1 Comprehensive Constrained Spherical Pendulum Equations

1.1 Variables

- $r(t)$ - pendulum length (variable due to pumping and COM shifts)
- $\theta(t)$ - polar angle (constrained to lower hemisphere)
- $\phi(t)$ - azimuthal angle
- α, β, γ - bob rotation angles around x, y, z axes respectively
- m - mass of the pendulum (including bob)
- g - gravitational acceleration
- I - moment of inertia of the bob
- $\varepsilon_x, \varepsilon_y, \varepsilon_z$ - COM offset from bob's geometric center

1.2 Equations

1.2.1 1. Pendulum motion equations:

$$\begin{aligned}
&(r + \varepsilon_x \sin\theta \cos\phi + \varepsilon_y \sin\theta \sin\phi - \varepsilon_z \cos\theta)\ddot{\theta} - \\
&(r + \varepsilon_x \sin\theta \cos\phi + \varepsilon_y \sin\theta \sin\phi - \varepsilon_z \cos\theta)\dot{\phi}^2 \sin\theta \cos\theta + \\
&g \sin\theta + 2\dot{r}\dot{\theta} + \\
&(\dot{\varepsilon}_x \cos\theta \cos\phi + \dot{\varepsilon}_y \cos\theta \sin\phi + \dot{\varepsilon}_z \sin\theta)\dot{\theta} + \\
&(\dot{\varepsilon}_x \sin\theta \sin\phi - \dot{\varepsilon}_y \sin\theta \cos\phi)\dot{\phi} = 0
\end{aligned}$$

$$\begin{aligned}
&(r + \varepsilon_x \sin\theta \cos\phi + \varepsilon_y \sin\theta \sin\phi - \varepsilon_z \cos\theta)^2 \ddot{\phi} \sin\theta + \\
&2(r + \varepsilon_x \sin\theta \cos\phi + \varepsilon_y \sin\theta \sin\phi - \varepsilon_z \cos\theta)^2 \dot{\theta}\dot{\phi} \cos\theta + \\
&2(r + \varepsilon_x \sin\theta \cos\phi + \varepsilon_y \sin\theta \sin\phi - \varepsilon_z \cos\theta)\dot{r}\dot{\phi} \sin\theta + \\
&2(\dot{\varepsilon}_x \cos\phi + \dot{\varepsilon}_y \sin\phi)(r + \varepsilon_x \sin\theta \cos\phi + \varepsilon_y \sin\theta \sin\phi - \varepsilon_z \cos\theta)\dot{\phi} \sin\theta = 0
\end{aligned}$$

1.2.2 2. Bob rotation equations:

$$\begin{aligned}
I\ddot{\alpha} + (I_z - I_y)\dot{\beta}\dot{\gamma} + mg(\varepsilon_y \cos\theta \cos\alpha - \varepsilon_z \sin\theta) &= \tau_x \\
I\ddot{\beta} + (I_x - I_z)\dot{\alpha}\dot{\gamma} + mg(\varepsilon_z \cos\theta \sin\alpha - \varepsilon_x \cos\theta) &= \tau_y \\
I\ddot{\gamma} + (I_y - I_x)\dot{\alpha}\dot{\beta} + mg(\varepsilon_x \sin\theta - \varepsilon_y \cos\theta \sin\alpha) &= \tau_z
\end{aligned}$$

1.2.3 3. COM shift equations:

$$
\begin{aligned}
\varepsilon_x &= A_x \sin(\omega_x t + \delta_x) + B_x \sin(\Omega t + \alpha) \\
\varepsilon_y &= A_y \sin(\omega_y t + \delta_y) + B_y \sin(\Omega t + \beta) \\
\varepsilon_z &= A_z \sin(\omega_z t + \delta_z) + B_z \sin(\Omega t + \gamma)
\end{aligned}
$$

1.2.4 4. Pumping action (variable length):

$$r(t) = r_0 + C \sin(\Omega_p t + \delta_p)$$

1.2.5 5. Constraint equation:

$$0 \leq \theta \leq \pi/2$$

Where:

- τ_x, τ_y, τ_z are external torques on the bob
- A_i, B_i are amplitudes of COM shift
- ω_i are frequencies of internal COM shift
- Ω is the frequency of COM shift due to bob rotation
- δ_i are phase shifts
- C is the amplitude of length change due to pumping
- Ω_p is the pumping frequency
- δ_p is the pumping phase shift