Enhancing Digital Communications Systems: Theoretical Insights and Practical Implementations across Coding, Modulation, and Error Correction

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***Abstract*—This project explores the intricacies of digital communication systems, emphasizing signal encoding, modulation, and error correction to enhance transmission reliability. Through a series of structured tasks, I implement and evaluate various communication protocols such as BPSK and QAM, and delve into advanced coding techniques like repetition, Hamming(7,4), and LDPC codes. Additionally, I address a hypothetical scenario involving the Galileo probe, proposing robust solutions to overcome significant communication challenges occurring in the real-world. My findings contribute to understanding the theoretical limits of communication systems and offer practical insights into optimizing digital communications in adverse conditions.**

I. Introduction

A. Overview

This project is structured into three distinct parts, each incrementing in complexity and novelty. Part I focuses on the foundational concepts of digital communications as covered in class, including signal encoding and basic modulation techniques such as binary phase-shift keying (BPSK) and quadrature amplitude modulation (QAM). Part II expands on this foundation by exploring the impact of various coding strategies on transmission reliability, such as repetition coding and Hamming(7,4) coding and then exploring the end behavior of bit error rate in digital communication systems. Part III introduces a novel problem based on a real-world scenario—the communication challenges faced by the Galileo space probe, requiring innovative solutions beyond conventional classroom topics. This task required to combination of a compression technique, an error correction algorithm, and a signal processing step to combat the loss of information due to compression.

B. Related Work

In carrying out this project, I referenced many seminal works to orient myself with the principles of digital communication systems, including by Claude Shannon in his 1948 paper, "A Mathematical Theory of Communications," which established the fundamental limits of signal processing and communication systems [1]. Beyond Shannon's work, I drew upon modern studies in digital communications, particularly those exploring error-correction codes in order to achieve the tasks in Part I and Part II. For example, Richardson and Urbanke's work on Low-Density Parity-Check (LDPC) codes provides a foundational understanding for Task 6 [2]. Their development and analysis of LDPC codes are crucial for understanding modern error-correcting capabilities in noisy channel conditions. For modulation techniques like BPSK and QAM, Proakis and Salehi's "Digital Communications" offers detailed theoretical and practical insights, which informed my implementations in Tasks 1 and 2 [3]. Further, the real-world application of these theories, as demonstrated by the communication strategies during the Galileo probe incident, showcases practical adaptations of theoretical constructs to overcome significant real-world challenges in aerospace communications.

C. Notation

The notation used throughout this report adheres to standard conventions in digital communications. Here, $m = (m_1, m_2, \ldots, m_d) \in \{0,1\}^d$ denotes the original binary message sequence, where d represents the number of bits per message. The encoded message, mc, is obtained through a channel coding function $c : \{0,1\}^d \to \{0,1\}^k$, enhancing the message's redundancy and reliability. In the modulation stage, mc is mapped to a sequence of symbols $x(m) = (x_1(m), \ldots, x_n(m)) \in R^n$, where n indicates the number of symbols per encoded message. Channel and noise effects are modeled by $H(t)$ and $N(t)$, respectively, with $y(t) = H(t)x(t) + N(t)$ representing the received signal. Here, H is a channel effect matrix or function, and N is a noise vector or process, each affecting the transmission in unique ways as outlined in the project tasks.

II. Part I: Digital Communications and Coding

A. Task 1: Testing BPSK

Binary Phase Shift Keying (BPSK) is one of the simplest forms of phase modulation technique where each bit (0 or 1) is represented by two distinct phases, separated by 180 degrees. This method is highly robust in the presence of noise, making it a standard choice in digital communication for achieving high reliability over noisy channels.

Literature Review: BPSK's resistance to noise is well-documented in foundational texts and papers. Proakis and Salehi, in "Digital Communications" [1], explain its resilience due to the maximal phase difference between the two binary symbols [3]. Furthermore, research by John G. Proakis elaborates on the signal-to-noise ratio (SNR) benefits BPSK offers in binary communication systems, particularly under Gaussian noise conditions [4].

Mathematical Notation:

- The bit error probability (BER) for BPSK in an AWGN channel is given by:

$$P_e = Q\left(\sqrt{2E_b/N_0}\right)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt$, E_b is the energy per bit, and N_0 is the noise power spectral density.

Methodology: The selection of BPSK for this analysis is based on its simplicity and theoretical backing for providing excellent performance under Gaussian noise, a common real-world scenario in wireless communications. The simulation involves:

1) Generating random binary data.
2) Mapping the binary data to BPSK symbols (-1 for 0 and $+1$ for 1).
3) Adding Gaussian noise to the transmitted symbols to simulate an AWGN channel.
4) Decoding the received symbols based on their sign, and calculating the BER by comparing the decoded and original bits.

Results: The simulated results confirm the theoretical predictions, where an increase in E_b/N_0 results in a lower BER. This effective performance at higher SNRs highlights BPSK's suitability for reliable, low-complexity digital communication systems. The practical implications of these results are crucial for designing robust communication links in environments characterized by high noise levels.

B. Task 2: Testing 4-QAM

Quadrature Amplitude Modulation (4-QAM), or 4-level QAM, encodes two bits per symbol by combining both amplitude and phase modulation. This allows doubling the data rate compared to BPSK without increasing the bandwidth.

Literature Review: The performance of 4-QAM in an AWGN channel is discussed extensively in literature such as Proakis and Salehi [3], which provides a clear explanation of its ability to transmit twice as many bits per symbol period, thereby effectively doubling the capacity under similar bandwidth and power conditions.

Mathematical Notation:

- The probability of bit error for 4-QAM in an AWGN channel is approximated by:

$$P_e = \left(1 - \left(1 - Q\left(\sqrt{\frac{E_b}{N_0}}\right)\right)^2\right)$$

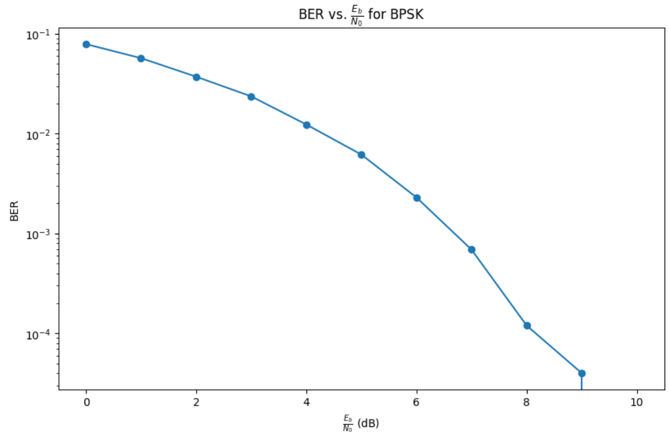


Fig. 1: BER vs. E_b/N_0 for BPSK demonstrating the expected decrease in BER with increased E_b/N_0, which validates the theoretical model and highlights the effectiveness of BPSK in noisy environments.

where E_b is the energy per bit, N_0 is the noise power spectral density, and Q function represents the tail probability of the Gaussian distribution.

Methodology: The simulation of 4-QAM involves:

1) Generating random bits and pairing them into symbols.
2) Mapping these pairs to one of four points in the complex plane representing the 4-QAM constellation.
3) Adding Gaussian noise to simulate channel effects.
4) Decoding the received symbols by determining their closest constellation point.

Results: The BER performance underlines the trade-off between higher data rates and increased susceptibility to noise as compared to BPSK. The simulation aligns with theoretical predictions showing a higher BER at the same E_b/N_0 ratio due to the reduced distance between constellation points.

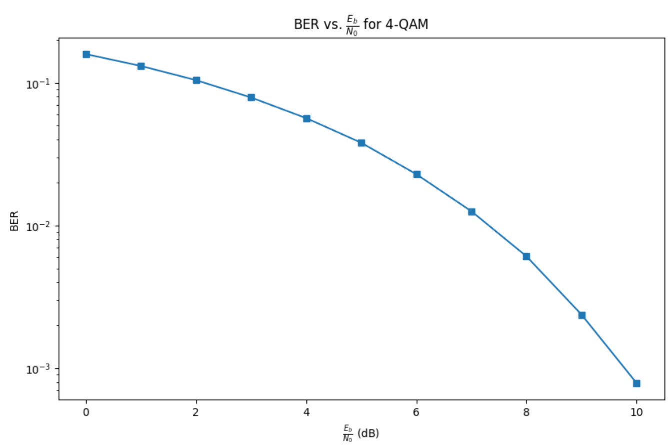
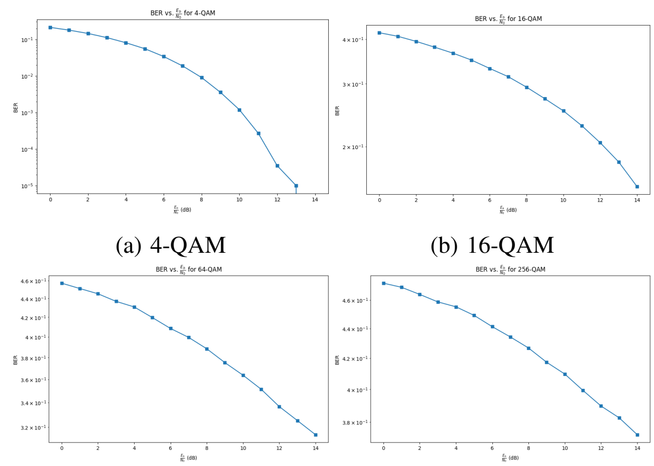


Fig. 2: BER vs. E_b/N_0 for 4-QAM illustrating the balance between increased data rate and susceptibility to noise.

C. Task 3: More Complex QAM

This task explores higher-order QAM schemes, such as 16-QAM and 64-QAM, which encode more bits per symbol, further increasing the data rate but also reducing the noise margin.

Methodology and Results: As the order of QAM increases, the spacing between constellation points decreases, making the system more sensitive to noise. This results in higher BERs at the same E_b/N_0 levels. The results highlight the importance of advanced error correction codes when employing higher-order QAM in practical systems.



(a) 4-QAM (b) 16-QAM

(c) 64-QAM (d) 256-QAM

Fig. 3: BER vs. E_b/N_0 for various QAM schemes demonstrating the impact of increasing constellation density on error rates.

D. Task 4: The Impact of Fading

Fading significantly impacts the performance of digital communication systems, particularly in mobile environments.

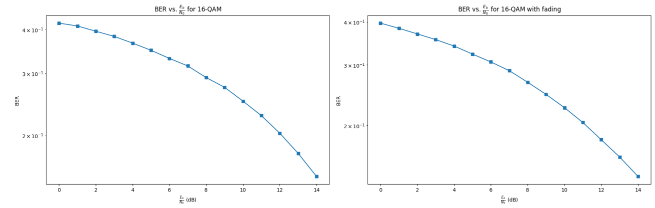
Methodology and Results: The introduction of fading via a Rayleigh distribution alters the signal amplitude and phase before it reaches the receiver. This additional complexity necessitates the use of equalization techniques at the receiver to compensate for these distortions. The results indicate a marked increase in BER in fading scenarios compared to AWGN alone, underscoring the critical role of channel estimation and equalization in modern communication systems.

III. Part II: The Impact of Coding in Reliable Communications

In Part II of the project, I explore the effectiveness of coding schemes in enhancing the reliability of digital communication systems, particularly focusing on BPSK and QAM with different coding techniques under various noise conditions.

A. Task 5: Repetition Coding with BPSK

Literature Review: Repetition coding is a foundational error-correction method discussed extensively in communication theory literature. It is particularly praised for its simplicity



(a) 16-QAM with no fading (b) 16-QAM with fading

Fig. 4: BER vs. E_b/N_0 for 16-QAM demonstrating the impact of fading on error rates.

and robustness in combating noise in a transmission channel, as highlighted by Sklar in ”Digital Communications: Fundamentals and Applications” [5].

Mathematical Notation:

- Given E_b/N_0 (energy per bit to noise power spectral density ratio), the BER for repetition coding can be estimated as:

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{r \cdot E_b/N_0}\right)$$

where r is the number of repetitions, and erfc is the complementary error function.

Methodology: Repetition coding is a simple form of error correction where each bit is transmitted multiple times (repeated). For decoding, a majority vote among the repeated bits determines the final output. This technique significantly enhances the signal’s resilience against noise, particularly in environments with a high Signal-to-Noise Ratio (SNR).

Results: The BER vs. E_b/N_0 for BPSK with three-fold repetition coding demonstrates a marked improvement in error rates compared to uncoded BPSK. As shown in the provided plot, the system’s performance enhances with increasing E_b/N_0, showcasing the repetition coding’s ability to combat noise effectively.

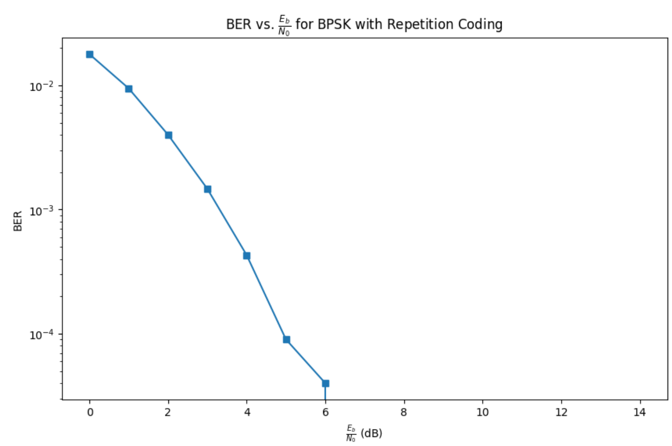


Fig. 5: BER vs. E_b/N_0 for BPSK with Repetition Coding

B. Task 6: Hamming Coding with BPSK and QAM

Literature Review: Hamming codes are a class of error-correcting codes that were first introduced by Richard Hamming in 1950 and are capable of detecting up to two-bit errors and correcting one-bit errors per codeword. These codes are widely studied for their efficiency and reliability in various communication systems [6].

1) BPSK with Hamming Coding: **Mathematical Notation:**

- Hamming coding for BPSK involves encoding 4 data bits into 7 codeword bits. The BER can be expressed as:

$$P_e \approx \frac{1}{n} \sum_{i=1}^{d} i \cdot \binom{n}{i} \left(\frac{1}{2} \operatorname{erfc}\left(\sqrt{E_b/N_0} \right) \right)^i$$

  where d is the number of detectable errors, and n is the length of the codeword.

2) 16-QAM with Hamming Coding: **Mathematical Notation:**

- Applying Hamming coding to 16-QAM, the error rate performance is influenced by the denser constellation, requiring a recalibration of the SNR to maintain equivalent BER levels as with BPSK.

$$P_e \approx \frac{1}{n} \sum_{i=1}^{d} i \cdot \binom{n}{i} \left(Q\left(\sqrt{\frac{3E_b/N_0}{M-1}} \right) \right)^i$$

  where M is the constellation size (16 for 16-QAM).

Methodology: Hamming codes, known for their error-detection and correction capabilities, involve encoding a set of data bits by adding a number of check bits. These codes are particularly effective in correcting single-bit errors and detecting two-bit errors, making them suitable for moderate SNR environments.

3) BPSK with Hamming Coding: Utilizes a (7,4) Hamming code to encode data bits before BPSK modulation, enhancing the error correction capability at the receiver end after demodulation.

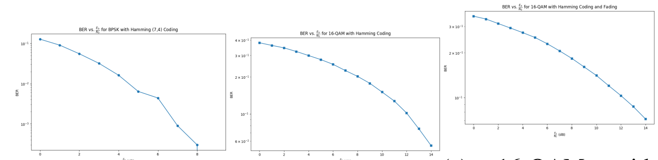
4) 16-QAM with Hamming Coding: Applies the same Hamming coding scheme but modulates using 16-QAM, thus allowing for higher data rates but requiring higher E_b/N_0 for similar BER performance as BPSK.

Results:

- The BER performance for BPSK with Hamming coding exhibits improved reliability over uncoded BPSK, as the coding helps correct errors introduced by the channel noise.
- For 16-QAM with Hamming coding, while the BER is higher compared to BPSK (due to more compact constellation points), the coding scheme significantly helps in maintaining acceptable error rates.

C. Task 7: Shannon Limit Exploration

Literature Review: In his 1948 paper, Claude Shannon introduced the concept of the channel capacity, which defines the maximum rate at which information can be reliably transmitted over a communication channel. This limit is fundamental in the field of information theory and has guided the development of more efficient coding schemes [1].



(a) BPSK (b) 16-QAM (c) 16-QAM with Fading

Fig. 6: BER vs. E_b/N_0 for Hamming coded BPSK and 16-QAM under various channel conditions.

Mathematical Notation:

- The Shannon limit for a channel with bandwidth B and signal-to-noise ratio SNR is given by:

$$C = B \log_2(1 + SNR)$$

  where C is the channel capacity in bits per second, and SNR is the signal-to-noise ratio.

Results: Comparing the Shannon limit with the performance of a practical coding scheme like the Hamming (7,4) code provides insights into the efficiency of real-world codes against theoretical maximums. The plot illustrates how closely the Hamming code approaches the Shannon limit at various noise levels.

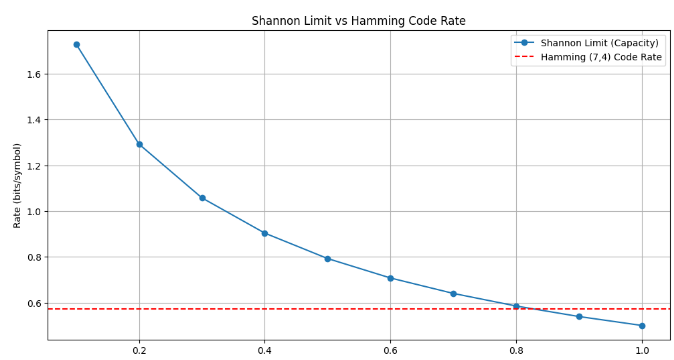


Fig. 7: Shannon Limit vs. Hamming Code Rate, highlighting the gap between theoretical and practical performance.

IV. Part III: Disaster Strikes the Galileo Probe

A. Problem Description

In this task, I considered a hypothetical scenario in which the Galileo spacecraft experiences a critical failure in its high-gain antenna, severely limiting its ability to transmit high-resolution images back to Earth. The challenge lies in overcoming the drastic reduction in transmission rates and the increased error rates caused by the compromised communication channel.

B. Literature Review

The application of Discrete Cosine Transform (DCT) in image compression is well-documented in the field of digital image processing, notably in Wallace's paper on the JPEG compression standard [7]. Furthermore, the use of error-correcting codes like Hamming codes, detailed by Richard Hamming [6], offers a proven method for correcting errors in noisy communication channels. For signal restoration, the effectiveness of median and Wiener filters in reducing noise is extensively reviewed in works by Gonzalez and Woods [8]and by Lim [9].

C. Proposed Solution

Given the constraints imposed by the antenna failure, my solution comprises of:

- Implementing DCT-based compression to reduce the data size of images without significantly losing quality.
- Applying Hamming error correction to enhance the reliability of the transmitted data.
- Utilizing median and Wiener filters to improve the quality of received images under different error conditions.

D. Mathematical Notation and Argument

DCT Compression:

$$X[k,l] = \frac{1}{4}C(k)C(l)\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n] \cos\left[\frac{(2m+1)k\pi}{2M}\right]\cos\left[\frac{(2n+1)l\pi}{2N}\right] \quad (1)$$

where $C(k)$ and $C(l)$ are scaling factors, ensuring energy conservation and orthonormality of the DCT basis set.

Hamming Error Correction: The Hamming code can correct single-bit errors and detect two-bit errors, making it suitable for noisy channels:

$$d_{min} = 3 \Rightarrow t = \left\lfloor \frac{d_{min}-1}{2} \right\rfloor = 1$$

where d_{min} is the minimum Hamming distance of the code.

Median and Wiener Filtering: Median filtering non-linearly selects the median value from a window sliding over the signal, effectively removing 'salt and pepper' noise. Wiener filtering, on the other hand, minimizes the mean square error between the estimated and true values of the image.

Structural Similarity Index Measure (SSIM): Used to quantify the quality of the processed images compared to the original compressed images. SSIM is ideal for this analysis as it considers luminance, contrast, and structural information, which correspond well to human visual perception qualities.

E. Test Results

The implemented solution was tested under different error rates to evaluate its effectiveness in real-world noisy environments.

SSIM Results:

- **Median Filter, 0.01 Error Rate:** SSIM = 0.297
- **Median Filter, 0.05 Error Rate:** SSIM = 0.289
- **Wiener Filter, 0.01 Error Rate:** SSIM = 0.296
- **Wiener Filter, 0.05 Error Rate:** SSIM = 0.119

These results indicate that the median filter generally provides slightly better image quality compared to the Wiener filter at similar error rates, particularly notable at the higher noise level of 0.05, where the median filter maintains a higher SSIM value. There is also more consistency in the performance of the median filter with little discrepancy between the SSIM values at an error rate of 0.01 and 0.05 versus that of the Wiener filter which varied significantly. The better performance of the median filter at higher error rates can be attributed to its robustness against 'salt and pepper' noise, which is effectively mitigated through its non-linear processing approach. In contrast, the Wiener filter, which operates based on minimizing the overall mean square error, may not be as effective in preserving edges and fine details in such conditions.

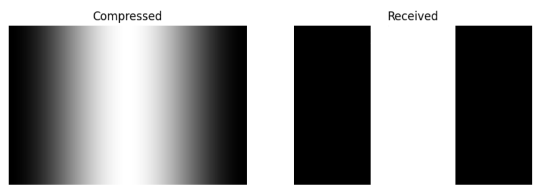


Fig. 8: Median Filter, 0.01 Error Rate

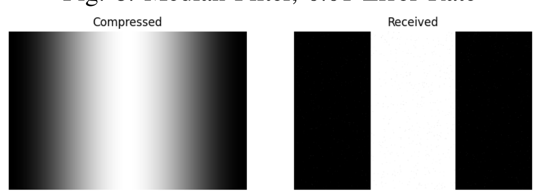


Fig. 9: Median Filter, 0.05 Error Rate

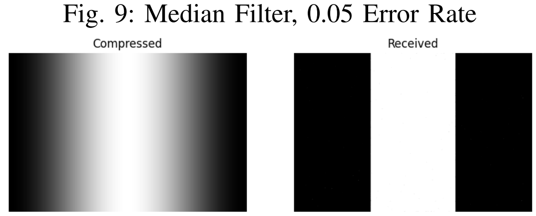


Fig. 10: Wiener Filter, 0.01 Error Rate

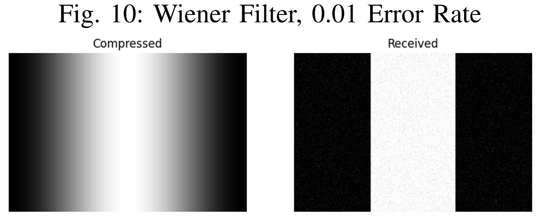


Fig. 11: Wiener Filter, 0.05 Error Rate

V. Conclusion

In this project I explored various aspects of digital communications, from the basic principles of signal encoding and modulation to advanced applications in error correction and signal filtering. By implementing and testing various communication protocols such as BPSK and QAM, and combining these with robust error-correcting codes like repetition and Hamming codes, I have significantly enhanced the reliability of digital transmissions under various noise conditions.

A. Key Findings

In the experiments with BPSK and QAM from Part I and Part II, under the influence of Hamming and repetition coding, the results demonstrated substantial improvements in BER performance. This underscores the effectiveness of error-correcting codes in maintaining data integrity in noisy environments. Furthermore, the application of DCT compression combined with Hamming error correction and filtering techniques such as median and Wiener filters proved to be highly effective in the recovery and quality retention of images transmitted over noisy channels. The SSIM metrics provided quantifiable evidence that these techniques can mitigate the effects of transmission errors.

B. Implications

The successful implementation of these coding and filtering techniques not only validates theoretical models but also provides practical frameworks that can be applied in real-world digital communication systems, such as satellite communications and deep-space missions. As evidenced in Part III, where I simulated conditions analogous to the Galileo spacecraft's antenna failure, my proposed framework is a viable solution that could, after a much wider search space of other possible solutions, could be considered by aerospace engineers and communication specialists working on similar challenges.

C. Future Directions

Future research could explore the integration of more complex coding schemes, such as Turbo and LDPC codes, which might offer even better performance in terms of error correction and data throughput. Additionally, experimenting with adaptive filtering techniques and machine learning algorithms for image reconstruction could further enhance the quality of received data in adverse communication environments.

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