Q1. *Algorithm Description*

We proceed by modeling the network of trails as a directed graph $G(V, E)$ where each edge $(u, v) \in E$ has an associated value $\mathbb{E}(u, v)$ that represents the expected number of slide occurrences while traversing (u, v) and visiting v. By modeling this way, we remove the weights on the vertices (and aggregate them on the edges) in such way that an edge (u, v) with terminal incidence at v has an assigned value of:

$$\mathbb{E}(u, v) = w(u, v) + w(v) \tag{1}$$

where w represents the probabilities (or weights) assigned to each site and route on the initial map. With this configuration in mind, we aim to find a path p such that the expectation on the number of slide occurrences on p is minimized. Let s be the source and t be the terminal, and let p be the sequence $(v_0, v_1, \cdots, v_k)$, where $v_0 = s$ and $v_k = t$, then:

$$p = \arg\min \left(w(s) + \sum_{i=1}^{k} \mathbb{E}(v_{i-1}, v_i) \right) \tag{2}$$

We use Dijkstra's algorithm to solve the single source shortest path problem on the converted. Using Equation 2 the following setup is assumed:

- *Vertices* — unchanged sites on the map with no associated probabilities nor weights (except for s which has an assigned weight $w(s)$ representing the expected number of slides in s).
- *Edges* — directed routes between different sites of the map with edge weights given by (1).

We set the source s to be the marked position on our friends on the map and initialize the $cost(s)$ to $w(s)$ as we must account for the expectation in s. Then we exhaust all paths going from s to the remaining vertices of G and (since edge weights encode expectations) we pick the smallest path with terminal at a trailhead.

Time Analysis

Let V represent the different sites on the map and E the routes between them. Then we can construct G in $O(V + E)$ time. A call to Dijkstra on G takes $O((E + V) \log V)$ and the reconstruction of p takes at most $O(V)$ with backtracking. This yields a total running time of $O((E + V) \log V)$.

Q2. Consider the following definition:

> **Definition** — Let $G = (V, E)$ be an undirected graph with edge-weights given by $E \to \mathbb{R}$. Assume that $w(e) \neq w(f)$ whenever e, f are distinct edges of G. We say that an edge is *treacherous* if it is the maximum weight edge for some cycle of G. On the other hand, an edge is *reliable* if it is not contained in any cycle of G.

> **Claim 1** — The minimum spanning tree of G contains every *reliable* edge.

Proof. Assuming G is connected, we prove the more general claim P that "an edge e is *reliable* in G if and only if it belongs to every spanning tree of G." Note Claim 1 follows as a direct consequence of P.

$\Longrightarrow$ (*Necessary Condition*) — Suppose e does not belong to every spanning tree of G. Let T be a spanning tree that does not contain e. Then the spanning tree T is a subgraph of $G \setminus \{e\}$. It follows then that for any arbitrary vertices u, v in $G \setminus \{e\}$ there is a unique (u, v) path in T. This is also a (u, v) path in $G \setminus \{e\}$ since $T \subseteq G \setminus \{e\}$. Thus, $G \setminus \{e\}$ is connected, and e is not *reliable* in G.

$\Longleftarrow$ (*Sufficient Condition*) — If e is not reliable (and part of cycle) in G then its removal causes $G \setminus \{e\}$ to remain connected. So $G \setminus \{e\}$ must have a spanning tree. But such a spanning tree would also be a spanning tree of G since it has the same vertex set, but it wouldn't contain e.

□

> **Claim 2** — The minimum spanning tree of G does not contain any *treacherous* edge.

Proof. Suppose (for the sake of contradiction) that there is a minimum spanning tree T containing a *treacherous* edge $e = (u, v)$ of some cycle of G. Let the cycle containing that edge e be C.

Let $T' = T \setminus \{e\}$. Since T is a spanning tree, e is *reliable* in T (not necessarily in G). Thus, T' has two connected components. Because e was part of cycle C, there is a path connecting u and v in G not containing e. This path must contain an edge with endpoints in different connected components of T'. Let such edge be e'. Because e' and e are in C and by assumption e is the edge with maximum weight in C, it follows that $w(e') < w(e)$.

Then define $T' = T \setminus \{e\} \cup \{e'\}$. Note T' is connected as e' created a path between the two connected components of $T \setminus \{e\}$. Additionally, T' has the exactly number of edges as T, so T' is a valid spanning tree of G. But its weight:

$$w(T') = w(T) - w(e) + w(e') < w(T)$$

Thus, T' is a spanning tree of G with smaller weight than T—the minimum spanning tree of G. But this contradicts the initial supposition of minimality of T in G. Hence if $e \in T$ then T cannot be the minimum weight spanning tree for G. □

Algorithm Description

We proceed as follows. We iteratively delete the highest *treacherous* edge until the resulting graph becomes acyclic. Since we only delete *treacherous* edges, the resulting graph stays connected. Since we stop when the graph becomes acyclic, the output will indeed be a spanning tree. Since the only edges removed were edges by Claim 2 not contained in any minimum weight spanning tree, the resulting tree must have minimum weight and be the MST of G. The pseudocode for this is provided below.

Algorithm Pseudocode

Algorithm 1 MINIMUM-SPANNING-TREE($G \leftarrow (V, E)$)

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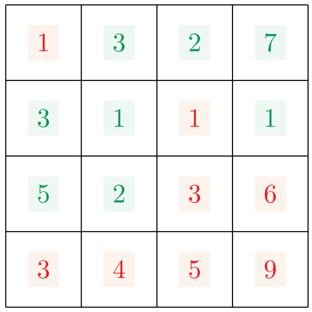
1: T ← REVERSE-SORT-BY-WEIGHT(E)
2: for each e ∈ T do
3:     if T \ {e} is connected then
4:         T ← T \ {e}
5:     end if
6: end for
7: return T
```

Time Analysis

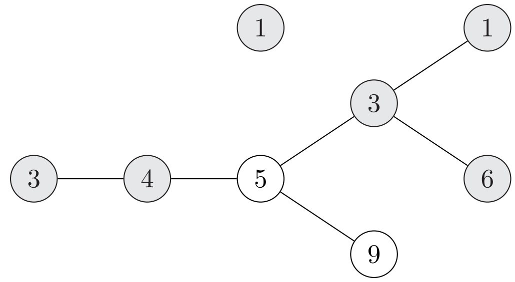
The implementation of Algorithm 1 begins by sorting the edges in $O(E \log E)$ time. For each edge we check whether $T \setminus \{e\}$ is connected and we do so via DFS. Each call to DFS takes $O(V + E)$, and doing for all entries of E takes $O(E(V + E))$. Since G is connected, E dominates V asymptotically, so the total running time is $O(E^2)$.

Q3. *Algorithm Description*

We proceed by modeling the game of Crusade as an undirected graph $G = (V, E)$ with vertices representing regions controlled by the current player and edges representing non-empty borders between these regions. We distinguish vulnerable regions from safe regions by coloring their corresponding vertices in G in *gray*. An exemplar of this construction is provided below.



(a) State of the game. Player 1 has their meeple colored in *red* and player 2 has theirs in *green*. We model player 1 defense strategy.



(b) A graph G capturing the state of the board for player 1. As can be inferred from the graph, vulnerable regions are colored in *gray*.

Figure 1. Graph construction/transformation.

After the construction of G, we query and extract the disconnected subgraphs of G containing at least one vertex colored in *gray*. These are precisely the connected regions of G we want to fortify its borders against invading neighbors. To perform this fortification, we use the MAX-FLOW algorithm to dictate how many meeple pieces to move from one region to another and fortify regions of most need. Here conservation constraints will guaranteed that no meeple gets stuck in intermediate regions of the board during a migration. We discuss the details of the algorithm below which boils down to solving the circulation with demands problem.

Firstly, we imagine the vertices of G having supplies and demands. That is, if a vertex has supply of 2 then it can spare 2 more meeples than it receives. By symmetry, if a vertex has demand of 2 then it wants to receive 2 more meeples than it sends to its neighbors. We assing each vertex v a value σ_v that represents the "excess" at v—i.e., the difference between incoming meeples and outgoing ones. Note that σ_v represents demand and when $\sigma_v < 0$ it represents supply. Then, the flow of conservation constraint could be replaced by the requirement that for each vertex v:

$$\sum_{e=(*,v)} f_e - \sum_{e=(v,*)} f_e = \sigma_v$$

We model this as a classical maximum flow problem by adding a source s and a sink t. Whenever, a vertex v has $\sigma_v > 0$, we add a directed edge $(v \to t)$ with capacity σ_v in the network flow. Likewise, whenever a vertex v has $\sigma_v < 0$, we add and edge $(s \to v)$ with capacity $-\sigma_v$. We want to saturate all the edges going to t to check if there is a way to satisfy all the demands in the network. If such demand is satisfied, then the maximum flow problem gives us the desired "excess" in each vertex, and consequently makes the weakest vulnarable region as strong as possible via the aforementioned demand and supply model.

Time Analysis

Let m be the number of rows in the board and n the number of columns. Below is an itemized list of operations and costs incurred by the aforementioned algorithm:

1. Constructing G takes $O(mn)$.
2. Extracting the connected components of G takes $O(V + E)$.
3. Filtering connected components that contain at least one vulnerable vertex takes $O(V)$.
4. Running MAX-FLOW and assigning integral capacities takes at most $O(VE^2)$ via Edmonds-Karp.

Hence the total running time of $O(VE^2)$.