ETH zürich

computer graphics laboratory

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Visual Computing - Exercise 8
Matrices & Quaternions

Exercise 1) Theory

a) **Short Questions**

I. *What are the three coordinate systems used to describe the scene?*

- Object coordinates
- World coordinates
- Camera coordinates

II. *State one advantage of using homogeneous coordinates.*

- All transformations can be expressed as matrix operations.
- An arbitrary number of affine and projective mappings, applied one after the other, can be combined in one single matrix.

III. *Given two homogeneous points $\boldsymbol{p_1} = [4\ 3\ 2\ 1]^T$ and $\boldsymbol{p_2} = [1\ 2\ 3\ 4]^T$, compute the Euclidian displacement vector $\boldsymbol{d} \in R^3$ from $\boldsymbol{p_1}$ to $\boldsymbol{p_2}$.*

To compare $\mathbf{p_1}$ and $\mathbf{p_2}$ in R^3, we must first make them homogeneous. This is done by simply dividing the first three components by the fourth component of the point vectors. The corresponding vectors in R^3 are $\widetilde{\mathbf{p_1}} = [4\ 3\ 2]^{\mathrm{T}}$and $\widetilde{\mathbf{p_2}} = (\frac{1}{4})[1\ 2\ 3]^{\mathrm{T}}$ and we therefore have $\mathbf{d} = \widetilde{\mathbf{p_2}} - \widetilde{\mathbf{p_1}} \quad = -\frac{1}{4}[15\ 10\ 5]^{\mathrm{T}}$ as the displacement vector.

b) **Homogenous Transformations**

I. *Decompose the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 & 47 \\ 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*into a linear transformation **L** and a translation **T**, so that **A** = **LT**. (Note: not **TL**!)*

Note that the structure of A indicates that A consists of a rotation and a translation (see lecture slide 14). Thus, if **TL** was asked (i.e., first rotation, then translation) the matrices can be constructed directly from **A** (upper left 3x3 matrix entries correspond to the rotation, and rightmost column corresponds to the translation, as written in the slides). In this exercise, however, the translation comes first and is affected by the rotation. Thus, the translation components in the rightmost column correspond to an already rotated initial translation vector. It suffices to rotate the translation vector back in the opposite direction to account for the rotation. Then, the same reasoning from above can be applied. Since **L** corresponds to a 90° rotation around the z-axis (see II.), we rotate the translation vector $[47\ 11\ 0]^T$ by -90° around the z-axis, resulting in the initial translation vector $[11\ -47\ 0]^T$.

$$\mathbf{L} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -47 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

II. *Please describe in words what the transformations **L** and **T** represent.*

L is a 90^0 rotation around the z axis, and **T** is a translation of $[11\ -47\ 0]^T$.

III. *In which order are **L** and **T** processed, if applied to a point in 3D?*

The point is translated by **T** and then rotated by **L**.

IV. *Let **n** be the normal of a plane H in "Hessian-Normalform". Thus, for all points $\boldsymbol{p} \in H$, we have $\boldsymbol{p}^T\boldsymbol{n} = 0$. The linear transformation A given above maps all $\boldsymbol{p} \in H$ to points $\boldsymbol{p}'$ in a second plane H' ($\boldsymbol{p}' = \boldsymbol{A}\boldsymbol{p} \in H'$). Compute a matrix $\boldsymbol{B}$ that maps **n** to the normal **n'** of the plane H' such that $\boldsymbol{n}' = \boldsymbol{B}\boldsymbol{n}$. (Tip: Compute $\boldsymbol{B}$ using the decomposition $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{T}$.)*

We have $\mathbf{n}'^{\mathbf{T}}\mathbf{p}' = \mathbf{n}'^{\mathbf{T}}\mathbf{A}\mathbf{p} = \mathbf{n}'^{\mathbf{T}}\mathbf{L}\mathbf{T}\mathbf{p} = (\mathbf{T}^{\mathbf{T}}\mathbf{L}^{\mathbf{T}}\mathbf{n}')^{\mathbf{T}}\mathbf{p} = \mathbf{0}$. Comparing this with $\mathbf{n}^{\mathrm{T}}\mathbf{p} = 0$ and noting that these are true for all $\mathbf{p}$, we see that $\mathbf{n} = \mathbf{T}^{\mathbf{T}}\mathbf{L}^{\mathbf{T}}\mathbf{n}'$ and thus $\mathbf{n}' = (\mathbf{L}^{\mathbf{T}})^{-1}(\mathbf{T}^{\mathbf{T}})^{-1}\mathbf{n} = \mathbf{L}(\mathbf{T}^{\mathbf{T}})^{-1}\mathbf{n}$ since **L** is a unitary matrix. Thus $\mathbf{B} = \mathbf{L}(\mathbf{T}^{\mathbf{T}})^{-1} =$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -11 & 47 & 0 & 1 \end{bmatrix}$$

c) **Quaternions**

I. *Assemble the unit quaternion **q** = c + xi + yj + zk which describes a rotation of 60^o around the rotation axis $[3\ 0\ 4]^T$. Simplify the quaternion as much as possible.*

The rotation of an arbitrary 3D-point **p** by the rotation angle φ around an axis **n** is described by the quaternion operation $\mathbf{R_q}(\mathbf{p}) = \mathbf{q}\mathbf{p}\bar{\mathbf{q}}$, where point **p** is represented by a pure quaternion and **q** is a unit quaternion. This unit quaternion **q** can be written as $\mathbf{q} = \mathbf{cos}\frac{\boldsymbol{\varphi}}{\mathbf{2}} + \mathbf{sin}\frac{\boldsymbol{\varphi}}{\mathbf{2}}\mathbf{n}$. Thus, if we write $\mathbf{n} = \frac{[3\ 0\ 4]^T}{\|[3\ 0\ 4]^T\|} = [0.6\ 0\ 0.8]$, this implies that we have $\mathbf{q} = \cos 30^{\mathrm{o}} + \sin 30^{\mathrm{o}}\ \mathbf{n} = \frac{\sqrt{3}}{2} + 0.3i + 0.4k$.

II. *Directly compute the quaternion of the inverse rotation from **q**.*

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{\|\mathbf{q}\|^2} = \bar{\mathbf{q}} = \frac{\sqrt{3}}{2} - 0.3i - 0.4k$$

III. *Which elementary quaternion operation have you used in II?*

Possible answers: Inversion, inversion of the unit quaternion, conjugation.

IV. *Given the quaternion* $\boldsymbol{r} = [c\ x\ y\ z]^T = [0\ 1\ 0\ 0]^T$. *What rotation does* $\boldsymbol{r}$ *correspond to?*

It corresponds to a rotation of 180º around the x-axis. We can see that by writing the quaternion in two different forms and find correspondences. On one hand, $\mathbf{r} = \mathbf{cos}\frac{\boldsymbol{\varphi}}{\mathbf{2}} + \mathbf{sin}\frac{\boldsymbol{\varphi}}{\mathbf{2}}\mathbf{n}$ where $\mathbf{n} = (\mathbf{1}\boldsymbol{i} + \mathbf{0}\boldsymbol{j} + \mathbf{0}\boldsymbol{k})$ from the last three components of r. On the other hand, $\mathbf{r} = \mathbf{0} + \mathbf{1}\boldsymbol{i} + \mathbf{0}\boldsymbol{j} + \mathbf{0}\boldsymbol{k} = \boldsymbol{i}$. Thus, we have $\mathbf{cos}\frac{\boldsymbol{\varphi}}{\mathbf{2}} + \mathbf{sin}\frac{\boldsymbol{\varphi}}{\mathbf{2}}\boldsymbol{i} = \boldsymbol{i}$, And therefore $\mathbf{cos}\frac{\boldsymbol{\varphi}}{\mathbf{2}} = \mathbf{0}$ and $\mathbf{sin}\frac{\boldsymbol{\varphi}}{\mathbf{2}} = \mathbf{1}$. This holds for $\boldsymbol{\varphi}$ = 180°. Of course, it also holds for all angles 180° + c 360° for an integer c, but we are interested in an angle in [0°, 360°). The rotation is around the x-axis.

V. *Please specify the rotation matrix that corresponds to* $\boldsymbol{r}$.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$