Furuta Pendulum Virtual Laboratory Experiment

Dr David Angeli, Dr Christos Gavriel, Dr Mario Sassano

Part I - Introduction

This experiment aims to demonstrate the design of several stabilizing controllers for a Furuta pendulum by means of a linear control tools. In implementing such a control system the following topics will be covered.

- Modelling the dynamics of an inverted pendulum using the Euler-Lagrange equation.
- Obtaining a linear state-space representation of the system.
- Creating a Simulink model to simulate the non-linear behaviour of the pendulum.
- Designing a state-feedback control law that improves damping for the pendulum in the downward position and balances it at its vertical upward position.
- Implementing the designed control law on the Simulink model and verifying its performance

The non-linear model is obtained using classical Lagrangian mechanics. The Lagrangian function, $L(\cdot, \cdot)$, is defined as

$$L(q(t), \dot{q}(t)) = T(q(t), \dot{q}(t)) - V(q(t)) \ , \tag{1}$$

where T and V are the kinetic and potential energies of the mechanical system, respectively and $(q, \dot{q})$ are the generalized coordinates.
The dynamics of the system can, then, be obtained through the Euler-Lagrange equation

$$\frac{d}{dt} L_{\dot{q}}(t, q(t), \dot{q}(t)) - L_q(t, q(t), \dot{q}(t)) = \tau \ , \tag{2}$$

where L_y denotes the partial derivative of the function L with respect to the variable y, and τ describes the non-conservative forces applied to the system with respect to the generalized coordinates, q.

Part II - Derivation of the model

For the considered pendulum, the generalized coordinates, $q(t)$, are taken to be the vector $(\theta(t), \alpha(t))$ representing the angles of the rotary arm and of the pendulum, respectively, see Figure 1.
The kinetic energy of the pendulum is given by (all parameters are defined in Table 1)

$$\begin{aligned} T = &\frac{1}{2}J_{arm}\dot{\theta}^2 + \frac{1}{2}J_p\dot{\alpha}^2 + \frac{1}{2}m_p\Big(-\cos(\theta)\sin(\alpha)\dot{\theta}l_p - \sin(\theta)\cos(\alpha)\dot{\alpha}l_p - \sin(\theta)\dot{\theta}r\Big)^2 + \\ &+ \frac{1}{2}m_p\Big(-\sin(\theta)\sin(\alpha)\dot{\theta}l_p + \cos(\theta)\cos(\alpha)\dot{\alpha}l_p + \cos(\theta)\dot{\theta}r\Big)^2 + \frac{1}{2}m_p\sin(\alpha)^2\dot{\alpha}^2l_p^2 \end{aligned} \tag{3}$$

and the potential energy by

$$V = -m_p\cos(\alpha)gl_p\ . \tag{4}$$

Moreover,

$$\tau = \begin{bmatrix} \tau_m - B_{arm}\dot{\theta}(t) \\ -B_p\dot{\alpha}(t) \end{bmatrix}\ , \tag{5}$$

where τ_m is the motor torque applied at the load gear, B_{arm} is the viscous friction force, B_p the damping coefficient of the pendulum and we neglected static friction terms acting on the shaft and the pendulum respectively. Since the pendulum is not actuated, the only force acting on the link is the damping. Considering the dynamics of the motor, the motor torque is expressed as

$$\tau_m = \frac{\eta_g K_g \eta_m K_t(V_m - K_g K_m\dot{\theta})}{R_m}\ . \tag{6}$$

The control variable is taken to be the motor voltage V_m.
The model is derived considering $(0, 0)$ as the downward position and $(0, \pi)$ as the upward position.

The non-linear model is given by

$$\begin{aligned} &2m_p\cos(\alpha)\dot{\alpha}l_p^2\dot{\theta}\sin(\alpha) - m_p\sin(\alpha)\dot{\alpha}^2l_pr + (m_pr^2 + m_pl_p^2 - m_pl_p^2\cos(\alpha)^2 + J_{arm})\ddot{\theta} + \\ &+ m_p\cos(\alpha)\ddot{\alpha}l_pr = \tau_m - B_{arm}\dot{\theta} \end{aligned} \tag{7}$$

and

$$-m_p\cos(\alpha)\dot{\theta}^2l_p^2\sin(\alpha) + m_p\cos(\alpha)\ddot{\theta}l_pr + (J_p + m_pl_p^2)\ddot{\alpha} + m_pg\sin(\alpha)l_p = -B_p\dot{\alpha}. \tag{8}$$

The dynamical equations can be rearranged in the matrix form

$$D\big(q(t)\big)\ddot{q}(t) + C\big(q(t), \dot{q}(t)\big)\dot{q}(t) + g\big(q(t)\big) = \tau. \tag{9}$$

Note that the matrix $C\big(q(t), \dot{q}(t)\big)$ is not uniquely defined by (9).

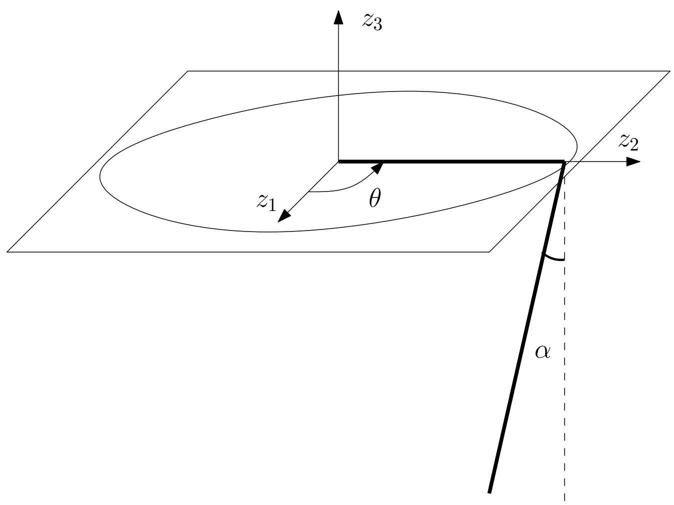

Figure 1: Kinematics of a single inverted pendulum system.

Q1) Show that the mechanical model can be expressed as in (7) and (8) using the Euler-Lagrange equation (2) and considering equations (3)-(6).

Q2) Rewrite the system of differential equations derived in Q1) in the matrix form (9).

Part III - Linear System Analysis

Q1) (a) Linearize the model (9) in the downward position, $(\theta, \alpha) = (0, 0)$.

(b) Repeat the procedure for the upward position, $(\theta, \alpha) = (0, \pi)$.

Q2) Obtain the state space representation of the linearized models derived in Q1), defining the state vectors $[x_1(t), x_2(t), x_3(t), x_4(t)]^T = [\theta(t), \alpha(t), \dot{\theta}(t), \dot{\alpha}(t)]^T$ and $[x_1(t), x_2(t), x_3(t), x_4(t)]^T = [\theta(t), \alpha(t) - \pi, \dot{\theta}(t), \dot{\alpha}(t)]^T$ for the downward and upward positions, respectively.

(*HINTS: (i) Get an expression for $\ddot{q}(t)$ and transform the two second-order differential equations into a system of four first-order differential equations. (ii) Since static friction is small, it can be ignored in the derivation of the linearised model.*)

Part IV - System Analysis

For this section consider the linearized state space models $\dot{x}(t) = Ax(t) + Bu(t)$ obtained in Part III.

Q1) Evaluate the expression for the matrices A, B (see Table 1 for the parameter values). Assess the stability of the linearized systems associated to both the upward and downward configurations.

Q2) Check controllability of the linearized systems.

Part V - Controller design and implementation

The aim of the controller is to stabilize the equilibrium point of the pendulum in the upward configuration and improve the damping of the pendulum in the downward. Assuming that all the states are measurable, the stabilization of the upward and downward equilibria of the pendulum is achieved through the implementation of a linear static state feedback law, K, of the form $u = K(x_d - x)$ where x_d is the desired reference input and x the actual state vector.

Q1) If the pair (A, B) is controllable assign the poles to a location of your choice that improves damping for the downward position and stabilises the upward. By using pole placement, obtain a pair of control laws corresponding to the poles you have chosen and comment on their success.

Q2) Compare and contrast your 'guess' with the pair of LQR control laws resulting from the weights:

$$Q_{down} = Q_{up} = \begin{bmatrix} 0.01 & 0_{1\times3} \\ 0_{3\times1} & I_{3\times3} \end{bmatrix}$$

and $R_{down} = 10, R_{up} = 100$.

Part V - Controller's simulation and testing

The first task is to implement and simulate the controllers developed for the downward position.

Q1) Simulate the open loop response of the Furuta pendulum from some non-trivial initial condition of the system. Connect and run the experiment. Plot a graph of $\alpha(t)$ showing the undamped oscillations. Now, include the linear control gain previously designed for the downwards equilibrium position. Repeat, plotting a graph for $\alpha(t)$. Compare the two results.

Q2) Stop the simulation and feed the system a square-wave reference signal, for the θ angle, while a constant 0 reference signal (downwards equilibrium) is kept for α. For instance the *set point* amplitude could be 45 degrees. What do you expect to observe? Comment on the response of the pendulum.

Q3) Compare the performance in simulation of the LQR design and the pole-placement design.

Q4) Repeat the previous simulations for the equilibrium corresponding to the upright pendulum. To this end, it is useful to first validate your control design on the linearized model. First apply a constant reference signal, and then a square-reference signal (for instance ± 45 degrees), while keeping a constant 180 degrees set-point for the θ angle.

Q5) Repeat the previous simulations for the nonlinear model (in closed loop with both the pole-placement and LQR controllers). Does the controller work for all possible initial conditions? Estimate the region of convergence of the controller in the α, θ space, when initial speeds are set to 0. You may grid the state-space accordingly and only add a colored dot for initial conditions which are stabilized by the controller.

Q6) Try a tracking experiment with a square-wave reference for the nonlinear system. Does the controller work well for all reference amplitudes? Can you interpret the difference with respect to the behaviour of the linearized model?

Appendix

Table 1: Parameter description and values

Symbol	**Description**	**Value**
R_m	Motor armature resistance	2.6 Ω
K_t	Motor torque constant	7.68e-3 $N \cdot m/A$
η_m	Motor efficiency	0.69
K_m	Back-emf constant	7.68e-3 $V/(rad/s)$
K_g	High-gear total gearbox ratio	70
η_g	Gearbox efficiency	0.9
m_p	Mass of the pendulum	0.127 Kg
l_p	Distance from pivot to center of gravity	0.1556 m
J_p	Pendulum moment of inertia	0.0012 $Kg \cdot m^2$
J_{arm}	Rotary arm moment of inertia	0.002 $Kg \cdot m^2$
B_p	Damping coefficient of the pendulum	0.0024 $N \cdot m \cdot s/rad$
B_{arm}	Viscous friction force	0.0024 $N \cdot m \cdot s/rad$
r	Length of the rotary arm	0.2159 m
g	Gravitational constant	9.81 m/s^2