Inverse Pricing for the Implied Volatility Surface with Physics-Informed Neural Networks

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The Inverse IV Surface

Goal: Recover a smooth, arbitrage-consistent implied volatility surface $\sigma(K, T)$ from sparse, noisy market option prices (K, T, C^{mkt}).

The Inverse Problem as a Minimization Task: Model the surface $\sigma_\theta(K, T)$ using a neural network and solve:

$$\min_\theta \mathcal{L}(\theta) = \underbrace{\sum_{(K,T)\in\mathcal{D}} \left(C_{\mathrm{BS}}(K, T; \sigma_\theta(K, T)) - C_{KT}^{\mathrm{mkt}}\right)^2}_{\text{Data Fitting (Pricing Error)}} + \underbrace{\lambda_{arb}\, \Phi_{\mathrm{arb}}(\sigma_\theta)}_{\text{Physics/Financial Constraints}}$$

Key Components:

- **Forward Map**: Black-Scholes formula $C_{\mathrm{BS}}(\sigma)$ bridges latent σ to observable Price.
- **Soft Constraints (PINN)**: Enforce no-arbitrage conditions via penalty terms (e.g., $\partial_T(\sigma^2 T) \geq 0$).

PINN Architecture & Training Strategy

PINN framework with a two-phase training strategy to ensure both accuracy and regularity.

1. Network Architecture (σ_θ):

- **Input**: Log-moneyness $\ln(K/S)$ and Time-to-maturity T.
- **Output**: Implied Volatility σ (Softplus activation ensures $\sigma > 0$).
- **Backbone**: MLP with 3 hidden layers (128, 128, 64 units), SiLU activation.

2. The "Warm-up" Training Strategy: A critical finding was that applying constraints too early leads to mode collapse (flat surfaces).

- **Phase A (Warm-up)**: Train with $\lambda_{arb} = 0$ for 20-30 epochs. Let the model learn the general market shape first.
- **Phase B (Refinement)**: Introduce penalties ($\lambda_{arb} = 1e^{-4}$) to smooth the surface and remove arbitrage violations without destroying the fit.

Results

Model on SPY options data (2023-01-03). The two-phase strategy successfully balanced data fitting with physical consistency.

Metric (Validation Set)	**Phase A (No Constraints)**	**Phase B (PINN Final)**	**Implication**
Price Loss (MSE)	~ 2.09	**3.44**	Retains high accuracy
Constraint Loss	N/A (High)	**1.91**	**Arbitrage nearly eliminated**
RMSE (Pricing)	-	**1.76** USD	**Excellent Fit**
Calendar Arb Violation	-	**0.00%**	**Physically Valid**

Key Result: The PINN achieves an RMSE of $1.76 while satisfying 100% of calendar arbitrage checks on the test grid, proving it is not just memorizing noise.

Results

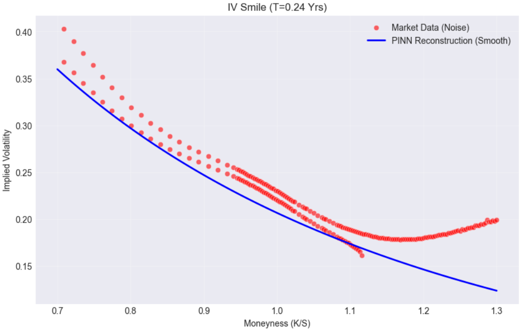

Figure 1: Learned IV Smile (T $\approx$ 0.24).
The blue line (PINN) smoothly interpolates the noisy market scatter points (red).

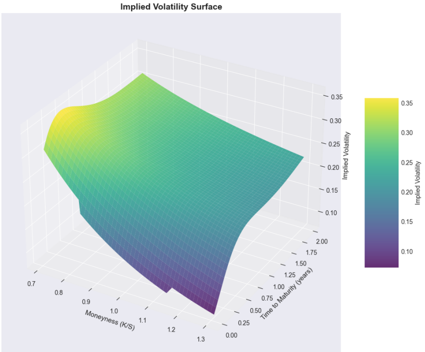

Figure 2: Reconstructed 3D IV Surface.
The surface is smooth and continuous across the entire (K, T) domain.

Generalization & Conclusions

Generalization to Future Markets: Test the 1/3 model on future dates (1/4, 1/5) to verify structural learning.

- **Zero-Shot**: RMSE $\approx 0.93 - 1.50$ on unseen days. The model captures the *fundamental structure* of the IV surface, not just daily noise.
- **Transfer Learning**: Fine-tuning for just **10 epochs** reduces RMSE by up to **7%**, enabling extremely efficient daily calibration.

Conclusions:

1. **Feasibility**: Deep Learning effectively solves the ill-posed inverse pricing problem.
2. **Physics Matter**: Soft constraints are essential for economic validity; the "Warm-up" strategy is key to training stability.
3. **Robustness**: The method is robust to sparse data and generalizes well over time.

References: [1] R. Cont et al., "Inverse problems in option pricing", 2005. [2] G.E. Karniadakis et al., "Physics-informed machine learning", 2021.