Furuta Pendulum Virtual Laboratory Experiment

Filippo Badalamenti (Occasional FT, CID: 01998569)

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This paper will follow the structure presented in the assignment. All the Matlab code and file used will be also provided through the following link: `https://github.com/fil-bad/Furuta_pendulum`.

Part II

Derivation of the model

1 From Lagrange equation to mechanical models

Without going much deeper, we have one main equation for the Lagrangian mechanics:

$$L\left(q\left(t\right),\dot{q}\left(t\right)\right)=T\left(q\left(t\right),\dot{q}\left(t\right)\right)-V\left(q\left(t\right)\right)$$

while the generalized coordinates in our case are: $(q,\dot{q})=\left(\begin{bmatrix}\theta\left(t\right) & \alpha\left(t\right)\end{bmatrix},\begin{bmatrix}\dot{\theta}\left(t\right) & \dot{\alpha}\left(t\right)\end{bmatrix}\right)$. In our case, the Lagrangian equation is:

$$L=\frac{1}{2}J_{arm}\dot{\theta}^{2}+\frac{1}{2}J_{p}\dot{\alpha}^{2}+\frac{1}{2}m_{p}\left(-\cos\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_{p}-\sin\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_{p}-\sin\left(\theta\right)\dot{\theta}r\right)^{2}+$$
$$+\frac{1}{2}m_{p}\left(-\sin\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_{p}+\cos\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_{p}+\cos\left(\theta\right)\dot{\theta}r\right)^{2}+\frac{1}{2}m_{p}\sin\left(\alpha\right)^{2}\dot{\alpha}^{2}l_{p}^{2}+m_{p}\cos\left(\alpha\right)gl_{p}$$

Finally, the Euler-Lagrange equation can be expressed as:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}-\frac{\partial L}{\partial q}=\tau$$

For this reason, we get two equations in column:

$$\begin{bmatrix}\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}\\ \\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}}-\frac{\partial L}{\partial\alpha}\end{bmatrix}=\tau$$

Then, we can calculate each partial derivative of the Lagrangian equation, that are:

$$\begin{aligned}\boldsymbol{\frac{\partial L}{\partial \theta}} = {} & m_p \left(-\cos(\theta)\sin(\alpha)\dot{\theta}l_p - \sin(\theta)\cos(\alpha)\dot{\alpha}l_p - \sin(\theta)\dot{\theta}r\right)\left(\sin(\theta)\sin(\alpha)\dot{\theta}l_p - \cos(\theta)\cos(\alpha)\dot{\alpha}l_p - \cos(\theta)\dot{\theta}r\right) + \\ & + m_p \left(-\sin(\theta)\sin(\alpha)\dot{\theta}l_p + \cos(\theta)\cos(\alpha)\dot{\alpha}l_p + \cos(\theta)\dot{\theta}r\right)\left(-\cos(\theta)\sin(\alpha)\dot{\theta}l_p - \sin(\theta)\cos(\alpha)\dot{\alpha}l_p - \sin(\theta)\dot{\theta}r\right) = \\ \dots = \mathbf{0} & \end{aligned}$$

$$\begin{aligned}\boldsymbol{\frac{\partial L}{\partial \dot{\theta}}} &= J_{\text{arm}}\dot{\theta} + m_p\left(r\cos(\theta) - l_p\sin(\alpha)\sin(\theta)\right)\left(r\cos(\theta)\dot{\theta} - l_p\sin(\alpha)\sin(\theta)\dot{\theta} + l_p\cos(\alpha)\cos(\theta)\dot{\alpha}\right) + \\ &+ m_p\left(r\sin(\theta) + l_p\sin(\alpha)\cos(\theta)\right)\left(r\sin(\theta)\dot{\theta} + l_p\cos(\alpha)\sin(\theta)\dot{\alpha} + l_p\sin(\alpha)\cos(\theta)\dot{\theta}\right) \\ \boldsymbol{\frac{d}{dt}}\left(\boldsymbol{\frac{\partial L}{\partial \dot{\theta}}}\right) &= J_{arm}\ddot{\theta} - m_p\left(r\cos(\theta) - l_p\sin(\alpha)\sin(\theta)\right)\left(r\sin(\theta)\dot{\theta}^2 - r\cos(\theta)\ddot{\theta} + l_p\sin(\alpha)\cos(\theta)\dot{\alpha}^2 - \right. \\ &- \left. l_p\cos(\alpha)\cos(\theta)\ddot{\alpha} + l_p\sin(\alpha)\cos(\theta)\dot{\theta}^2 + l_p\sin(\alpha)\sin(\theta)\ddot{\theta} + 2l_p\cos(\alpha)\sin(\theta)\dot{\theta}\dot{\alpha}\right) + \\ &+ m_p\left(r\sin(\theta) + l_p\sin(\alpha)\cos(\theta)\right)\left(r\cos(\theta)\dot{\theta}^2 + r\sin(\theta)\ddot{\theta} - l_p\sin(\alpha)\sin(\theta)\dot{\alpha}^2 + \right. \\ &+ \left. l_p\cos(\alpha)\sin(\theta)\ddot{\alpha} - l_p\sin(\alpha)\sin(\theta)\dot{\theta}^2 + l_p\sin(\alpha)\cos(\theta)\ddot{\theta} + 2l_p\cos(\alpha)\cos(\theta)\dot{\theta}\dot{\alpha}\right)\end{aligned}$$

Once calculated, we can assemble the two equations, and simplifying them we get:

$$\boldsymbol{\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}} - \cancel{\boldsymbol{\frac{\partial L}{\partial \theta}}} = J_{arm}\ddot{\theta} - m_p l_p^2\cos(\alpha)^2\ddot{\theta} + m_p l_p^2\ddot{\theta} + m_p l_p r\cos(\alpha)\ddot{\alpha} + m_p r^2\ddot{\theta} - m_p\sin(\alpha)l_p r\dot{\alpha}^2 + 2m_p\sin(\alpha)\dot{\theta}l_p^2\cos(\alpha)\dot{\alpha}$$

The same can be done for the $\alpha(t)$ variable, and reducing the amount of term for each step leads to the equations:

$$\boldsymbol{\frac{\partial L}{\partial \alpha}} = -l_p m_p\left(-\frac{1}{2}l_p\sin(2\alpha)\dot{\theta}^2 + r\sin(\alpha)\dot{\theta}\dot{\alpha} + g\sin(\alpha)\right)$$

$$\begin{aligned}\boldsymbol{\frac{\partial L}{\partial \dot{\alpha}}} &= m_p l_p^2\dot{\alpha} + m_p r\cos(\alpha)l_p\dot{\theta} + J_p\dot{\alpha} \\ \boldsymbol{\frac{d}{dt}}\left(\boldsymbol{\frac{\partial L}{\partial \dot{\alpha}}}\right) &= m_p l_p^2\ddot{\alpha} + m_p r\cos(\alpha)l_p\ddot{\theta} + J_p\ddot{\alpha} - m_p r\sin(\alpha)\dot{\theta}l_p\dot{\alpha}\end{aligned}$$

$$\boldsymbol{\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}}} - \boldsymbol{\frac{\partial L}{\partial \alpha}} = \left(m_p l_p^2 + J_p\right)\ddot{\alpha} + m_p r\cos(\alpha)l_p\ddot{\theta} - m_p\cos(\alpha)\sin(\alpha)l_p^2\dot{\theta}^2 + g m_p\sin(\alpha)l_p$$

Equalling the vector:

$$\tau = \begin{bmatrix}\tau_m - B_{arm}\dot{\theta}(t) \\ -B_p\dot{\alpha}(t)\end{bmatrix}, \tau_m = \frac{\eta_g K_g \eta_m K_t\left(V_m - K_g K_m\dot{\theta}\right)}{R_m}$$

to the results found above, finally leads to the nonlinear model expressed in the equations (7) and (8) of the assignment.

2 Matricial model

Our model can be rewritten in the form:

$$D(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = \tau$$

where the $C(q(t), \dot{q}(t))$ matrix is not uniquely defined due to mixed products in the equations; a possible representation is given by:

$$D = \begin{bmatrix} m_p r^2 + m_p l_p^2 - m_p l_p^2 \cos(\alpha)^2 + J_{arm} & m_p \cos(\alpha) l_p r \\ m_p \cos(\alpha) l_p r & J_p + m_p l_p^2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2m_p \cos(\alpha) \dot{\alpha} l_p^2 \sin(\alpha) & -m_p \sin(\alpha) \dot{\alpha} l_p r \\ -m_p \cos(\alpha) \dot{\theta} l_p^2 \sin(\alpha) & 0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ m_p g \sin(\alpha) l_p \end{bmatrix}$$

Part III
Linear System Analysis

1 Linearizing the model

The matricial equation is not linear; in fact, each matrix depends from the coordinates $q(t)$ (the angular position of each section of the pendulum) and/or from their derivative $\dot{q}(t)$ (the angular speed). For this reason, we can linearize the model in its two equilibrium points (as we will see from further analysis), the *downward* position and the *upward* position.

We can then proceed in two ways:

- Neglecting any term with order higher than linear. This can be done using the formula $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, that leads for the trigonometric functions to:

  $$\begin{cases} \sin(x)|_{x_0=0} \approx \sin(0) + \cos(0)(x-0) = x \\ \cos(x)|_{x_0=0} \approx \cos(0) + \sin(0)(x-0) = 1 \end{cases} \quad \begin{cases} \sin(x)|_{x_0=\pi} \approx \sin(\pi) + \cos(\pi)(x-\pi) = -x+\pi \\ \cos(x)|_{x_0=\pi} \approx \cos(\pi) + \sin(\pi)(x-\pi) = -1 \end{cases}$$

  considering the two points later involved.

- Linearize the system around the equilibrium points; in fact, it is legit to assume that both angular speeds are equal to zero – being the position a constant value –, thus allowing us to linearize the rewritten system $\dot{x} = f(x, u)$ around x_e, following the formula:

  $$\dot{\delta x} = \frac{\partial}{\partial x} f(x_e, u_e)\, \delta x + \frac{\partial}{\partial u} f(x_e, u_e)\, \delta u$$

  where $\delta x, \delta u$ are the increments, and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}$ are the Jacobian matrices with respect to the state and input vectors, and defining:

  $$A = \left. \frac{\partial}{\partial x} f(x, u) \right|_{x=x_e} \quad B = \left. \frac{\partial}{\partial u} f(x, u) \right|_{x=x_e}$$

  we get a linear system in the form $\dot{x} = Ax + Bu$.

Due to a greater familiarity with the second method, the further dissertation will follow the latter one.

1.1 Downward position, $(\theta, \alpha) = (0, 0)$

For this position, we can define the equilibrium point $\begin{pmatrix}\theta & \alpha & \dot{\theta} & \dot{\alpha}\end{pmatrix}^T = \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix}^T$. Moreover, in order to linearize around this point, we have to rewrite the equation in the form:

$$\dot{x} = \begin{pmatrix}\dot{q} \\ \ddot{q}\end{pmatrix} = \begin{pmatrix}\begin{bmatrix}\dot{\theta} \\ \dot{\alpha}\end{bmatrix} \\ D^{-1}\left(\tau - g - C\begin{bmatrix}\dot{\theta} \\ \dot{\alpha}\end{bmatrix}\right)\end{pmatrix}$$

However, the matrix D must be invertible, and this can be easily verified if $\det(D) \neq 0$:

$$\det(D) = \left(m_p r^2 + m_p l_p^2 - m_p l_p^2 \cos(\alpha)^2 + J_{arm}\right)\left(J_p + m_p l_p^2\right) - \left(m_p \cos(\alpha) l_p r\right)^2$$

but if we consider that all the terms have physical sense when they are strictly positive numbers, we can do some considerations:

- for $\alpha = \pm\frac{\pi}{2}$, the determinant is always sign definite (for everything that makes sense);
- for $\alpha \neq \pm\frac{\pi}{2}$ (and assuming it to be w.l.o.g. $= k\pi,\ k \in \mathbb{Z}$, so that we consider the most negative case), we get:

  $$\left(m_p r^2 + J_{arm}\right)\left(J_p + m_p l_p^2\right) - \left(m_p l_p r\right)^2$$

  so it is enough that the equation, once replaced the real values, it is not fullfilled at the equality with zero.

Once ensured this, we can then write down the state space equations[1]:

$$\begin{bmatrix}x_1(t) & x_2(t) & x_3(t) & x_4(t)\end{bmatrix}^T = \begin{bmatrix}\theta(t) & \alpha(t) & \dot{\theta}(t) & \dot{\alpha}(t)\end{bmatrix}^T$$

so that we can rewrite the equation as:

$$\dot{x} = \begin{pmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4\end{pmatrix} = \begin{pmatrix}x_3 \\ x_4 \\ D^{-1}\left(\tau - g - C\begin{bmatrix}x_3 \\ x_4\end{bmatrix}\right)\end{pmatrix}$$

where:

$$D^{-1} = \frac{1}{\det(D)}\begin{bmatrix}D_{22} & -D_{12} \\ -D_{21} & D_{11}\end{bmatrix}$$

and D_{ij} denote the component of the matrix D in the i-row, j-column position.

We can then linearize the system through the Jacobian operator, and then evaluating the resulting matrices in the equilibrium point x_{down} (thanks to the *subs* function in Matlab).

Without writing the full matrix equations (they are difficult to fit in a page), the resulting matrices of the linearized system are:

[1] For simplicity, I will treat (Q1) and (Q2) of Part III at the same moment, writing directly the state space representation.

$$A=\left.\frac{\partial}{\partial x}f\left(x,u\right)\right|_{x=x_{down}}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{gl_p^2m_p^2r}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & -\frac{\left(m_pl_p^2+J_p\right)\left(B_{arm}+\frac{\eta_g\eta_mK_g^2K_mK_t}{R_m}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & \frac{B_pl_pm_pr}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p}\\ 0 & -\frac{gl_pm_p\left(m_pr^2+J_{arm}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & \frac{l_pm_pr\left(B_{arm}+\frac{\eta_g\eta_mK_g^2K_mK_t}{R_m}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & -\frac{B_p\left(m_pr^2+J_{arm}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p}\end{bmatrix}$$

$$B=\left.\frac{\partial}{\partial u}f\left(x,u\right)\right|_{x=x_{down}}=\begin{bmatrix}0\\ 0\\ \frac{\eta_g\eta_mK_gK_t\left(m_pl_p^2+J_p\right)}{R_m\left(J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p\right)}\\ -\frac{\eta_g\eta_mK_gK_tl_pm_pr}{R_m\left(J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p\right)}\end{bmatrix}$$

1.2 Upward position, $(\theta,\alpha)=(0,\pi)$

Following the same approach, we have as equilibrium point $\begin{pmatrix}\theta & \alpha & \dot{\theta} & \dot{\alpha}\end{pmatrix}^T=\begin{pmatrix}0 & \pi & 0 & 0\end{pmatrix}^T$. Since the state vector that we have to find out is $\begin{bmatrix}x_1\left(t\right) & x_2\left(t\right) & x_3\left(t\right) & x_4\left(t\right)\end{bmatrix}^T=\begin{bmatrix}\theta\left(t\right) & \alpha\left(t\right)-\pi & \dot{\theta}\left(t\right) & \dot{\alpha}\left(t\right)\end{bmatrix}^T$. This means that we can change the coordinates for α so that, when we will linearize, we do it aound the new origin. Another approach is instead to leave the equations unchanged and then linearize around the equilibrium point expressed as function of θ and α.

If we proceed as seen above, the system becomes:

$$A=\left.\frac{\partial}{\partial x}f\left(x,u\right)\right|_{x=x_{up}}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{gl_p^2m_p^2r}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & -\frac{\left(m_pl_p^2+J_p\right)\left(B_{arm}+\frac{\eta_g\eta_mK_g^2K_mK_t}{R_m}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & -\frac{B_pl_pm_pr}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p}\\ 0 & \frac{gl_pm_p\left(m_pr^2+J_{arm}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & -\frac{l_pm_pr\left(B_{arm}+\frac{\eta_g\eta_mK_g^2K_mK_t}{R_m}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p} & -\frac{B_p\left(m_pr^2+J_{arm}\right)}{J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p}\end{bmatrix}$$

$$B=\left.\frac{\partial}{\partial u}f\left(x,u\right)\right|_{x=x_{up}}=\begin{bmatrix}0\\ 0\\ \frac{\eta_g\eta_mK_gK_t\left(m_pl_p^2+J_p\right)}{R_m\left(J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p\right)}\\ \frac{\eta_g\eta_mK_gK_tl_pm_pr}{R_m\left(J_{arm}m_pl_p^2+J_pm_pr^2+J_{arm}J_p\right)}\end{bmatrix}$$

Part IV
System Analysis

1 Evaluating A, B matrices and assessing stability

1.1 Downward position

Using the values given in the Table 1 of the assessment, and substituting it in the two matrices, we get:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{5169259020477411}{97834235525000} & -\frac{15506119043916576}{794903163640625} & \frac{2559865848}{3913369421} \\ 0 & -\frac{38382920592829641}{391336942100000} & \frac{15475651065023064}{794903163640625} & -\frac{4751896122}{3913369421} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 52.837 & -19.507 & 0.654 \\ 0 & -98.082 & 19.469 & -1.214 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{44598579129216}{1271845061825} \\ -\frac{44510947365024}{1271845061825} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ 35.066 \\ -34.997 \end{bmatrix}$$

Now, to assess the stability property, we have to remember for a linearized system, we have to look at the eigenvalues of the matrix $A = \frac{\partial}{\partial x} f(x_e, u_e)$, for which three possible cases arise:

1. The matrix $\frac{\partial}{\partial x} f(x_e, u_e)$ is Hurwitz. This means that:

$$\forall \lambda_i \in eigs\left(\frac{\partial}{\partial x} f(x_e, u_e)\right) = \{\lambda_1, \ldots, \lambda_n\} : Re\{\lambda_i\} < 0$$

   In this case, the equilibrium is locally asymptotically stable.

2. The matrix $\frac{\partial}{\partial x} f(x_e, u_e)$ is such that:

$$\exists \lambda_i \in eigs\left(\frac{\partial}{\partial x} f(x_e, u_e)\right) = \{\lambda_1, \ldots, \lambda_n\} : Re\{\lambda_i\} > 0$$

   In this case, the equilibrium is unstable.

3. The matrix $\frac{\partial}{\partial x} f(x_e, u_e)$ is such that:

$$\begin{cases} \forall \lambda_i \in eigs\left(\frac{\partial}{\partial x} f(x_e, u_e)\right) = \{\lambda_1, \ldots, \lambda_n\} : Re\{\lambda_i\} \leq 0 \\ \exists \lambda_j : Re\{\lambda_j\} = 0 \end{cases}$$

   In this case, we cannot assert anything; it is logical since we don't have a single point with minimum energy; instead, we have a minimum for every value of θ. The only thing we could eventually assess is that it would be Lyapunov stable, but not asymptotically convergent (in fact, the point of convergence depends from the specifical initial condition).

For the downward position, the eigenvalues are:

$$eigs(A_{down}) = \left\{ \begin{array}{c} 0 \\ -\frac{1222738973915481}{70368744177664} \\ -\frac{3766174288345861}{2251799813685248} + j\frac{976194666914269}{140737488355328} \\ -\frac{3766174288345861}{2251799813685248} - j\frac{976194666914269}{140737488355328} \end{array} \right\} \approx \left\{ \begin{array}{c} 0 \\ -17.376 \\ -1.673 + j6.936 \\ -1.673 - j6.936 \end{array} \right\}$$

From the rules analyzed above, we cannot assess any form of stability from linearization.

1.2 Upward position

From the point of view of simply substituting values, we can use the results found for the downward position and change the signs appropriately. Hence, we have:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{5169259020477411}{97834235525000} & -\frac{15506119043916576}{794903163640625} & -\frac{2559865848}{3913369421} \\ 0 & \frac{38382920592829641}{391336942100000} & -\frac{15475651065023064}{794903163640625} & -\frac{4751896122}{3913369421} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 52.837 & -19.507 & -0.654 \\ 0 & 98.082 & -19.469 & -1.214 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{44598579129216}{1271845061825} \\ \frac{44510947365024}{1271845061825} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ 35.066 \\ 34.997 \end{bmatrix}$$

Now, assessing stability, and applying the same rules as the previous point, we get:

$$eigs(A_{up}) = \begin{Bmatrix} 0 \\ -\frac{1607239695991555}{70368744177664} \\ \frac{8300623629294261}{1125899906842624} \\ -\frac{5914786364422933}{1125899906842624} \end{Bmatrix} \approx \begin{Bmatrix} 0 \\ -22.840 \\ 7.372 \\ -5.253 \end{Bmatrix}$$

As we expected, the upward position is an unstable equilibrium.

2 Checking controllability

We can verify the controllability of a linear system checking if the $R = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nr}$ matrix has full row rank, that is $rank\,(R) = n$. Moreover, we can verify directly with Matlab through the *ctrb* function.

2.1 Downward position

For this position, we have:

$$R_{down} = \begin{bmatrix} 0 & \frac{4935107427201581}{140737488355328} & -\frac{6218166413817175}{8796093022208} & \frac{6825292479804947}{549755813888} \\ 0 & -\frac{4925410432840623}{140737488355328} & \frac{6378776113532615}{8796093022208} & -\frac{770400999661797}{68719476736} \\ \frac{4935107427201581}{140737488355328} & -\frac{6218166413817175}{8796093022208} & \frac{6825292479804947}{549755813888} & -\frac{3628355444977149}{17179869184} \\ -\frac{4925410432840623}{140737488355328} & \frac{6378776113532615}{8796093022208} & -\frac{770400999661797}{68719476736} & \frac{6328762140136585}{34359738368} \end{bmatrix}$$

and we have $rank\,(R_{down}) = 4$, so the system in this configuration is controllable.

2.2 Upward position

In this case, we have:

$$R_{up} = \begin{bmatrix} 0 & \frac{4935107427201581}{140737488355328} & -\frac{6218166413817175}{8796093022208} & \frac{8858444698715643}{549755813888} \\ 0 & \frac{4925410432840623}{140737488355328} & -\frac{6378776113532615}{8796093022208} & \frac{4968681040599551}{274877906944} \\ \frac{4935107427201581}{140737488355328} & -\frac{6218166413817175}{8796093022208} & \frac{8858444698715643}{549755813888} & -\frac{3130720068146841}{8589934592} \\ \frac{4925410432840623}{140737488355328} & -\frac{6378776113532615}{8796093022208} & \frac{4968681040599551}{274877906944} & -\frac{1747114645631177}{4294967296} \end{bmatrix}$$

and again, we have $rank\,(R_{up}) = 4$, thus allowing the possibility of designing a control for the system.

Part V
Controller design and implementation