Coursework DES

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The system that will be analysed is composed by a robot that can move in a specific environment (possible case of path planning, delegating all the control theory stuff to the single state in an hypotetic hybrid system).

1 Modeling the map

Since map and transitions between rooms are provided , we can rapidly derive our finite deterministic automaton (FDA)[1]:

[1] We use the following online tool to rapidly generate an SVG file of the automaton; however, due to his intrinsic limitation, the initial state will be represented by a single arrow that doesn't start from any state.

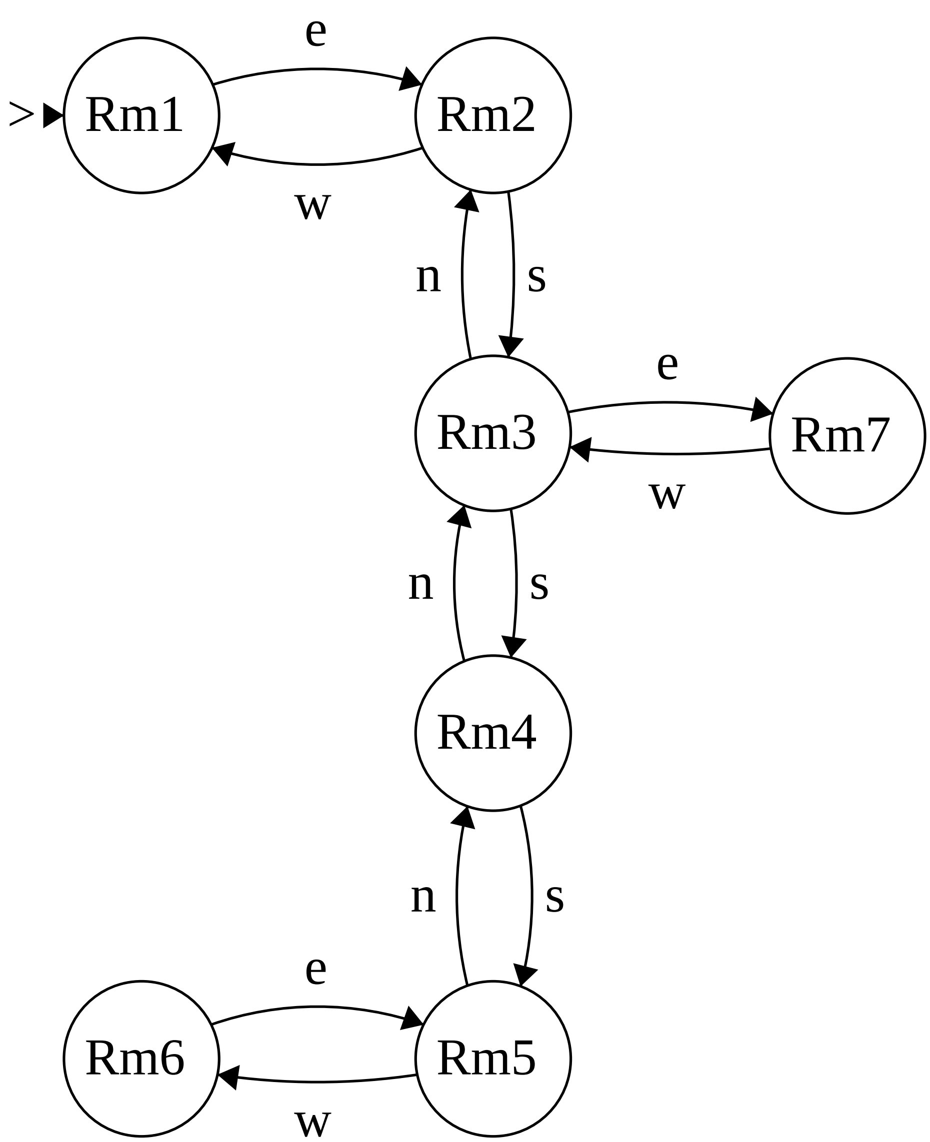


Figure 1: G_M automaton.

The system has two variables with a second order derivative, so we can define the state space $x \in X \subseteq \mathbb{R}^4$ as:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ r \\ \dot{r} \end{bmatrix} \implies \dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{r} \\ \ddot{r} \end{bmatrix}$$

Rearranging the equation in a convenient form, we get:

$$\begin{cases} \ddot{\theta}(t) = \frac{u_1(t) - 2r(t)\dot{r}(t)\dot{\theta}(t)}{(r^2(t)+1)} \\ \ddot{r}(t) = u_2(t) + r(t)\dot{\theta}^2(t) \\ \quad y(t) = \begin{bmatrix} \dot{\theta}(t) \\ \dot{r}(t) \end{bmatrix} \end{cases} \implies \dot{x} = \begin{bmatrix} x_2 \\ \frac{u_1 - 2x_3x_4x_2}{(x_3^2+1)} \\ x_4 \\ u_2 + x_3x_2^2 \end{bmatrix}, y = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

Furthermore, looking at the equation, we can identify a *control affine* form so, arranging further the form of our equations and introducing a matrix $G(x) \in \mathbb{R}^{n \times m}$, we can write:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-2x_3x_4x_2}{(x_3^2+1)} \\ x_4 \\ x_3x_2^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{(x_3^2+1)} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

This form is going to be useful in the next point of the coursework.

2 Show the system is passive from u to y

Since the *storage function* must be $S(x) \geq 0$ for passivity, as suggested by the assignment we will use:

$$V(x) = \frac{1}{2}(r^2+1)\dot{\theta}^2 + \frac{1}{2}\dot{r}^2 \Rightarrow \frac{1}{2}(x_3^2+1)x_2^2 + \frac{1}{2}x_4^2$$

which is a semi-definite positive function ($V(\underline{x}) = 0, \underline{x} \neq 0$, and in particular $x_{2,4} = 0 \wedge x_{1,3} \in \mathbb{R}$). So the system is *passive* iff:

$$\begin{cases} \frac{\partial V(x)}{\partial x} \cdot f(x) \leq 0 \\ \frac{\partial V(x)}{\partial x} \cdot G(x) = h(x)^T \end{cases} \rightarrow \begin{cases} \begin{bmatrix} 0 & x_2(x_3^2+1) & x_3x_2^2 & x_4 \end{bmatrix} \begin{bmatrix} x_2 \\ \frac{-2x_3x_4x_2}{(x_3^2+1)} \\ x_4 \\ x_3x_2^2 \end{bmatrix} \leq 0 \\ \begin{bmatrix} 0 & x_2(x_3^2+1) & x_3x_2^2 & x_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{(x_3^2+1)} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_2 & x_4 \end{bmatrix} \end{cases}$$

Carrying out all the calculations, we get:

$$\begin{cases} -2x_3x_4x_2^2 + x_3x_2^2x_4 + x_4x_3x_2^2 = 0 \leq 0 \\ \begin{bmatrix} x_2 & x_4 \end{bmatrix} = \begin{bmatrix} x_2 & x_4 \end{bmatrix} \end{cases}$$

thus verifying the property.

3 Show the overall feedback system being GAS

Rewriting the input according to state space representation:

$$u_f = \begin{cases} u_1 = -\dot{\theta} - (\theta - \theta_d) \\ u_2 = -\dot{r} - (r - r_d) \end{cases} \rightarrow \begin{cases} u_1 = -x_2 - (x_1 - \theta_d) \\ u_2 = -x_4 - (x_3 - r_d) \end{cases} \xrightarrow[\theta_d=0, r_d=0]{} \begin{cases} u_1 = -x_2 - x_1 \\ u_2 = -x_4 - x_3 \end{cases}$$

Leads the expression $\dot{x} = f(x, u_f)$ to be:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-x_2 - x_1 - 2x_3x_4x_2}{(x_3^2+1)} \\ x_4 \\ -x_4 - x_3 + x_3x_2^2 \end{bmatrix}$$

Again, as suggested by the assignment, we can modify the Lyapunov function adding the potential energy (that is expression of $\frac{1}{2}\theta^2$ and $\frac{1}{2}r^2$, function of position), so that it becomes:

$$\widetilde{V}(x) = \frac{1}{2}(x_3^2 + 1)x_2^2 + \frac{1}{2}x_4^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_3^2$$

Note that in this way, the $\widetilde{V}(x)$ is positive *definite* ($\widetilde{V}(x) = 0 \Leftrightarrow x = 0$), and radially unbounded. By calculating the function:

$$\dot{\widetilde{V}}(x) = \frac{\partial V(x)}{\partial x} \cdot f(x) = \begin{bmatrix} x_1 & x_2(x_3^2+1) & x_3(x_2^2+1) & x_4 \end{bmatrix} \begin{bmatrix} x_2 \\ \frac{-x_2 - x_1 - 2x_3x_4x_2}{(x_3^2+1)} \\ x_4 \\ -x_4 - x_3 + x_3x_2^2 \end{bmatrix} =$$

$$= x_1x_2 - x_2^2 - x_1x_2 - 2x_3x_4x_2^2 + x_3x_2^2x_4 + x_3x_4 - x_4^2 - x_4x_3 + x_4x_3x_2^2 =$$

$$= -x_2^2 - x_4^2 \leq -(\delta_1 + \epsilon_2)\begin{bmatrix} x_2 + x_1 & x_4 + x_3 \end{bmatrix} \begin{bmatrix} x_2 + x_1 \\ x_4 + x_3 \end{bmatrix} - (\delta_2 + \epsilon_1)\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}\begin{bmatrix} x_2 & x_4 \end{bmatrix} \leq 0$$

At this point we can make some considerations about the feedback system; in fact it can be rewritten as a *MIMO linear system*, where:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

however, since the variables are completely decoupled, it can be analyzed as two parallel SISO systems (with regard to θ and r respectively), with

$$A_1 = 0, B_1 = 1, C_1 = 1, D_1 = 1$$

as matrices and where each one is *strictly input passive*, as their transfer function $G(s) = C(sI - A)^{-1}B + D$ verify the propriety $Re\{G(s)\} = 1 > 0, \forall s$.

For this reason, we can choose as coefficients:

$$\delta_1 = \epsilon_2 = \epsilon_1 = 0, \delta_2 \in (0, 1]$$

thus proving that $\dot{\tilde{V}}(x)$ is semi-definite negative (x_1 and x_3 could assume any value).
To prove Global Asymptotic Stability, we can use LaSalle invariance criterion, verifying that $\{0\}$ is the largest invariant set contained in $Ker\left\{\dot{\tilde{V}}\right\} := K_0$.

But $Ker\left\{\dot{\tilde{V}}\right\} = \left\{x : -x_2^2 - x_4^2 = 0\right\}$ can be seen as

$$-\begin{bmatrix} x_2 & x_4 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = -\left\|y_1\right\|^2 = 0$$

and, by the vector norm propriety, $\left\|y_1\right\|^2 = 0 \Leftrightarrow y_1 = 0$. For this reason, we can work with a simplified expression $K_0 = \left\{\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \{x_2 = 0 \wedge x_4 = 0\} = \mathbb{R}^2$ ($x_1 x_3$ plane). Using Lie derivatives, it is possible to calculate:

$$K_1 = \left\{ x : K_0 \wedge \mathcal{L}_f^1 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \frac{\partial \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}{\partial x} f(x) = 0 \right\} = \left\{ K_0 \wedge \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ \frac{-x_2 - x_1 - 2x_3 x_4 x_2}{(x_3^2+1)} \\ x_4 \\ -x_4 - x_3 + x_3 x_2^2 \end{bmatrix} = 0 \right\}$$
$$= \left\{ K_0 \wedge \begin{bmatrix} \frac{-x_2 - x_1 - 2x_3 x_4 x_2}{(x_3^2+1)} \\ -x_4 - x_3 + x_3 x_2^2 \end{bmatrix} = 0 \right\}$$

and using the relations found in the previous iteration, we could simplify the expression:

$$\left\{ K_0 \wedge \begin{bmatrix} \frac{-x_1}{(x_3^2+1)} \\ -x_3 \end{bmatrix} = 0 \right\} = \left\{ K_0 \wedge \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = 0 \right\} = \{x_2 = 0 \wedge x_4 = 0 \wedge x_1 = 0 \wedge x_3 = 0\} = \{0\} = K_1$$

So the largest invariant set is $\Omega \subset K_1 = \{0\} \Rightarrow \{0\}$ is GAS.

4 Simulate the system for different constant values of r_d and θ_d

In order to show the results achieved in a way conceptually similar to reality, the simulation has been carried out in Simulink. The built model is the following one, where:

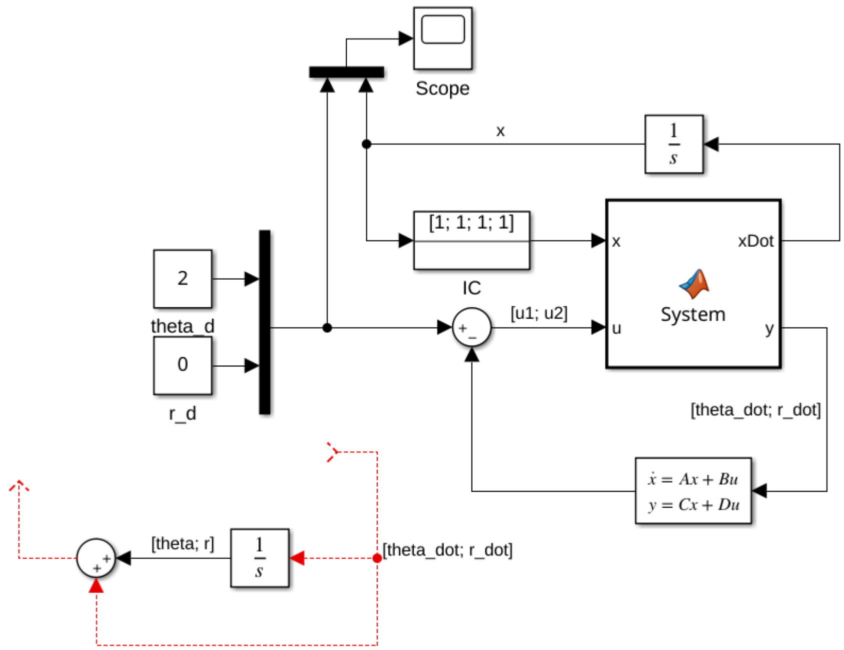


Figure 2: Simulink Model.

- the System block is described by the following Matlab function:

```
function [xDot,y] = System(x,u)
    xDot=zeros(4,1); % defining size of xDot
                     % (required for simulation)
    xDot(1) = x(2);
    xDot(2) = (u(1)-2*x(3)*x(4)*x(2))/((x(3)^2)+1);
    xDot(3) = x(4);
    xDot(4) = u(2)+x(3)*(x(2)^2);

    y=zeros(2,1); % defining size of y
    y(1)=x(2);
    y(2)=x(4);
end
```

- the IC block allows to set a specific initial condition (in our case, it's sufficient to not be the origin, since it is an equilibrium).
- the feedback loop is built from the output y and accordingly to the equation given by the assignment.
- the scope is attached to the state, showing that all its components converge to a specific value.

Another way to describe the system is through a Matlab function that would be integrated through ode45:

```
function [xDot] = sys(t,x)
    global theta_d r_d;

    u(1)=-x(2)-( x(1) - theta_d );
    u(2)=-x(4)-( x(3) - r_d );

    xDot(1,1) = x(2);
    xDot(2,1) = (u(1)-2*x(3)*x(4)*x(2))/((x(3)^2)+1);
    xDot(3,1) = x(4);
    xDot(4,1) = u(2)+x(3)*(x(2)^2);
end
```

where the global variables are set into an external script to be easily accessible and editable.

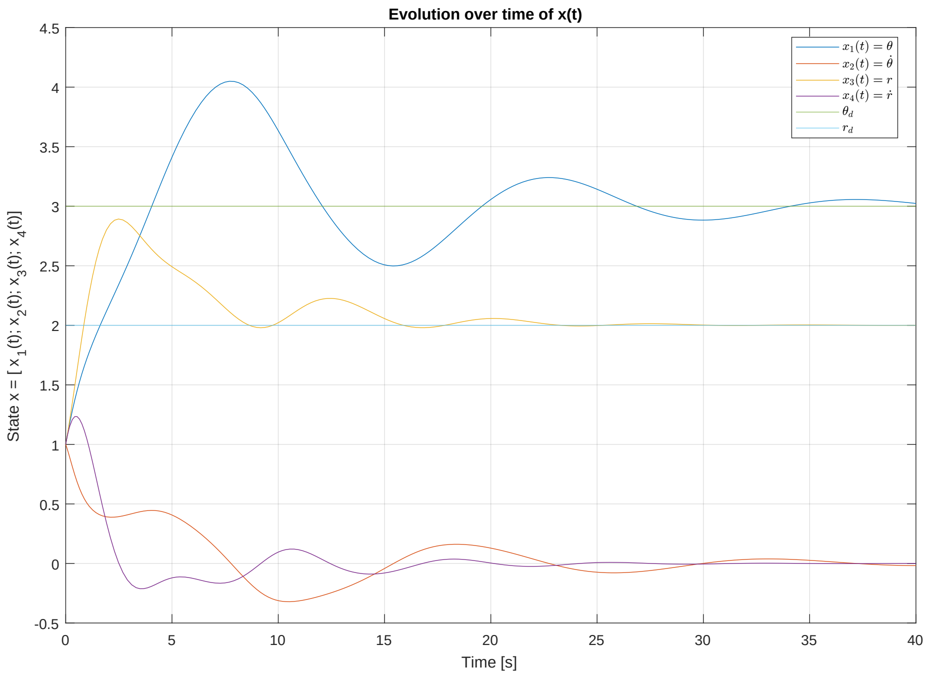


Figure 3: Simulation for $\theta_d = 3, r_d = 2$, Matlab Figure.

In the simulations - we refer to the first plot due to its smaller scale and better resolution -, once the θ_d and r_d value are fixed and the system begins to evolve, as we can see the x_1 and x_3 components of the state (θ and r respectively) converge to the reference values, while x_2 and x_4 (that are $\dot{\theta}$ and $\dot{r}$) reduce their amplitude towards zero as the state approaches the asymptotic value.

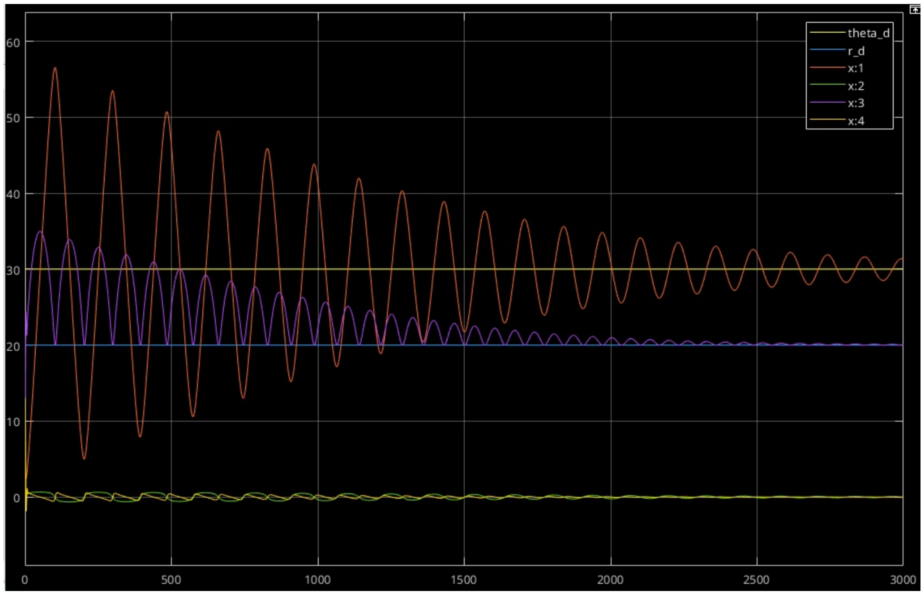


Figure 4: Simulation for $\theta_d = 30, r_d = 20$, Simulink scope.

With higher reference value, the overshoot of both the components of the state increases, but eventually, after a long transient, the state converges to:

$$x_\infty = \begin{bmatrix} \theta_d & 0 & r_d & 0 \end{bmatrix}^T$$

In conclusion, we can claim that the controller achieves *asymptotic tracking for constant values*, with the special case of $\theta_d = 0, r_d = 0$, where the converging point is the origin.

5 Prove that the closed-loop system is not ISS

5.1 Build an appropriate input signal

Unfortunately, this point was not really clear for me. Due to the hint to use the Lyapunov function and the form of $k(x(t)) : \mathbb{R}^4 \longrightarrow \mathbb{R}$ for the disturbance, I initially thought that the function was exactly $\widetilde{V}(x)$, or a similar function; however, this cannot be possible, since a sign definite function would simple lead to a constant value tracking.

On the other hand, since the idea is to maximize the power, we could touch the variables that are involved with velocity (x_2 or x_4), and since $\theta_d$appears in x_2, and according to the simulation for constant values, the best approach would be to invert the input signal every time the x_2 crosses zero. As we would expect, x_1 has an oscillatory unbounded solution (it is directly influenced by x_2, so that each time the solution changes direction, it is accelerated by the new value of derivative). Moreover, the disturbance should have a rather large constant compared to initial condition, being the only positive term for $\dot{x_2}$.

Another consideration could be done using the derivative of Lyapunov function with this specific disturbance. In fact, doing some calculations, for $\theta_d = c \cdot sign\left(k\left(x\left(t\right)\right)\right), r_d = 0$, we obtain:

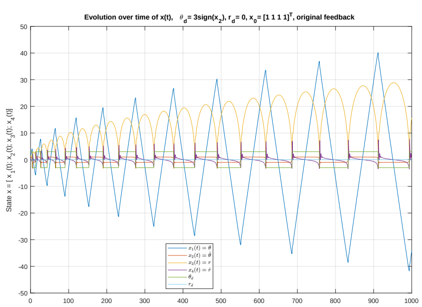
$$\dot{\tilde{V}}(x,d) = -x_2^2 - x_4^2 + x_2 \cdot c \cdot sign\left(k\left(x\left(t\right)\right)\right)$$

and choosing $k\left(x\left(t\right)\right) = x_2$, we would get:

$$\dot{\tilde{V}}(x,d) = -x_2^2 - x_4^2 + |x_2| \cdot c$$

that not only is not definite negative, it is also not sign defined for $c > 0$. Even with some manipulations (rewriting $\dot{\tilde{V}}(x,d)$ as $-\frac{x_2^2}{2} - x_4^2 + \frac{\theta_d^2}{2}$, or find $\chi\left(|\theta_d|\right) = 2\left|\theta_d\right| = 2\left|c \cdot sign\left(k\left(x\left(t\right)\right)\right)\right| = 2\left|c\right|$) we cannot use this Lyapunov function as an ISS one of either types; another proof is given by Theorem 19 on Lecture Notes.

5.2 Show the resulting simulation



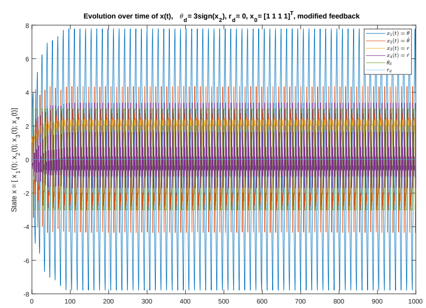


Figure 5: Simulation with $\theta_d = 3sign\left(x_2\right), r_d = 0$ and $x_0 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$ for both original and modified feedback.

As noticeable by the figure and by their description, we have chosen the same exact conditions, while using different feedback, where the second one is the same used in the next point analysis. In the first one, both x_1 and x_3 are diverging and, while x_2 becomes somewhat the same at each cycle of its waveform, x_4 continues to increase its peak, pushing higher x_3 and consequently x_1.

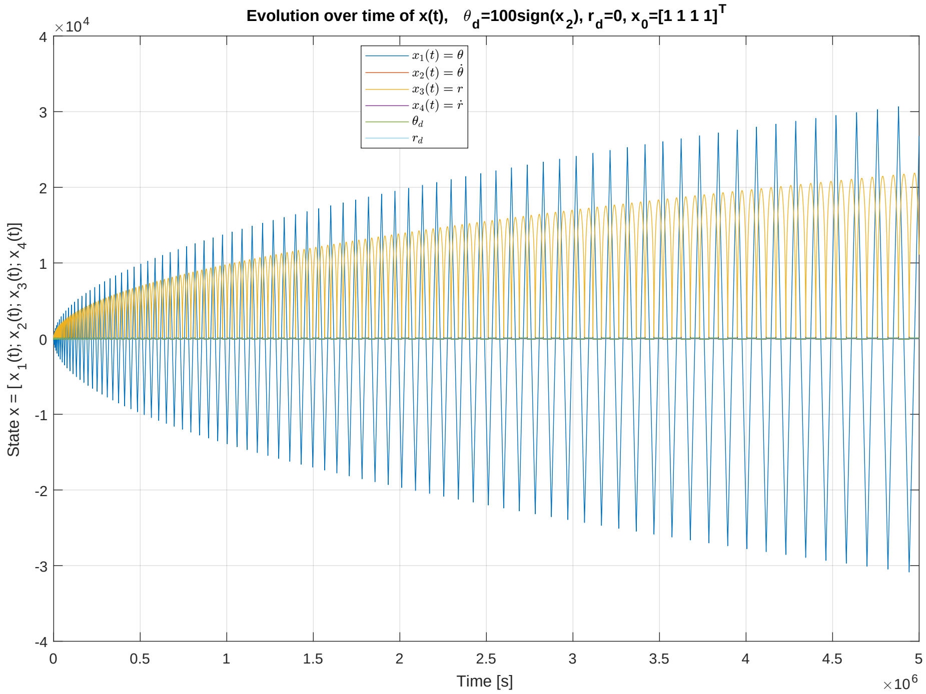


Figure 6: Simulation with $\theta_d = 100sign\left(x_2\right), r_d = 0$ and $x_0 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, original feedback.

Even for larger times, the state continues to increase, following an $\sqrt{x}$ form and thus being unbounded, while having a bounded input.

6 Show that the modified feedback fulfills the ISS properties

With the new pair of inputs:

$$u_N = \begin{cases} u_1 = -\dot{\theta} - (\theta - \theta_d) \\ u_2 = -\dot{r} - (r - r_d) - (r^3 - r_d^3) \end{cases} \quad \rightarrow \begin{cases} u_1 = -x_2 - (x_1 - \theta_d) \\ u_2 = -x_4 - (x_3 - r_d) - (x_3^3 - r_d^3) \end{cases}$$

the state space representation becomes:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-x_2 - (x_1 - \theta_d) - 2x_3 x_4 x_2}{(x_3^2 + 1)} \\ x_4 \\ -x_4 - (x_3 - r_d) - (x_3^3 - r_d^3) + x_3 x_2^2 \end{bmatrix}$$

Then, doing some computations, we can show how this system fulfills the typical ISS properties.

6.0.1 Input-to-State Stable Definition

A system is ISS if and only if:

$$|\phi(t, x_0, d)| \leq \max\left\{\beta(|x_0|, t), \gamma\left(\|d\|_\infty\right)\right\}, \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$$

Due to this definition, we have a shape in which our solution is bounded, and for large time, it almost solely depends on the second term.

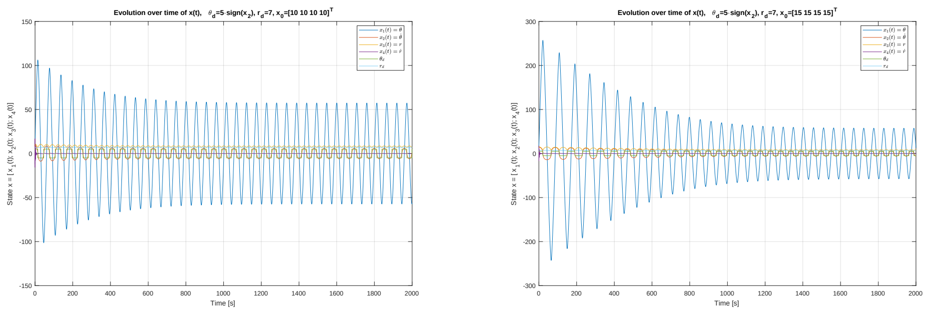


Figure 7: Simulation for $\theta_d = 5sign(x_2), r_d = 7$, and $x_{0_1} = \begin{bmatrix}10 & 10 & 10 & 10\end{bmatrix}^T$,$x_{0_2} = \begin{bmatrix}15 & 15 & 15 & 15\end{bmatrix}^T$respectively.

As we can notice from the two figures above, when x_0 increases the transient does the same, while the disturbance, being the same, leaves the second part of the bound (given by γ for large time) unaltered.

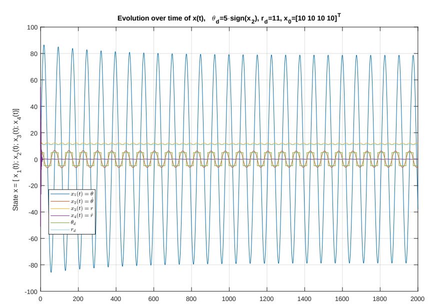


Figure 8: Simulation for $\theta_d = 5sign(x_2), r_d = 11$, and $x_0 = \begin{bmatrix}10 & 10 & 10 & 10\end{bmatrix}^T$.

When the disturbance increases (only one component in our example for simplicity of analysis), we can note how the γ part increases, leading to a steady state value bigger according to $\mathcal{K}_\infty$ function definition.

6.0.2 Converging-Input Converging-State (CICS)

To prove this property, we could choose a suitable disturbance that goes to zero for $t \to +\infty$. For simplicity of analysis $r_d$is fixed to zero, while $\theta_d = 3e^{-\frac{t}{10}}$, an $\mathcal{L}$-type function (again, one of the simplest possible).

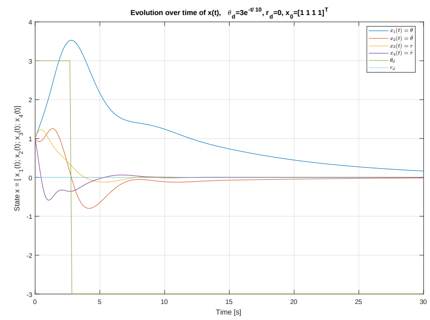


Figure 9: Simulation for $\theta_d = 3e^{-\frac{t}{10}}, r_d = 0$, and $x_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

As we can notice from the figure, after a transient somewhat "chaotic", the system is bounded by the γ part of the ISS definition, thus the solution goes down exponentially to zero.

6.0.3 0-GAS

The Global Asymptotic Stability in absence of disturbance ($d = 0$), means that the solution is bounded by the β part of the ISS definition, an $\mathcal{L}$-type function that eventually goes to zero for $t \to +\infty$.

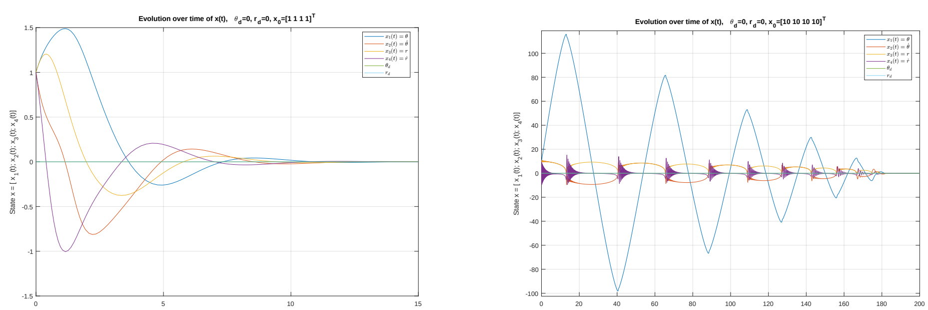


Figure 10: Simulation for $\theta_d = 0, r_d = 0$, and $x_{0_1} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$,$x_{0_2} = \begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}^T$respectively.

As we can see by the figures, the system goes to zero regardless of initial condition and, after an eventually long transient, it converges to the origin.