Furuta Pendulum Virtual Laboratory Experiment

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This paper will follow the structure presented in the assignment. All the Matlab code and file used will be also provided through the following link: `https://github.com/fil-bad/Furuta_pendulum`.

Part II

Derivation of the model

1 From Lagrange equation to mechanical models

Without going much deeper, we have one main equation for the Lagrangian mechanics:

$$L\left(q\left(t\right),\dot{q}\left(t\right)\right)=T\left(q\left(t\right),\dot{q}\left(t\right)\right)-V\left(q\left(t\right)\right)$$

while the generalized coordinates in our case are: $(q,\dot{q})=\left(\begin{bmatrix}\theta\left(t\right) & \alpha\left(t\right)\end{bmatrix},\begin{bmatrix}\dot{\theta}\left(t\right) & \dot{\alpha}\left(t\right)\end{bmatrix}\right)$. In our case, the Lagrangian equation is:

$$L=\frac{1}{2}J_{arm}\dot{\theta}^2+\frac{1}{2}J_p\dot{\alpha}^2+\frac{1}{2}m_p\left(-\cos\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_p-\sin\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_p-\sin\left(\theta\right)\dot{\theta}r\right)^2+$$
$$+\frac{1}{2}m_p\left(-\sin\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_p+\cos\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_p+\cos\left(\theta\right)\dot{\theta}r\right)^2+\frac{1}{2}m_p\sin\left(\alpha\right)^2\dot{\alpha}^2l_p^2+m_p\cos\left(\alpha\right)gl_p$$

Finally, the Euler-Lagrange equation can be expressed as:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}-\frac{\partial L}{\partial q}=\tau$$

For this reason, we get two equations in column:

$$\begin{bmatrix}\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}\\ \\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}}-\frac{\partial L}{\partial\alpha}\end{bmatrix}=\tau$$

For this reason, we can calculate each partial derivative of the Lagrangian equation, that are:

$$\boldsymbol{\frac{\partial L}{\partial\theta}}=m_p\left(-\cos\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_p-\sin\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_p-\sin\left(\theta\right)\dot{\theta}r\right)\left(\sin\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_p-\cos\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_p-\cos\left(\theta\right)\dot{\theta}r\right)+$$
$$+m_p\left(-\sin\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_p+\cos\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_p+\cos\left(\theta\right)\dot{\theta}r\right)\left(-\cos\left(\theta\right)\sin\left(\alpha\right)\dot{\theta}l_p-\sin\left(\theta\right)\cos\left(\alpha\right)\dot{\alpha}l_p-\sin\left(\theta\right)\dot{\theta}r\right)=$$
$$\ldots=-2l_pm_p\sin\left(\alpha\right)\left(2\sin\left(\theta\right)^2-1\right)\left(r\dot{\theta}+l_p\cos\left(\alpha\right)\dot{\alpha}\right)\dot{\theta}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot\theta} &= J_{\text{arm}}\dot\theta + m_p\left(r\cos(\theta)+l_p\sin(\alpha)\sin(\theta)\right)\left(r\cos(\theta)\dot\theta + l_p\sin(\alpha)\sin(\theta)\dot\theta + l_p\cos(\alpha)\cos(\theta)\dot\alpha\right) + \\
&+ m_p\left(r\sin(\theta)+l_p\sin(\alpha)\cos(\theta)\right)\left(r\sin(\theta)\dot\theta + l_p\cos(\alpha)\sin(\theta)\dot\alpha + l_p\sin(\alpha)\cos(\theta)\dot\theta\right) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) &= J_{\text{arm}}\ddot\theta + l_p^2 m_p\ddot\theta + m_p r^2\ddot\theta - l_p^2 m_p\sin(2\theta)\dot\alpha^2 - l_p^2 m_p\cos(\alpha)^2\ddot\theta - l_p m_p r\sin(\alpha)\dot\alpha^2 + l_p m_p r\cos(\alpha)\ddot\alpha - \\
&- 4l_p m_p r\sin(\alpha)\dot\theta^2 + 8l_p m_p r\sin(\alpha)\cos(\theta)^2\dot\theta^2 + 4l_p^2 m_p\cos(\alpha)^2\cos(\theta)\sin(\theta)\dot\alpha^2 + 4l_p^2 m_p\cos(\alpha)\sin(\alpha)\cos(\theta)^2\dot\theta\dot\alpha + \\
&+ 4l_p m_p r\sin(\alpha)\cos(\theta)\sin(\theta)\ddot\theta + 2l_p^2 m_p\cos(\alpha)\sin(\alpha)\cos(\theta)\sin(\theta)\ddot\alpha + 4l_p m_p r\cos(\alpha)\cos(\theta)\sin(\theta)\dot\theta\dot\alpha
\end{aligned}
$$

Once calculated, we can assemble the two equations to get:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} &= J_{\text{arm}}\ddot\theta + l_p^2 m_p\ddot\theta + m_p r^2\ddot\theta - l_p^2 m_p\sin(2\theta)\dot\alpha^2 - l_p^2 m_p\cos(\alpha)^2\ddot\theta - l_p m_p r\sin(\alpha)\dot\alpha^2 + l_p m_p r\cos(\alpha)\ddot\alpha - \\
&- 2l_p m_p r\sin(\alpha)\dot\theta^2 + l_p^2 m_p\sin(2\alpha)\dot\theta\dot\alpha + 4l_p m_p r\sin(\alpha)\cos(\theta)^2\dot\theta^2 + 4l_p^2 m_p\cos(\dot\alpha)^2\cos(\theta)\sin(\theta)\dot\alpha^2 + \\
&+ 4l_p m_p r\sin(\alpha)\cos(\theta)\sin(\theta)\ddot\theta + 2l_p^2 m_p\cos(\alpha)\sin(\alpha)\cos(\theta)\sin(\theta)\ddot\alpha + 4l_p m_p r\cos(\alpha)\cos(\theta)\sin(\theta)\dot\theta\dot\alpha
\end{aligned}
$$

The same can be done for the $\alpha(t)$ variable, leading to the equation:

$$
\begin{aligned}
\frac{\partial L}{\partial \alpha} &= m_p\sin(\alpha)l_p^2\cos(\alpha)\dot\theta^2 + 2m_p r\cos(\theta)\sin(\theta)l_p\cos(\alpha)\dot\theta^2 + 4m_p\cos(\theta)\sin(\theta)l_p^2\cos(\alpha)^2\dot\theta\dot\alpha - \\
&- 2m_p\cos(\theta)\sin(\theta)l_p^2\dot\theta\dot\alpha - m_p r\sin(\alpha)l_p\dot\theta\dot\alpha - g m_p\sin(\alpha)l_p \\
\frac{\partial L}{\partial \dot\alpha} &= J_p\dot\alpha + l_p^2 m_p\dot\alpha + l_p m_p r\cos(\alpha)\dot\theta + \frac{1}{2}l_p^2 m_p\sin(2\alpha)\sin(2\theta)\dot\theta \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\alpha}\right) &= J_p\ddot\alpha + l_p^2 m_p\ddot\alpha + l_p m_p r\cos(\alpha)\ddot\theta + l_p^2 m_p\cos(\alpha)\sin(\alpha)\sin(2\theta)\ddot\theta + \\
&+ l_p^2 m_p\sin(2\theta)\left(2\cos(\alpha)^2-1\right)\dot\theta\dot\alpha - l_p m_p r\sin(\alpha)\dot\theta\dot\alpha + 2l_p^2 m_p\cos(\alpha)\sin(\alpha)\cos(2\theta)\dot\theta^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot\alpha} - \frac{\partial L}{\partial \alpha} &= J_p\ddot\alpha + l_p^2 m_p\ddot\alpha - \frac{1}{2}l_p^2 m_p\sin(2\alpha)\dot\theta^2 + g l_p m_p\sin(\alpha) + l_p^2 m_p\sin(2\alpha)\cos(2\theta)\dot\theta^2 + \\
&+ l_p m_p r\cos(\alpha)\ddot\theta + \frac{1}{2}l_p^2 m_p\sin(2\alpha)\sin(2\theta)\ddot\theta - l_p m_p r\cos(\alpha)\sin(2\theta)\dot\theta^2
\end{aligned}
$$

If we equal the $\tau = \begin{bmatrix}\tau_m - B_{arm}\dot\theta(t) & -B_p\dot\alpha(t)\end{bmatrix}^T$ vector to the results found above, we finally get the non linear model expressed in the equations (7) and (8) of the assignment.

2 Matricial model

Our model can be rewritten in the form:

$$
D(q(t))\ddot q(t) + C(q(t),\dot q(t))\dot q(t) + g(q(t)) = \tau
$$

where the $C(q(t),\dot q(t))$ matrix is not uniquely defined due to mixed products in the equations; a possible representation is given by:

$$
D = \begin{bmatrix} m_p r^2 + m_p l_p^2 - m_p l_p^2\cos(\alpha)^2 + J_{arm} & m_p\cos(\alpha)l_p r \\ m_p\cos(\alpha)l_p r & J_p + m_p l_p^2 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 2m_p\cos(\alpha)\dot\alpha l_p^2\sin(\alpha) & -m_p\sin(\alpha)\dot\alpha l_p r \\ -m_p\cos(\alpha)\dot\theta l_p^2\sin(\alpha) & 0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ m_p g\sin(\alpha)l_p \end{bmatrix}
$$

Part III

Linear System Analysis

1 Linearising the model

The matricial equation is not linear; in fact, each matrix depends from the coordinates $q(t)$ (the angular position of each section of the pendulum) and/or from their derivative $\dot{q}(t)$ (the angular speed). For this reason, we can linearize the model in its two equilibrium points (as we will see from further analysis), the *downward* position and the *upward* position. The resulting system is a linear 2-DOF one that can be analysed through the classical approach for mechanical systems.

1.1 Downward position, $(\theta, \alpha) = (0, 0)$

Substituting the values in the matrices, they become:

$$D = \begin{bmatrix} m_p r^2 + J_{arm} & m_p l_p r \\ m_p l_p r & J_p + m_p l_p^2 \end{bmatrix} \; C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \; g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1.2 Upward position, $(\theta, \alpha) = (0, \pi)$

Substituting the values in the matrices, they become:

$$D = \begin{bmatrix} m_p r^2 + J_{arm} & -m_p l_p r \\ -m_p l_p r & J_p + m_p l_p^2 \end{bmatrix} \; C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \; g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2 State space representation

Due to the linearization around equilibrium points, both C and g matrices become null ones. In particular, the equations become:

$$D_{(0,0)} \ddot{q}(t) = \tau \quad , \quad D_{(0,\pi)} \ddot{q}(t) = \tau$$

For this reason, if the matrix D is invertible – that means $\det(D) \neq 0$ –, we can invert it and get an expression for the second order derivatives of $q(t)$:

$$\begin{cases} \ddot{q}_{down}(t) = D^{-1}_{(0,0)} \tau = \begin{pmatrix} \frac{J_p + l_p^2 m_p}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} & -\frac{r m_p l_p}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} \\ -\frac{r m_p l_p}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} & \frac{r^2 m_p + J_{arm}}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} \end{pmatrix} \begin{pmatrix} \tau_m - B_{arm} \dot{\theta}(t) \\ -B_p \dot{\alpha}(t) \end{pmatrix} \\ \ddot{q}_{up}(t) = D^{-1}_{(0,\pi)} \tau = \begin{pmatrix} \frac{J_p + l_p^2 m_p}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} & \frac{r m_p l_p}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} \\ \frac{r m_p l_p}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} & \frac{r^2 m_p + J_{arm}}{l_p^2 J_{arm} m_p + r^2 m_p J_p + J_{arm} J_p} \end{pmatrix} \begin{pmatrix} \tau_m - B_{arm} \dot{\theta}(t) \\ -B_p \dot{\alpha}(t) \end{pmatrix} \end{cases}$$

We can now introduce the state vectors $\begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T = \begin{bmatrix} \theta(t) & \alpha(t) & \dot{\theta}(t) & \dot{\alpha}(t) \end{bmatrix}^T$ and $\begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T = \begin{bmatrix} \theta(t) & \alpha(t) - \pi & \dot{\theta}(t) & \dot{\alpha}(t) \end{bmatrix}^T$ for downward and upward positions respectively.

In this case, the systems become (we keep the matrices D in the shortest form, since they have only constants):

$$down: \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ D^{-1}_{(0,0)} \begin{pmatrix} \tau_m - B_{arm} x_3 \\ -B_p x_4 \end{pmatrix} \end{bmatrix} \quad , \quad \tau_m = \frac{\eta_g K_g \eta_m K_t \left(u - K_g K_m x_3\right)}{R_m}$$

$$D^{-1} g = \begin{pmatrix} -\frac{r g \cos(\alpha) \sin(\alpha) m_p^2 l_p^2}{\left(l_p^2 m_p + J_p\right)\left(r^2 m_p + l_p^2 m_p - l_p^2 \cos^2(\alpha) m_p + J_{arm}\right) - r^2 \cos^2(\alpha) m_p^2 l_p^2} \\ \frac{g \sin(\alpha) m_p l_p \left(r^2 m_p + l_p^2 m_p - l_p^2 \cos^2(\alpha) m_p + J_{arm}\right)}{\left(l_p^2 m_p + J_p\right)\left(r^2 m_p + l_p^2 m_p - l_p^2 \cos^2(\alpha) m_p + J_{arm}\right) - r^2 \cos^2(\alpha) m_p^2 l_p^2} \end{pmatrix}$$