# Modern Portfolio Theory

# & Capital Asset Pricing Model

# (Introduction)

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BMETE15MF78
github.com/fij/btup

# About Citi

| | | |
|---|---|---|
| CEO | Jane Fraser | 220 000+ colleagues globally |
| Managing Director,<br>Citi Country Officer, and<br>Banking Head for Hungary | Veronika Spanarova | 3 000+ in Budapest |

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# About myself

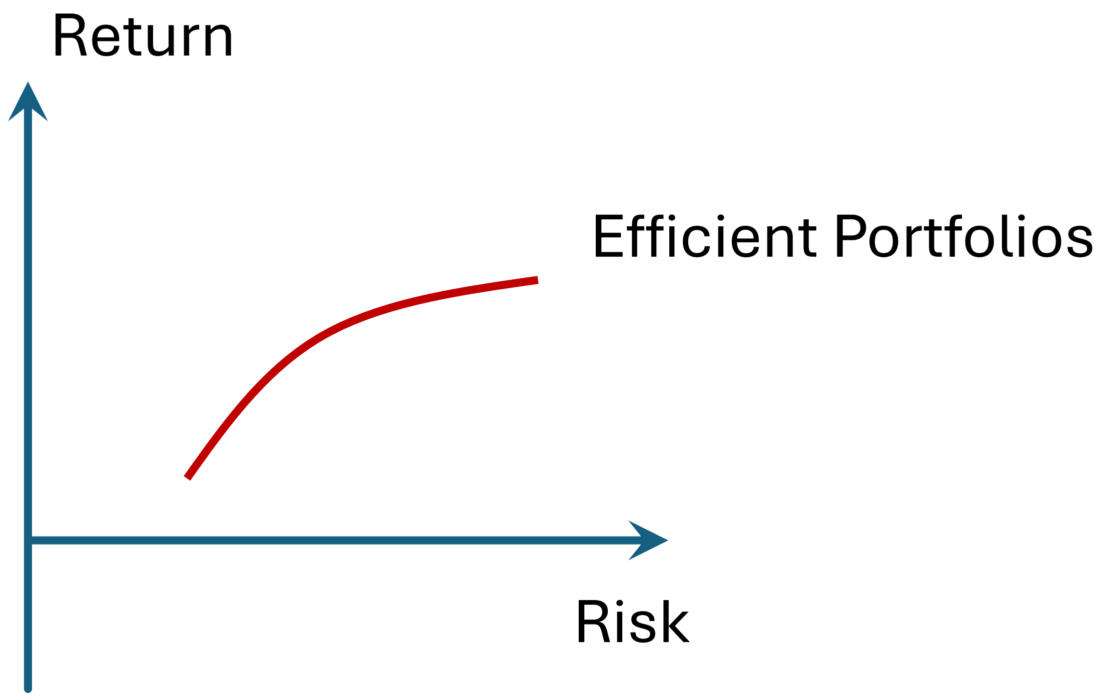
| | |
|---|---|
| 2004 | Physics PhD at Eotvos University, Advisor: Tamas Vicsek |
| – 2017 | Eotvos Univ. / Hung. Acad. of Sci.<br>Habilitation, DSc |
| 2017 – | Citi HU MQA (Markets Quantitative Analysis) |

# Agenda

A brief introduction to return, risk, and portfolios

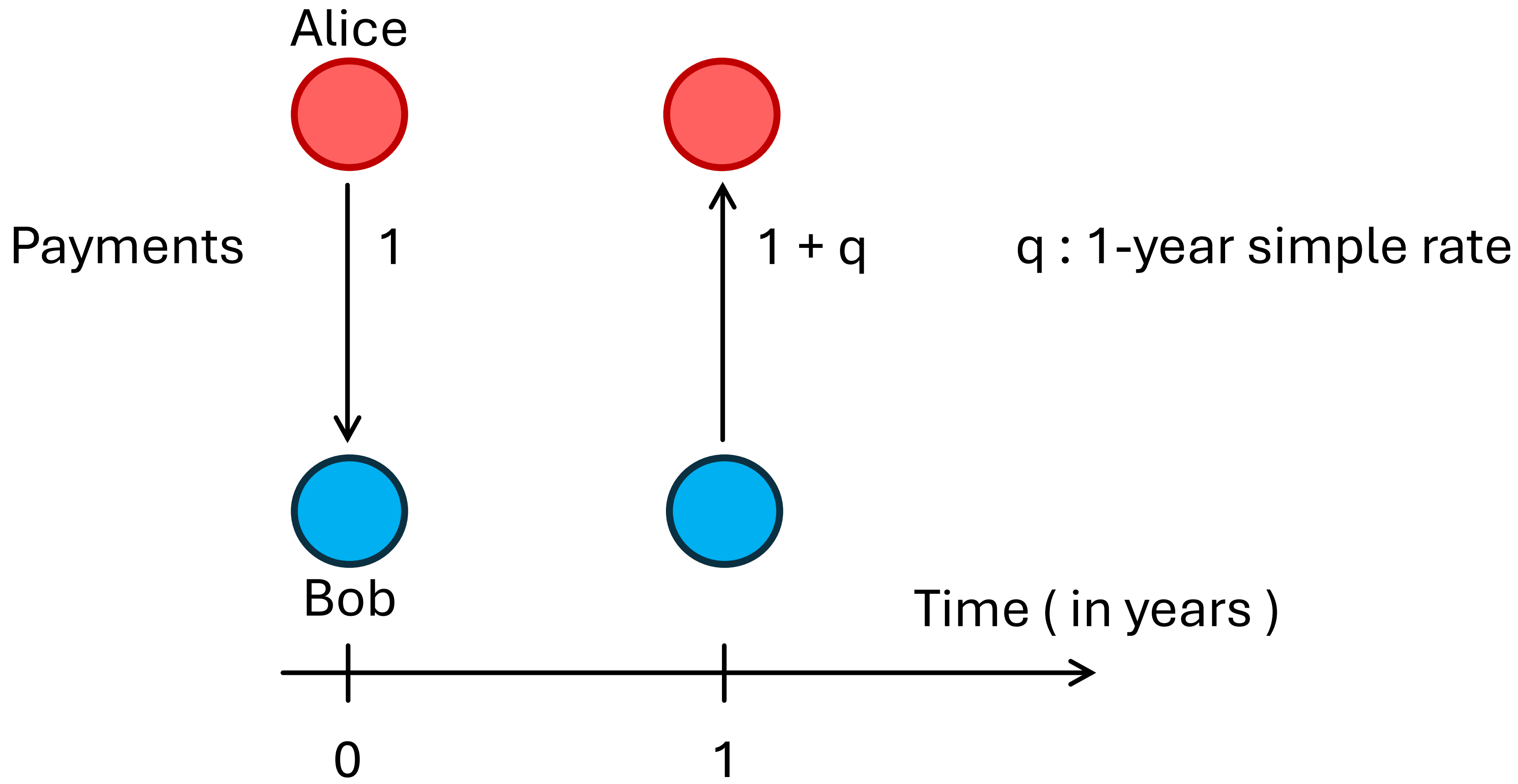
Modern Portfolio Theory (MPT)

Capital Asset Pricing Model (CAPM)



Introduction

**Return**



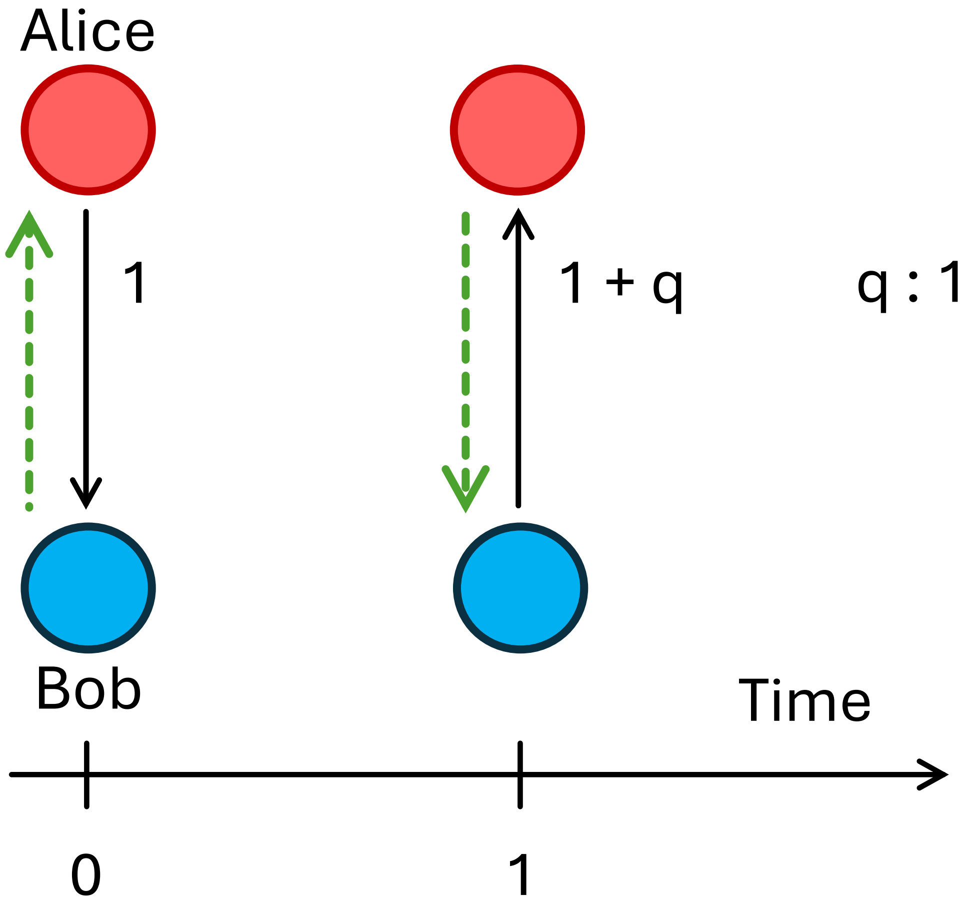
# Introduction

**Asset**

something that has value

Payments

Usually at both payments an asset is transferred in the opposite direction




q : 1-year simple rate

**Portfolio**

A linear combination of assets

For example, 1 unit of MSFT and 1 unit of AAPL

Normalization to sum of weights = 1 gives : 0.5 MSFT + 0.5 AAPL

# Introduction

**Risk**
( for a single asset )

Any contract contains numbers
that will be known only in the future

Risk quantifies the **uncertainty** of a **future** value,
for example, the future price of an asset

We don’t know the future,
but we can **forecast**

A simple forecast for the uncertainty of an asset’s
future price is the same asset’s **past** **volatility**

**Why forecast the return ?**

$q$ can be used for discounting

discounting can be applied to – for example – bond pricing

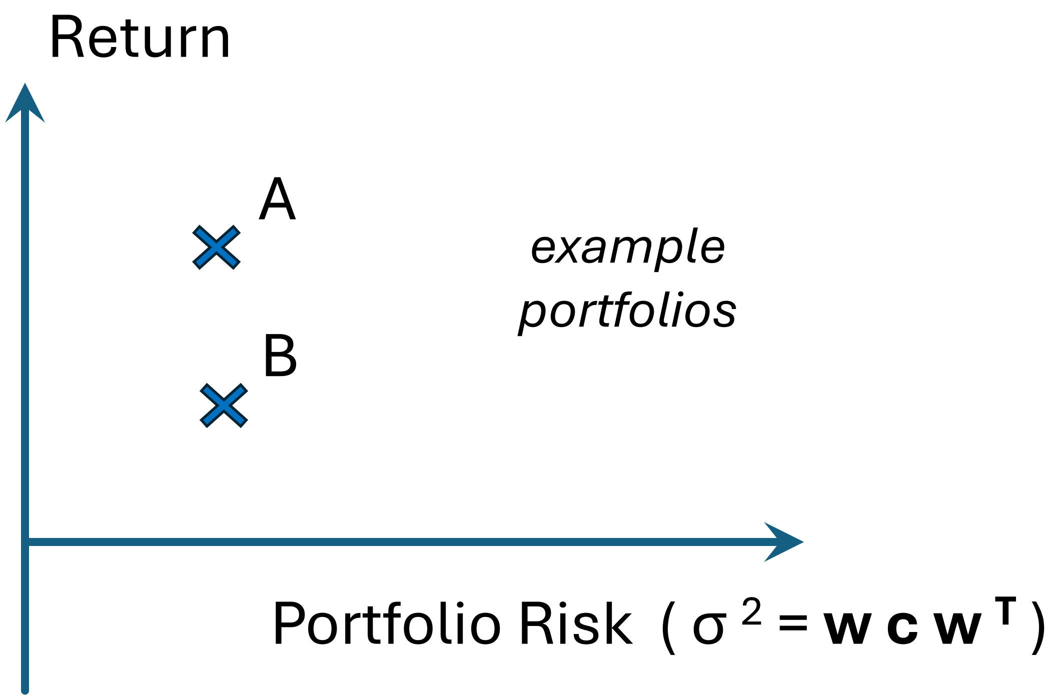
# Modern Portfolio Theory (MPT)

*N* **risky** assets :

**Variable** weights $w_i$ — **w** — row vectors
**Fixed** yearly returns $r_i$ — **r**
and covariance $w_{ij}$ — **c** — matrix

What is the max q ?
( q does have a max, for example, q ≤ max $r_i$ )

Equivalently : What is the min $\sigma^2$ ?



# What is the min $\sigma^2$ ?

Fixed: covariance matrix and return. Variable: weight vector.

Find min $\sigma^2$ with 2 constraints : sum of weights is 1 , portfolio return is q.

# Lagrange method

$$\searrow \quad \sigma^2(q) \longrightarrow q(\sigma^2) \xrightarrow[\text{Variable transformation}]{\text{Add risk-free asset}} q(\sigma)$$

Risky assets only

# What is the min $\sigma^2$ ?

Following the description of the task and the Lagrange method, let's apply two new scalar variables, $-\lambda_1$ and $-\lambda_q$ , and `minimize the Lagrange function` $\mathcal{L}\left(\mathbf{w},\, \lambda_1,\, \lambda_q\right) = \mathbf{w}\,\mathbf{c}\,\mathbf{w}^{\mathrm{T}} - \lambda_1\left(\mathbf{w}\,\mathbf{1}^{\mathrm{T}} - 1\right) - \lambda_q\left(\mathbf{w}\,\mathbf{r}^{\mathrm{T}} - q\right)$ .

The necessary $\mathbf{0} = \boldsymbol{\nabla}\mathcal{L}$ condition for a $\left(\mathbf{w},\, \lambda_1,\, \lambda_q\right)$ vector to be the location of a local minimum has three parts :

$$(1) \quad \mathbf{0} = \frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{c}\,\mathbf{w}^{\mathrm{T}} - \lambda_1\,\mathbf{1}^{\mathrm{T}} - \lambda_q\,\mathbf{r}^{\mathrm{T}}$$

$$(2) \quad 0 = \frac{\partial \mathcal{L}}{\partial \lambda_1} = 1 - \mathbf{w}\,\mathbf{1}^{\mathrm{T}}$$

$$(3) \quad 0 = \frac{\partial \mathcal{L}}{\partial \lambda_q} = q - \mathbf{w}\,\mathbf{r}^{\mathrm{T}}$$

Modern Portfolio Theory (MPT)

## What is the min $\sigma^2$ ? Equivalently : max q ?

$$A = \begin{pmatrix} \mathbf{1}\,\mathbf{c}^{-1}\,\mathbf{1}^{T} & \mathbf{r}\,\mathbf{c}^{-1}\,\mathbf{1}^{T} \\ \mathbf{r}\,\mathbf{c}^{-1}\,\mathbf{1}^{T} & \mathbf{r}\,\mathbf{c}^{-1}\,\mathbf{r}^{T} \end{pmatrix}$$

$$\frac{1}{K^2} = \frac{1}{A_{22} - (A_{12})^2 / A_{11}}\,, \quad q_0 = \frac{A_{12}}{A_{11}}\,, \quad \sigma_0^2 = \frac{1}{A_{11}}$$

$$\sigma^2(q) = \frac{1}{K^2}(q - q_0)^2 + \sigma_0^2$$

$$q(\sigma^2) = q_0 + K\sqrt{\sigma^2 - \sigma_0^2}$$

Efficient Frontier (EF) of the risky assets only

Numerical issues

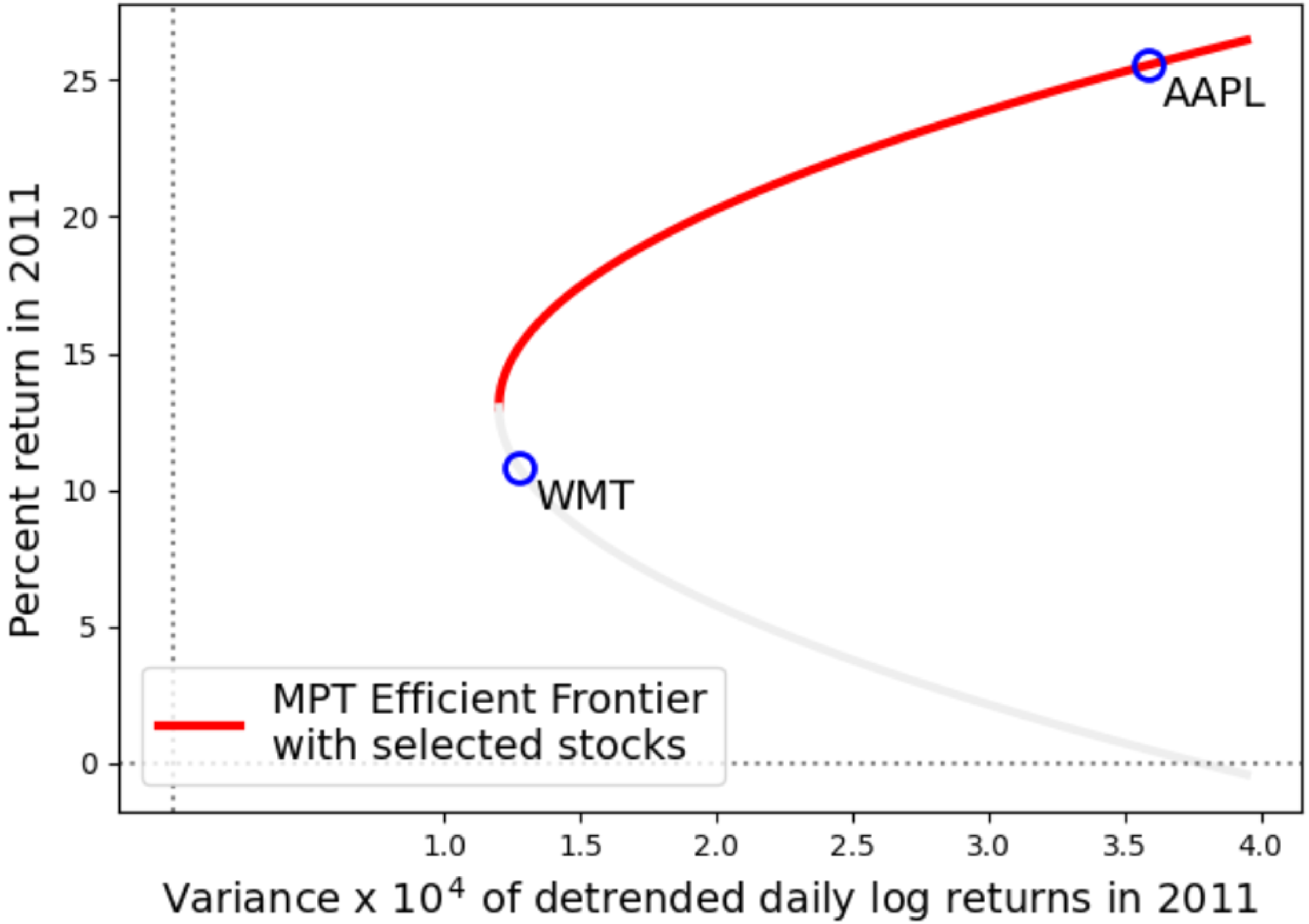
$\mathbf{c}^{-1}$

$\mathbf{A}^{-1}$

# Modern Portfolio Theory (MPT)

Example :

Efficient Frontier
of 2 risky assets

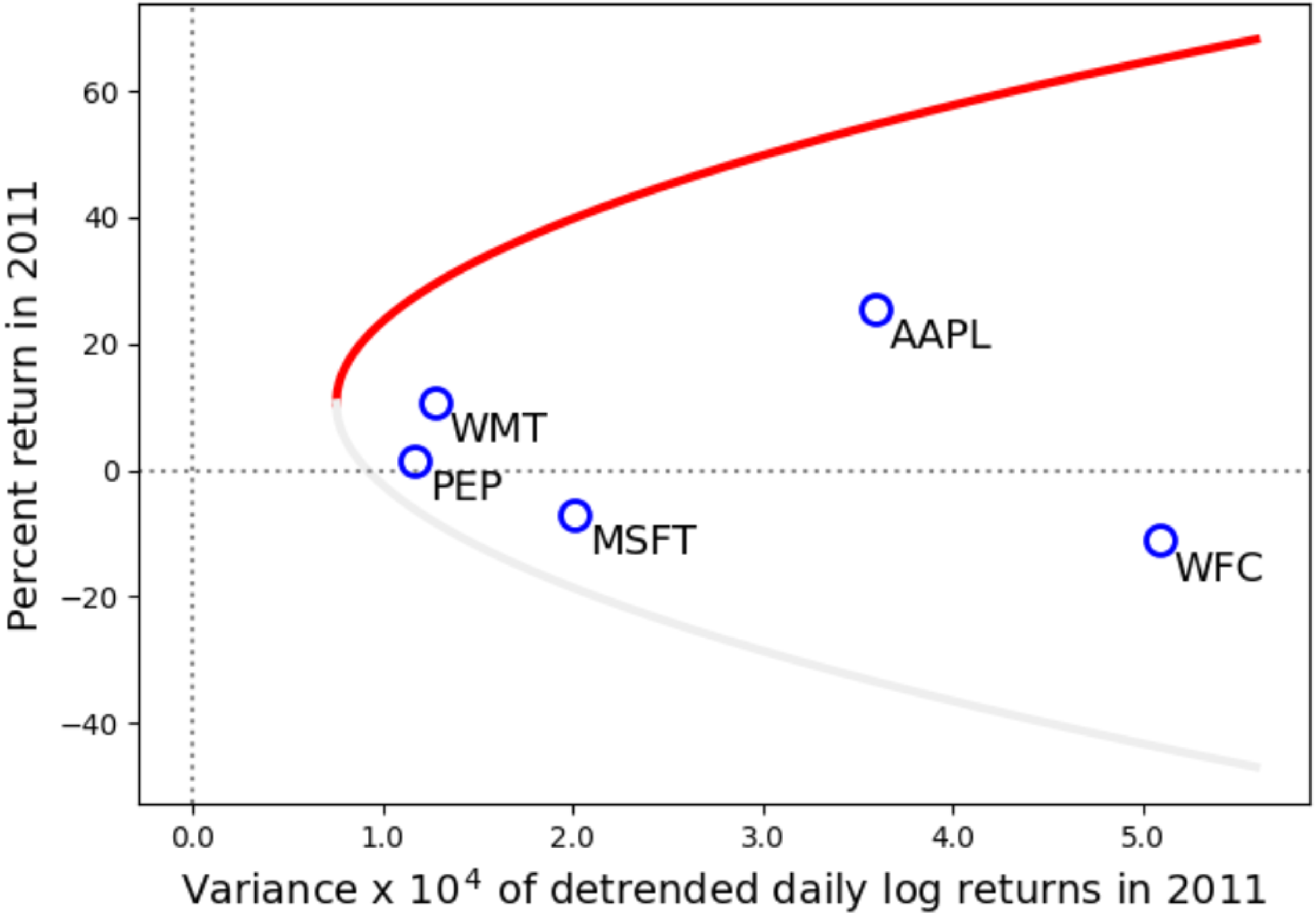



Data source:
`kaggle.com/dgawlik/nyse`

Modern Portfolio Theory (MPT)

Example :

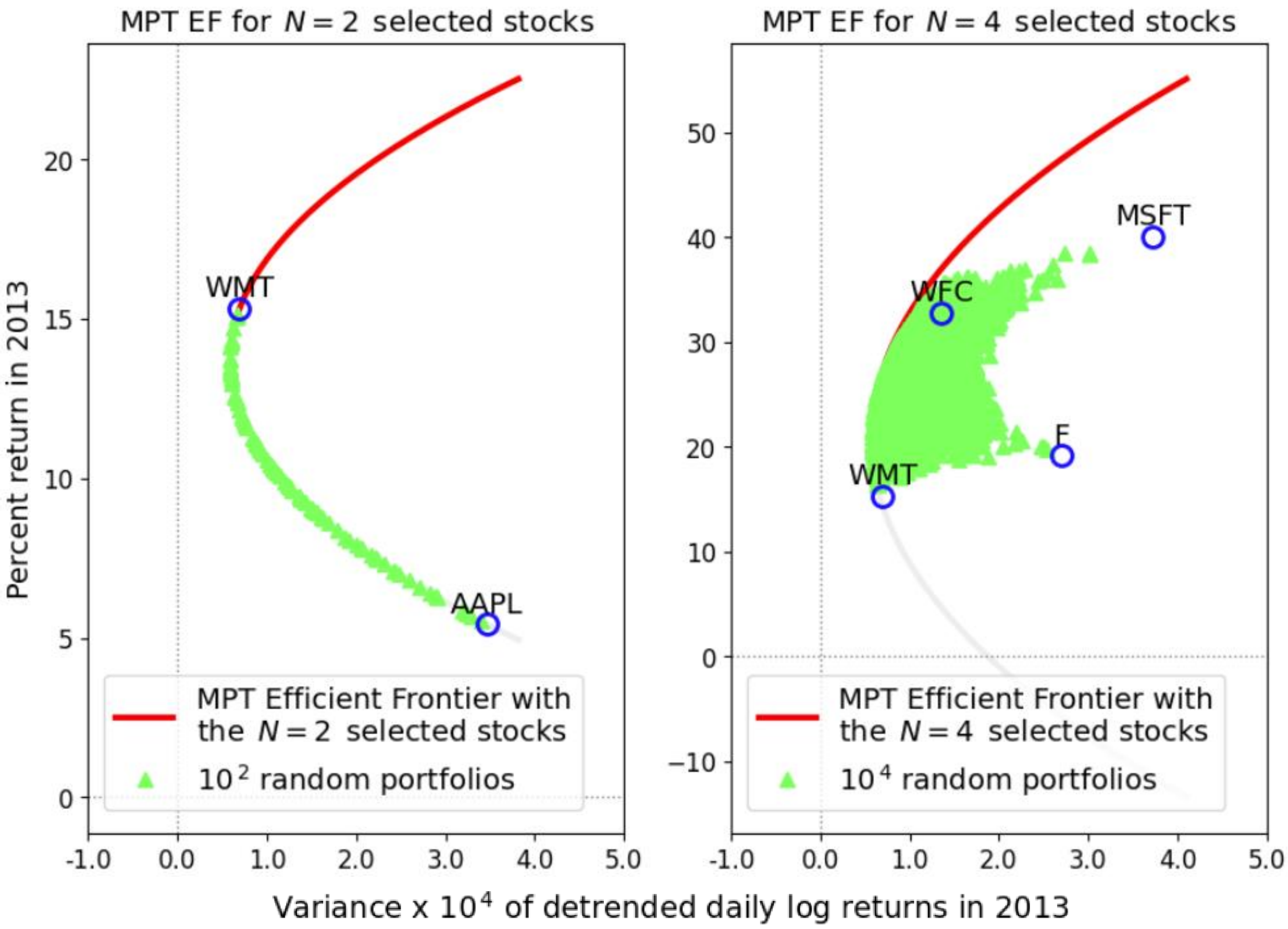
Efficient Frontier of 5 risky assets



# Modern Portfolio Theory (MPT)

Example :

Rnd portfolios
of risky assets



# Modern Portfolio Theory (MPT)

## Variable change : $\sigma^2 \rightarrow \sigma$

$$q(\sigma) = q_0 + K\sqrt{\sigma^2 - \sigma_0^2} \quad \text{with} \quad \sigma^2 \geq \sigma_0^2$$

$$\Downarrow$$

$$\frac{\sigma^2}{\sigma_0^2} - \frac{(q - q_0)^2}{K^2 \sigma_0^2} = 1$$

Recall that in the MPT only the $(\sigma > \sigma_0,\ q > q_0)$ quadrant of the hyperbola is used.

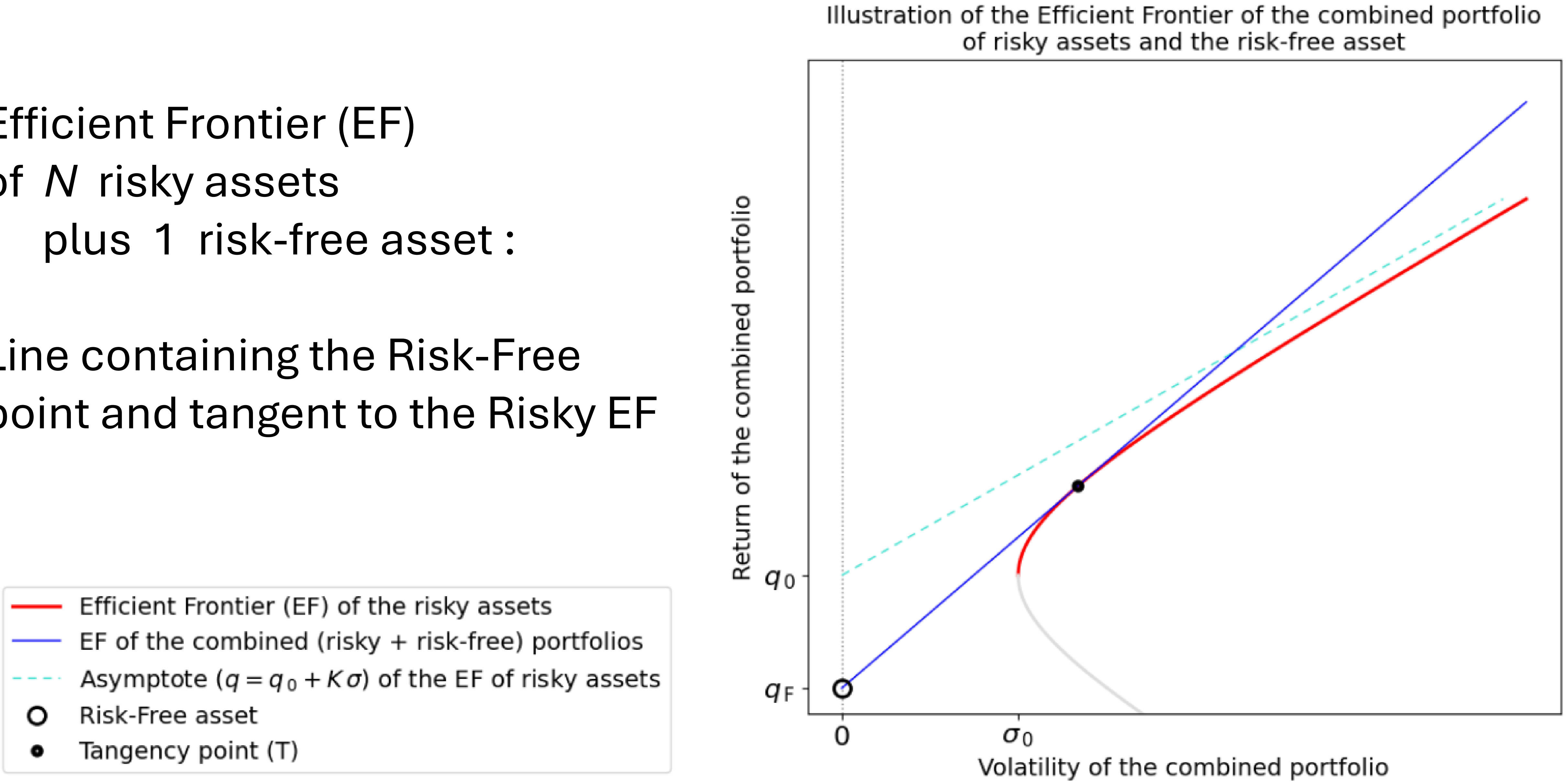
From the above standard equation form note that the `center of the hyperbola` is at $(\sigma = 0,\ q = q_0)$.

Take the $\sigma \rightarrow \infty$ limit (meaning also $q \rightarrow \infty$) to see that the `asymptote` of the hyperbola is $q = q_0 + K\sigma$.

Modern Portfolio Theory (MPT)

# Efficient Frontier (EF) of $N$ risky assets plus 1 risk-free asset :

## Line containing the Risk-Free point and tangent to the Risky EF



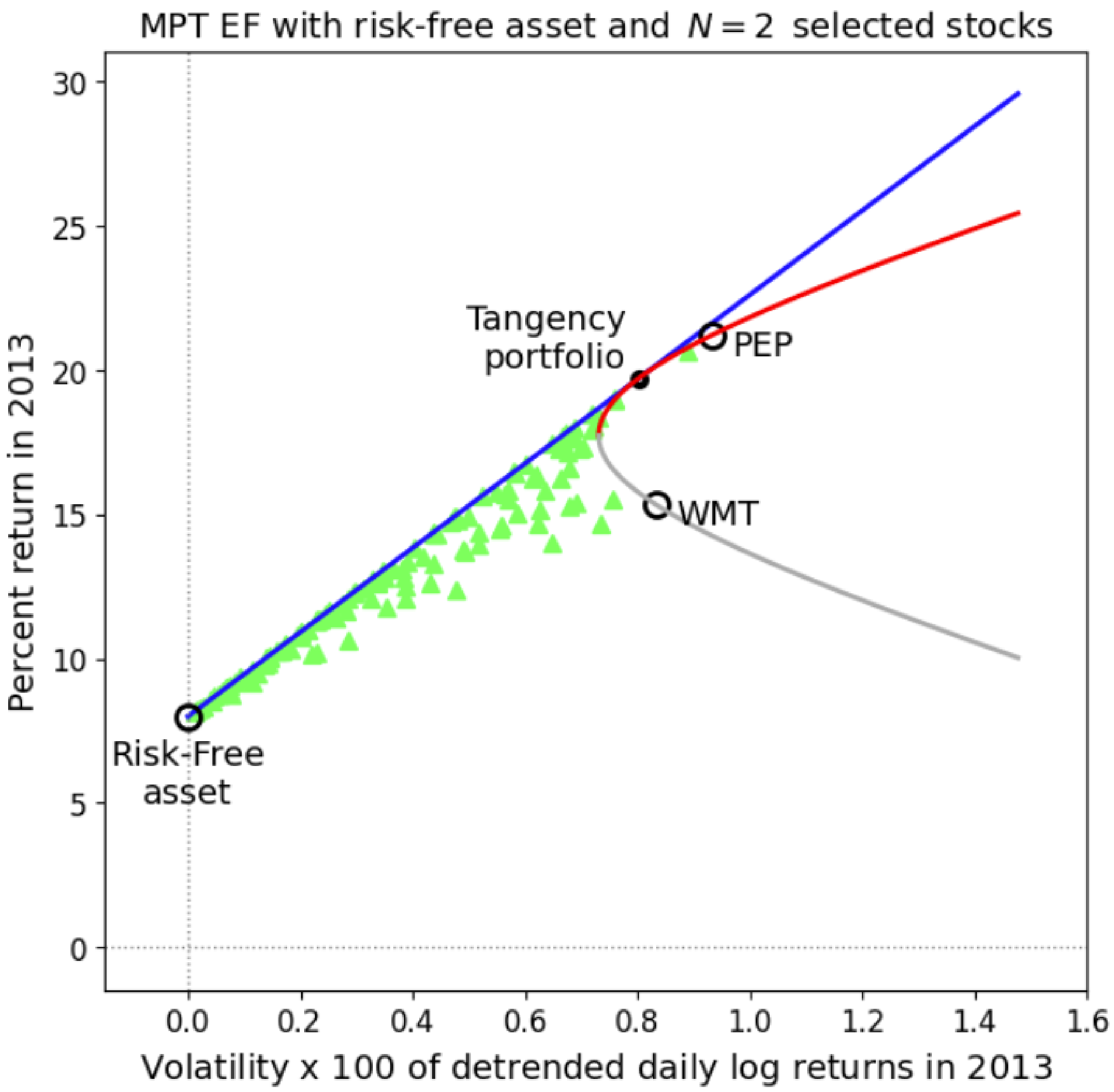
## Modern Portfolio Theory (MPT)

Example:

risk-free asset + 2 risky assets

Efficient Frontier

Random portfolios



# Capital Asset Pricing Model (CAPM)

| | | |
|---|---|---|
| MPT | $\sigma$ | full risk |
| CAPM | $\beta$ | non-diversifiable risk |

# Capital Asset Pricing Model (CAPM)

***What does the MPT do ?***

`Modern Portfolio Theory` (MPT) uses the Efficient Market Hypothesis (EMH) to calculate
(i) the Efficient Frontier (EF) of portfolios containing only risky assets
(ii) and the Capital Market Line (CML), which is the EF of the combined (risk-free + risky) portfolios.

The EF of risky assets and the CML are tangent at the `tangency point`.

The coordinates of the tangency point are the volatility and return of the `tangency portfolio`.

The tangency portfolio has no risk-free asset. It has relative risky asset weights equal to those of the market portfolio.

The Capital Market Line (CML) compares return with the `full risk`, which is quantified as the volatility.

***What does the CAPM do ?***

The `Capital Asset Pricing Model` was developed in the 1960s by Jack Treynor and others based on the MPT.

The CAPM compares a risky portfolio's return with the amount of its `non-diversifiable risk` (also called: systematic risk).

*Notes*

1. This non-diversifiable risk will quantified as the $\beta$ value defined below.
2. Diversifiable risk is also called specific risk, or unsystematic risk, or idiosyncratic risk.

# Capital Asset Pricing Model (CAPM)

## β of a portfolio P

(P can be a single asset)

## Equivalent definitions

(M is the market portfolio)

$$\beta_P = \frac{Cov\,(\,P, M\,)}{Var\,(\,M\,)}$$

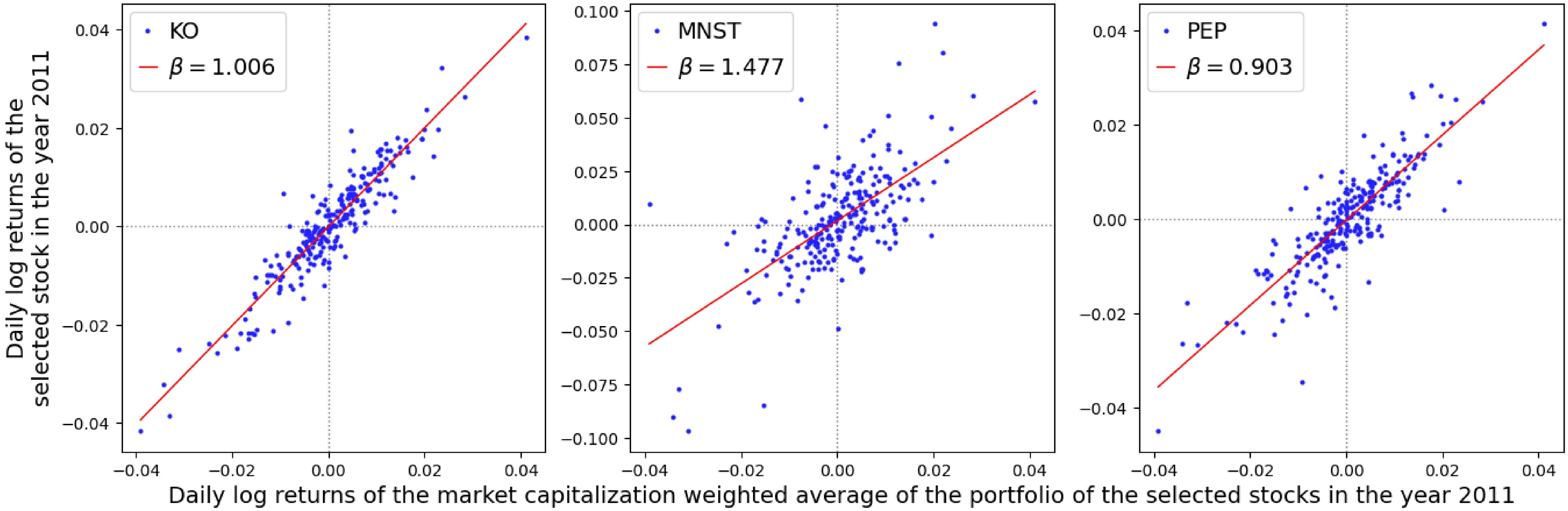
$\beta_P$ is the slope of the linear fit to the scatter plot displaying P (vertical coordinate) vs M (horizontal)

$\beta_P$ is the non-diversifiable (also called: systematic) risk of the portfolio P

# Capital Asset Pricing Model (CAPM)

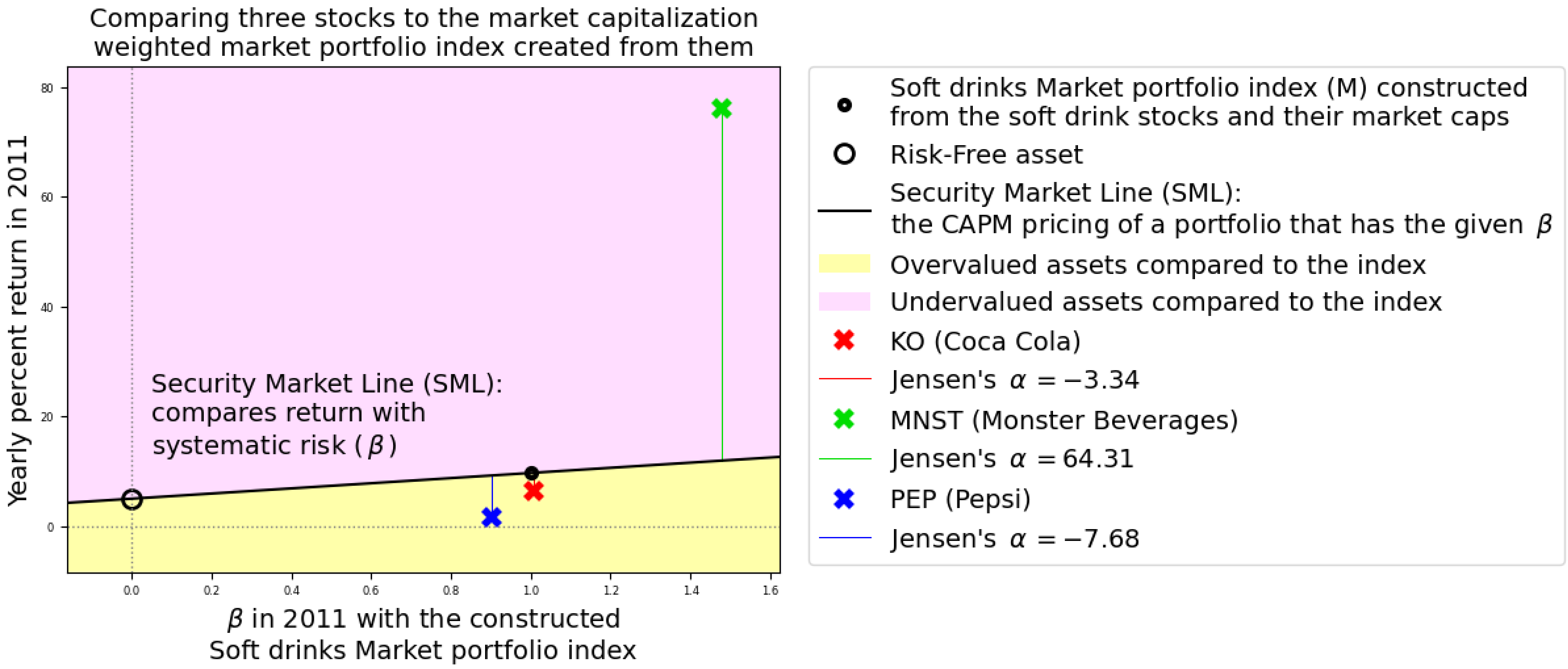
## Example: Market is 3 stocks

### Pepsi, Coke, Monster Beverages



# Capital Asset Pricing Model (CAPM)

## Example: Market is 3 stocks



# Thank you

Questions