HUL320: Topics in IO

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Contents

Chapter 1

Introduction

1.1 Some Examples

Industrial Organization studies industries in which there are at least some firms with large market shares, that have market power. By market power, we will mean that the firm has the ability to set the prices of its products above (or well above) what it costs to produce them. Market power can result in 'deadweight loss', a welfare loss to society as a whole; on the other hand, in certain contexts, market power can be correlated with faster innovation. We will discuss both, but we'll not be obsessed with only these aspects of industrial organization. Instead, we'll explore much more broadly how technology, consumer tastes, the potential for strategic behaviour by firms, and yes, regulation, influence what products get innovated and produced, how markets get organized and how well they function and deliver. Here are a few preliminary examples of industries, firms and issues that our course content will talk to.

Construction is a very important sector in India, contributing more than 5 % to GDP, and is a huge employer. Two large industries that feed into it are cement and steel. Take cement - it has a high C4 of well over 50% (aggregate market share of top 4 firms). Ultra Tech owned by the Aditya Birla group is the market leader with ¿ 25% market share, followed by Ambuja and ACC (together maybe close to Ultra Tech, followed by Shree).

The market is concentrated, and the Competition Commission of India (CCI) also accuses the big players of price fixing. In fact, CCI has pending cases against some of them: for instance, against Ambuja and ACC, it imposed penalties of over Rs. 2000 cr. Despite the size of the firms and alleged collusion, mergers or large cross-holdings have been permitted from time to time. For instance, the Swiss firm Holcim and Ambuja bought majority stake in ACC around 2005; then Holcim bought majority stake in Ambuja in 2006; with a common management, did Ambuja and ACC display signs of becoming a merged entity? Recently, Holcim exited India. Adani bought out its stakes in Ambuja and ACC, beating bids of Ultra Tech and others; in response, Ultra Tech is investing in increasing its capacity. Our analyses will consider the ramifications of firm-size, price-fixing, mergers, entry of new players or firms (such as Adani here), and responses of incumbent firms.

If 2 firms in the cement industry merge, we call that a *horizontal* merger, since the product category of the firms is the same. There are also vertical mergers, or firms that are vertically integrated across product categories. In early August 2022 (a few days ago), the market capitalization of AMD overshot that of Intel, to reach around \$ 160 billion. Is there any difference in the AMD and Intel business models that is possibly giving some current advantage to AMD, over the long time market leader Intel?

Possibly. While Intel is vertically integrated across chip design and manufacturing, AMD is now a pure design firm. Its designs are fabricated by (perhaps mostly) the Taiwanese giant pure play foundry TSMC, which does only chip manufacturing. While Intel manufactures chips designed internally by Intel alone, TSMC manufactures chips based on designs from AMD *and* other firms. Does this give it access to a larger market size than Intel manufacturing? If so, it is possibly more successful right now in spreading its huge fixed cost of R&D and manufacturing across this larger market size. We will see that in products with large fixed cost and low marginal cost, such as software, but even chip manufacturing, a large market size/share is key to competitive advantage. Note that 'marginal cost' is the cost of producing one extra unit of a product. For instance, making one additional copy of Windows OS is the marginal cost of producing this OS: it is practically zero: the upfront, high fixed costs pertain to creating, developing and testing software.

An example of a high fixed cost, low marginal cost setting with potentially far-reaching impact was the entry of Reliance Jio into the Indian Telecom industry. Existing players such as Airtel and Vodafone had spent a lot buying 3G spectrum and were more focused on developing that, when Jio bid in the 4G Auction and entered. With 4G, it leapfrogged over 3G technology; 4G does not have circuit-switching; all is data, including voice calls. Jio incurred very large fixed costs in winning spectrum in the 4G auction, and laying an India-wide 4G network. Having done that, its marginal cost of providing bandwidth was relatively low, and it priced extremely competitively, to the point that the commentariat speculated on possible exits of all players other than Jio.

But Jio had plans to not be just a telecom firm; it was working to provide platforms, and more generally, to be at the centre of digital innovation in the future. For instance, for Jio Platforms to take on Amazon in India. We will study platforms in the last part of the course. When you log into the Ola app, that is a platform that intermediates ride demands of consumers and ride supplies from taxis. When you pay after your ride, it is through a fintech payment gateway that connects your bank to a final receiving bank through a network of connections that enable and authenticate the transaction. When you're watching YouTube, the Google Ads platform connects firms that advertise with potential consumers like you. What Platforms do, how they operate, how intermediation differs from the textbook firm in a first Micro course, what is the role of search frictions in our lives and what do platforms do to alleviate these: we will try to spend some time on these issues.

Issues of technology, fixed costs, market structure, search frictions, platforms, will come up in the latter part of the course. We will begin with plain vanilla demand and supply, and perfect competition and monopoly, as for a bunch of you this is your first exposure to a formal economics course. Perfect competition, and demand and supply diagrams are a natural setting to discuss the notion of allocative efficiency; and a monopoly is a natural setting to discuss the notion of deadweight loss. We will also discuss the fascination problem of price (and quality) discrimination in the context of a monopoly. Consider an airline. How does it decide the prices and proportions of its business class and economy class tickets? This is a great early application of information economics, with strong connections to the field of Mechanism Design.

We will then discuss standard, benchmark models of oligopolies - the industries with a few large firms that form the mainstay of IO. These models include the Cournot, Bertrand and repeated Cournot and Bertrand models of collusion. At this juncture, we may also treat some of the important non-price strategies of firms, such as product differentiating, branding and advertising (although this happens later in the prescribed textbook by Cabral).

And then we will move to considerations of entry, technology, market structure. Followed by search frictions, and platforms.

1.2 Discrete Demand and Supply

Two friends A, B meet. A has an IPL ticket but doesn't really want to go for the match. She is willing to sell the ticket for any price greater than or equal to INR 500. We say her willingness to accept (WTA) equals 500. B is keen to go and watch the match, and is willing to pay up to INR 2000 (WTP). The ticket is allocated efficiently if it is allocated to the person who values it more: so if A gives the ticket to B, the allocation is efficient. Since A loses something worth 500 to her, and B gains something worth 2000, in the aggregate, there is a gain whose money equivalent is $2000 - 500 = 1500$.

If A sells the ticket to B for some price p, $500 \leq p \leq 2000$, then both persons are better off than before (if $500 < p < 2000$). We call the original situation as one from where we can make a Pareto improvement: i.e., make at least one person better off without making the other worse off.

What is the money equivalent gain from such a trade? The money equivalent gain for B equals $2000 - p$, and that for A equals $p - 500$. In the aggregate, the gain is therefore

$$(2000 - p) + (p - 500) = 1500$$

as before.

More generally, with two agents, agent A valuing an object she possesses at s and agent B valuing it at v, with $v - s > 0$, $v - s$ is the potential gains from trading the object. Any price p such that $s \leq p \leq v$ results in a Pareto-improving trade, with both agents being at least weakly better off.

Now consider a setting in which there are N potential sellers each with 1 IPL ticket for the match, and WTAs being $s_1, s_2, \ldots, s_N$; and N potential buyers interested in 1 buying 1 ticket each; with WTPs $v_1, v_2, \ldots, v_N$. To fix ideas, let's use the following values:

$$(v_1, v_2, \ldots, v_N) = (4400, 4000, 3600, 3000, 2600, 2000, 1200, 800)$$

$$(s_1, s_2, \ldots, s_N) = (3800, 3400, 2800, 2400, 2000, 1400, 1000, 600)$$

How might a naïve planner (or a platform) match the demand and supply of tickets here? At first blush, the planner in his naïveté might match high WTP buyers to high WTA sellers: so, buyer i and seller i trade, at some price p_i with $s_i < p_i < v_i,\ i = 1, \ldots, n$. This matching will result in gains from trade equal to

$$\sum_{i=1}^{N} (v_i - s_i) = 4200$$

since from each match i, the gains from trade equal the sum of buyer i's and seller i's payoffs:

$$(v_i - p_i) + (p_i - s_i) = v_i - s_i$$

.

A more experienced or clever planner notices that this matching does not maximize gains from trade. One matching that does maximize this is obtained by asking: If we could implement only 1 trade, which agents would we want to trade to maximize gains from trade? This would be buyer 1 with $v_1 = 4400$ and seller N with $s_N = 600$, leading to gains $v_1 - s_N = 3800$. Next, proceed to match the buyer with the second highest value, v_2, with the seller with the second from lowest WTA, s_{N-1}, giving a gain from trade of $4000 - 1000 = 3000$. Already, the gains from these two trades sum to $3800+3000 = 6800$, much larger than the *total* achieved by the naïve planner. After 5 trades matching buyer i and seller $N+1-i$ in this fashion, 3 buyers and 3 sellers remain, with no (or negative) gains to be made from any trade. The clever planner stops here, and you can calculate that the gains from these 5 trades exceeds 10,000.

Note that this sum equals

$$\sum_{i=1}^{5}(v_i - s_{N+1-i}) \tag{1.1}$$

This allocation, where tickets are ultimately in the hands of buyers 1,2,3,4,5 and sellers 6,7,8, satisfies the *utilitarian criterion* that the 8 tickets should be allocated to the agents who value them the most: indeed,

$(v_1, v_2, s_8, v_3, s_7, v_4, s_6, v_5) = (4400, 4000, 3800, 3600, 3400, 3000, 2800, 2600)$

are the 8 highest valuations for the tickets in this setting. So, allocating tickets to these 8 agents achieves *allocative efficiency*: it maximizes the aggregate value to the economy from these 8 tickets.

Here is one way in which Adam Smith's Invisible Hand of the market is modelled: it uses Walras' notion of a fictitious auctioneer and a certain honest behaviour by the buyers and sellers to motivate the textbook demand and supply diagram. The buyers and sellers, with WTPs and WTAs privately known to them, assemble in a market with an auctioneer. The auctioneer starts with a very low price quote or very high price quote. Let's try this procedure with our WTP/WTA numbers above.

Suppose the auctioneer starts by asking 'who all want to buy and sell at price p=100?'. Suppose all buyers i with WTP (v_i) greater than 100 raise their hands, and all sellers j with $p > s_j$ raise their hands (i.e., all agents who profit/get positive/non-negative payoff from trading at $p = 100$ raise their hands). Since even the lowest v_i, i.e. $800 > p = 100$, all 8 buyers raise their hands; and since all WTAs / s_j's are greater than $p = 0$, no seller raises their hand. So, at price $p = 100$, there is *excess demand* for IPL tickets.

So the auctioneer raises the price, in small increments (say of 200; the auctioneer does not know the WTPs and WTAs). At $p = 1100$, 2 sellers raise their hand as they are willing to supply; and 7 buyers raise their hands. The auctioneer raises the price further. When the price is 2300, 4 sellers are willing to supply and 5 buyers are willing to buy; then when the price is incremented to $p = 2500$, 5 sellers are willing to sell and 5 buyers are willing to buy. Demand equals Supply; the auctioneer implements all trade at $p = 2500$ and clears the market.

What are the total gains from trade here? They equal:

$$\sum_{i=1}^{5}(v_i - 2500) + \sum_{i=4}^{8}(2500 - s_i)$$

It is easy to see this maximizes gains from trade; it adds up to exactly the sum in Equation (1.1). See Figure 1. It is *as if* buyer i is matched to seller $N + 1 - i$, for $i = 1, \ldots, 5$.

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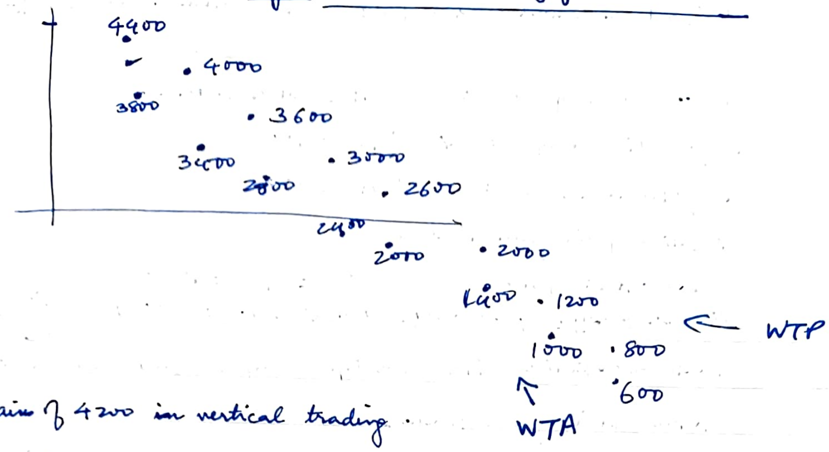
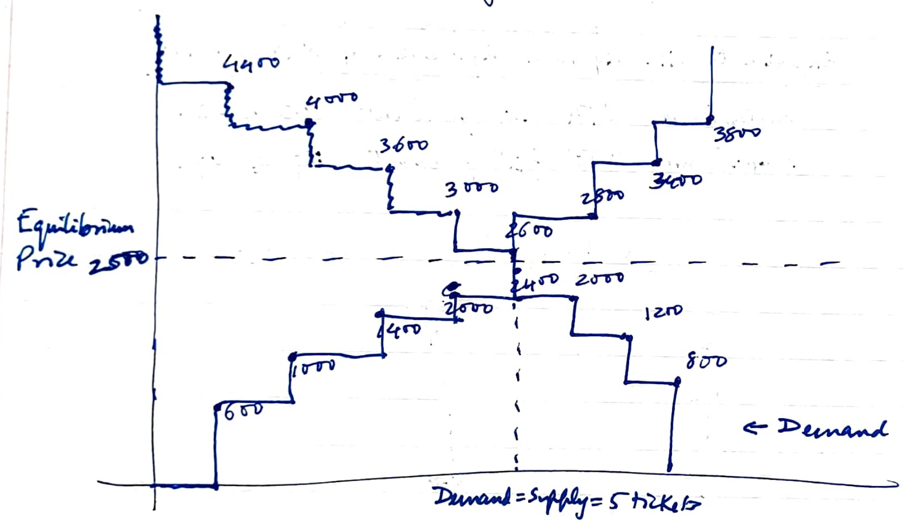
Fig 1 : Demand & Supply of IPL tickets



Gains of 4200 in vertical trading.



Gains : 3800 + 3000 + 2200 + 1000 + 200 = 10,200 .

: Does the Double Auction replicate the allocative efficiency of the Walrasian Auctioneer ?

Finally, we did a class experiment with the above WTPs and WTAs assigned to 8 buyers and sellers respectively (or students role-playing as buyers and sellers); the WTPs and WTAs were privately known. In a version of Gode and Sunder (1993), the agents shouted out bids and asks, and settled in any buyer-seller pair whenever both agreed to do so. They were free to determine when to agree etc. The only restriction was that sellers not ask below their WTA and buyers not bid above their WTP. Trading was opened for 3 minutes.

There was one 'error', in which the agreed price was below the seller's WTA. Otherwise, pretty much the top 5 buyers according to WTP in descending order, and sellers from WTA in ascending order, settled. Among these, note that the lowest WTP of 2600 is greater than the highest WTA of 2400. Observe that no matter what prices different pairs settled at between their WTP and WTA, the gains from trade are exactly

$$\sum_{i=1}^{5} v_i - \sum_{i=4}^{8} s_i$$

i.e. it maximizes gains from trade in this economy à la Equation (1.1).

1.3 Production and Supply

We discuss now the standard economics underlying supplying and demand curves more generally, starting with supply.

The supply curve for a product tells us how much output the firms in an industry together wish to produce (supply) at each price. In an industry with large firms, this is usually not a well-posed question, as a large firm knows that its choice of output will change the price itself, substantially. So the existence of a supply curve is best analyzed in industries that have a large number of small firms, each with very little influence on market price as their individual production volumes are too low relative to the size of the market.

Good examples include agricultural markets, where the firms are individual farmers or households. So to begin with, consider the production problem of a single farmer who wishes to produce a crop, say toor dal. The farmer has access to a technology that converts inputs to output of dal. We will call such a technology a *production function*. To produce toor dal, various inputs are used: land, labour, capital (tractors and other machinery), water, seeds, fertilizer, pesticides; and indeed, with these multiple inputs, multiple outputs can be produced: early-maturing toor dal sown in North India has enough space between rows to grow a second crop (e.g. maize); some of the inputs such as irrigation, could be common; I don't know whether nitrogen-fixation by toor dal nodules would also provide nutrients for maize. But you get the picture.

We capture this with an n-input, m-output production function $f : \mathbb{R}^n_+ \to \mathbb{R}^m_+$. For simplicity, we will work with $m = 1$, a many-input, one-output production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$. Agricultural production is subject to weather shocks, and we can model this by assuming that if the level of n inputs is the vector $(x_1, \ldots, x_n)$, then the output equals $f(x_1, \ldots, x_n)\epsilon$, where ϵ is a random variable.

1.3.1 Profit Maximization and Supply with a Single Input

For now, we abstract from all of this, and start with a very simple production function, in which the farmer has a fixed amount of land, and uses only labour to produce his/her output of toor dal: the production function gives the output $y = f(L)$ in some units (say, quintals), from an application of L units of labour.

We will look at the farmer's problem twice: first, as a problem of choosing L to maximize profits; then, as a problem of choosing output y to maximize profits. The two ways are equivalent here, because we will assume that the production function f is strictly increasing: so there is a bijection between L and y over the range of f.

The farmer takes the market price or expected market price of output, p, as given; and also takes the unit of price of labour, w, as given. From hiring L units of labour, the farmer's profit equals

$$\pi(L) = pf(L) - wL$$

where $pf(L) \equiv py$ is the revenue obtained from selling the y units of toor dal produced, and wL is the amount of money paid out in wages to L units of labour. Profit is the difference between total revenue and total cost.

The farmer's problem is to choose the amount of L that will maximize profit $\pi(L)$, given p and w. That is,

$$\max \pi(L) = pf(L) - wL$$

over all $L \geq 0$.

We assume the following about the production function f: It is twice continuously differentiable, and

$f' > 0, f'' < 0, f(0) = 0, f'(0) = \infty, f'(\infty) = 0.$

Notice that $f'' < 0$, or strict concavity of f, translates to strict concavity of π, since

$\pi''(L) = pf''(L) < 0$ for all L. So, a solution to

$\pi'(L) \equiv pf'(L) - w = 0$ is a global profit-maximizing L.

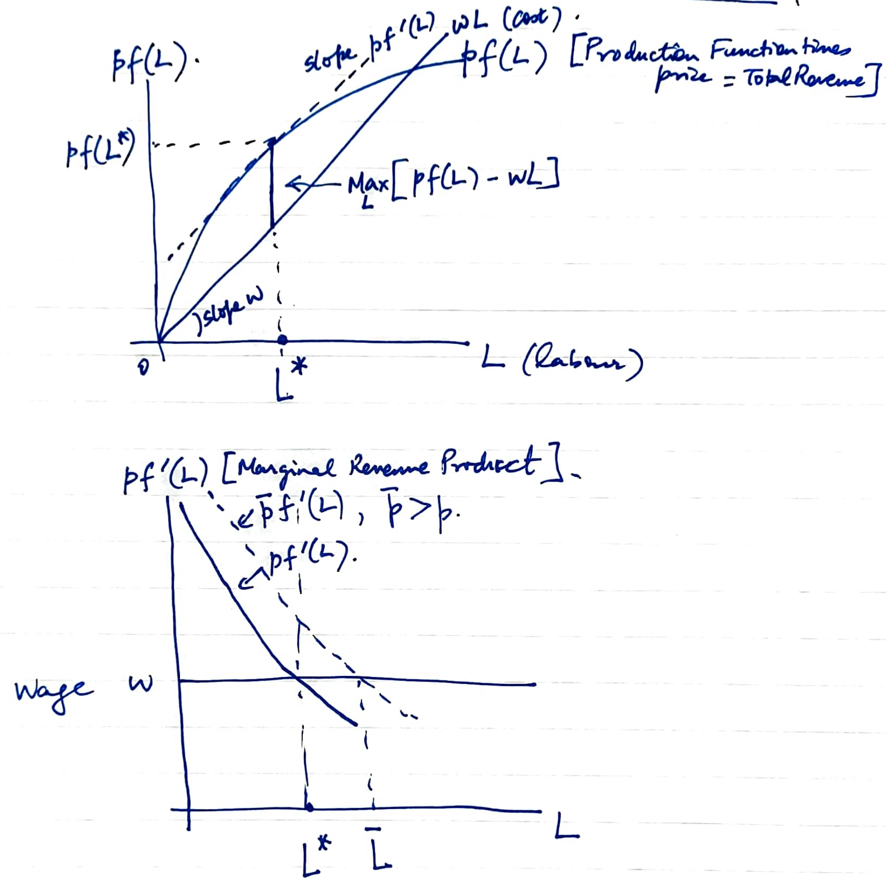
Since $pf'(0) = \infty$ and $pf'(\infty) = 0$, and $f'' < 0$, $pf'(L)$ is strictly decreasing; and decreases from a very high positive number for small L to close to 0 for large L. So by continuity of f' (and the Intermediate Value Theorem), there is a *unique* L s.t.

$$pf'(L^*) - w = 0. \tag{1.2}$$

You can see the picture in Fig. 2.

Fig. 2.

Single Input; Profit Maximizing Farmer, in terms of input L



$pf'(L)$ is the slope of the revenue curve $pf(L)$, and w is the slope of the total cost wL, with respect to L. $f'(L)$ is called the *marginal product of labour*: in common parlance, this is the change in output that results from a small increase in labour L ; this sort of captures the limit argument that underlies the derivative $f'(L)$. $pf'(L)$ is called the *value of marginal product of labour* or *marginal value product of labour*. The economic intuition behind the first-order condition (Eq. (1.2)) is this: we consider increasing L in small increments, starting from low or zero L. $pf'(L)$ is the addition to revenue for a small increase in labour, (call this a 'unit' increase in labour), and w is the increase in cost for a unit increase in labour. At L close to 0, $pf'(L)$ is extremely high; much higher than w; so it makes sense to increase L, adding more to revenue than to cost. As L is increased gradually, the additional revenue increases at lower rates, since $f'' < 0$, so the additional revenue minus additional cost, i.e. $pf'(L) - w$ gap starts to close. When $pf'(L) - w = 0$, the gap has closed. Any further increase in L implies $pf'(L) - w < 0$; so it is best to stop when $pf'(L) - w = 0$.

The second part of Fig. 2 directly plots pf' and w as functions of L. w is a constant, and $pf'(L)$ decreases in L; where these two lines intersect captures $pf'(L) = w$.

We also answer now: at price p (and given wage w), what output y will the farmer supply? The answer is: the supply will be $y^* = f(L^*)$, where L^* is the profit maximizing amount of labour that the farmer will hire.

How does output y change as price p increases?

The second part of Fig. 2 plots $pf'(L)$ and $\bar{p}f'(L)$, where $\bar{p} > p$. We see that the profit maximizing amount of labour increases from L^* to $\bar{L}$; correspondingly, output increases with the increase in price.

In terms of the logic, to 'start out', we had $pf'(L^*) = w$. Now suppose price increases to $\bar{p}$. At the labour level L^*, we now have

$$\bar{p}f'(L^*) > w$$

.

To re-equilibrate the first-order condition, L needs to go up, until $pf'(L)$ decreases to re-equal w; this happens at $L = \bar{L}$.

In terms of math, how L (and therefore $y = f(L)$) responds to an increase in output price p can be seen by applying the Implicit Function Theorem. When the *parameters* (p, w) are fixed in the farmer's profit maximization exercise, and L is the variable that is chosen optimally, and so solves Eq. (1.2), let's write

$$F(p, w, L^*) \equiv pf'(L^*) - w = 0$$

Applied to this setting, where F is twice continuously differentiable since we assume f is so, the implicit function theorem will say that if the partial derivative F_L (i.e. $\partial F(p, w, L)/\partial L$; also written DF_L) is not zero, then there is an open set $A \subseteq \mathbb{R}^2$ containing (p, w), and an open set $B \subseteq \mathbb{R}$ containing L, and a unique (implicit) function $L(\bar{p}, \bar{w}) : A \to B$ that is twice continuously differentiable, such that

$$F(\bar{p}, \bar{w}, L(\bar{p}, \bar{w})) \equiv \bar{p}f'(L(\bar{p}, \bar{w})) - \bar{w} = 0 \tag{1.3}$$

for all $(\bar{p}, \bar{w}) \in A$; that is if we vary price and wage by a small amount around (p, w), $L(p, w)$ gives the value of L that will re-equate the two sides of equation (1.2). This exercise includes the possibility of changing *only* p or only w as well. The response of L to changes in p and w can be worked out using differentiation.

For instance, if we perturb p and $L(p, w)$ adjusts to re-equate Eq. (1.2), then $F(p, w, L(p, w))$ will stay at 0; i.e. differentiating Eq. (1.3) w.r.t. p, and evaluating at the 'initial' point (p, w, L^*), we have

$$F_p(p, w, L^*) + F_L(p, w, L^*)L_p(p, w) = 0$$

where F_p, F_L, L_p are partial derivatives, and the stuff inside the parentheses is the point (p, w, L^*) at which these partial derivatives are evaluated. This boils down to

$$f'(L^*) + pf''(L^*)L_p(p,w) = 0 \tag{1.4}$$

or

$$L_p(p,w) = \frac{-f'(L^*)}{pf''(L^*)}$$

which is positive since $f' > 0, f'' < 0$.

So, L, and the supply of $y = f(L)$, increase as p increases.

To complete the discussion, we can 'sign' $L_w(p,w)$, the response of profit maximizing L to a change in the wage w. Differentiating Eq. (1.3) w.r.t. w, and evaluating at the point (p,w,L^*), we have

$$F_w(p,w,L^*) + F_L(p,w,L^*)L_w(p,w) \equiv -1 + pf''(L^*)L_w(p,w) = 0 \tag{1.5}$$

So $L_w(p,w) = \frac{1}{pf''(L^*)} < 0$.

Note. We can put together the above discussion more compactly in the language of the Implicit Function Theorem stated in the appendix (optional). For this, we consider the function $F(p,w,L(p,w)) \equiv pf'(L(p,w)) - w$ as a composition of 2 functions h and g, in which g maps from $(p,w) \mapsto (p,w,L(p,w))$ and h maps from $(p,w,L) \mapsto pf'(L) - w$.

So we can treat F as a function of the parameters (p,w), and taking matrix derivatives, get

$DF_{(p,w)}(p,w,L) = (F_p(p,w,L) + F_L(p,w,L)L_p(p,w) \quad F_w(p,w,L)L_w(p,w)) = (0 \quad 0)$

which is exactly what we get by stacking Equations (1.4) and (1.5) in the same row.

Seen more tediously, since

$g(p,w) \equiv (g_1(p,w), g_2(p,w), g_3(p,w)) = (p,w,L(p,w))$ goes from $\mathbb{R}^2$ to $\mathbb{R}^3$,

its derivative $Dg(p,w)$ is a 3×2 matrix of partial derivatives: Row 1 is the matrix $Dg_1(p,w) = (1 \quad 0)$, Row 2 is the matrix $Dg_2(p,w) = (0 \quad 1)$, and

Row 3 is $Dg_3(p,w) = (L_p(p,w) \;\; L_w(p,w))$.

Since $h(p,w,L) = pf'(L) - w$, its derivative is the 1×3 matrix
$Dh(p,w,L) = (h_p(p,w,L)\; h_w(p,w,L)\;\; h_L(p,w,L)) = (f'(L) \;\; -1 \;\; pf''(L))$
Since $F(p,w,L(p,w)) = h(g(p,w))$, we have by the Chain Rule:
$DF(\;\;) = Dh(g(p,w))Dg(p,w))$

$$\begin{pmatrix} f'(L) & -1 & pf''(L) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ L_p(p,w) & L_w(p,w) \end{pmatrix} \tag{1.6}$$

This is exactly the left hand sides of Equations (1.4) and (1.5) stacked in a row. We equate this to $(0 \;\; 0)$ and solve for $(L_p(p,w) \;\; L_w(p,w))$.

The appendix to this chapter states the Implicit Function Theorem more generally, for those interested.

Choosing Output to Maximize Profit

What is the *total cost* $C(y)$ of producing quantity y of output? Total cost is measured as a function of *output*. Suppose that it takes labour quantity L to produce the specified quantity y, so $y = f(L)$. Since $f' > 0$, f is invertible, so $L = f^{-1}(y)$. In this single input example, the total cost of producing this specified y is the labour cost wL, where $L = f^{-1}(y)$. Therefore, total cost

$$C(y) = wf^{-1}(y)$$

.

Marginal Cost at y is defined as the derivative $C'(y)$; and in words, as the increment to total cost $C(y)$ that results from a small increase in output.

So in this single input case,

$$C'(y) = \frac{w}{f'(f^{-1}(y))}$$

by the inverse differentiation rule. This makes sense since looking at Fig. 2, which plots L on the horizontal axis and y on the vertical axis, so $dy/dL = f'(L)$, the inverse f^{-1} is seen by visualizing y on the horizontal axis and L on the vertical axis. So, its derivative is just going to be the reciprocal of $f'(L)$. We can also see this from

$$y = f(L) = f(f^{-1}(y))$$

and differentiate both sides w.r.t. y and use the Chain Rule on the right-hand side.

Notice that $C''(y) = \frac{-wf''(f^{-1}(y))/f'(f^{-1}(y))}{(f'(f^{-1}(y)))^2} > 0$.
That is, total cost $C(y)$ is a strictly convex function of y.

The farmer chooses y to Maximize

$$\pi(y) = py - C(y)$$

the difference between the total revenue py from the sale of y units of output, and the total cost $C(y)$ of producing y. Since py is linear in y and $-C(y)$ is strictly concave in y, $\pi(y)$ is strictly concave in y; so setting its first derivative equal to 0 and solving for y characterizes the profit maximizing y:

$$\pi'(y) = p - C'(y) = 0$$

Since $C'' > 0$, $C'(y)$ is strictly increasing in y. See Fig. 3. Fig. 3 therefore says that for any given output price p, to see what is the profit maximizing output y, look at where the Marginal Cost curve $C'(y)$ becomes equal to p.

In fact, the supply of y equals $y(p) = C'^{-1}(p)$. **This is in fact** the supply curve of the farmer: it gives, for each price p, what output he will supply.

Now consider the entire set of farmers growing Toor Dal. They have conceivably different production relationships, if only because they may be cultivating land of different sizes. Let's capture this by saying that there are N farmers (N is very large); and Farmer i has production function $y_i = f_i(L_i), i = 1, 2, \ldots, N$.

Correspondingly, farmer i has total cost function $C_i(y_i)$ and Marginal Cost function $C_i'(y_i)$.

At each price p, for farmer i, the profit maximizing supply is

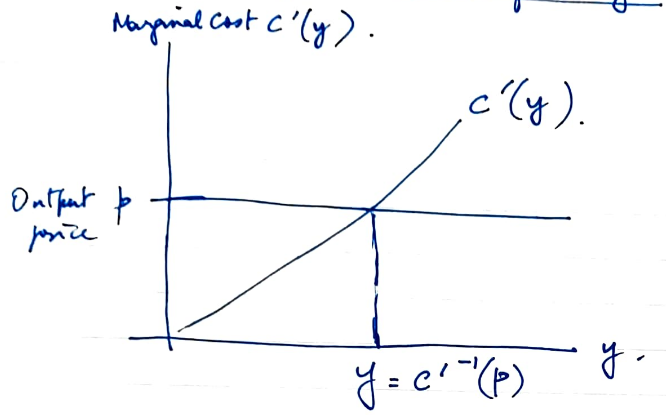
$$y_i(p) = C_i'^{\ -1}(p)$$

The **industry supply** at price p equals $\sum_{i=1}^{N} y_i(p)$.

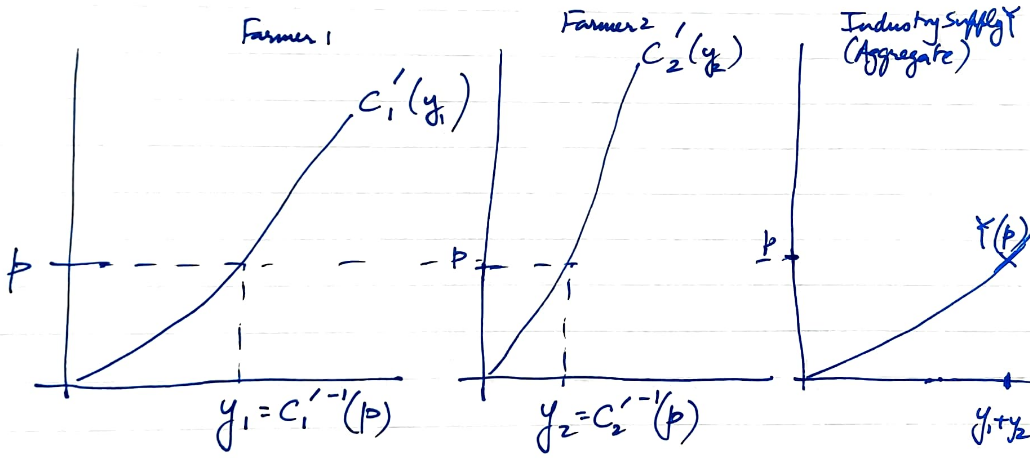
The second panel in Fig. 3 shows the usual horizontal summation of individual farm's/firm's supply curves to get the industry supply curve.

Fig. 3.

Profit Maximizing Farmer, in terms of Output y.



Industry supply if $N = 2$ farmers



1.3.2 Production and Supply with Multiple Inputs

We can make the more realistic assumption that for our farmer, even though land may be fixed, labour, capital (tractors and other machines), fertilizers, etc. are not fixed (and maybe he or she can lease in or lease out land, so even that is not fixed). To correspond to standard exposition, we take the case here of a small firm (or farm) for which all of its inputs can vary: at given input prices, the farmer can hire whatever amounts of the inputs he or she deems fit.

To fix ideas, take a simple case of 2 variable inputs, capital K and labour L, and let the production function be

$$y = f(K, L) = AK^a L^b, a > 0, b > 0$$

A could be land quality for a farmer, as before, or more generally how productively he or she can use inputs. This functional form is called Cobb-Douglas. What happens if both inputs are scaled (multiplied) by a factor λ? We have:

$$f(\lambda K, \lambda L) = A(\lambda K)^a(\lambda L)^b = \lambda^{a+b} f(K, L)$$

Output gets scaled by λ^{a+b}.

If $(a+b) = 1$, doubling inputs doubles output; if $(a+b) > (<)1$, doubling inputs more than, or less than, doubles output. We say:

The production function exhibits *constant, increasing or decreasing returns to scale* respectively, as

$$a + b \begin{cases} = & 1 \\ > & 1 \\ < & 1 \end{cases}$$

The partial derivatives $f_K(K, L), f_L(K, L)$ evaluated at any (K, L) are respectively called the **marginal product of capital** and **marginal product of labour** at (K, L); (often written MP_K and MP_L). For example, we may say informally that MP_L evaluated at (K, L) is the extra output obtained if we increment L by a small amount, *holding constant* the level of K.

Unlike the single input case earlier, we can see that *any output level* y can be produced with different combinations of K and L. For ease of notation, let $A = 1$ so $y = K^a L^b$. Fix y at any positive level. We can ask: What is the set of all input combinations that can produce y? This is the set

$$S(y) = \{(K, L) \in \mathbb{R}^2_+ | y = f(K, L) = K^a L^b\}$$

This is the the *level curve* or *contour set* of the production function $f(K, L)$ at the level y. See Fig. 4. This is called an **isoquant** in economics.

Fix any y. For various levels of L, we can implicitly solve for $K(L)$ such that

$$f(K(L), L) = y$$

provided f is continuously differentiable and $f_K(K, L) \neq 0$. This is the implicit function theorem applied here. For our function $y = K^a L^b$, with $0 < a < 1, 0 < b < 1$, we can indeed solve explicitly for $K(L)$. Anyway, let's stick to the implicit representation

$$(K(L))^a L^b = y$$

and differentiate w.r.t. L. We have

$$(K(L))^a b L^{b-1} + L^b a (K(L))^{a-1} K'(L) = 0$$

so $K'(L) = -\frac{bK(L)}{L} < 0$

and differentiating once more w.r.t. L (and substituting for $K'(L)$ from above, we have

$$K''(L) = -\frac{b}{a} \left(\frac{\frac{-b}{a} K - K}{L^2} \right) > 0$$

So, the level curve $S(y)$ is downward-sloping and convex. From the equation $K^a L^b = y$, we see also that it has the horizontal and vertical axes as asymptotes.

Now, our farmer wishes to choose K, L, and therefore $y = f(K, L)$ to maximize profit. To do so, whatever y he or she ends up producing *must be produced using a combination of* (K, L) *that minimizes the cost of producing this* y: otherwise, he/she is not maximizing profits, as the same y can be produced at lower cost.

So we will study profit maximization in 2 steps.

Step 1: First, for every possible output y, what is the *minimum cost* $C(y)$ at which it can be produced, by a judicious combination of K and L?

Step 2: Second, given this *cost function*, or *minimum cost function* $C(y)$, choose y to Maximize profit $\pi(y) = py - C(y)$.

Step 1 (A): Cost Minimization - Theory

The farmer can hire each unit of labour at wage w, and each unit of capital at capital rental rate r.

Step 1: $C(y) = \min(rK + wL)$ subject to $f(K, L) = y$.

In other words, given a target output level y, choose $(K, L) \in S(y)$, the level set of y, that minimizes cost $wL + rK$. Higher target y will mean higher cost $C(y)$, and so on. This is a *constrained optimization* problem. We will first solve it graphically, in Fig. 4.

Consider the equation $rK + wL = c$, for $K \geq 0, L \geq 0$. This is a level curve of the function $g(K, L) \equiv rK + wL$ at level c. It is a straight line with slope $-w/r$. It is called an **isocost curve**: combinations of K, L that cost the same amount c. As we increase c, the isocost curve moves farther Northeast from the origin.

The isoquant $S(y)$ in Fig. 4 has all (K, L) that produce y. To get the (K, L) that does so at minimum cost, we find the isocost curve closest to the origin that contains (K, L) that can produce output y. You can see that this happens on the isocost curve that is *tangent to the isoquant* $S(y)$.

At the tangency, the slopes of the isocost and isoquant curves are equal. For any production function $f(K, L)$, on an isoquant $f(K, L) = y$, if $f_K \neq 0$ there exists a unique implicit function $K(L)$ s.t. $f(K(L), L) = y$, at (K, L) and in an open set around L. Differentiating both sides w.r.t. L, we have

$$f_K(K,L)K'(L) + f_L(K,L) = 0, \qquad \text{or}$$

$$K'(L) = -\frac{f_L(K,L)}{f_K(K,L)}$$

the (negative of) the ratio of MP_L and MP_K at (K,L).

The slope of the isocost $g(K,L) = rK + wL = c$ is similarly

$$-\frac{\partial g(K,L)/\partial L}{\partial g(K,L)\partial K} = -\frac{w}{r}$$

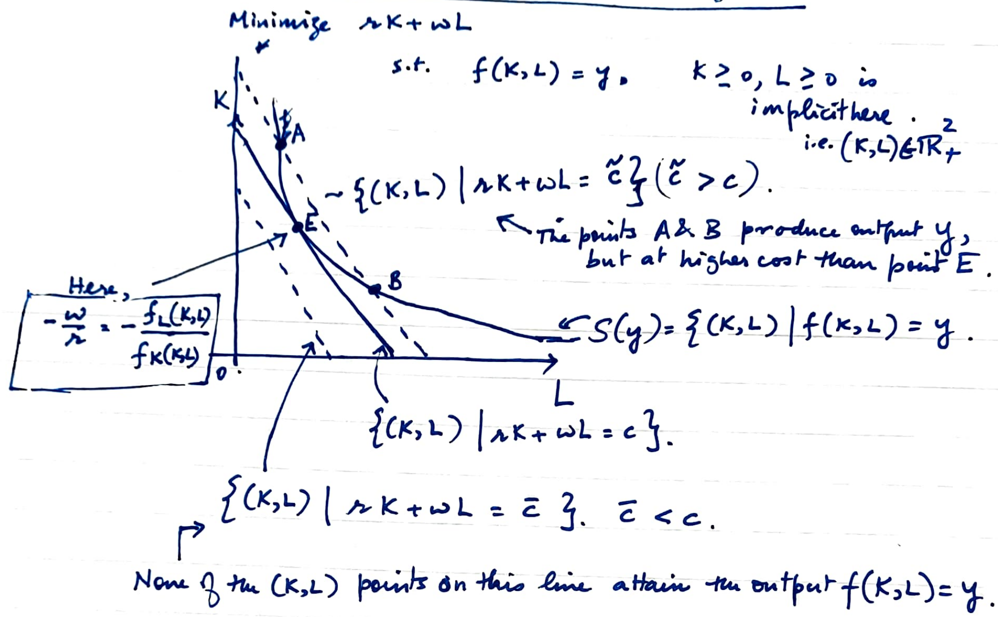
the (negative of) the ratio of the 'factor prices' for labour and capital. So, at a cost minimum for producing output y, *the ratio of factor prices equals the ratio of marginal products*:

$$\frac{f_L(K,L)}{f_K(K,L)} = \frac{w}{r} \tag{1.7}$$

(1.7) is one equation in the 2 unknowns (K,L). $f(K,L) = y$ is the second equation in these unknowns. We can solve these simultaneously for the cost minimizing levels of K and L, and get the minimum cost $C(y)$ accordingly. But we postpone this for now and look more closely at (1.7) first.

Fig. 4.

Isoquant, Isocost lines, and Cost Minimization



Here is some intuition for Eq.(1.7) in terms of an 'arbitrage' argument. Suppose we are at the point (K, L) that minimizes cost $rK+wL$ s.t. $f(K, L) = y$. Then, increasing or decreasing L by a small amount dL, and adjusting K by dK to maintain $f(K + dk, L + dL) = y$, should not reduce cost any further. Work with a first-order Taylor approximation:

$$f(K + dK, L + dL) - f(K, L) \approx f_K(K, L)dK + f_L(K, L)dL = 0$$

so $dK = -\frac{f_L dL}{f_K}$, where I have suppressed the argument (K, L) in f_K and f_L.

These changes should not reduce cost: So,
$rdK + wdL = -r\frac{f_L dL}{f_K} + wdL = \left(-\frac{rf_L}{f_K} + w\right) dL \geq 0$
This implies:
If $dL > 0$, then $\left(-\frac{rf_L}{f_K} + w\right) \geq 0$; and
if $dL < 0$, then $\left(-\frac{rf_L}{f_K} + w\right) \leq 0$.
Together, $\left(-\frac{rf_L}{f_K} + w\right) = 0$. This is (1.7).

A small unit increase or decrease in L costs, or saves cost, of w; capital must adjust by $-f_L/f_K$ to maintain output at y; so that saves, or costs, rf_L/f_K. Since we are at a cost minimum, net-net there should be no change; so $w = rf_L/f_K$.

We want to give one last equation from which we can derive (1.7). Note we will use $g(K, L) = rK + wL$, so that the derivative matrix $Dg(K, L) \equiv (g_K \quad g_L) = (r \quad w)$. We will also feel free to instead write these partial derivatives in a *vector*, the *gradient* of g, and call it $\nabla g(K, L)$, which is regarded as a column vector.

Similarly, $\nabla f(K, L) = (f_K(K, L) \quad f_L(K, L))^T$, where superscript T stands for transpose. The tangency condition $f_L/f_K = w/r$ in (1.7) can be obtained from the following equation, which we state as a little result: it is just a special case of the Theorem of Lagrange.

Theorem 1. *Suppose that the vector (K, L) solves the problem of minimizing $g(K, L) = rK + wL$ s.t. $f(K, L) = y$ and $K \geq 0, L \geq 0$; and g and f are* *continuously differentiable. Suppose this $K > 0, L > 0$ and $f_K(K, L) \neq 0$. Then there exists a number $\lambda \neq 0$ such that*

$$\nabla g(K, L) = \lambda \nabla f(K, L) \tag{1.8}$$

This Theorem says that if we have an interior cost minimum at (K, L), then the vectors $\nabla g(K, L)$ and $\nabla f(K, L)$ are *linearly dependent*. **Appendix B sketches a proof of this Theorem, for those interested**.

Writing (1.8) as 2 equations, noting that $\nabla g(K, L) = (r, w)^T$, we have:

$$\begin{aligned} r &= \lambda f_K(K, L) \\ w &= \lambda f_L(K, L) \end{aligned} \tag{1.9}$$

Dividing the second of the equations by the first gives $w/r = f_L/f_K$, which was our (1.7).

This suggests the following method for solving for the cost-minimizing (K, L) using first derivative equals 0 type of *unconstrained conditions*. Set up the Lagrangean function:

$$\mathcal{L}(K, L, \lambda) = g(K, L) - \lambda(f(K, L) - y)$$

.

Get the partial derivatives of $\mathcal{L}(\ )$ w.r.t. K, L and λ respectively, equate them each to 0, and solve. Since the theorem gives a necessary condition, we may need to ensure that our solution is actually a cost minimum. Carrying this out, we have:

$$\begin{aligned} r - \lambda f_K(K, L) &= 0 \\ w - \lambda f_L(K, L) &= 0 \\ f(K, L) - y &= 0 \end{aligned} \tag{1.10}$$

These are the **first-order conditions** for this constrained optimization problem. Let us apply this to $f(K, L) = K^a L^b$.

Step 1 (B)- Application: Cost Minimization with Cobb-Douglas Production Function

$C(y) = \min(rK + wL)$ subject to $K^aL^b = y$ and $K, L \geq 0$.

Here, with positive output y, both K and L need to be positive in order that $f(K,L) \equiv K^aL^b = y$. Also, $f_K(K,L) = aK^{a-1}L^b > 0$. So, the above theorem of Lagrange applies. We set up the Lagrangean, write the first-order conditions, and solve. We will find a unique solution, so that must be the cost minimizing (K,L).

$$\mathcal{L}(K,L,\lambda) = rK + wL - \lambda(K^aL^b - y)$$

.

First-order conditions:

$$\begin{aligned} r - \lambda a K^{a-1}L^b &= 0 \\ w - \lambda b K^a L^{b-1} &= 0 \\ K^aL^b - y &= 0 \end{aligned} \tag{1.11}$$

Division gives

$w/r = bK/aL$, or $K = (wa/rb)L$.

Substitute this into the constraint:

$((wa/rb)L)^a \, L^b = y$ so

$L^{a+b} = (wa/rb)^{-a}y$ or

$L = (wa/rb)^{-a/(a+b)}y^{1/(a+b)}$

$K = (wa/rb)(wa/rb)^{-a/(a+b)}y^{1/(a+b)}$ so

$K = (wa/rb)^{b/(a+b)}y^{1/(a+b)}$.

Since the cost-minimizing K, L must be solutions to the first-order conditions (1.10), and since we see here that there is a *unique* (K,L) solution, this K and L must be the cost-minimizing K and L.

To get the minimum cost $C(y)$, we substitute the above cost-minimizing choices of K, L in the expression $rK + wL$. We have:

$C(y) = r(wa/rb)^{b/(a+b)}y^{1/(a+b)} + w(wa/rb)^{-a/(a+b)}y^{1/(a+b)}$

This simplifies to

$$C(y) = Br^{a/(a+b)}w^{b/(a+b)}y^{1/(a+b)} \tag{1.12}$$

where $B = \left((a/b)^{b/(a+b)} + (a/b)^{-a/(a+b)}\right)$, a constant.

To use just the fact that total cost $C(y)$ depends on y, let us suppress the w, r parameters and write

$$C(y) = \bar{B} y^{1/(a+b)}$$

where $\bar{B} = B r^{a/(a+b)} w^{b/(a+b)}$.

Marginal Cost is then

$$C'(y) = \frac{\bar{B}}{a+b} y^{(1-a-b)/(a+b)}$$

We see that Marginal Cost $C'(y) > 0$. So, the *total cost* $C(y)$ is increasing; the *total cost curve*, which plots $C(y)$ against y, is upward sloping.

What about the Marginal Cost curve, that plots $C'(y)$ against y? To see whether it is upward-sloping or not, consider

$$C''(y) = \frac{\bar{B}(1-a-b)}{(a+b)^2} y^{(1-2a-2b)/(a+b)}$$

Notice that

$$C''(y) = \begin{cases} > 0 & if (a+b) < 1 \\ = 0 & if (a+b) = 1 \\ < 0 & if (a+b) > 1 \end{cases}$$

So, if $(a+b) < 1$, i.e. when $f(K, L) = K^a L^b$ has *diminishing returns to scale*, $C''(y) > 0$,the Marginal Cost curve is upward-sloping, and the Total cost curve is convex.

If $(a+b) = 1$, i.e. when $f(K, L) = K^a L^b$ has *constant returns to scale*, $C''(y) = 0$,the Marginal Cost curve is flat since Marginal cost $C'(y)$ is constant, and the Total cost curve is linear.

If $(a+b) > 1$, i.e. when $f(K, L) = K^a L^b$ has *increasing returns to scale*, $C''(y) < 0$,the Marginal Cost curve is downward sloping, and the Total cost curve is concave.

See Fig. 5.

Fig. 5

Returns to Scale & MC, TC curves.

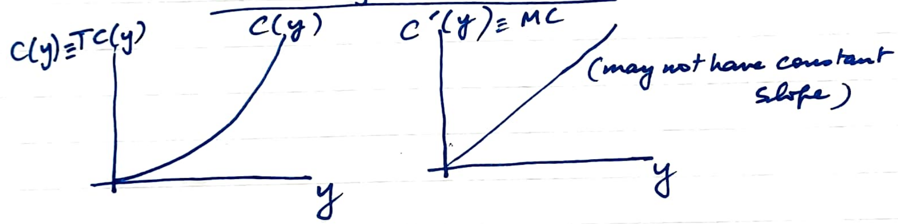
- $y = f(K, L)$ is homogeneous of degree k if $\lambda^{k} y = f(\lambda K, \lambda L)$.

e.g. $y = K^{a} L^{b}$ is homogeneous of degree $(a+b)$.

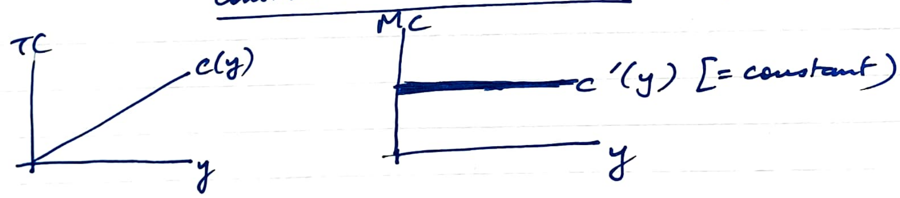
We can show: $C(y) = \underbrace{\bar{B}(w, r)}_{\text{some function of } w \,\&\, r} . \, y^{k}$

Accordingly, we have:

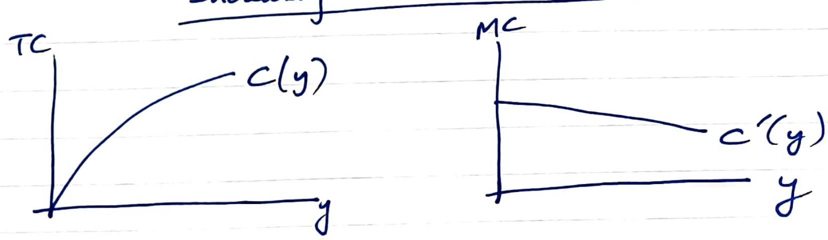
Diminishing returns to scale : $k < 1$



Constant Returns to scale $k = 1$.



Increasing Returns to Scale $k > 1$.



Step 2. Profit Maximization

Having obtained his or her minimum cost $C(y)$ of producing any output level y, our small farmer, taking the crop output price p as given, is ready to choose the output level y that maximizes profit (Revenue minus Cost). This amounts to solving

$$\max \pi(y) = (py - C(y))$$

If $\pi(y)$ is strictly concave, the unique y that solves the first-order condition

$$\pi'(y) = p - C'(y) = 0$$

maximizes profit. For $\pi(y)$ to be strictly concave, we require that $C(y)$ be strictly convex: to get this strict convexity with the Cobb-Douglas production function constraint $f(K, L) = K^a L^b = y$, for example, this requires that $(a + b) < 1$, or f displays diminishing returns to scale.

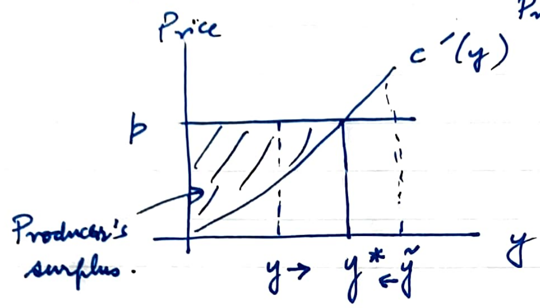
Let's analyze this $C(y)$ strictly convex case first.

There is a unique y such that $\pi'(y) = 0$ or $p = C'(y)$. Call this output y^*, to give it a specific name here. I reproduce this in the first panel of Fig. 6, from Fig. 3. Notice the *economic logic* behind the profit maximizing y^*. Starting at $y = 0$, the farmer can ask whether producing a small unit of y is profitable: the incremental revenue from this equals p, the sale price of this unit of y, and the incremental cost is $C'(y)$. For $y = 0$, and indeed for any $y < y^*$, we see from Fig. 6 that $p > C'(y)$. So producing this incremental unit is profitable for any $y < y^*$.

For $y > y^*$, *reducing* output a little is profitable, because here $p < C'(y)$: selling one less unit implies p less in revenue, but $C'(y)$ less in cost, and this cost reduction is larger. And so profit is maximized at $p = C'(y)$.

Fig. 6.

Profit Maximization (with Diminishing Returns to Scale Production Function)



In Fig. 6, the area between the price equals p line, and the $C'(y)$ curve, upto $y = y^*$, equals the maximum profit $\pi(y^*) = py^* - C(y^*)$. Indeed, this area equals:

$$\int_0^{y^*} (p - C'(t))\, dt$$

By the Fundamental Theorem of Calculus, this integrates to:

$$py^* - C(y^*) + C(0)$$

$= \pi(y^*)$, as $C(0) = 0$.

In economics, the area is actually called **Producer's Surplus**, and can be subtly different from profit depending on whether the firm/farmer incurs any fixed cost even if he or she produces zero output.

As before, the supply curve of the farmer answers: what outputs y will the farmer produce and sell at each price p? We can read these off from the price and marginal cost diagram in Fig. 6; so in fact, the marginal cost curve **is** the farmer's supply curve here, if s/he has decided to produce any positive amount at all. In other words, supply

$$y(p) = C'^{-1}(p)$$

as before, and the sum of all these over all farmers producing Toor Dal is the industry supply curve of Toor Dal.

Constant returns to scale and profit maximization

We saw that under constant returns to scale $(a + b) = 1$ in $f(K, L) = K^a L^b$, marginal cost $C'(y)$ is a constant, say $C'(y) = k$.

So, $\pi'(y) = p - k$.

If $p > k$, each incremental unit of output earns positive profit, so the farmer/firm will want to supply an infinitely large amount of output y. If $p < k$, the farmer would want to supply zero output.

The analysis is therefore interesting only if $p = k$, in which case the farmer is willing to supply any output at all, at zero profit. In this case, aggregating over all farmers' supply decisions, the supply curve is flat at $p = k$.

Increasing returns to scale and profit maximization

Increasing returns does not sit well with the price-taking behaviour of firms in perfect competition. For our Cobb-Douglas production function, with $a+b>1$ we have that $C'(y)$ is downward sloping and goes to zero; so if a small firm takes output price p as given, then for y high enough, $p > C'(y)$, and beyond this, expansion of output indefinitely only adds to profit.

1.4 Consumption and Demand

The supply of a product at a price p is modelled as the sum over the supply at this price by each individual firm; and the supply of an individual firm is determined by the firm choosing supply to maximize its profit, given price p and input prices.

To model demand for a product at a price p, we similarly add over the demand by each individual consumer; and the demand by an individual consumer is determined by the consumer choosing demand to maximize 'utility', subject to being constrained by the amount of income or wealth that he or she has to spend across all goods and services.

We consider a consumer who can purchase bundles (vectors) of L goods, at prices $p_1, p_2, \ldots, p_L$. To keep things uncluttered, we take $L = 2$, say toor dal (good 1) and roti (or kapRa)(good 2). The *starting point* is that the consumer can make binary comparisons between any 2 vectors of goods, and can rank one over the other. Suppose the two bundles are $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ (where x_i, y_i are quantities of good i in the two bundles). We say $\mathbf{x} \succeq \mathbf{y}$ if the consumer thinks $\mathbf{x}$ is at least as good as $\mathbf{y}$. Suppose X is the set of all pairs/bundles/vectors of the two goods that the consumer has available to consider, to make a decision about which bundle to choose. We will later assume $X = \mathbb{R}^2_+$ (and more generally equals $\mathbb{R}^L_+$). But keep that aside for a bit.

We say $\succeq$ is *complete* on X if for all $\mathbf{x}, \mathbf{y} \in X$, either $\mathbf{x} \succeq \mathbf{y}$, or $\mathbf{y} \succeq \mathbf{x}$, or both, are true.

If $\mathbf{x} \succeq \mathbf{y}$ is true and $\mathbf{y} \succeq \mathbf{x}$ is not true, we say $\mathbf{x}$ is *strictly preferred to* $\mathbf{y}$ and write $\mathbf{x} \succ \mathbf{y}$.

If both $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$ are true, we say the consumer is *indifferent between* $\mathbf{x}$ and $\mathbf{y}$, and write $\mathbf{x} \sim \mathbf{y}$.

We will also write $\mathbf{x} \succeq \mathbf{y}$ as $\mathbf{y} \preceq \mathbf{x}$; here, too, we can say $\mathbf{x}$ is at least as good as $\mathbf{y}$.

We say $\succeq$ is *transitive* if for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$, $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$ both holding implies $\mathbf{x} \succeq \mathbf{z}$.

People were for long interested in whether we could 'represent' such a consumer's preferences numerically, i.e. using a real-valued function. Could there be a real-valued function $u : X \to \mathbb{R}$ such that $u(\mathbf{x}) \geq u(\mathbf{y})$ if, and only if, $\mathbf{x} \succeq \mathbf{y}$ is true? One basic result is this:

Theorem 2. *Suppose X is a finite set and suppose $\succeq$ defined on X is complete and transitive. Then there exists a function $u : X \to \mathbb{R}$ that represents $\succeq$; i.e., for all $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \succeq \mathbf{y}$ if, and only if, $u(\mathbf{x}) \geq u(\mathbf{y})$. Moreover, if $g : \mathbb{R} \to \mathbb{R}$ is any increasing function, then $g \circ u$ also represents $\succeq$.*

Proof. Since X is finite, it has K elements, for some positive integer K. Let's index the elements (vectors/bundles) in X any which way, and write them as $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$.

Since $\succeq$ is complete, each pair of successive elements $\mathbf{x}_i, \mathbf{x}_{i+1}$ satisfies either $\mathbf{x}_i \succeq \mathbf{x}_{i+1}$ or $\mathbf{x}_i \preceq \mathbf{x}_{i+1}$. We want to rearrange the elements in the list to look like (after renaming elements)

$\{\mathbf{y}_1 \succeq \mathbf{y}_2 \succeq \mathbf{y}_3 \succeq \cdots \succeq \mathbf{y}_K\}$

We will sort by induction. Suppose $K = 1$. It is then trivial to sort this singleton set. Now suppose we have a fully sorted set of $K = n-1$ elements:

$\{\mathbf{z}_1 \succeq \mathbf{z}_2 \succeq \cdots \succeq \mathbf{z}_{n-1}\}$.

Consider the nth element $\mathbf{z}_n$. If $\mathbf{z}_{n-1} \succeq \mathbf{z}_n$, just tack on this last element on the right, to the sorted list of $n-1$ elements. If $\mathbf{z}_{n-1} \prec \mathbf{z}_n$, and $\mathbf{z}_{n-2} \succeq \mathbf{z}_n$, put the nth element second to last and the $(n-1)$th element last; if $\mathbf{z}_{n-1} \prec \mathbf{z}_n$, and $\mathbf{z}_{n-2} \prec \mathbf{z}_n$, continue by comparing $\mathbf{z}_{n-3}$ and $\mathbf{z}$, and so on.

In at most $(n-1)$ comparisons, we will end with a sorted list of n elements from our sorted list of $(n-1)$ elements and the element $\mathbf{z}_n$.

Essentially, this is insertion sort, and we will have to make at most $(K-1)K/2$ comparisons to get

$\{\mathbf{y}_1 \succeq \mathbf{y}_2 \succeq \mathbf{y}_3 \succeq \cdots \succeq \mathbf{y}_K\}$.

Then just construct a function $u : X \to \mathbb{R}$ by, say, having $u(\mathbf{y}_K) = 0$ and $u(\mathbf{y}_i) = (K-i)/(K-1)$ for all i.

Notice that assigning utility numbers $u(\mathbf{y})$ for $\mathbf{y} \in X$ can be done in a zillion ways, *as long as* the ascending order of the numbers preserves the preference comparison list

$\{\mathbf{y}_1 \succeq \mathbf{y}_2 \succeq \mathbf{y}_3 \succeq \cdots \succeq \mathbf{y}_K\}$,

starting from $\mathbf{y}_K$. ■

Note.

We call u above a **utility function**, and we talk about consumers and demand in terms of their utility functions, keeping preferences in the background. Notice that if u represents $\succeq$ on X, then so does $g \circ u$, where $g : \mathbb{R} \to \mathbb{R}$ is an increasing function. Indeed, $u(\mathbf{x}) \geq u(\mathbf{y})$ if and only if $g(u(\mathbf{x})) \geq g(u(\mathbf{y}))$.

Notice also that for any pair of vectors/bundles, $\mathbf{x} \succ \mathbf{y}$ if and only if $u(\mathbf{x}) > u(\mathbf{y})$. Indeed, $\mathbf{x} \succ \mathbf{y}$ implies $\mathbf{x} \succeq \mathbf{y}$, (so $u(\mathbf{x}) \geq u(\mathbf{y})$), but not $\mathbf{y} \succeq \mathbf{x}$. This implies $u(\mathbf{y}) \geq u(\mathbf{x})$ cannot be true; or else we would have the implication $\mathbf{y} \succeq \mathbf{x}$. Together, $u(\mathbf{x}) \geq u(\mathbf{y})$), but not $u(\mathbf{y}) \geq u(\mathbf{x})$ implies $u(\mathbf{x}) > u(\mathbf{Y})$.

By a similar argument, if both $\mathbf{x} \succeq \mathbf{y}$, and $\mathbf{y} \succeq \mathbf{x}$ are true, then $u(\mathbf{x}) = u(\mathbf{y})$.

What if the set X of choices is **uncountably infinite**? In the standard model, $X = \mathbb{R}^L_+$, so X has the cardinality of the reals. It turns out that $\succeq$

being complete and transitive is not sufficient for a numeric representation to exist. I'm making an optional, technical aside here: additionally, the strict preference relation $\succ$ needs to have a property that $>$ has in the set of real numbers: namely, that for any reals x, y with $x > y$, there is a number z from a countable subset of $\mathbb{R}$, such that $x > z > y$. We know that for the real numbers, the rational numbers form a countable subset that have this property: due to this, we say that the set of rational numbers is a *dense subset* of the set of real numbers. Analogously, for the consumption set X, and for the strict preference $\succ$, there needs to be some countable subset Q such that for all consumption bundles $\mathbf{x}, \mathbf{y}$, there is a $\mathbf{z} \in Q$ such that $\mathbf{x} \succ \mathbf{z} \succ \mathbf{y}$: we then say that the preference relation is *order-dense.*

In Microeconomics, a more intuitive assumption is used to clinch utility representation: that $\succeq$ should be *continuous* on X. I will not define continuity of $\succeq$ here. The result is stated below.

Theorem 3. *(Debreu (1953)): Suppose $X = \mathbb{R}^L_+$, and suppose $\succeq$ is complete, transitive and continuous. Then there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succeq$.*

In addition to these assumptions, we will also assume that preferences satisfy *monotonicity* and *strict convexity*:

Monotonicity (*more is better*): If $\mathbf{x} \geq \mathbf{y}$, then $\mathbf{x} \succeq \mathbf{y}$; and '$>$' implies '$\succ$'.

The utility function u that represents the preferences $\succeq$ gives rise to a very useful object called an *indifference curve.* An indifference curve corresponding to a specific utility level, say $\bar{u}$, is the set of all bundles of goods that give this level of utility: let's call this

$$I(\bar{u}) = \{\mathbf{y} \in X | u(\mathbf{y}) = \bar{u}\}$$

So, an indifference curve is just a level set or contour set of the real-valued function u.

If $\succeq$ satisfies Monotonicity, then indifference curves must be downward-sloping. This is illustrated in Panel 2 of Fig. 7: all bundles in set S NorthEast of A, are strictly preferred to the bundle A, because they have more of both goods. So, all those bundles give higher utility than $u(A)$; similarly, $u(A)$ is a higher utility number when compared to utilities obtained from bundles in the set $\bar{S}$ SouthWest of A. So, bundles giving the same utility as A must have one of the goods is larger quantity than in bundle A, and the other good in smaller quantity: they must lie NorthWest and SouthEast of A. The set of all such bundles with utility equal to $u(A)$ is the indifference curve $I(u(A))$. Panel 2 of Fig. 7 depicts this as a downward-sloping, squiggly curve.

Strict Convexity: If given $\mathbf{x} \succeq \mathbf{y}$, then for every point $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ with $\lambda \in (0,1)$, $\mathbf{z} \succ \mathbf{y}$.

Note that $\mathbf{z}$ above is a point on the line segment joining $\mathbf{x}$ and $\mathbf{y}$.

In particular, this means that if $\mathbf{x}$ and $\mathbf{y}$ are on the same indifference curve, so $\mathbf{x} \sim \mathbf{y}$ (or $u(\mathbf{x}) = u(\mathbf{y})$), the points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ give strictly greater utility than $u(\mathbf{x}) = u(\mathbf{y})$.

This is illustrated in Panel 3 of Fig. 7. This property of preferences means that the consumer prefers averages or weighted averages of more extreme bundles. The property of a utility function representing this is that it is '*strictly quasiconcave*'. In terms of preferences, it captures preferences of a consumer who strictly prefers 'averages' or weighted averages of two bundles on any indifference curve, to the bundles themselves. She or he likes a more even mix of the two goods. Note that as in the second panel of Fig. 7, weighted averages of the bundles A and B on the indifference curve all lie on the line segment joining A and B (the set of all 'convex combinations'); and these points all lie on 'higher' indifference curves, that yield higher utility.

1.4.1 Utility Maximization and Demand

We will assume that the consumer's preferences on $X = \mathbb{R}^L_+$ can be described by a utility function $u : X \to \mathbb{R}$ that is twice continuously differentiable, strictly quasiconcave, and increasing. The consumer has income or wealth $w > 0$, faces positive prices $(p_1, \ldots, p_L)$ for the goods, and chooses non-negative quantities $(y_1, \ldots, y_L)$ to

$$\max u(y_1, \ldots, y_L) \quad s.t. \quad \sum_{i=1}^{L} p_i y_i \leq w$$

Because utility is increasing in $\mathbf{y}$, at the utility maximum, it must be that the 'budget constraint' holds with equality: $\sum_{i=1}^{L} p_x y_i = w$.

We will assume $L = 2$ so that we can draw stuff in $\mathbb{R}^2$. We will represent the solution to the utility maximization problem graphically, as well as set up the Lagrangean method; and solve for the specific utility function $u(y_1, y_2) = y_1^a y_2^b$.

For $\mathbf{y} \geq \mathbf{0}$, $\sum_{i=1}^{2} p_i y_i \equiv \mathbf{p} \cdot \mathbf{y} = w$ is the line segment in Fig. 7, Panel 1; here, $\mathbf{p} \cdot \mathbf{y}$ is the dot product or inner product of the price vector $\mathbf{p}$ and the goods or commodity vector/bundle $\mathbf{y}$. It is called the **budget line**. It is the set of all combinations (y_1, y_2) that cost exactly w. The slope of the budget line equals $-p_1/p_2$; in absolute value, this is the ratio of the prices of the two goods. If I am at some point on the budget line, and then I buy 1 more unit of good 1, I have to spend p_1; I need to give up p_1/p_2 units of good 2 so that I can stay on the budget line: as $1(p_1) - (p_1/p_2)p_2 = 0$. So, along the budget line, I can substitute p_1/p_2 units of good 2 by an extra unit of good 1. We could say this is the rate of substitution of the 2 goods that *the market* permits.

Notice also that if $\mathbf{y}$ and $\mathbf{z}$ are both on the budget line, then $\mathbf{p} \cdots \mathbf{y} = \mathbf{p} \cdots \mathbf{z} = w$, $\mathbf{p} \cdots (\mathbf{y} - \mathbf{z}) = 0$. That is, $\mathbf{p} \perp (\mathbf{y} - \mathbf{z})$. So in Panel 1 on Fig. 7, I have drawn the vector $\mathbf{p} = (p_1, p_2)$ orthogonal to the budget line.

Now consider an indifference curve and a point $\mathbf{y} = (y_1, y_2)$ on it. Since the curve is downward-sloping, y_2 is an implicit function of y_1 on the curve. In other words,

$$u(y_1, y_2(y_1)) = \bar{u}$$

,

so differentiating w.r.t. y_1, we have:
$u_1(y_1, y_2) + u_2(y_1, y_2) y_2'(y_1) = 0$, or

$y_2'(y_1) = -u_1(y_1, y_2)/u_2(y_1, y_2)$.

Here, the notation $u_j(y_1, y_2) \equiv \partial u(y_1, y_2)/\partial y_j,\ j = 1, 2$.

$y_2'(y_1)$ tells us by what rate y_2 must decrease, if y_1 increases, for the goods bundle to stay on the indifference curve. Restating the implicit function theorem in terms of small changes in the two goods while staying on the same indifference curve, we can say this: if dy_1 and dy_2 are small changes in y_1 and y_2 such that we stay on the indifference curve, then the first-order change is close to zero:

$$du = u_1(y_1, y_2)dy_1 + u_2(y_1, y_2)dy_2 \approx 0$$

or $dy_2/dy_1 \approx -u_1(y_1, y_2)/u_2(y_1, y_2)$.

This is called the *marginal rate of substitution* of good 2 for good 1: if I increment good 1, then good 2 must adjust at this rate for me to stay at the given utility level / stay on the given indifference curve.

Which of the combinations on the budget line maximizes utility? Suppose it is some vector $\mathbf{y}^* = (y_1^*, y_2^*)$. Changing good 1 consumption by dy_1, and adjusting dy_2 so that we stay on the budget line means

$$p_1 dy_1 + p_2 dy_2 = 0$$

So, $dy_2 = -p_1 dy_1/p_2$.

This should not lead to a first-order increase in utility, since we are at a utility max:

$$u_1(y_1^*, y_2^*)dy_1 + u_2(y_1^*, y_2^*)dy_2 = u_1(\ )dy_1 + u_2(\ )(-p_1 dy_1/p_2) \leq 0$$

That is

$$(u_1(\ ) - u_2(\ )(p_1/p_2))\, dy_1 \leq 0$$

If $dy_1 > 0$, it must be that $(u_1(\ ) - u_2(\ )(p_1/p_2)) \leq 0$;

If $dy_1 < 0$, it must be that $(u_1(\ ) - u_2(\ )(p_1/p_2)) \geq 0$.

Putting these together,

$$u_1(y_1^*, y_2^*) - u_2(y_1^*, y_2^*)(p_1/p_2) = 0$$

So, at the utility maximum,

$$u_1(y_1^*, y_2^*)/u_2(y_1^*, y_2^*) = p_1/p_2$$

.

This says the marginal rate of substitution (MRS) between goods 1 and 2 must equal the price ratio p_1/p_2. Let's belabour this point a bit more. Suppose $p_1/p_2 = 2$, so good 1 is twice more expensive than good 2: on my budget line, to increase y_1 by 1 unit, I need to reduce y_2 by 2 units; and if I reduce y_1 by 1 unit, I can increase y_2 by 2 units.

On the other hand, suppose that at (y_1, y_2), my MRS is 3. That is, I can increase y_1 by 1 unit, while reducing y_2 by 3 units, to stay at the same utility level. So if I do increase y_1 by 1 unit, giving up 2 units of y_2 in order to stay within my budget, I must have *increased* my utility. So, I was not maximizing utility. Hence it is necessary that $MRS = p_1/p_2$ at the utility max.

To get this condition using a diagram, in Fig. 7, we draw level sets of the utility function u: for a specified utility level $\bar{u}$, this is the set of all combinations (y_1, y_2) such that $u(y_1, y_2) = \bar{u}$. Recall that such a level set is called an **indifference curve**, as the consumer is indifferent between all these (y_1, y_2) combinations. Note that the '*more is better*', or '*monotonicity of preferences*' assumption implies that an indifference curve is downward sloping. If from a point $u(y_1, y_2) = \bar{u}$, and $z_1 > y_1$, then $u(z_1, y_2) > u(y_1, y_2)$; so (z_1, y_2) cannot be on the same indifference curve as (y_1, y_2). There must be a $z_2 < y_2$ such that $u(z_1, z_2) = \bar{u}$. Since (z_1, z_2) lies to the SouthEast of (y_1, y_2), this establishes that an indifference curve is downward sloping in our setting.

Again, because 'more is better', indifference curves fanning out North-East represent progressively higher utilities. So, utility is maximized at the point at which an indifference curve is *tangent* to the budget line, as in the third panel of Fig. 7. At this point, the slopes of the budget line and the indifference curve are equal: $p_1/p_2 = u_1/u_2$.

Finally, notice that at the utility maximizing $(y_1^*, y_2^*) \equiv \mathbf{y}^*$, the gradient $\nabla u(\mathbf{y}^*) = (u_1(y_1^*, y_2^*), u_2(y_1^*, y_2^*))$ (the vector of partial derivatives of the utility function at $\mathbf{y}^*$), is perpendicular to the tangent to the indifference curve at that point. To see this, take a differentiable curve $\mathbf{y}(s)$ along the indifference curve containing $\mathbf{y}^*$. Let t be such that $\mathbf{y}^* = \mathbf{y}(t)$, and $\mathbf{y}(t+h)$ be to the SouthEast of $\mathbf{y}^*$ on this indifference curve, as shown in Fig. 8. As h becomes closer to 0, the vector

$$\frac{\mathbf{y}(t+h) - \mathbf{y}(t)}{h}$$

tends to the tangent to the indifference curve at $\mathbf{y}^*$. So the derivative

$$\mathbf{y}'(t) = \lim_{h \to 0} \frac{\mathbf{y}(t+h) - \mathbf{y}(t)}{h}$$

points in the direction of the tangent.

Now, on the indifference curve, utility $u(\mathbf{y}(t)) = \bar{u}$; it is constant at $\bar{u}$. So, the derivative of this function with respect to t is 0:

$$u_1(\mathbf{y}^*)y_1^{'}(t) + u_2(\mathbf{y}^*)y_2^{'}(t) = 0$$

or

$$\nabla u(\mathbf{y}^*) \cdot \mathbf{y}'(t) = 0$$

.

So, just like the price vector (p_1, p_2), the gradient $\nabla u(\mathbf{y}^*) \equiv (u_1(\mathbf{y}^*), u_2(\mathbf{y}^*))$ is also orthogonal to the tangent to the indifference curve. So, it is perpendicular to the budget line. In other words, there is a nonzero scalar λ such that

$$\nabla u(\mathbf{y}^*) = \lambda(p_1, p_2)$$

or separating the two equations:

$$u_1(y_1^*, y_2^*) = \lambda p_1,\ u_2(y_1^*, y_2^*) = \lambda p_2$$

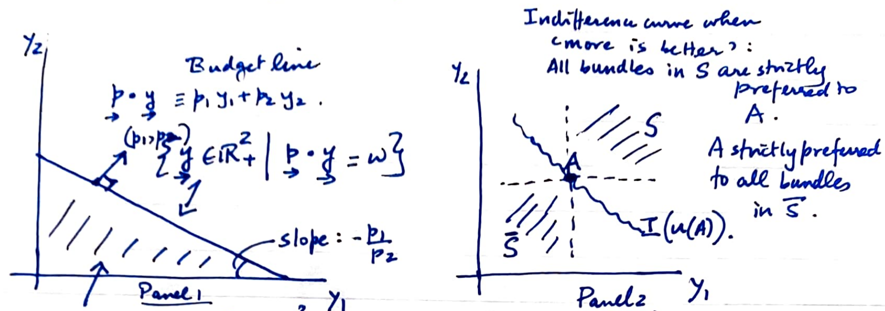
Dividing the first equation by the second, $u_1/u_2 = p_1/p_2$.

And so, we can frame the utility maximization subject to the budget constraint in terms of the Langrangean method.

IO Topics.

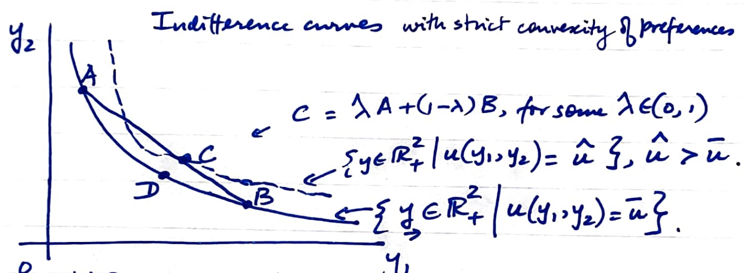
Fig. 7

Utility Maximization.



Budget Set: $\{\vec{y} \in \mathbb{R}^2_+ \mid \vec{p}\cdot\vec{y} \leq w\}$.

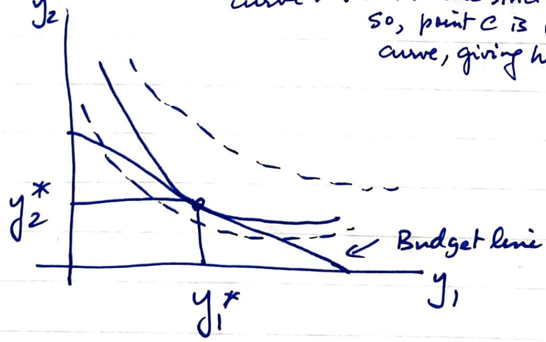
Panel 3



A & B are on the same indifference curve.

(All bundles on the line segment joining A & B are strictly preferred to A & to B: so, point C is on a higher indifference curve, giving higher utility).

Panel 4



Panel 4.

Lagrangean Method

Let us use this method for the utility function $u(y_1, y_2) = y_1^a y_2^b$.

$$\mathcal{L}(y_1, y_2, \lambda) = u(y_1, y_2) - \lambda(p_1 y_1 + p_2 y_2 - w)$$

First-order conditions:

$$\partial\mathcal{L}/\partial y_1 = u_1(y_1, y_2) - \lambda p_1 = 0 \quad (i)$$

$$\partial\mathcal{L}/\partial y_2 = u_2(y_1, y_2) - \lambda p_2 = 0 \quad (ii)$$

$$\partial\mathcal{L}/\partial \lambda = p_1 y_1 + p_2 y_2 - w = 0 \quad (iii)$$

Since $u_1 > 0, u_2 > 0$, $\lambda > 0$, so dividing (i) by (ii),

$$u_1/u_2 = p_1/p_2 \quad (iv)$$

For the given utility function, $u_1 = a y_1^{a-1} y_2^b,\ u_2 = b y_1^a y_2^{b-1}$.
Plugging these in (iv), we get:

$$a y_2 / b y_1 = p_1/p_2$$

or, $p_2 y_2 = (b/a) p_1 y_1$. Plug this in (iii), the budget constraint:

$$p_1 y_1 + (b/a) p_1 y_1 = w$$

So,

$$p_1 y_1 = (a/(a+b))w, \quad y_1 = (a/(a+b))(w/p_1)$$

Similarly,

$$p_2 y_2 = (b/(a+b))w, \quad y_2 = (b/(a+b))(w/p_2)$$

At least 3 things are worth noticing here.

(1) These equations are special to Cobb-Douglas utility functions: They are saying that the *budget shares* of the 2 goods, as a function of income/wealth w, are *constant*. Dividing the second equation by the first and rearranging, we get:

$$\frac{y_2}{y_1} = \frac{b}{a}\frac{p_1}{p_2}$$

Imagine a situation in which people's income/wealth is increasing, but the prices are constant. Then, their budget lines are shifting NorthEast in parallel fashion. The above equation says that with Cobb-Douglas preferences, the ratio at which y_2 and y_1 are consumed stays unchanged (Fig. 8, Panel 2).

If good 1 is food, say, then this pattern is **not** borne out by data. In the data, consumers allocate a smaller proportion of their budget to food as they get richer: this observation is called Engel's Law, after the 19th century German statistician Ernst Engel.

(2) Next, take any of the goods, say good 1. It's *demand*, or Marshallian or Walrasian demand function, is given by dividing its budget share equation by p_1. We have:

$$y_1(p_1, p_2, w) = \frac{a}{a+b}\frac{w}{p_1}$$

For the **special case** of Cobb-Douglas utility, the demand depends only on wealth and own price p_1, not on other prices.

We can hold w fixed, and draw this relationship in the (y_1, p_1) plane: Fig. 8, Panel 3: this is the *demand curve.* Not surprisingly, demand is downward-sloping: holding w constant, a reduction in p_1 increases y_1. Obtaining this result using only the assumptions on utility/preferences without using Cobb-Douglas utility involves subtlety, and the notion of substitution and income effects, something we will not go into right now. Note also that increasing w **shifts the demand curve** to the right. If demand were also a function of p_2, then a change in p_2 would also shift the demand curve.

If there are N consumers, all with identical Cobb-Douglas utility functions, but with different wealths $w_i, i = 1, \ldots, N$, then the individual demands are

$$y_1^i = \frac{a}{a+b}\frac{w_i}{p_1},\ \ i = 1, \ldots, N$$

Add these up to get market demand for good 1:

$$Y_1(p_1, p_2, \mathbf{w}) = \frac{a}{a+b}\frac{\sum_{i=1}^N w_i}{p_1}$$

or

$$p_1 = \frac{a}{a+b}\frac{\sum_{i=1}^N w_i}{Y_1}$$

(3) **Consumer Surplus (CS)**

We can compute Consumer Surplus at market quantity and price $(\hat{y}_1, \hat{p}_1)$ as the amount consumers are willing to pay over and above what they actually pay at $(\hat{y}_1, \hat{p}_1)$. We use our demand curve as derived above (i.e. the Marshallian demand curve) for this, although *strictly speaking, this is not right, as real wealth is not held fixed as price changes*; as p_1 decreases, purchasing power (real wealth) increases, and this can have an effect on WTP. We ignore this point now.

As in Fig. 9, we can think of CS as the sum of differences between WTP and price $\hat{p}_1$, for all units that are purchased. For our model in which quantity includes all real numbers and not just integer units, we need to take limits over all partitions of $[0, \hat{y}_1]$ and measure the relevant *area under the demand curve and above* $\hat{p}$; so, and removing the subscript 1 for convenience, we have:

$CS(y, p) = \int_0^{\hat{y}} (p(y) - \hat{p})\, dy = \int_0^{\hat{y}} p(y) dy - \hat{p}\hat{y}$

where $p(y)$ is the inverse of the demand curve.

1.5 Putting Supply and Demand together in Perfect Competition

1.5.1 Total Surplus

Fig. 9, Panel b, is an example where we put together the sum of consumer surplus and producer surplus (see Fig 6 and the discussion there), at $(\hat{y}, \hat{p})$, where $\hat{p}$ is the price at which the output $\hat{y}$ is demanded by the market. We have:

$$\begin{aligned} S(\hat{y}) &= CS(\hat{y}) + PS(\hat{y}) \\ &= \int_0^{\hat{y}} (p(y) - \hat{p})\, dy + \int_0^{\hat{y}} (\hat{p} - C'(y))\, dy \\ &= \int_0^{\hat{y}} (p(y) - C'(y))\, dy \end{aligned}$$

Notice that the derivative

$$S'(\hat{y}) = p(\hat{y}) - C'(\hat{y})$$

If the Total Surplus is at a maximum, this derivative must equal zero, or

$$p(\hat{y}) = C'(\hat{y})$$

which happens at the intersection of the demand and supply curves (Fig. 9, Panel c). We had made this point in a different way when we traded IPL tickets in class. You can see that in Fig. 9, Panel b, where $\hat{y}$ is below the market-clearing quantity, there are 'unrealized' gains from trade: the WTP for an additional unit is greater than the marginal cost of supplying it.

1.5.2 Demand-Supply Examples

(i) Excess demand and excess supply

Consider a situation in which the government enforces that the price of a good is lower than the market price or market-clearing price; this is called a *price ceiling.* (Fig. 10, Panel a). The justification could be a perception that the market price is 'too high'; so some essential consumers get left out. For instance, this could be the market for some essential medicine.

Such an example is not totally appropriate, because usually, high-priced medicines happen because there is a patent for the medicine, giving the manufacturing firm a temporary monopoly to recoup its costs of R & D that was needed to make the drug discovery. But for us, this is just a top example to illustrate two issues. First, notice that there is a deadweight loss, in the sense of maximum surplus minus the surplus that the consumers and producers actually make, at this lower-than-market price.

Second, at this price, demand is greater than supply. There is a shortage of the good (medicine) at this price. If it is not easy to enforce the price, buyers and sellers would be tempted to bid up the price. If, on the other hand, the price is enforced, it is not clear that the most 'deserving' would necessarily get to buy: if demand is, say, 100 units and supply is 50 units at this price, and the good is sold in shops, it is not clear which of the customers demanding the 100 units will actually get to buy. For example, it could be first come, first served, and then the store shelves are empty. Or, it could be a lottery. If there are poorer consumers lower down on the demand curve, it is not clear that either of these ways to allocate the good will get the good across to them.

Two of the ways in which this sort of problem is addressed are these: (1) instead of putting a price ceiling, give some lumpsum income to supplement low-income consumers, and then let the market equilibrium price be the one at which transactions are done. (2) Put a price ceiling, but sell some of the good from 'ration shops', to which access is restricted by giving out 'ration cards' only on evidence of low income.

Now consider a situation where the government believes the market-determined price is too low, and so fixes a price higher than the market price. This is called a *price floor*. For example, the government might decide that the market price for wheat is too low for the producers (farmers) and specify a *minimum support price* (MSP) that is higher. As we see from Fig. 10, Panel b, this leads to a situation of *excess supply*. The government in this case probably stands ready to purchase the amount of grain that is in excess supply. It can then either distribute this grain at a subsidized price through the public distribution system, or store it, or both.

(ii) A sales tax

Suppose the govt. levies a tax of Rupees t per unit of a good sold. If before tax a perfectly competitive firm's supply curve was given by $p = C'(y)$, meaning that the firm is willing to supply y units if it gets a price p, then after the tax is levied, the firm is willing to supply y units if it gets a price that equals $p + t$, so that it retains p after paying the govt. t rupees. So, at every output y, the price at which this output will be supplied shifts up by t. Since this happens for all firms in the industry, the industry supply curve shifts up by t. See Fig. 11, Panel a.

Suppose, as in Fig. 11, Panel b, that the pre-tax equilibrium was $(\bar{y}, \bar{p})$. If the demand curve is downward-sloping, then in the post-tax equilibrium, the equilibrium price $p^* < \bar{p} + t$. The firms *cannot* 'pass on' the entire tax to consumers. The consumers pay p^*, the firm receives $p^* - t$; so, relative to the pre-tax price, the consumer pays $p^* - \bar{p}$ more (this is the *consumer's tax burden*) and the firm receives $\bar{p} - (p^* - t)$ less (which we call the firm's tax burden).

Suppose there is a tax to begin with, but the govt. wants to reduce the tax, say by Rupees s. For example, this happened a number of times with GST. Note, though, that we are doing per-unit taxes here, not percentage taxes on value or value-added tax which is more close to GST. Anyway, as Fig. 11, Panel c shows, the supply curve will now shift down by s units. However, with downward-sloping demand, in the new equilibrium, we see that not all the reduction is passed on to consumers.

This point got lost when a fine was imposed on firms for 'profiteering' from tax cuts and not passing on the entire reduction to consumers. As in the figure, from the initial equilibrium p^*, at price $p^* - s$, there is *excess demand*; the market will *bid the price up* to re-equilibrate demand and supply, so the new equilibrium price *must be greater than* $p^* - s$.

1.5.3 Cost, Returns to Scale and Perfect Competition

The traditional theory of perfect competition makes a key distinction between the short- and the long-runs: in the short run, capital, which takes time to build and install, maybe fixed; and firms choose other inputs such as labour optimally, to maximize profits in the short run. They could make profits, or they could be making losses but nevertheless be operating, because their revenue may be bigger than the variable costs they incur from hiring the variable inputs such as labour; (in this case, they may be making losses because their fixed costs, such as what they pay for the capital that is fixed in the short run, may, when added to variable costs, be greater than their revenue).

In the long run, capital is not fixed; it can be chosen just as labour and other inputs, in a profit-maximal way. And, in the long run, it is also assumed that if firms were making short run profits, then more firms will enter the industry in search of profitable business; while if firms were making losses, some of them would exit the industry. In perfect competition, entry and exit are assumed to be costless.

The considerations sketched above are somewhat unsatisfactory because they consider the notion of making decisions *over time*, without putting these into a formal model of intertemporal decision-making. Literature starting way back in the 1980s began to change this, and one of the influential papers (Jovanovic (1982)) is discussed in Cabral's textbook.

We have skirted any distinction between short- and long-run so far, by discussion profit maximization when a firm can choose both capital and labour. If you take slightly seriously our reference to decisions on crop inputs by small farmers, it is true *land* may be fixed there, but in a sense land is going to remain fixed, especially as we are not considering a discussion here of improvements in technology that improve land productivity. In any case, the decision-making of the firm we discussed was one with all sensible inputs variable.

We made the assumption of DRS (diminishing returns to scale) to give us upward-sloping marginal cost curves and therefore upward-sloping supply curves. In such a setting, incremental output has higher marginal cost, so to produce it, a firm has to be compensated via a higher price (and hence the supply curve is upward-sloping).

We had touched upon profit maximization and perfect competition with constant returns to scale, and observed that the only price consistent with this was that equal to the constant marginal cost that results from constant returns to scale. So, in this case, the industry supply curve is horizontal at this marginal cost: firms will supply any amount at this price, and zero at any lower price.

And we had noted that the price equals marginal cost condition does not identify a profit maximum if the marginal cost curve is sloping downward (the case of increasing returns to scale); so increasing returns are not consistent with perfect competition.

Bringing Average Cost into the Discussion

We now extend this discussion by explicitly bringing in the related notion of **Average Cost**, that is, $C(y)/y$, where $C(y)$ is total cost.

Note that

$$\frac{dAC}{dy} = \frac{yC'(y) - C(y)}{y^2}$$

So, $\frac{dAC}{dy} > (=, <) 0$ according as $C'(y) > (=, <) \frac{C(y)}{y}$.

So, if the marginal cost $MC > AC$ at y, then AC is increasing in y; and vice-versa. Think of it as a batting average: if someone with a T20 batting **average** of about 52, makes 122 in the current match (**marginal** score, then his or her batting average increases; if the player has had a series of scores less than his or her average earlier, then their average must have been decreasing.

Fig. 12 illustrates this relationship between AC and MC for homogeneous production functions of degree k. Note that for **any** such function (including Cobb-Douglas), we can show that

$$C(y) = \bar{B} y^{1/k}$$

$$\frac{C(y)}{y} = \bar{B}y^{(1-k)/k}$$

$$C'(y) = \frac{\bar{B}}{k}y^{(1-k)/k}$$

where $\bar{B}$ depends on parameters and input prices. That is, output y factors out in an exponential term. *Reminder to myself to write down an optional proof here, later.*

The expressions above clearly follow the $MC > AC \implies AC$ increasing, $MC < AC \implies AC$ decreasing relationship, illustrated for the different returns to scale cases in Fig. 12.

Drawing AC and MC in a profit-maximization figure (i.e. a figure in which we illustrate the first-order condition $p = C'(y)$ tells us whether the firm is making a profit or a loss at the intersection of the price and MC curves. Indeed:

$$\pi(y) = py - C(y) = \left(p - \frac{C(y)}{y}\right)y = (p - AC)y$$

So, we can interpret profits as the product of the *profit per unit sold*, $(p - AC)$, times y, the total number of units sold.

So, consider Fig. 13, Panel a, where there are decreasing returns to scale, so AC is increasing. But this means that MC lies above AC. Now, a profit maximum occurs where $p = MC$, so we have $p = MC > AC$ at the y where the profit maximum occurs. And so, at this y, $p - AC > 0$, so profits are positive, and equal to the shaded rectangle representing $(p - AC)y$.

The other panels in Fig. 13 show that with increasing returns to scale, producing y such that $p = MC$ results in losses; and that with constant returns to scale, profits equal zero.

U-Shaped Average Cost Curve

It may be unrealistic to imagine that a technology displays increasing returns to scale irrespective of how large the scale of production is. A U-shaped average cost curve tries to capture settings in which:

(i) To start with, there are indivisibilities, so inputs are better utilized at some scale. For example, in Adam Smith's pin factory, one or a few workers would not produce efficiently, but having an assembly-line of 10 workers would optimize the process of producing a pin by breaking it up into 10 different tasks.

This implies increasing returns to scale: upto 10 workers, doubling workers and materials more than doubles output. Correspondingly, average cost decreases.

(ii) Then, the efficient scale of 10 workers plus materials could be replicated; double workers and materials from 10 to 20 workers and doubling of materials, would double output. This is a constant returns to scale stretch in the production process. Correspondingly, average cost is constant.

(iii) But it may be difficult to replicate indefinitely within a single firm. Monitoring 20 workers is different from monitoring 10,000; at some point, managerial inefficiencies would set in, and doubling workers will not double output. This stretch is a decreasing returns to scale one. Correspondingly, average cost increases.

$C(y) = F + ky^2$ is a stylized Total Cost function which yields a U-Shaped AC curve. Indeed, Average Cost $C(y)/y = F/y + ky$, whose first and second derivatives are, respectively, $\left(yC'(y) - C(y)\right)/y^2$ and $2F/y^3$. Since $y > 0$, the AC curve is therefore convex, and has a unique minimum at $y = (F/k)^{1/2}$, which we call the *minimum efficient scale* (MES) of production.

Notice from FIg. 14 that the MC is, as it should be, below AC where AC is decreasing, and above it where AC is increasing; so that the two curves intersect where AC is at its minimum. It follows that a firm producing $y < y_{eff}$ will make losses: in the long run, no firm will produce such an output. Also, in the long run, if firms produce $y > y_{eff}$, they make positive profit; this should spur entry of new firms, and so shift industry supply curve to the right, and bring down the market price until no firm makes any profit and there is no entry (or exit).

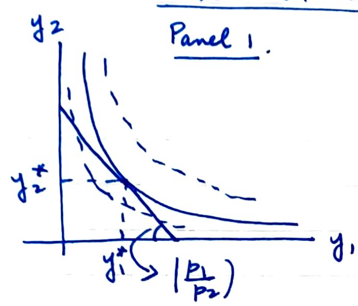
There is a small caveat to this if the number of firms is an integer value rather than any real number, as seen in this example. Suppose the zero profit price $\hat{p}$ equal minimimum AC is INR 100. And at this price, market demand is 1025 units. And $y_{eff} = 50$. Then, if 20 firms operate at y_{eff}, the supply equals 1000, and if 21 firms operate at y_{eff}, it is 1050. To supply exactly 1025 and clear the market, with 21 firms at least some firm(s) will have to produce less than the minimum efficient scale y_{eff}; but we know such an output will make losses as $MC < AC$ at this output. So, 20 firms will operate, with at least some producing slightly more than y_{eff} and making some profit, without provoking any entry.

ID Topics

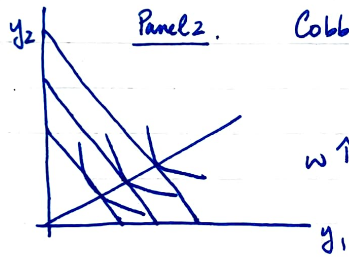
Fig. 8.

Demand with Cobb-Douglas Utility.

Panel 1.



Panel 2. Cobb-Douglas "expansion path"



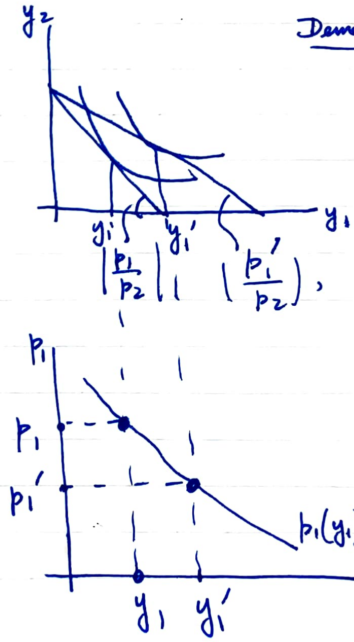
$w \uparrow$es, budget lines shift parallely.

$\frac{p_1}{p_2}$ does not change.

$p_1 y_1$, $p_2 y_2$ are constant proportions of w.

Demand

Panel 3



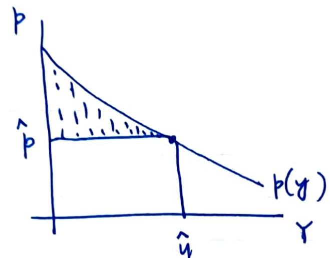
$p_1' < p_1$

$p_1(y_1)$ [Inverse demand function].

IO Topics.

Fig. 9

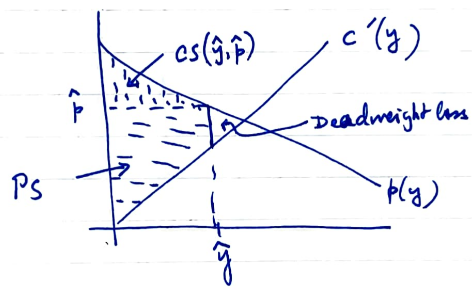
Panel (a) Consumers' Surplus.



$$CS(\hat{y}) = \int_0^{\hat{y}} (p(y) - \hat{p})\, dy$$

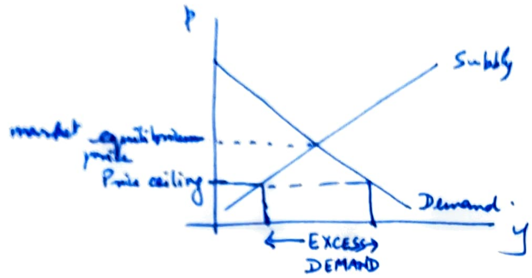
← sum (integral) of vertical distances $(p(y) - \hat{p})$.

Panel (b) Consumer Surplus (CS) + Producer Surplus (PS)

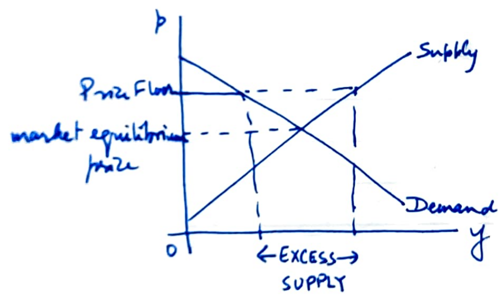


IO Topics
Fig. 10

Panel (a) Price Ceiling



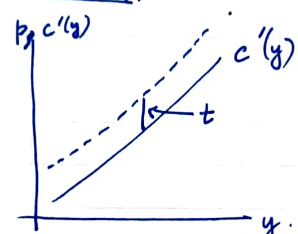
Panel (b) Price Floor



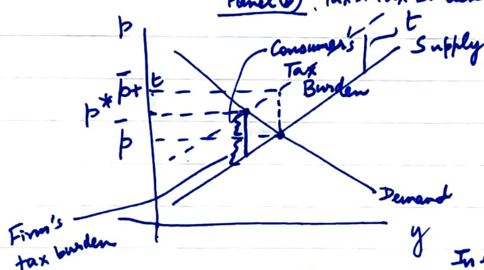
IO Topics.

Fig. 11 Sales Tax per unit of good

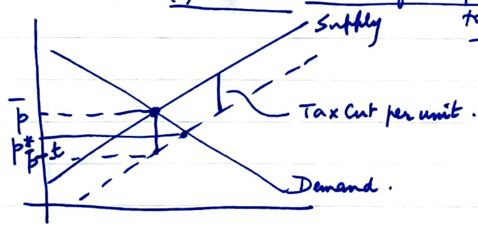
Panel (a).



Panel (b). Tax & Tax Burden



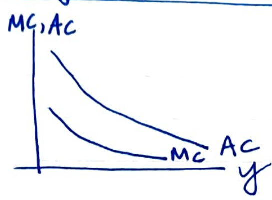
Panel (c): Tax Cut: In equilibrium, Not all of it is passed on to Consumer



IO Topics

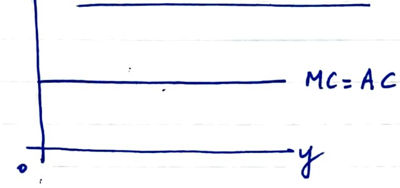
Fig 12 : Homogeneous Production Functions, MC, AC.

Increasing Returns to Scale



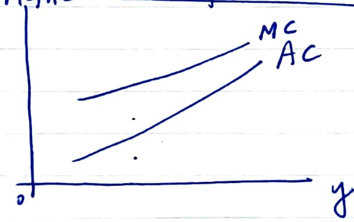
MC, AC

Constant Returns to Scale



Decreasing Returns to Scale

MC, AC

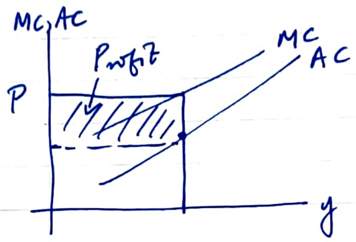


IO Topics

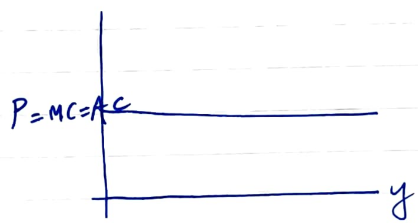
Fig. 13

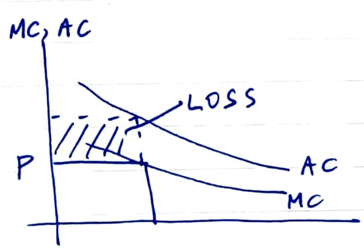
Perfect Competition, Profits, Returns to Scale

Panel (a) Decreasing Returns to Scale



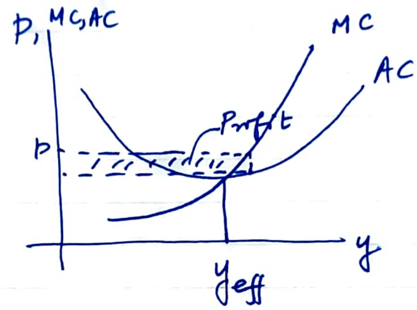
Panel (b) Constant Returns to Scale





IO Topics

Fig 14 MC, AC for U-shaped AC curve.



1.6 Appendix A0: Derivatives in $\mathbb{R}^n$

This begins the Appendices, which are completely optional reading and peripheral to the IO course. The objective of Appendix A0 is to introduce notation for the derivative of a multivariable real-valued function $f : \mathbb{R}^n \to R$ (and more generally for a vector-valued function $g : \mathbb{R}^n \to \mathbb{R}^m$).

The definition of a derivative uses the notion of the limit of a function, so let's review this notion first. x is an *accumulation point* or *limit point* of a set S if for every $\epsilon > 0$, $B(x, \epsilon)$ contains a point from S that is distinct from x itself.

Let's restrict ourselves first to $g : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$. Let x be a limit point of S. We say $\lim_{y \to x} g(y) = L$ if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $|y - x| < \delta$, $y \neq x$ implies $|g(y) - L| < \epsilon$.

We can show that this is equivalent to saying that for every $(x_n) \to x$, $x_n \neq x$ for any n, $g(x_n) \to L$.

Definition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is defined to be differentiable at x if there exists $a \in \Re$ s.t.*

$$\lim_{y \to x} \left(\frac{f(y) - f(x)}{y - x} - a \right) = 0 \tag{A0.1}$$

By limit equal to 0 as $y \to x$, we require that the limit be 0 w.r.t. all sequences (y_n) s.t. $y_n \to x$. a turns out to be the unique number equal to the slope of the tangent to the graph of f at the point x. We denote a by the notation $f'(x)$. We can rewrite Equation (1) as follows:

$$\lim_{y \to x} \left(\frac{f(y) - f(x) - a(y - x)}{y - x} \right) = 0 \tag{A0.2}$$

Note that this means the numerator tends to zero faster than does the denominator.

Note also that it says that for y close to x, $f(y)$ is well-approximated by $f(x) + a(y - x)$. And also that $(y, f(x) + a(y - x))$ is on the tangent line through $(x, f(x))$. *Insert picture.*

We can use this way of defining differentiability for more general functions.

Definition 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. f is* **differentiable at** *x if there is an $m \times n$ matrix A s.t.*

$$\lim_{y \to x} \left(\frac{||f(y) - f(x) - A(y - x)||}{||y - x||} \right) = 0$$

In the one variable case, the existence of a gives the existence of a tangent; in the more general case, the existence of the matrix A gives the existence of tangents to the graphs of the m component functions $f = (f_1, ..., f_m)$, each of those functions being from $\Re^n \to \Re$. In other words this definition has to do with the 'best' linear affine approximation to f at the point x. To see this in a way equivalent to the above definition, put $h = y - x$ in the above definition, so $y = x + h$. Then in the 1-variable case, from the numerator, $f(x+h)$ is approximated by the affine function $f(x)+ah = f(x)+f^{'}(x)h$. In the general case, $f(x + h)$ is approximated by the affine function $f(x) + Ah$.

It can be shown that (w.r.t. the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$), the matrix A equals $Df(x)$, the $m \times n$ matrix of partial derivatives of f evaluated at the point x.

$Df(x)$ is known as the *derivative* of f at x:

$$Df(x) = \begin{pmatrix} \partial f_1(x)/\partial x_1 & \dots & \partial f_1(x)/\partial x_n \\ \dots & \dots & \dots \\ \partial f_m(x)/\partial x_1 & \dots & \partial f_m(x)/\partial x_n \end{pmatrix}$$

Here, row i contains the partial derivatives of the component function f_i w.r.t. the variables $x_1, x_2, \dots, x_n$. Since there are m component functions, there are m rows of partial derivatives.

By this convention, if $f : \mathbb{R}^n \to \mathbb{R}$, $Df(x)$ is the row matrix:

$Df(x) = (\partial f(x)/\partial x_1 \cdots \partial f(x)/\partial x_n)$.

On the other hand, if $f : \mathbb{R} \to \mathbb{R}^m$, this says that f has m component functions $f_1, \ldots, f_m$, each of them mapping $\mathbb{R} \to \mathbb{R}$. So the derivative matrix $Df(x)$ now has m rows, one for each component function, but only one column:

$Df(x) = (f_1'(x) \ldots f_m'(x))^T$, where the superscript T refers to transpose; we have written it this way to save space.

Let us recall the notion of a partial derivative. Let $f : \mathbb{R}^n \to \mathbb{R}$. The partial derivative $\partial f(x)/\partial x_j$ is defined as a number a_j that satisfies

$$\lim_{t \to 0} \frac{f(x_1, .., x_j + t, .., x_n) - f(x_1, .., x_j, .., x_n) - a_j t}{t} = 0$$

In other words:

$$\partial f(x)/\partial x_j = a_j = \lim_{t \to 0} \frac{f(x_1, .., x_j + t, .., x_n) - f(x_1, .., x_j, .., x_n)}{t}$$

We also want to talk about partial derivatives and directional derivatives using less clunky notation.

Let $e_j = (0, .., 0, 1, 0, ..., 0)$ be the unit vector in $\mathbb{R}^n$ on the j^{th} axis. Then,

$$\frac{\partial f(x)}{\partial x_j} = \lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t}$$

That is, the partial of f w.r.t. x_j, evaluated at the point x, is looking at essentially a function of 1-variable: we take the $(n-1)$ dimensional surface of the function f, and slice it parallel to the j^{th} axis, s.t. point x is contained on this slice/plane; we'll get a function pasted on this plane; it's derivative is the relevant partial derivative.

To be more precise about this one-variable function pasted on the slice/plane, note that the single variable $t \in \Re$ is first mapped to a vector $x + te_j \in \mathbb{R}^n$, and then that vector is mapped to a real number $f(x + te_j)$. So, let $\phi : \Re \to \Re^n$ be defined by $\phi(t) = x + te_j$, for all $t \in \Re$. Then the one-variable function we're looking for is $g : \Re \to \Re$ defined by $g(t) = f(\phi(t))$, for all $t \in \Re$; it's the composition of f and ϕ.

In addition to slicing the surface of functions that map from $\Re^n$ to $\Re$ in the directions of the axes, we can slice them in *any* direction and get a function pasted on the slicing plane. This is the notion of a *directional derivative.*

Recall that if $x \in \Re^n$, and $h \in \Re^n$, then the set of all points that can be written as $x + th$, for some $t \in \Re$, comprises the line through x in the direction of h.

Insert picture.

Definition 3. *The directional derivative of a function $f : \Re^n \to \Re$ at $x \in \Re^n$, in the direction $h \in \Re^n$, denoted $Df(x; h)$, is*

$$\lim_{t \to 0+} \frac{f(x + th) - f(x)}{t}$$

If $t \to 0+$ is replaced by $t \to 0$, we get the 2-sided directional derivative.

We now show that (w.r.t. the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$), the matrix A in differentiability Definition 7 equals $Df(x)$. To see this, take the slightly less general case of a function $f : \mathbb{R}^n \to \mathbb{R}$. If f is differentiable at x, there exists a $1 \times n$ matrix $A = (a_1, \ldots, a_n)$ satisfying the definition above: i.e.

$$\lim_{h \to \theta} \frac{||f(x + h) - f(x) - Ah||}{||h||} = 0$$

In particular, the above must hold if we choose $h = (0, .., 0, t, 0, .., 0)$ with $h_j = t \to 0$. That is,

$$\lim_{t \to 0} \frac{||f(x_1, .., x_j + t, .., x_n) - f(x_1, .., x_j, .., x_n) - a_j t||}{t} = 0$$

But from the limit on the LHS, we know that a_j must equal the partial derivative $\partial f(x)/\partial x_j$.

For a function $F\mathbb{R}^n \to \mathbb{R}^m$, we will use f_i to denote the ith component function. Then

$$\frac{\partial f_i(x)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(x_1, .., x_j + t, ..., x_n) - f_i(x_1, .., x_j, ..., x_n)}{t}$$

A function that is *differentiable on a set* S if it is differentiable at all points in S. f is *continuously differentiable* if it is differentiable and all partial derivatives are continuous.

1.7 Appendix A1: Implicit Function Theorem

This optional digression is here for rounding things out. For anyone interested, this stuff can be found in various introductory real analysis texts, such as the one by Apostol.

Theorem 4. *Suppose* $F : \Re^{n+m} \rightarrow \Re^m$ *is* C^1 *(continuously differentiable), and suppose* $F(x^*, y^*) = 0$ *for some* $y^* \in \Re^m$ *and some* $x^* \in \Re^n$*. Suppose also that* $DF_y(x^*, y^*)$ *has rank* m*. Then there are open sets* U *containing* x^* *and* V *containing* y^* *and a* C^1 *function* $f : U \rightarrow V$ *s.t.*

$$F(x, f(x)) = 0 \ \forall x \in U$$

.

Moreover,

$$Df(x^*) = -[DF_y(x^*, y^*)]^{-1} DF_x(x^*, y^*)$$

Note that we could alternatively look at the equation $F(x, y) = c$, for some given $c \in \Re^m$, without changing anything. The proof of this theorem starts going 'deep', so will not be part of this course. The proof for the $n = m = 1$ case, however, is provided at the end of this Chapter. But notice, that applying the Chain Rule to differentiate

$$F(x^*, f(x^*)) = 0$$

yields

$$DF_x(x^*, y^*) + DF_y(x^*, y^*)Df(x^*) = 0 \tag{*}$$

whence the expression for $Df(x^*)$.

More tediously in terms of compositions, if $h(x) = (x, f(x))$, then $Dh(x) = \left(\frac{I}{Df(x)}\right)$,

whereas $DF(.) = (DF_x(.)|DF_y(.))$, so matrix multiplication using partitions yields Eq.(*).

Application: Comparative Statics of Cournot Duopoly.

Firms 1 and 2 have constant unit costs c_1 and c_2, and face the twice continuously differentiable inverse demand function $P(Q)$, where $Q = q_1 + q_2$ is industry output. So profits are given by

$$\pi_1 = P(q_1 + q_2)q_1 - c_1 q_1$$

and

$$\pi_2 = P(q_1 + q_2)q_2 - c_2 q_2$$

If profits are concave in own output, then the first-order conditions below characterize Cournot-Nash equilibrium (q_1^*, q_2^*).

$$\partial\pi_1/\partial q_1 = P'(q_1^* + q_2^*)q_1^* + P(q_1^* + q_2^*) - c_1 = 0$$

$$\partial\pi_2/\partial q_2 = P'(q_1^* + q_2^*)q_2^* + P(q_1^* + q_2^*) - c_2 = 0$$

The concavity of profit w.r.t. own output conditions follow from the condition below: For all q_1, q_2

$$\partial^2\pi_1/\partial q_1^2 = P''(q_1 + q_2)Q + 2P'(q_1 + q_2) \leq 0, i = 1, 2$$

The two first-order conditions can be written as the vector equation

$$F(q_1^*, q_2^*, c_1, c_2) = \mathbf{0}$$

.

We want to know: How do the Cournot outputs change as a result of a change in unit costs? If c_1 decreases, for instance, does q_1 increase and q_2 decrease? The implicit function theorem says that if $DF_{q_1,q_2}(q_1^*, q_2^*, c_1, c_2)$ is of full rank (rank=2), then, locally around this solution, $q = (q_1, q_2)$ is an implicit function of $c = (c_1, c_2)$, with $F(f(c), c) = 0$. And

$$Df(c) = -\left[DF_q(q^*, c)\right]^{-1} DF_c(q^*, c)$$

Note that
$DF_q(.) = \begin{pmatrix} \partial F_1/\partial q_1 & \partial F_1/\partial q_2 \\ \partial F_2/\partial q_1 & \partial F_2/\partial q_2 \end{pmatrix}$

For brevity, let P' and P'' be the derivative and second derivative of $P(.)$ evaluated at the equilibrium. Then
$DF_q(.) = \begin{pmatrix} P''q_1 + 2P' & P''q_1 + P' \\ P''q_2 + P' & P''q_2 + 2P' \end{pmatrix}$

The determinant of this matrix works out to be
$(P')^2 + P'(P''(q_1 + q_2) + 2P') > 0$ since $P' < 0$ and the concavity in own output condition is assumed to be met. So the implicit function theorem can be applied

Notice also that
$DF_c(.) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Thus we can work out $Df(c)$, the changes in equilibrium outputs as a result of changes in unit costs. It would be a good exercise for you to work these out, and **sign** these.

1.8 Appendix A2: Lagrange's Theorem

Theorem 5. *Let $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$, ($n \geq 2$) be continuously differentiable, and $y \in \mathbb{R}$ lie in the range of f. Suppose U is an open set in* *the domain of* g*, and suppose there is a minimum of* g *on the set* $S = \{\mathbf{x} \in \mathbb{R}^n | f(\mathbf{x}) = y\}$ *intersected with* U*, at point* $\mathbf{x}^*$*.*

Suppose $Df(\mathbf{x}^*) \neq \mathbf{0}$*. Then there exists a real number* $\lambda \neq 0$ *such that*

$$Dg(\mathbf{x}^*) = \lambda Df(\mathbf{x}^*)$$

Proof (sketch). Consider a differentiable curve $\mathbf{x} : [a, b] \to \mathbb{R}^n$ in the set S, such that $\mathbf{x}^* = \mathbf{x}(t)$ at some t. From Fig. A2, notice that the vector $\mathbf{x}'(t)$ is tangent to the surface S, at the point $\mathbf{x}(t) = \mathbf{x}^*$.

Define the composition function h as follows:

$$h(t) = f(\mathbf{x}(t))$$

By the Chain Rule, $h'(t) = Df(\mathbf{x}(t))\mathbf{x}'(t)$.

Since the value $f(\mathbf{x}(t)) = y$ over all t on this curve, this derivative $h'(t) = 0$. Also note that $\mathbf{x}^*$ is the point that the curve $\mathbf{x}(\ )$ visits at t. So, from

$Df(\mathbf{x}^*)\mathbf{x}'(t) = 0$, we have

$Df(\mathbf{x}^*) \perp \mathbf{x}'(t) \quad (A2.1)$.

Now, consider the function values for g on the same curve. Since g attains a minimum in the set S at x^*, the composition function $\phi(\ ) = g(\mathbf{x}(\ ))$ attains a minimum at $\phi(t)$, so we have

$\phi'(t) = 0$. By the Chain Rule,

$\phi'(t) = Dg(\mathbf{x}^*)\mathbf{x}'(t)(= 0)$, so

$Dg(\mathbf{x}^*) \perp \mathbf{x}'(t) \quad (A2.2)$.

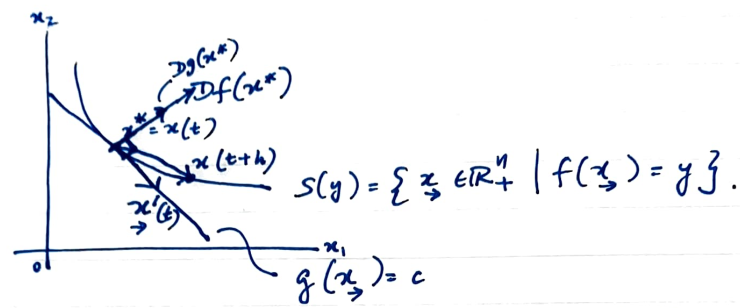
So, combining $(A2.1)$ and $(A2.2)$,

$$Dg(\mathbf{x}^*) = \lambda Df(\mathbf{x}^*)$$

■

Fig A2.

Minimize $g(\vec{x})$ s.t. $f(\vec{x}) = y$.



$$\vec{x}'(t) = \lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$$

Chapter 2

Monopoly

2.1 Introduction

Why study Monopoly? Aren't single-firm industries rare?

Maybe so. But there are and historically have been various monopolies: in infrastructure - railways, railroads, telephone, national airlines, financial clearing houses, patented drugs. There are highly concentrated industries (airlines, telecom) in which the precarious financial position of some of the players lead to fears that all but one player may exit.

Even when an industry has a few large firms rather than a single firm, there are possibilities of collusion among them to try and replicate a monopoly-like outcome in term of joint profits.

Features of the economic landscape that tend to lead to concentration and monopoly include

(i) a system of patenting new products to allow firms a temporary monopoly to recoup R & D costs.

(ii) Technology such that the cost function has the *subadditivity* property:

$$C(\sum_{i=1}^{m} q_i) < \sum_{i=1}^{m} C(q_i) \tag{2.1}$$

If (2.1) holds, then the cost that a single firm can produce output at a lower cost than m firms producing the same aggregate output. It is said that this setting is therefore one of a *natural monopoly*.

A leading example of a subadditive cost function is that which arises from an increasing returns to scale technology. We've seen earlier that increasing returns to scale is not consistent with perfect competition, as price equals marginal cost is less than average cost, which decreases with output. So in any case, we expect some form of oligopoly or monopoly to operate.

(iii) A related point: entry into an industry is typically not free. The sunk costs of entry can be high. Sunk costs are costs in industry- or firm-specific inputs/technology that have much less value outside of this firm or industry, and so cannot be recovered by, say, selling these when exiting. Hence the term 'sunk': these costs cannot be recovered.

The prospect of high sunk costs can deter entry. On the other hand, for a firm that has incurred these costs and entered, the additional, marginal costs of incrementing output could be relatively low in several industries. For example, think of software, in which the major cost is related to R & D and then marketing, but where the cost of making one more unit of the same software is near zero, involving making a copy. Then, we expect a cost function which would be some version of

$$C(q) = F + cq$$

so that average cost $C(q)/q$ decreases with output q.

In less extreme versions, it could still be that the average cost curve for a technology while being U-shaped, has a very high minimum efficient scale. Then, if the market demand is limited, then for practical purposes it may still be that a single firm can operate at lower cost than multiple firms sharing the same output.

(iv) An industry with strong network effects could also exhibit a trend toward becoming a monopoly. For instance, lots of businesses share spreadsheets, so the utility of a user of the spreadsheet does not depend only on their taste for it, but also on how many of his or her collaborators use the same spreadsheet. This is how Quattro, a popular spreadsheet app in the 1990s got tipped over by Microsoft Excel. At that time, Microsoft was accused of bundling-in Excel and ms-Word with its operating system and creating an environment where a network effect would tip-in lots of customers.

2.2 Monopoly Output and Deadweight Loss

Suppose the monopolist faces the demand curve $p = P(q)$ and has the cost function $C(q)$. Assume $C(0) = 0, C' > 0, C'' > 0$. So, the marginal cost $C'(q)$ increases in output q. Actually, $P(q)$ is called the inverse demand function or the inverse of the demand function; where $q = D(p)$ is the demand function specifying that at each price p, consumers are willing to buy $D(p)$ units of the good.

Given the inverse demand curve $P(q)$, if the monopolist produces some quantity q, then the market clears at price $p = P(q)$. If the monopolist chooses to sell at some price p, the demand will be $D(p)$. Whichever way we look at it, the monopolist's problem is really to choose some (q, p) pair on the demand curve that will maximize its profit.

Suppose first that the monopolist chooses q to maximize

$$\pi(q) = qP(q) - C(q)$$

$C(\ )$ is strictly convex; if additionally $P(q)$ is (weakly) concave, or even 'not too convex', then $\pi(q)$ is strictly concave and the solution to the first-order condition characterizes the profit maximum:

$$\pi'(q) = qP'(q) + P(q) - C'(q) = 0 \tag{2.2}$$

Letting $R(q) \equiv qP(q)$ denote Revenue, we let $R'(q) \equiv qP'(q) + P(q)$ be called *marginal revenue*; the extra revenue from a small increase in q. This says that if we increase output by one small unit to greater than q, the first-order effect is getting price $P(q)$ for this extra unit of output, but also, this unit decreases the price by $P'(q)$ for all q units sold; the sum of these is the marginal or incremental revenue.

Notice that $R''(q) = qP''(q) + 2P'(q)$. We assume that this is negative, so marginal revenue $R'(q)$ is decreasing in q. Then, (2.2) says that at the profit maximizing q, the downward sloping (w.r.t. q) marginal revenue curve cuts the upward sloping marginal cost $C'(q)$ curve. As long that incremental revenue exceeds incremental cost, increase output, and stop when these increments equal each other.

As an economist, we could ask whether, starting from some q and profit $R(q) - C(q)$, a small increment $\epsilon > 0$ in output will increase profit.

That is, is $R(q + \epsilon) - C(q + \epsilon) - (R(q) - C(q)) > 0$?

The first-order (Taylor series) difference above equals

$$(R'(q) - C'(q))\epsilon$$

At a profit max, this must equal zero.

Fig. 2.1 illustrates this, with a straight-line demand curve; for this, the marginal revenue curve is located at half the distance, as you can check by working out the algebra. Notice that monopoly output is lower than output in perfect competition. If the monopoly were to behave like a perfectly competitive firm, it would take market price as given (rather than set market price and quantity), and produce that output at which price equals marginal cost. That is, q such that $P(q) = C'(q)$, the intersection of the demand curve and the marginal cost curve in Fig. 2.1. Recall that at this point, the sum of consumer and producer surplus is maximized.

A monopoly will produce less. Because, increasing output does not increase revenue by $P(q)$; the monopolist takes into account that the higher output causes price to decrease by $P'(q)$ at the margin, for *all* q units sold. Because $R'(q) < P(q)$, it becomes equal to $C'(q)$ at a lower quantity.

Notice from Fig. 2.1, therefore, that under monopoly, there are unrealized gains from trade: at the margin, consumers are willing to pay higher than the monopolist's marginal cost, and yet, the monopolist will not increase output, because even though it gets around the marginal WTP for the additional unit of production, the price drops on all units of output, resulting in no marginal revenue greater than marginal cost.

2.3 Monopoly Markup and Elasticity

A monopoly's profit maximizing price (simply called the monopoly price), is a markup over its marginal cost, and the extent of the markup depends on the *price elasticity of demand*.

2.3.1 Price Elasticity of Demand

2.4 Price Discrimination

2.4.1 Third degree Price Discrimination

2.5 First degree Price Discrimination

2.6 Second degree Price Discrimination

Chapter 3

Strategic Form Games and Nash Equilibrium

3.1 Introduction

Rousseau, in his "Origin and Basis of Equality among Men", describes the following situation.

"A group of hunters out to take a stag have to remain faithfully at their posts to succeed: but if a hare happens to pass near one of them ... and he pursued it without qualm, and ... caught his prey, he cared very little whether or not he had made his companions miss theirs."

This is a situation in which what one hunter does affects the well-being of *all* of them. Game Theory models such interactive settings between people/agents: where the *outcomes* of the interaction, and so people's payoffs from these outcomes, depend on not just their own actions or strategies/plans, but on the actions/plans of *all* involved in the interaction.

Introductions should be more substantial, and perhaps this one will be, too, in the fullness of time. For now, we end it here, with the comforting thought that many examples of games will follow soon enough.

3.2 Strategic Form Games

A shorthand for describing such interactive situations is the Strategic Form. In this, we specify the set of Players, their possible strategies, and the consequences for each player, when players choose any of the possible strategy profiles.

We will describe this a little expansively first, and then cut to the standard description. In the expansive description, a strategic game is comprised of:

- a set N of players.
- a set S_i for each Player i; $S \equiv \times_{i \in N} S_i$ is the set of strategy profiles.
- a set Y of outcomes.
- an outcome function $g : S \to Y$.
- and for each Player i, a von Neumann - Morgenstern utility function $v_i : Y \to \mathbb{R}$ representing i's preferences on Y.

In a 2-player Stag Hunt, we could model this as:

- $N = \{1, 2\}$.
- $S_1 = S_2 = \{St, H\}$; i.e. 2 strategies each, St=hunt for stage, H=hunt for hare.
- Outcome set $Y = \{Stag, H_1, H_2, H_{12}\}$, referring to, respectively, the stag being caught and divided equally betwen the hunters; 1 catching a hare and 2 getting nothing, 2 catching a hare and 1 getting nothing, and both catching a hare each.
- Outcome function $g : S \to Y$ defined by: $g(St, St) = Stag$, $g(St, H) = H_2$, $g(H, St) = H_1$, $g(H, H) = H_{12}$. We see that the outcome function crucially ties the interaction and its consequences or outcomes.
- For each Player $i = 1, 2$, we could specify the vn-M utility function, say, with $v_i(Stag) = 2, v_i(H_{12}) = 1, i = 1, 2$; $v_1(H_1) = v_2(H_2) = 1, v_1(H_2) = v_2(H_1) = 0$.

We can interpret the $v_1(), v_2()$ numbers in this game as roughly suggesting that the players' preferences for deer and hare are sort of the same, and that half a deer is worth twice a hare, etc. In other games, the outcomes may be monetary outcomes and the payoffs then could just be money payoffs. But at least at this initial juncture, let's discuss the strategic form game model more fully by discussing outcomes and vn-M utility functions, if only by way of linking to what you've covered in compulsory Micro.

Let the set of outcomes Y be finite, so the discussion can avoid technical details. Each player i has preferences over lotteries or probability distributions over the outcome set Y, given by $\succsim_i$. Call the set of probability distributions over Y, $\Delta(Y)$. We permit compound lotteries, or probability distributions over probability distributions, or compound lotteries; and assume that agents evaluate these as the resultant lotteries over the outcomes in Y (this is sometimes called the Axiom of Simplification). Preferences over compound lotteries are needed to make sense of continuity and independence of $\succsim_i$.

$\succsim_i$ satisfies the vN-M Axioms below.

(1) $\succsim_i$ is complete and transitive.

(2) Continuity: For all lotteries p, q, r with $p \succsim_i q \succsim_i r$, there is a probability $\theta_i \in [0,1]$ s.t. $q \approx_i \theta_i p + (1-\theta_i)r$.

(3) Monotonicity: For all lotteries p, q s.t. $p \succ_i q$ and probabilities $\alpha, \beta \in [0,1]$, $\alpha p + (1-\alpha)q \succsim_i \beta p + (1-\beta)q$ if and only if $\alpha \geq \beta$.

(4) Independence: If L is a compound lottery $\alpha p + (1 - alpha)q$, and $q \approx_i r$, then $L \approx_i \alpha p + (1 - alpha)r$.

The continuity axiom has bite if $p \succ_i q \succ_i r$; if we combine this axiom with Monotonicity, there is in fact a *unique* probability θ_i s.t. we get indifference between q and the compound lottery $\theta_i p + (1-\theta_i)r$. (Because if there is indifference using θ_i, then for any probability number $\alpha > \theta_i$, Monotonicity tells us $q \approx_i \theta_i p + (1-\theta_i)r \prec_i \alpha p + (1-\alpha)r$.

Let's use the notation that a lottery p puts probability $p(y)$ on outcome $y \in Y$.

The von Neumann - Morgenstern Expected Utility Theorem says that if $\succsim_i$ satisfies Axioms 1 - 4 above, then there is a function $v_i : Y \to \mathbb{R}$, s.t. for any pair of lotteries p, q, $p \succsim_i q$ if and only if $\sum_{y \in Y} v_i(y)p(y) \geq \sum_{y \in Y} v_i(y)q(y)$.

Notice that v_i is only unique upto a positive affine transformation.

With a finite set Y of outcomes, a complete and transitive preference relation ensures that there is at least one worst outcome W and one best outcome B, and we can construct the vN-M utility function by, say, letting $v_i(W) = 0, v_i(B) = 1$, (or indeed any two numbers with $v_i(W) < v_i(B)$); and for any outcome y with $B \succsim_i y \succsim_i W$, and $\theta)_y$ being the unique number in $[0, 1]$ s.t.

$y \approx_i \theta_y B + (1 - \theta_y)W$,

we let $v_i(y) = \theta v_i(B) + (1 - \theta_y)v_i(W)$. As should be intuitively clear, this will work.

Let's interpret the payoff numbers we wrote down for the stag hunt. Take any of the players, say Player 1. His or her preferences $\succsim_1$ are clearly:

$Stag \succ_1 H_1 \approx_1 H_{12} \succ_1 H_2$.

So we can arbitrarily assign $v_1(Stag) = 2, v_1(H_2) = 0$ to her best and worst outcomes.

Then if $H_1 \approx_1 \theta_1 Stag + (1 - \theta_1)H_2$,

we let $v_1(H_1) = \theta_1 v_1(Stag) = 2\theta_1$.

Our $v_1(H_1) = 1$ indicates that Player 1 is indifferent between the outcome that gives her a hare, and a lottery that gives her half a stag with probability $\theta_1 = 1/2$ and nothing with probability $(1 - \theta_1) = 1/2$.

Notice that numbers that are obtained from v_1 and v_2 by any positive affine transformation will work just as well to represent these preferences.

One reason to specify an outcome set and preferences, rather than jump to the standard representation in terms of payoff functions on strategy profiles (below), is to highlight that in principle, we can admit various considerations: for example, Player 1 could prefer catching a hare to getting half a stag: $H_1 \succ_1 \approx_1 H_{12} \succ_1 Stag \succ_1 H_2$. Or, she could be very altruistic, and prefer that Player 2 catch a hare rather than she catching a hare: $Stag \succ_1 H_{12} \succ_1 H_2 \succ_1 H_1$.

But why require preferences not just on the set of outcomes, but on *lotteries* (and lotteries over lotteries) or probability measures on the set of outcomes? This is because of uncertainty. Even if there is no extrinsic uncertainty, a player can be uncertain about what strategies the other players will choose; and therefore, of the outcomes that can result given his own choice of strategy. But such uncertainty may not always be best described in a von Neumann-Morgenstern setting of objectively known probabilities: given a player's own strategy, this is akin to saying that there are objective probabilities that can be placed on other players' choices of strategies. We can then think of the situation as one where players are *Bayesian*: they form *beliefs* about how others will play and maximize expected utility given these beliefs.

The standard setting to model this is one of *subjective expected utility*. These are *state space models* and take us too far afield here. The classic references are Savage (1954) and Anscombe and Aumann (1963). A more recent classic that discusses the Savage and Anscombe and Aumann models is Kreps (1988); and Gilboa (2009) is a fairly recent book that includes discussions of departures from Savage's model that now form large literatures of their own, such as the literature on ambiguity. In the Savage framework, a decision-maker describes uncertainty in terms of states, and has preferences over actions which are maps from states to outcomes. Axioms on preferences are given such that a utility function *as well as* probabilities to the states that describe the decision-maker's uncertainty are simultaneously obtained.

Having laid out an expanded setting, we turn to the standard, more compact, definition of a strategic form game. In this, there is no separate discussion of the set of outcomes: that is implicit; as is the outcome function. We can think of this as saying that for each player i, there is a *payoff function* $u_i : S \to \mathbb{R}$ given by $u_i(s) = v_i(g(s))$, for each strategy profile $s \in S$. We define:

Definition 4. *A Strategic Form Game is a triple $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consisting of a set N of players, a set S_i of strategies for each Player i, and a payoff function $u_i : S \to \mathbb{R}$ for each player i.*

		2: St	2: H
1	St	$2,2$	$0,1$
	H	$1,0$	$1,1$

Table 3.1: Stag Hunt

		2: L	2: R
1	L	$1,1$	$0,0$
	R	$0,0$	$1,1$

Table 3.2: Drive Left or Right

We can represent the 2-player stag hunt strategic form game using a Box diagram, as in Table 1.1.

The Stag Hunt is a coordination game in which there is a Pareto-ordering of Nash equilibria. Another coordination game is "drive left or dirve right" on a road, played by two drivers coming from opposite directions (Table 1.2). Either coordination is equally good, and we see both norms across countries; but failure to coordinate leads to disaster.

In the Battle of the Sexes (Table 1.3), Player 2 prefers to go to an Opera (O), Player 1 prefers to go to a Wrestling match (W); over the other activity; but they also like to do these activities together; alone, they derive no utility from either activity.

The next example is a Prisoners' Dilemma game with an effort and shirk story. 2 players jointly undertake production. Each can put in high (E) or

		2: W	2: O
1	W	$2,1$	$0,0$
	O	$0,0$	$1,2$

Table 3.3: Battle of the Sexes

		2: E	2: $\hat{S}$
1	E	$1,1$	$-1,2$
1	$\hat{S}$	$2,-1$	$0,0$

Table 3.4: Prisoners' Dilemma

low ($\hat{S}$) effort ('effort'and 'shirk'). So, $N = \{1,2\}$, $S_i = \{E, \hat{S}\}$, $i = 1,2$. For an easily described setting, we assign numbers $\hat{S} = 0$ and $E = 1$ in some effort units. The value of output $= 4(s_1 + s_2)$, effort cost is $c(0) = 0, c(1) = 3$. The payoff functions are: $u_i(s_i, s_j) = 4(s_i + s_j) - c(s_i),\ i, j = 1,2, i \neq j$.

The strategic form game can then be represented as in Table 1.4.

As a final introductory example, we consider a game with more than 2 players, and strategy sets that are infinite.

Example 1. *There are n farmers in a village; each farmer i chooses $s_i \in \mathbb{R}_+$ cows. So $S_i = \mathbb{R}_+$. The $M = \sum_{i \in N} s_i$ cows graze on the commons, and the revenue from each one is $v(M)$, where $v' < 0, v'' \leq 0$, and $v(M) > 0$ if $M < \bar{M}$, and is otherwise 0. It is instructive to draw $v()$.*

The cost of a cow is $c > 0$, and $v(0) > c$.

Farmer i's payoff function is $u_i(s_i, s_{-i}) = s_i v(\sum_{j=1}^{n} s_j) - cs_i$. So, an additional cow purchased by Farmer i reduces the per cow revenue of all other farmers as well, making this setting a non vacuous game.

A short note here may be in place. The notion of a strategy is far more involved than that of an action, but in the above examples, these two things seem to be conflated. We will separate the two when we model a game in terms of an *extensive form*; which is a rich description that can involve players moving multiple times, an order of moves, players knowing different information, and so forth.

		2: L	2: R
1	L	$1, -1$	$-1, 1$
	R	$-1, 1$	$1, -1$

Table 3.5: Penalty Kick

3.2.1 Mixed Strategies

Definition 5. *A mixed strategy σ_i of Player i is a probability measure over the set S_i of pure strategies.*

We will usually (but not always) use mixed strategies in games with finite S_i; so, we'll avoid using measure-theoretic notions of probability. And $\sigma(s_i)$ will represent the probability with which Player i chooses the pure strategy s_i, if she is using the mixed strategy σ_i.

Interpretations of Mixed Strategies:

(i)In some games, players may wish to randomize among their pure strategies in order to keep their opponents guessing. For example, in a penalty kick, the shooter may randomize between kicking left or right, and the goalkeeper between diving left or right. The game would look like in Table 1.5.

(ii) Players may be uncertain about what the other players intend to play. Then, $\sigma_j(s_j)$ can represent the *common beliefs* of all players other than j, about what j intends to play: here, uncertainty gets described by a subjective probability distribution. Why the beliefs should be common is another story.

(iii) Players may be uncertain about each others' *payoffs*. This is the classic Harsanyi interpretation in the context of games of incomplete information. Player j plays a pure strategy which depends on his payoff function: if others are uncertain about this payoff function and put a probability distribution over possible payoff functions, then we will get a probability distribution over Player j's optimal pure strategy that depends on the exact payoff function.

We will assume that players' mixed strategies are statistically independent. We will give a justification for this assumption when we discuss play in extensive form games. In particular, choosing mixed strategies indepen-

1 \ 2	L	M	R
U	$4,3$	$5,1$	$6,2$
M	$2,1$	$8,4$	$3,6$
D	$3,0$	$9,6$	$2,8$

Table 3.6: Illustrating Payoffs

dently does not rule out possibilities of players consulting each other and then choosing mixed strategies: but, this consultation has to be modelled in an extensive form game, and the strategic form to be analyzed must be derived from this larger extensive form game.

So, under a mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, the probability that pure strategy profile $s = (s_1, \ldots, s_n)$ occurs equals $\prod\limits_{j \in N} \sigma_j(s_j)$.

So, Player i's expected payoff from σ is:

$u_i(\sigma) = \sum\limits_{s \in S} u_i(s) \prod\limits_{j \in N} \sigma_j(s_j)$.

Note that since S_i is finite for all i, if $|S_i| = k_i$, then we can represent each mixed strategy σ_i as a point in the unit simplex $\Delta(S_i)$ in $\mathbb{R}^{k_i}$. We denote the set of all mixed strategies by Σ_i or $\Delta(S_i)$.

The handwritten notes have a couple of instructive pictures, that may also be drawn here in a lecture.

Consider the game in Table 1.6. We want to evaluate payoffs from a mixed strategy profile.

Suppose the players play the following mixed strategies:

$\sigma_1 \equiv (\sigma_i(U), \sigma_1(M), \sigma_1(D)) = (1/3, 1/3, 1/3)$

$\sigma_2 = (0, 1/2, 1/2)$.

Then:

$u_1(\sigma_1, \sigma_2) = 1/3(0(4) + 1/2(5) + 1/2(6)) + 1/3(0(2) + 1/2(8) + 1/2(3)) + 1/3(0(3) + 1/2(9) + 1/2(2)) = 11/2$.

The factoring above shows that

$u_1(\sigma_1, \sigma_2)$ is a weighted average of payoffs from playing the pure strategies U, M, D.

More generally, with n players,

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s \in S} u_i(s_i, s_{-i}) \prod_{j \in N} \sigma_j(s_j)$$
$$= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \sigma_i(s_i) \prod_{j \neq i} \sigma_j(s_j)$$
$$= \sum_{s_i \in S_i} \sigma_i(s_i) \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \prod_{j \neq i} \sigma_j(s_j)$$
$$= \sum_{s_i \in S_i} u_i(s_i, \sigma_{-i}) \sigma_i(s_i)$$

This is a weighted average of Player i's payoffs from her pure strategies s_i, weighted by the probabilities $\sigma_i(s_i)$; or, the expectation of $u_i(s_i, \sigma_{-i})$ w.r.t. the probability measure σ_i.

Notice that we had defined u_i as part of the strategic form game, as a mapping $u_i : S \to \mathbb{R}$; i.e. on the domain of all pure strategy profiles. We are using the same u_i notation now in extended fashion, as the domain has been extended to the set of all distributions on S.

While dealing with mixed strategies, of course, it suffices to have $u_i : \underset{j \in N}{\times} \Sigma_j \to \mathbb{R}$. Notice that with players choosing mixed strategies independently, not all joint distributions over the pure strategy profiles are attainable. For example, in a 2-player, 2-strategies each game (say, $S_i = \{L, R\}, i = 1, 2$, a joint distribution that puts probability 1/2 each on $(L, L), (R, R)$ is not attainable. For, this would require σ_1, σ_2 to place positive probabilities on L and R; but then, $(L, R), (R, L)$ would also be played with positive probability.

Players who mix independently can nevertheless be able to correlate their strategies by conditioning their play on a publicly observed signal. In the above example, the two players can achieve the above probability distribution on the strategy profiles if they can condition their play on a publicly observed coin toss. Each can play L if Heads occurs and R if Tails occurs. We can then call the distribution above a **correlated strategy**. Whether the players have an incentive to actually follow the *diktat* of a publicly observed signal (here, coin toss), leads to the notion of a *correlated equilibrium*.

3.2.2 Dominated Strategies

What is a good way for a rational player to choose a strategy? Well, she will certainly not play a strategy that gives her less than another pure or mixed strategy that she has, no matter what strategy profile others choose.

Definition 6. *A pure strategy* s_i *is* **strictly dominated** *for Player* i *if there exists a mixed strategy* $\sigma_i^{'} \in \Sigma_i$ *s.t.*

$u_i(\sigma_i^{'}, s_{-i}) > u_i(s_i, s_{-i}), \ \forall s_{-i} \in S_{-i}.$

s_i *is* **weakly dominated** *if the above holds with* $` \geq'$ *for all* $s_{-i} \in S_{-i}$*, and with* $` >'$ *for at least one* s_i*.*

There is also the notion of a *strictly dominant* strategy, and a *weakly dominant* strategy. A weakly dominant strategy gives a player at least as large a payoff as that from playing any other strategy, no matter what others play; and a strictly higher payoff against some strategy profile that others choose.

In the game in Table 1.6, M is strictly dominated for Player 2, since

$u_2(s_1, M) < u_2(s_1, R), \ \forall s_1 \in S_1.$

A rational Player 2 will not play M. If Player 1 believes Player 2 is rational, Player 1 should look at the *reduced* game in which $\hat{S}_2 = \{L, R\}$. But in this game, M, D are both strictly dominated strategies for Player 1. Furthermore, if Player 2 believes that Player 1 believes that Player 2 is rational, Player 2 should look at the reduced game in which $\hat{S}_1 = \{U\}, \hat{S}_2 = \{L, R\}$. Now, R is stricty dominated for Player 2. So, this *iterated* removal of strictly dominated strategies leads to the unique prediction (U, L); and is backed up by a hierarchy of beliefs about rationality. The single strategy profile that remains is said to survive *iterated strict dominance.*

Such a game is called *dominance solvable.* The Prisoners' Dilemma is also dominance solvable: there, in fact, both players have strictly *dominant* strategies.

The definition above highlights the fact that a pure strategy can be dominated by a mixed strategy, even if it is not dominated by any other pure strategy. The partly drawn game in Table 1.7 shows that for Player 1, L is dominated by a mixed strategy $\sigma_1^{'}$ that places probability 1/2 each on U and D. We have there:

$u_1(M, L) < u_1(\sigma_1^{'}, L = 1/2, \ u_1(M, R) < u_1(\sigma_1^{'}, R) = 1/2.$

Sometimes, a principle of *cautiousness* may suggest a setting in which even though a game is dominance solvable, iterated strict dominance is not

		2	
		L	R
1	U	$2,$	$-1,$
	M	$0,$	$0,$
	D	$-1,$	$2,$

Table 3.7: Domination by Mixed Strategy

		2	
		L	R
1	U	$8, 10$	$-1000, 9$
	D	$7, 6$	$6, 5$

Table 3.8: ISD and Cautious Play

completely persuasive. Consider the game in Table 1.8. The strategy profile (U, L) is the unique survivor of iterated strict dominance. But if Player 1 is even a little unsure about whether Player 2 will never play R, he may not choose to play U with probability 1. One way out is to say that we should model such uncertainty in Player 1's mind, say by modeling whether he believes that Player 2 is not fully rational, or that he does not know exactly what Player 2's payoffs are. That is, analyze an allied game of incomplete information.

3.3 Nash Equilibrium

Many games are not dominance solvable. We are then back to the question of how players should play. When he wrote his 1928 paper, John von Neumann explained the notion of strategy in detail, which we will do only in our discussion of extensive form games; suffice it to say that in an extensive form that completely describes all moves available to players, including possible communication with other players; and a strategy for a Player is a complete plan of what to do along the game. Now once strategies and a strategic form is obtained from an extensive form, von Neumann says a player must choose her strategy in complete ignorance of what others will do; because whatever she knows or has perhaps discussed with other players has already been incorporated into the extensive form game, from which the strategic form game has been derived; in the strategic form, therefore, he argued that Player i's choice of strategy is done in ignorance of what the others do. I will repeat and elaborate on this in the extensive games part. And so, von Neumann proposes the conservative notion of minimax play for players.

In game theory today, on the other hand, we adopt the notion of players maximizing utility subject to their beliefs. So the question is, what should players believe about what others do, when they choose their strategy? Nash proposes a concept of play that in a sense says that players beliefs about what others will play are correct. It is an *equilibrium* concept.

Consider a strategic form game $G = \langle N, (S_i), (u_i) \rangle$.

Definition 7. *A strategy profile* $\sigma^* \equiv (\sigma_1^*, \ldots, \sigma_n^*)$ *is a* **Nash Equilibrium** *if for all players* $i \in N$, $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i^{'}, \sigma_{-i}^*), \forall \sigma_i^{'} \in \Sigma_i$.

A pure strategy profile $s^* \equiv (s_1^*, \ldots, s_n^*)$ is a N.E. if for all $i \in N$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i^{'}, s_i^*), \forall s_i^{'} \in S_i$.

In words, for each Player i, s_i^* maximizes her payoff $u_i(s_i, s_{-i}^*)$ over all s_i, holding fixed s_{-i} at s_{-i}^*.

This suggests that in order to check whether a strategy profile $(s_1, \ldots, s_n)$ is a N.E., we must check, for each player i, that given the others' strategies s_i, there is no other strategy $s_i^{'}$ that Player i can 'deviate to', from s_i, that gives her a higher payoff than $u_i(s_i, s_{-i})$. We mean 'deviate to' only in a contemplative sense; operationally, it means only that you, the game theory analyst, can check to see whether there are such profitable deviations; and **not** that (s_i, s_{-i}) has already been chosen and played before Player i contemplates a deviation.

Interpretations:

- Rational players here anticipate correctly what others will play, and choose a strategy to maximize their payoffs given these correct beliefs. Note that if players are rational and they anticipate s^* will be played, then each Player i will play s_i^* - the beliefs are self-fulfilling.

- If Players play a profile s' that is NOT a N.E., then at least one Player i is playing an $s_i^{'}$ that is NOT a best response to $s_{-i}^{'}$: $u_i(s_i^{'}, s_{-i}^{'}) < u_i(s_i, s_{-i}^{'})$, for some $s_i \in S_i$. Either Player i is not rational, or her belief about what other players play is incorrect (or both).

- A Nash Equilibrium can be a steady state of some dynamical process: including learning and evolution.

- A mixed strategy Nash Equilibrium could represent a situation in which there are n populations of players, one for each player i: in each population, different proportions of players play different pure strategies in S_i; and in the strategic game, in each period one player is randomly drawn from each population to play the game. If σ^* is a Nash equilibrium, it could be that Player i plays one of the pure strategy best responses to σ^*_{-i}; and together, the proportions played by players representing Player i constitute σ^*_i.

- Mixed strategies in a M.S.N.E. could represent equilibrium beliefs about what other players will play.

- Harsanyi's Purification Theorem interpretation referred to earlier and to be done later.

Let's find the N.E. in the Battle of the Sexes, reproduced here.

Indeed, in terms of the N.E. definition: $u_1(W, W) > u_1(O, W)$, and $u_2(W, W) > u_2(W, O)$. So, no player has a 'profitable deviation' from (W, W); so, (W, W) is a Nash equilibrium. Similarly, (O, O) is a N.E. These are the only pure strategy N.E. Check that from the other 2 strategy profiles, at least one player i can deviate to another strategy, holding fixed player j's strategy. For instance, from (O, W), Player 1 can deviate to W, holding fixed Player 2's strategy at W, and improve her payoff.

Another way to get to N.E. is to use the notion of a *best response*, which we will look at in more detail soon. By fixing Player 2's pure strategy at W, we see that Player 1's best response to this is to play W and get 2, rather than play O and get 0. We basically do some underlining of the largest payoff that Player 1 can achieve, having fixed Player 2's strategy at W; and then at

		2	
		W	O
1	W	$2,1$	$0,0$
	O	$0,0$	$1,2$

Table 3.9: Battle of the Sexes

O. And similarly fix Player 1's strategy and find what strategy of Player 2 maximizes 2's payoff. A completely underlined box indicates strategies that comprise a pure strategy N.E.

What about Mixed Strategy N.E.?

Suppose (σ_1^*, σ_2^*) is a mixed strategy N.E. in which both players mix. Player 1's payoff equals: $u_1(\sigma_1^*, \sigma_2^*) = \sigma_1^*(W)u_1(W, \sigma_2^*) + \sigma_1^*(O)u_1(O, \sigma_2^*)$.

This is a weighted average of $u_1(W, \sigma_2^*)$ and $u_1(O, \sigma_2^*)$. If one of these is greater than the other, then putting positive probabilities on both cannot be payoff-maximizing: playing the pure strategy that gives the higher payoff is the payoff-maximizing strategy choice for Player 1. So, since σ_1^* is payoff-maximizing against σ_2^*, *it must be that*

$u_1(W, \sigma_2^*) = u_1(O, \sigma_2^*)$. Similarly, it must be that

$u_2(\sigma_1^*, W) = u_1(\sigma_1^*, O)$.

The first of these *indifference equations* gives:

$2\sigma_2^*(W) = 1\sigma_2^*(O) = 1(1 - \sigma_2^*(W))$; so, $\sigma_2^*(W) = 1/3$.

The second equation yields:

$\sigma_1^*(W) = 2(1 - \sigma_1^*(W))$; so $\sigma_1^*(W) = 2/3$.

The (σ_1^*, σ_2^*) numbers above comprise the mixed strategy N.E. for the Battle of the Sexes.

Nash Equilibrium expressed in terms of Best Responses

Definition 8. *A strategy* $\sigma_i \in \Sigma_i$ *is a* **best response** *to* $\sigma_{-i} \in \underset{j \neq i}{\times} \Sigma_j$ *if*

$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i^{'}, \sigma_{-i}),\ \forall \sigma_i^{'} \in \Sigma_i$. *Given any* σ_{-i}, *let*

$B_i(\sigma_{-i}) = \{\sigma_i \in \Sigma_i | u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i^{'}, \sigma_{-i}),\ \forall \sigma_i^{'} \in \Sigma_i\}$.

So, $B_i(\sigma_{-i})$ is the set of all best responses of Player i to σ_{-i}. It is a set because Player i could have multiple best responses to σ_{-i}. Note also that if

S_i is finite, then we can order the payoffs $u_i(s_i, \sigma_{-i}),\ s_i \in S_i$. There is at least one pure strategy s_i corresponding to the highest of these payoffs: so that $s_i \in B_i(\sigma_{-i})$; so, $B_i(\sigma_{-i})$ is nonempty. There may be other pure strategies that give this highest payoff as well: if so, then any mix of all these pure strategies that give the highest payoff is also a mixed strategy best response to σ_{-i}.

Definition 9. *Let C and D be sets. A* **correspondence** $F : C \rightrightarrows D$ *assigns, to every $c \in C$, a subset $F(c) \subseteq D$.*

We define $B_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ as Player i's *best response correspondence*. It assigns, to every $\sigma_{-i} \in \Sigma_{-i}$, the set $B_i(\sigma_{-i}) \subseteq \Sigma_i$ of Player i's best responses to σ_{-i}. We can define a Nash equilibrium using the language of best responses.

Definition 10. *A strategy profile $\sigma^* \equiv (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash Equilibrium of G if for all players $i \in N$, $\sigma_i^* \in B_i(\sigma_{-i}^*$.*

In other words, in a Nash equilibrium, each strategy in the strategy profile is a best response to the other strategies. The process of underlining payoffs for best responses in the BoS example, gave us pure strategy Nash equilibria characterized by completely underlined boxes: these captured strategy profiles (s_1, s_2) that were mutual best responses: $s_1 \in B_1(s_2), s_2 \in B_2(s_1)$.

Let us write down, more generally, the best response correspondences in BoS. Note that:

$2\sigma_2(W) = u_1(W, \sigma_2) > (=, <) u_1(O, \sigma_2) = 1 - \sigma_2(W)$ as
$\sigma_2(W) > (=, <) 1/3$.
So:

$$B_1(\sigma_2) = \begin{cases} O, & \text{if } \sigma_2 \ s.t.\ \sigma_2(W) < 1/3 \\ \Sigma_1, & \text{if } \sigma_2 \ s.t.\ \sigma_2(W) = 1/3 \\ W, & \text{if } \sigma_2 \ s.t.\ \sigma_2(W) > 1/3 \end{cases}$$

Similarly:
$u_2(\sigma_1, W) = \sigma_1(W) > (=, <) 2(1 - \sigma_1(W)) = u_2(\sigma_1, O)$ as
$\sigma_1(W) > (=, <) 2/3$. The larger probability on W needed for Player 2 to be indifferent reflects her relatively less taste for W. We have:

$$B_2(\sigma_1) = \begin{cases} O, & \text{if } \sigma_1 \ s.t. \ \sigma_1(W) < 2/3 \\ \Sigma_2, & \text{if } \sigma_1 \ s.t. \ \sigma_1(W) = 2/3 \\ W, & \text{if } \sigma_1 \ s.t. \ \sigma_1(W) > 2/3 \end{cases}$$

Since $B_i(\sigma_j)$ is a subset of the unit simplex in $\mathbb{R}^2$, we can represent it using just the probabilities that the best responses put on $\sigma_i(W)$. A figure is instructive, and is either drawn here or will be pencilled in with a graphics tablet in the lecture. The intersections of $B_1(\sigma_2)$ and $B_2(\sigma_1)$ are points (σ_1, σ_2) s.t. $\sigma_1 \in B_1(\sigma_2)$ and simultaneously $\sigma_2 \in B_2(\sigma_1)$, i.e., Nash equilibria.

Now, here is a Proposition on a point we have been belabouring.

Proposition 1. *σ^* is a mixed strategy Nash Equilibrium of a finite strategic form game if and only if for all players i, every pure strategy s_i in the support of σ_i^* is in $B_i(\sigma_{-i}^*)$.*

Proof. Suppose every s_i in the support of σ_i^* satisfies $s_i \in B_i(\sigma_{-i}^*)$; i.e., maximizes i's payoff, given σ_{-i}^*. Since $u_i(\sigma_i^*, \sigma_{-i}^*)$ is just a weighted average of this maximum payoff (weighted by the $\sigma_i^*(s_i)$ probabilities, this payoff is also the maximum payoff. So, $\sigma_i^* \in B_i(\sigma_{-i}^*)$.

Conversely, suppose that for some Player i, there is a strategy $s_i^{'}$ in the support of σ_i^* that is *not* a best response to $B_i(\sigma_{-i}^*)$. It should be clear that not playing $s_i^{'}$ and instead assigning the probability $\sigma_i^*(s_i^{'})$ to any best response will increase her payoff, so σ_i^* is not a best response to σ_{-i}^*. ■

All this suggests a procedure/algorithm to compute mixed strategy Nash equilibria in finite games. This procedure is discussed in Myerson (Section 3.3) with a game that is given here is Table 1.10. The complexity of this procedure increases exponentially with the size of the strategy-profile space; so this discussion is more for firming up the idea of a M.S.N.E. than for studying questions of computing Nash equilibria for finite strategic games more generally.

So here is the procedure.

(1) Guess supports $\bar{S}_i \subseteq S_i$, $i = 1, \ldots, n$.

(2) Solve the system of equations:

		2		
		L	M	R
1	T	$7,2$	$2,7$	$3,6$
	B	$2,7$	$7,2$	$4,5$

Table 3.10: Myerson Section 3.3 game: Computing Nash Equilibrium

$$\begin{array}{rcll} \sum_{s_{-i}\in\bar{S}_{-i}} u_i(s_i,s_{-i})\prod\limits_{j\neq i}\sigma_j(s_j) & = & w_i,\ \forall i\in N, s_i\in\bar{S}_i & (2.1)\\ \sigma_i(s_i^{'}) & = & 0,\ \forall s_i^{'}\in\bar{S}_i^c, \forall i\in N & (2.2)\\ \sum_{s_i\in\bar{S}_i}\sigma_i(s_i) & = & 1,\ \forall\in N & (2.3)\end{array}$$

(2.1) says the payoffs from the pure strategies of a player that are played with positive probability are all equal; (2.2) just assigns 0 probability to the strategies not played (not in $\bar{S}_i$; and (2.3) assigns probability 1 to the guessed supports $\bar{S}_i$. (2.1),(2.2),(2.3) have, respectively, $|\bar{S}_i|$, $|\bar{S}_i^c|$, and 1 equations for each player; so $\left(\sum_{i\in N}|S_i|\right)+n$ equations in all. The variables are all the probability numbers $\sigma_i(s_i)$, and the n w_i's. So the number of equations and variables is the same; (2.1) has equations that are multilinear in the σ_i's, and the other equations are linear.

But having obtained a solution to (2.1)-(2.3) does not suffice. We have to check:

(3) $\sigma_i(s_i)\geq 0,\ \forall i,\ \forall s_i\in\bar{S}_i$.

And, it must not be that any player is not playing some strategy that will yield a higher payoff than the strategies played.

(4) $\sum_{s_i\in\bar{S}_{-i}} u_i(s_i^{'},s_{-i})\prod\limits_{j\neq i}\sigma_j(s_j)\leq w_i,\ \forall i,\forall s_i^{'}\in\bar{S}_i^c$.

3.3.1 Baby Applications

(1) n-player Stag Hunt

$N=\{1,\ldots,n\}$. $S_i=\{St,H\}$. If the Stag is caught, it is divided equally, and each player gets a payoff of k/n, a number greater than 1, the payoff from catching a hare.

$$u_i(s_i, s_{-i}) = \begin{cases} k/n & \text{if} \quad s_j = St \ \forall \ j \in N \\ 1 & \text{if} \quad s_i = H \\ 0 & \text{if} \quad s_i = St \ \& \ s_j = H \text{ for some } j \neq i \end{cases}$$

It is easy to see that for all i, $s_i = St$ is a best response to $s_{-i} = (St, \ldots, St)$, giving $k/n > 1$, so $(St, \ldots, St)$ is a N.E. Similarly, $(H, \ldots, H)$ is a N.E.; and these are the two pure strategy N.E.

M.S.N.E.: Suppose $(\sigma_1, \ldots, \sigma_n)$ is a M.S.N.E. The indifference equations for Players 1 and 2 are:

$\sigma_2(St)\sigma_3(St) \ldots \sigma_n(St)(k/n) = 1$

$\sigma_1(St)\sigma_3(St) \ldots \sigma_n(St)(k/n) = 1$

and for these to hold, all probabilities on the LHS must be positive. So dividing the first by the second, we have $\sigma_1(St) = \sigma_2(St)$. Extending this argument, $\sigma_1 = \sigma_2 = \cdots = \sigma_n$.

So let $\sigma_i(St) = x$ for all i, and plug this into the first indifference equation, say, and solve.

We get: $x = (n/k)^{1/(n-1)}$.

(2) Discrete Public Goods

We discuss the model of Palfrey and Rosenthal (1984) here; we do the 'no refund' case, which is easier to interpret as a more general collective action problem (a literature that within economics is sometimes dated back to Olson (1965), and also includes early work by Tullock).

To take a recent example, Cantoni, Yang, Yuchtman and Zhang (2019)(QJE) ran a framed field experiment around the annual July 1 mass anit-authoritarianism protest in Hong Kong. They note that mass protests, from Tiananmen Square to Tahrir Square (and we may add the mass protests that we have seen around us in the past years), raise the question: what drives individuals' decisions to participate in political protests?

The older literature (Olson, Tullock, Palfrey and Rosenthal) suggests that since participation is costly and the outcome of successful action is a public good for the protesters, there is incentive to free-ride: so a player is less likely to show up if she believes that lots of people will show up. But recent theoretical work stresses complementarities: such as the cost of participation being decreasing in the size of the protest. Cantoni et al elicit beliefs about how many will participate in the July 1 protest, then provide information on intention to participate, permitting people to update their prior beliefs; then they observe whether people participate or not. On balance, they find evidence of free riding.

In Palfrey and Rosenthal, who cast their game as one of providing a discrete public good, there are n players, and each has strategy set $S_i = \{0, 1\}$; interpreted as "do not contribute" and "contribute". In the language of a protest movement, "do not participate" and "participate".

It takes $w \in \{2, \ldots, n-1\}$ players to contribute for the public good to be produced (or the protest to succeed); and if it is produced, the gross utility equals 1, for each player.

Contribution is costly: it costs $c \in (0, 1)$.

Let $1(s)$ be a function from strategy profiles $s \in S$ to $\{0, 1\}$, defined by $1(s) = 1 \ if \ \sum_{j \in N} s_j \geq w$, and $1(s) = 0 \ if \ \sum_{j \in N} s_j < w$. $1(s)$ captures the condition under which the public good is produced.

So, Player i's payoff from the strategy profile s is: $u_i(s_i, s_{-i}) = 1(s) - c \ if \ s_i = 1$; and $u_i(s_i, s_{-i}) = 1(s) \ if \ s_i = 0$.

First, let's get the pure strategy N.E.

If none of the other players contribute, then i gets 0 from not contributing and $-c$ from contributing. So, do not contribute is her unique best response. So $(s_1, \ldots, s_n) = (0, \ldots, 0)$ is a N.E.

Now, consider a strategy profile s in which exactly w players contribute; so the public good is produced. Each contributor i has payoff $1 - c > 0$; if she changes to be a non-contributor, then given s_i, the good is not produced, and her payoff reduces to 0. So, each contributor i is playing a best response to s_i by contributing. On the other hand, for any non-contributor j, who's getting 1, contributing reduces his payoff to $1 - c$. So for any non-contributor j, not contributing is a best response. So every such strategy profile s is a N.E.

There are $\binom{n}{w}$ such N.E. They are efficient. Please show that any strategy profile in which less than w or more than w players contribute is not a N.E.

In these equilibria, $(n-w)$ of the players get to free ride.

For studying mixed strategy N.E. in this model, we divide N into 3 subsets, according to whether they contribute or not in a mixed strategy profile: G^1 is the set of all players that contribute for sure; G^2 is the set of all players that do not contribute for sure; G^3 is the set of all players who contribute with probability q. So, Palfrey and Rosenthal look at mixed strategy equilibria in which the *randomizers* (not randomistas) all mix with the same probability distribution.

Let $|G^1| = k$, $|G^2| = j$, and let m^3 be the number of players in G^3 who *ex post* end up contributing (choosing $s_j = 1$); i.e. the number of players in this subset for whom the randomization/mixing ends up selecting $s_j = 1$. Finally, let m_i^3 be the number of players in $G^3 - \{i\}$ who end up contributing ex post.

For $i \in G^1$, the payoff from contributing is (weakly) larger:

1.$Pr(m^3 \geq w-k) - c \geq Pr(m^3 \geq w-k+1)$,

because if i contributes, then the good is provided if m^3 turns out to be at least $w-k$; if i does not contribute, then m^3 needs to be at least $w-(k-1)$.

Rearranging, $Pr(m^3 \geq w-k) - Pr(m^3 \geq w-k+1) \geq c$.

But the difference between the two events on the left is the event that m^3 exactly equals $w-k$; and this can happen in $\binom{n-k-j}{w-k}$ ways, as $|G^3| = n-k-j$.

So, we have:

(1) $c \leq \binom{n-k-j}{w-k} q^{w-k}(1-q)^{n-w-j}$

(since $n-k-j-(w-k) = n-w-j$).

Similarly, for $i \in G^2$, we must have:

$Pr(m^3 \geq w-k) \geq Pr(m^3 \geq w-k-1) - c$.

This reaaranges to $c \geq Pr(m^3 = w-k-1)$, or

(2) $c \geq \binom{n-k-j}{w-k-1} q^{w-k-1}(1-q)^{n-w-j+1}$.

Finally, any player $i \in G^3$ must be indifferent between not contributing (LHS) and contributing (RHS):

$Pr(m_i^3 \geq w-k) = Pr(m_i^3 \geq w-k-1) - c$, or

$c = Pr(m_i^3 = w-k-1)$. So

(3) $c = \binom{n-k-j-1}{w-k-1} q^{w-k-1}(1-q)^{n-w-j}$.

Some general results of the features of this model, of when (1),(2) and (3) or a subset of these are satisfied, are available in Palfrey and Rosenthal (1984). Here, we discuss an example.

Example. $n = 4, w = 2, c = 0.096$.

There are various possibilities for MSNE, depending on how many players play pure strategies of contributing or not contributing. Note that if $k = 2$, that's it; no one else will contribute, we're already at a pure strategy N.E. So:

Case I. $k = 0, j = 0$. For $i \in G^3$, we need:

$c = 0.096 = \binom{n-k-j-1}{w-k-1} q^{w-k-1}(1-q)^{n-w-j} = \binom{3}{1} q^1 (1-q)^2$

There are 2 relevant solutions: $q = 0.8$ and $q \sim 0.0343$. In the first of these, there is overprovision of effort for the public good since the expected number of contributors is $0.8(4) = 3.2 > 2$; in the second of these, there is underprovision.

Case II. $k = 1, j = 0$. For the 3 randomizers in G^3, the indifference equation translates to:

$c = 0.096 = \binom{2}{0} q^0 (1-q)^2$.

In the $0 < q < 1$ range, this quadratic has the solution $q \sim 0.69$. So, the expected number of contributors is $1 + 0.69(3) > 3$; which is more than the efficient number of 2.

Check that the incentives for the player in G^1 are met; that is contributing for sure is at least as good as not contributing at all.

Work out Case III. $k = 0, j = 1$ and Case IV. $k = 1, j = 1$.

Also, does a case where $j = 2$ need to be worked out?