**Sampling Mean and Variance**

**General Concept of Sampling Mean and Variance**

1. **Sampling Mean**:
   * The sampling mean (Xˉ\bar{X}) is the average of observations in a sample drawn from a population.
   * Formula: Xˉ=1n∑i=1nXi\bar{X} = \frac{1}{n} \sum\_{i=1}^n X\_i where XiX\_i are the sample observations and nn is the sample size.
   * The sampling mean is a random variable itself, derived from the population's mean.
2. **Sampling Variance**:
   * Variance measures the dispersion of the sample data from its mean.
   * Two types:
     + **Corrected Sampling Variance (Sample Variance)**: S2=1n−1∑i=1n(Xi−Xˉ)2S^2 = \frac{1}{n-1} \sum\_{i=1}^n (X\_i - \bar{X})^2 Dividing by n−1n-1 corrects the bias, making it an unbiased estimator of the population variance.
     + **Uncorrected Sampling Variance**: S2=1n∑i=1n(Xi−Xˉ)2S^2 = \frac{1}{n} \sum\_{i=1}^n (X\_i - \bar{X})^2 Used primarily for populations.
3. **Features of Their Distributions**:
   * **Sampling Mean**:
     + Follows a normal distribution (Central Limit Theorem) for sufficiently large nn, even if the population distribution is not normal.
     + Mean of the sampling distribution: μXˉ=μ\mu\_{\bar{X}} = \mu (population mean).
     + Variance of the sampling mean: σXˉ2=σ2n\sigma^2\_{\bar{X}} = \frac{\sigma^2}{n} (population variance divided by sample size).
   * **Sampling Variance**:
     + Follows a chi-square distribution with n−1n-1 degrees of freedom.
     + The mean of the sampling variance: Equal to the population variance.
     + Variance of the sampling variance: Depends on the fourth central moment of the population distribution.

**Lebesgue–Stieltjes Integration**

**General Idea**

* The Lebesgue–Stieltjes integral extends the concept of integration beyond the traditional Riemann integral.
* It's particularly useful in **probability theory** and **measure theory**, dealing with cases where the distribution of a random variable is not necessarily continuous.

**Definition:**

* The integral of a function ff with respect to a function gg is given by: ∫abf(x) dg(x)\int\_a^b f(x) \, dg(x) where g(x)g(x) is a **monotonic function**, often related to the cumulative distribution function (CDF) in probability theory.

**Applications in Probability Theory:**

1. **Expected Value (Discrete and Continuous Distributions)**:
   * For a random variable XX with CDF F(x)F(x), the expected value can be expressed as: E[X]=∫−∞∞x dF(x)\mathbb{E}[X] = \int\_{-\infty}^{\infty} x \, dF(x)
   * This unifies the treatment of discrete and continuous random variables.
2. **Cumulative Distributions**:
   * Lebesgue–Stieltjes integration provides a mathematical framework to work with cumulative distribution functions (CDFs), even for mixed or step distributions.
3. **Change of Measure**:
   * Widely used in stochastic processes and financial mathematics (e.g., Girsanov's theorem), where measures are transformed via integrals.

**Applications in Measure Theory:**

1. **Probability Spaces**:
   * The integral allows defining expectations and variances directly from the probability measure, unifying discrete, continuous, and mixed distributions.
2. **Lebesgue Measure**:
   * Helps in defining probability measures for continuous distributions (e.g., Gaussian).
3. **Handling Singular Functions**:
   * Can integrate functions with respect to singular measures, such as Cantor's function.

**Practical Insights:**

* **Why Important for Probability?**:
  + Allows modeling of real-world random variables that may not follow simple continuous or discrete patterns.
* **Why Important for Measure Theory?**:
  + Provides a foundation for modern probability theory and complex integrations needed in advanced mathematical fields like ergodic theory and functional analysis.

A graph on a white background

Description automatically generated

This output represents a numerical evaluation of the **Riemann** and **Lebesgue integrals** for a specified function f(x)f(x), over the interval [0, 1].

**Key Points:**

1. **Function**:
   * The plot illustrates the **standard normal distribution** f(x)=12πe−x2/2f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, or a similar continuous function, based on your configuration.
2. **Interval**:
   * The integration is performed over the interval **[0, 1]**.
3. **Riemann Integral**:
   * The **Riemann integral** approximates the area under f(x)f(x) by dividing the interval into subintervals, calculating function values at specific points (e.g., midpoints), and summing up rectangles.
   * Result: **0.1558**
4. **Lebesgue Integral**:
   * The **Lebesgue integral** computes the same area by considering the range (values of f(x)f(x)) and summing contributions from sets of xx-values that correspond to specific function ranges.
   * Result: **0.1558**
5. **Comparison**:
   * Both integration methods produce identical results, confirming that for this function and interval, the **Riemann** and **Lebesgue** integrals are consistent, as expected theoretically for well-behaved functions.
6. **Plot**:
   * The blue curve visually confirms the function f(x)f(x). The flatness towards x=1x=1 reflects the decrease in the function value.

This validates the numerical implementation of both integration methods and shows their equivalence for continuous and integrable functions over a closed interval.