pricing process, it is more interesting to have the survival term structure. We derive the SDE's under different constraints or assumptions on the estimations of parameters. We consider our assumptions to be fair and generous enough for real data fitting because we keep enough number of degrees of freedom.

The detailed discussions on both the separation priciple and the term structure equations in different circumstances could be found in the following.

3.5.1 The estimation of θ

The estimation of θ itself is the simplest case of our filtering problem. The $\theta(t)$ estimation SDE is the direct application of our main result in the previous section.

We start from the simplest case when the θ function $g(\theta, t, T)$ is linear on θ. In this case, the conditional expectation on $g(\theta, t, T)$ is directly related to the conditional expectation of θ. Denote the conditional expectation of $\theta(t)$ under $\mathcal{M}_t$ as $\hat{\theta}(t) = E\{\theta(t)|\mathcal{M}_t\} = \pi(\theta, t)$, then the rough estimations are given by

$$\pi(g, t; T) = g(\hat{\theta}(t), t, T) \qquad \textit{and} \qquad \pi(G, t; T) = G(\hat{\theta}(t), t, T);$$

$$f(t, T) = -\frac{g(\hat{\theta}(t), t, T)}{G(\hat{\theta}(t), t, T)} \qquad \textit{and} \qquad \lambda(t) = -\frac{g(\hat{\theta}, t, t)}{G(\hat{\theta}, t, t)}.$$

The differential equation (3.54) applies to any measurable L^2 function of $\theta(t)$. So does it to $\theta(t)$ itself. Application of the formula (3.54) indicates, at present time $t \in [0, T]$,

$$\begin{aligned} \pi(\theta, t) &= \theta([t]) + \int_{[t]}^{t} \big[\pi(\mu^\theta, u) - \pi(\gamma\theta, u) + \pi(\theta, u)\pi(\gamma, u)\big] du \\ &\quad + \int_{[t]}^{t} \big[\pi(l\theta, u) - \pi(\theta, u)\pi(l, u)\big] d\bar{W}_u^S + \int_{[t]}^{t} [\frac{\pi(\gamma\theta, u-)}{\pi(\gamma, u-)} - \pi(\theta, u-)] dY_u. \end{aligned} \tag{3.66}$$

Let us consider a simple case of above SDE. Denote the conditional covariance of $l(t)$ and $\theta(t)$ and the covariance of $\gamma(r)$ and $\theta(t)$ as:

$$C^l(u) = E\{l(u)\theta(u)|\mathcal{M}_u\} - E\{l(u)|\mathcal{M}_u\}E\{\theta(u)|\mathcal{M}_u\}$$

and

$$C^\gamma(u) = E\{\gamma(u)\theta(u)|\mathcal{M}_u\} - E\{\gamma(u)|\mathcal{M}_u\}E\{\theta(u)|\mathcal{M}_u\}.$$

Thus we rewrite the solution to the θ-SDE as:

$$\begin{aligned} \hat{\theta}(t) &= \pi(\theta, t) \\ &= \theta([t]) + \int_{[t]}^{t} [\pi(\mu^\theta, u) - C^\gamma(u)] du + \int_{[t]}^{t} C^l(u) d\bar{W}_u^S \\ &\quad + \int_{[t]}^{t} \frac{C^\gamma(u)}{\pi(\gamma, u-)} dY_u. \end{aligned} \tag{3.67}$$

Thus we have found the integration formula for $\hat{\theta}(t)$. Parameter functions $C^l(u)$ and $C^\gamma(u)$ can be considered as the conditional covariance between $l(\theta(u), u)$ and $\theta(u)$

and between $\gamma(\theta(u), u)$ and $\theta(u)$ respectively.

In case when $g(\theta(t), t, T)$ is nonlinear on θ, we can estimate it by applying the density function of $\theta(t)$. Denote $\rho(t, x)$ as the density function of $\theta(t)$ under the filtration $\mathcal{M}_t$, and the càdlàg function $h(\theta(t), x)$ as: $h(\theta(t), x) = \mathbf{1}_{\{\theta(t)\leq x\}}$, then we have:

$$\rho(t, x) = \frac{\partial}{\partial x} P(\theta(t) \leq x|\mathcal{M}_t) = \frac{\partial}{\partial x}\pi(h, t) = \pi(\frac{\partial}{\partial x}h, t). \tag{3.68}$$

Then we can write the estimation of a general θ function $g(\theta(t), t, T)$ as:

$$\pi(g, t, T) = \int_{[t]}^{t} g(x, t, T) \cdot \rho(t, x)dx. \tag{3.69}$$

The following Proposition gives a detailed property of the density function $\rho(t, x)$:

Proposition 3.5.1 *Let $\rho(t, x)$ be the density function of $\theta(t)$ we defined in (3.68), then $\rho(t, x)$ is the solution of the SDE:*

$$\begin{cases} \frac{\partial}{\partial t}\rho(t, x) = & \tilde{\mathcal{A}}^\theta \rho(t, x) + \rho(t, x)\big[l(x) - \int_{-\infty}^{\infty} l(y)\rho(t, y)dy\big]d\bar{W}_t^S \\ & +\rho(t, x)\big[\frac{\gamma(x)}{\int_{-\infty}^{\infty}\gamma(y)\rho(t, y)dy} - 1\big]d\bar{Y}_t \\ \rho([t], x) = & \mathbf{1}_{\{\theta([t])=x\}}. \end{cases} \tag{3.70}$$

where $\bar{W}_t^S$ is defined in (3.55), $\bar{Y}_t$ is the compensated Poisson process under the filtration $\mathcal{M}_t$ and

$$\tilde{\mathcal{A}}^\theta \rho(t, x) = -\frac{\partial}{\partial x}(\mu^\theta(x, t)\rho(t, x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}({\sigma^\theta}^2(x, t)\rho(t, x)).$$

Proof The proof of the Proposition is the application of Theorem 3.4.9. Plug the function $h(\theta(t), x) = \mathbf{1}_{\{\theta(t)\leq x\}}$ into the SDE in (3.4.9), we have:

$$\begin{aligned} \pi(h, t) &= P(\theta \leq x|\mathcal{M}_t) \\ &= \pi(h, [t]) + \int_{[t]}^{t}\big[\int_{-\infty}^{x} \mathcal{A}^\theta h^x(y, u)\rho(u, y)dy\big]du \\ &\quad + \int_{[t]}^{t}\big[\int_{-\infty}^{x} l(y, u)\rho(u, y)dy - \big(\int_{-\infty}^{\infty} l(y, u)\rho(u, y)dy\big)\big(\int_{\infty}^{x}\rho(u, y)dy\big)\big]d\bar{W}_u^S \\ &\quad + \int_{[t]}^{t}\big[\frac{\int_{-\infty}^{x}\gamma(y, u)\rho(u, y)dy}{\int_{-\infty}^{\infty}\gamma(y, u)\rho(u, y)dy} - \int_{-\infty}^{x}\rho(u, y)dy\big]d\bar{Y}_u, \end{aligned}$$

where $\mathcal{A}^\theta$ is the θ operator we defined in Theorem 3.4.7. Denote

$$Ah(t,x) = \int_{[t]}^{t} \big[\int_{-\infty}^{x} \mathcal{A}^\theta h(y,u)\rho(u,y)dy \big] du,$$

then apply the forward Kolmogorov equation to $Ah(t,x)$, we have:

$$\frac{\partial}{\partial t}\big[\frac{\partial}{\partial x}Ah(t,x)\big] = -\frac{\partial}{\partial x}(\mu^\theta(x,t)\rho(t,x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^{\theta 2}(x,t)\rho(t,x)) = \tilde{\mathcal{A}}^\theta \rho(t,x).$$

Now from (3.68), by taking double derivatives with respect to t and x on both sides of above equation, we have $\rho(t,x)$ as the solution of SDE (3.70). ■

Thus all estimation problems in our article is simplified by solving the SDE (3.70) only.

3.5.2 A simplified example of term structure equations

In what follows, we study a simplified detailed version of our model, in which we assume that the the coefficients are linear θ functions. Then we derive the corresponding forms of the equations.

To do the simplification, we modify our model in the following way:
The total asset process X_t in equation (3.4) has the following geometric Brownian motion form with relative constant coefficients,

$$X_T^t = x_{[t]} + \int_{[t]}^{T} X_u^t \cdot a \cdot \theta(t)du + \int_{[t]}^{T} X_u^t \sqrt{\theta(t)}dB_u, \quad T \in [0,\bar{T}], \tag{3.71}$$

where a is a constant.

The bounded stochastic factor $\theta(t)$ in equation (3.3) also enjoys a stochastic differential equation with linear θ coefficients. It is the solution of the SDE:

$$\theta(t) = \theta_{[t]} + \int_{[t]}^{t} (a_1 + a_2\theta(u))du + \int_{[t]}^{t} b_1\theta(u)dW_u^\theta \qquad t \in [0,\bar{T}], \tag{3.72}$$

for constant numbers a_1, a_2 and b_1.

The stochastic barrier $\nu(\theta(t),t,u)$ is assumed to be constant with initial condition:

$$x_{[t]} > \nu(\theta([t]),[t],[t]).$$

The observable stock price S_t is a geometric Brownian motion in (3.7). The drift of S_t carries the information in a form of a linear function of θ :

$$S_t = s_0 + \int_0^t S_u[\mu_1^S\theta(u) + \mu_0^S]du + \int_0^t S_u\sigma^S(u)dW_u^S, \tau \in [0, \bar{T}], \tag{3.73}$$

where μ_1^S and μ_2^S are constant, $\sigma^S(t)$ is a bounded function of t.

The observable pure jump process Y_t is a counting process as we set up in (3.8). The intensity is also assumed to be a linear function of θ: $\gamma(\theta(t)) = \gamma_1\theta(t) + \gamma_0$ and initial value $Y_0 = 0$.

All the coefficients in equations (3.71) to (3.73) satisfies the conditions in assumptions (**A1**) to (**A3**).

Now we follow the main steps in previous section to derive the detail forms of the functions.

By definition, $l(\theta(t), t)$ is of the form:

$$l(\theta(t), t) = \frac{\mu^S(\theta(t), t)}{\sigma^S(t)} = \frac{\mu_1^S}{\sigma^S(t)}\theta(t) + \frac{\mu_0^S}{\sigma^S(t)},$$

Note here we are assuming that the parameters of the X_u^t SDE are constant. So the above survival formula can be simplified as the following:

$$\begin{aligned} G(\theta(t), t, T) &= \Psi\big((a - \frac{1}{2})\theta(t), \sqrt{\theta(t)}, \nu, T - [t]\big) \\ &= N\big(d_2(a\theta(t), \sqrt{\theta(t)}, x_{[t]}, \nu, \nu, T - [t])\big) \\ &\quad - \exp((2a - 1)(\log[\nu/x_{[t]}]))N\big(d_1(a\theta(t), \sqrt{\theta(t)}, x_{[t]}, \nu, \nu, T - [t])\big) \end{aligned}$$

and

$$\begin{aligned} g(\theta(t), [t], T) &= \frac{\partial}{\partial T}G(\theta(t), [t], T) \\ &= \frac{\log[\nu/x_{[t]}]}{T - [t]}n\big(d_1(a\theta(t), \sqrt{\theta(t)}, x_{[t]}, \nu, \nu, T - [t])\big). \end{aligned}$$

In above formulas,

$$d_1(a\theta(t), \sqrt{\theta(t)}, x_{[t]}, \nu, \nu, \Delta T) = \frac{(a - \frac{1}{2})\Delta T\theta(t) + \log[\nu/x_{[t]}]}{\sqrt{\theta(t)\Delta T}}$$

and

$$d_2(a\theta(t), \sqrt{\theta(t)}, x_{[t]}, \nu, \nu, \Delta T) = d_1(a\theta(t), \sqrt{\theta(t)}, x_{[t]}, \nu, \nu, \Delta T) - \frac{2(\log[\nu/x_{[t]}])}{\sqrt{\theta(t)\Delta T}}.$$

Some direct calculation also indicate the following results of the generators on above functions:

$$\begin{aligned} \mathcal{A}^\theta G &= \big[\big(a_2 - \frac{3}{4}b_1^2\big)(T-[t]) + \big(a_1(T-[t]) - \frac{1}{4}b_1^2\nu^2\big)\theta^{-1} \\ &\quad + \frac{1}{4}(a-\frac{1}{2})^2(T-[t])^2 b_1^2\theta(t)\big]G_T(\theta(t), t, T)) \\ &= \big[A_1(T) + A_2(T)\theta^{-1} + A_3(T)\theta(t)\big]G_T(\theta(t), t, T), \end{aligned} \tag{3.74}$$

where $A_1(T)$ $A_2(T)$ and $A_3(T)$ correspond to the coefficients on the right hand side of the first equivalence in the formula.

$$\begin{aligned} \mathcal{A}^\theta g &= \big[\frac{a_1(T-[t])^{\frac{1}{2}}\nu}{4}\theta^{-\frac{3}{2}}(t) - \frac{a_1(T-[t])}{2}\theta^{-1}(t) + \big(\frac{a_2(T-[t])^{\frac{1}{2}}\nu}{4} \\ &\quad - \frac{b_1^2(T-[t])^{\frac{1}{2}}}{4}\big)\theta^{-\frac{1}{2}}(t) + \big(\frac{3}{8}b_1^2\tilde{\nu} - \frac{1}{2}a_2(T-[t])\big) \\ &\quad + \big(\frac{b_1^2(T-[t])^{\frac{1}{2}}\tilde{\nu}}{8} - \frac{a_1(a-\frac{1}{2})^2(T-[t])^{\frac{5}{2}}}{2\tilde{\nu}}\big)\theta^{\frac{1}{2}} - \frac{a_2(a-\frac{1}{2})^2(T-[t])^{\frac{5}{2}}}{2\tilde{\nu}}\theta^{\frac{3}{2}} \\ &\quad - \frac{(a-\frac{1}{2})^2 b_1^2(T-[t])^{\frac{5}{2}}}{4\tilde{\nu}}\theta^{\frac{5}{2}}\big]G_T(\theta(t), t, T) \\ &= \big[B_1(T)\theta^{-\frac{3}{2}}(t) + B_2(T)\theta^{-1}(t) + B_3(T)\theta^{-\frac{1}{2}}(t) + B_4(T) + B_5(T)\theta^{\frac{1}{2}}(t) \\ &\quad + B_6(T)\theta^{\frac{3}{2}}(t) + B_7(T)\theta^{\frac{5}{2}}(t)\big]G_T(\theta(t), t, T). \end{aligned} \tag{3.75}$$

Similar as before, we correspond the deterministic coefficients of powers of $\theta(t)$ to the terms $B_i(T)$'s.

Now we apply Theorem 3.4.9 and Proposition 3.4.10 to calculate the estimations terms.

First, the estimation of $\theta(t)$ could be derived in the same way as we did in last section. Let $\hat{\theta}(t) = \pi(\theta, t)$, then

$$\hat{\theta}(t) = \theta([t]) + \int_{[t]}^t \big\{a_1 + a_2\hat{\theta}(u) - \gamma_1 C^\theta(u)\big\}du + \int_{[t]}^t \mu_1^S C^\theta(u)d\bar{W}_u^S + \int_{[t]}^t \frac{\gamma_1 C^\theta(u)}{\pi(\gamma, u-)}dY_u, \quad t \in [0, T],$$

where the $C^\theta(t)$ is denoted similarly as the before: $C^\theta(t) = \pi(\theta^2, t) - \pi^2(\theta, t)$. Thus by now, we are able to derive the intensity by plugging the estimator $\hat{\theta}$ into the G and g functions.

Second, the derivation of the $\pi(G, t, T)$ and $\pi(g, t, T)$ equations are also from the general formulae. Direct calculation gives us the following results:

$$\begin{aligned}
\pi(G, t, T) \quad = \quad & G(\theta([t]), [t], T) + \int_{[t]}^{t} \big[A_1(T)\pi(g, u, T) + A_2(T)\pi(\theta^{-1}g, u, T) \\
& + A_3(T)\pi(\theta g, u, T) + \pi(\gamma G, u, T) - \pi(\gamma, u, T)\pi(G, u, T)\big] du \\
& + \int_{[t]}^{t} [\pi(lG, u, T) - \pi(l, u, T)\pi(G, u, T)] d\bar{W}_u^S \\
& + \int_{[t]}^{t} \frac{\pi(\gamma G, u, T) - \pi(\gamma, u, T)\pi(\Gamma, u, T)}{\pi(\gamma, u-, T)} dY_u
\end{aligned}$$

and

$$\begin{aligned}
& \pi(g, t, T) \\
= \quad & g(\theta([t]), [t], T) + \int_{[t]}^{t} \big[B_1(T)\pi(\theta^{-\frac{3}{2}}g, u, T) + B_2(T)\pi(\theta^{-1}g, u, T) \\
& + B_3(T)\pi(\theta^{-\frac{1}{2}}g, u, T) + B_4(T)\pi(g, u, T) + B_5(T)\pi(\theta^{\frac{1}{2}}g, u, T) \\
& + B_6(T)\pi(\theta^{\frac{3}{2}}g, u, T) + B_7(T)\pi(\theta^{\frac{5}{2}}g, u, T) + \pi(\gamma g, u, T) - \pi(\gamma, u, T)\pi(g, u, T)\big] du \\
& + \int_{[t]}^{t} [\pi(lg, u, T) - \pi(l, u, T)\pi(g, u, T)] d\bar{W}_u^S \\
& + \int_{[t]}^{t} \frac{\pi(\gamma g, u, T) - \pi(\gamma, u, T)\pi(g, u, T)}{\pi(\gamma, u-, T)} dY_u,
\end{aligned}$$

where the coefficients $A_1(T)$, $A_2(T)$ and $A_3(T)$ are defined in (3.74). $B_1(T)$ to $B_7(T)$ are defined in (3.75).

Finally we have the forward intensity $f(t, T)$ to be the solution of the SDE:

$$\begin{aligned}
& f(t, T) \qquad\qquad (3.76) \\
= \quad & f([t], T) \\
& + \int_{[t]}^{t} \big[A(u, T)f(u, T) + B(u, T)\big] du + \int_{[t]}^{t} \big[C(u, T)f(u, T) + D(u, T)\big] d\bar{W}_u^S \\
& - \int_{[t]}^{t} \Big[\frac{\gamma_1\pi(\theta g, u, T) + \gamma_0\pi(g, u, T)}{\gamma_1\pi(\theta G, u, T) + \gamma_0\pi(G, u, T)} + f(u-, T)\Big] dY_u
\end{aligned}$$

where

$$
\begin{aligned}
A(t,T) &= \big(\frac{\mu_1^S}{\sigma^S}\big)^2 \nu(\theta G,t,T)[\nu(\theta G,t,T) - \pi(\theta,t)] + \gamma_1 \nu(\theta G,t,T) - \nu(\mathcal{A}^\theta G,t,T) \\
B(t,T) &= (\frac{\mu_1^S}{\sigma^S})^2 \nu(\theta g,t,T)[\nu(\theta G,t,T) - \pi(\theta,t)] + \gamma_1 \nu(\theta g,t,T) - \nu(\mathcal{A}^\theta g,t,T) \\
C(t,T) &= -\frac{\mu_1^S}{\sigma^S} \nu(\theta G,t,T) \\
D(t,T) &= -\frac{\mu_1^S}{\sigma^S} \nu(\theta g,t,T).
\end{aligned}
$$

And $\mathcal{A}^\theta G$ and $\mathcal{A}^\theta g$ are defined in (3.74) and (3.75).

4. SUMMARY

We derived the term structure equations for single firm forward default intensity and spot default intensity in general cases and special cases under different assumptions.

The default term structure equations are of both theoretical and practical importance. One can compare the forward and spot intensity with the forward and spot interest rate in the rate models. In practice, the term structure equations can make improvement on the default discount on the legs of the default swaps, thus improve the pricing of CDS's. The term structure equations can also be applied to the pricing of CDS options. Practically, people deal with the pricing of CDS options by finding out the correlation between constant intensity and the equity process. Comparing with our method, we consider our's more intrinsic.

In this work, the term structure equations are based on a single defaultable asset, where we only assumed one random factor θ. As we pointed in the beginning, our model is a general version of Frey and Runggaldier's work [16] on single defaultable firm setting. The model can be expanded to the basket default model with multi random factors. Instead of starting from the relations between the intensities, our consideration on the total asset of the firm give more intrinsic connections to the default contagion.

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