Maximizing Quasi-Linear Utility Function

Max $a \ln x + y$ s.t. $x \geq 0, y \geq 0, p_x x + p_y y = M$.

$$\mathcal{L}(x, y, \lambda) = a \ln x + y + \lambda(M - p_x x - p_y y).$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{a}{x} - \lambda p_x \leq 0, \quad x \geq 0 \text{ with CS} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - \lambda p_y \leq 0, \quad y \geq 0 \text{ with CS} \quad (2)$$

$$M - p_x x - p_y y = 0. \quad (3)$$

$x = 0 \Rightarrow y = \frac{M}{p_y}$; $y > 0 \Rightarrow 1 - \lambda p_y = 0$

$\Rightarrow \lambda = \frac{1}{p_y}$;

This in (1)

$\Rightarrow \frac{a}{x} - \frac{p_x}{p_y} < 0$ not possible for $x = 0$.

Case 1) $x > 0, y = 0$. $\Rightarrow x = \frac{M}{p_x}$.

(1) $\Rightarrow \frac{a}{M/p_x} = \lambda p_x \Rightarrow \frac{a}{M} = \lambda$

(2) $\Rightarrow 1 - \frac{a}{M} p_y \leq 0 \Rightarrow M \leq a p_y$.

Case 2) $x > 0, y > 0$.

$\Rightarrow \frac{a}{x} = \lambda p_x$

$1 = \lambda p_y$

$\Rightarrow \frac{a}{x} = \frac{p_x}{p_y}$.

$\Rightarrow x = a \frac{p_y}{p_x}$

$\Rightarrow p_x \left(a \frac{p_y}{p_x}\right) + p_y y = M$

$\Rightarrow p_y y = M - a p_y$

$\boxed{y = \frac{M - a p_y}{p_y}}$, $x = \frac{a p_y}{p_x}$.

As $M \uparrow$es from 0 to $a p_y$, consume $x = \frac{M}{p_x}$; then keep it fixed at $x = \frac{a p_y}{p_x}$ & consume y with ~~of~~ remaining income.

Recall the arbitrage argument:

If I give Re 1 to good x, I buy $\frac{1}{p_x}$ unit & get approximate utility

$$dU = \frac{\partial u}{\partial x} \cdot \frac{1}{p_x} = \frac{a}{x} \cdot \frac{1}{p_x};$$

& in good y, I get $\frac{\partial u}{\partial y} \cdot \frac{1}{p_y} = \frac{1}{p_y}$.

$$\frac{a}{x} \cdot \frac{1}{p_x} > \frac{1}{p_y} \text{ as long as}$$

$$x < \frac{a p_y}{p_x} \Longleftrightarrow p_x x \lessgtr a p_y.$$

More generally for $u(x, y)$:

$$dU = u(x+dx, y+dy) - u(x, y) \simeq u_x \, dx + u_y \, dy$$

~~At an optimum~~

Feasibility $\Rightarrow$ sell dy, get income $p_y \, dy$, buy ~~let~~ $\frac{p_y \, dy}{p_x}$ units of good x, & get

$$u_x \cdot \frac{p_y \, dy}{p_x} - u_y \, dy = 0$$

$$\Rightarrow u_x \cdot \frac{p_y}{p_x} - u_y = 0$$

$$\Rightarrow \frac{u_x}{u_y} = \frac{p_x}{p_y}.$$