Visual Computing - Exercise 9
Matrices & Quaternions

Exercise 1) Theory

a) **Short Questions**

I. *What are the three coordinate systems used to describe the scene?*

- Object coordinates
- World coordinates
- Camera coordinates

II. *State one advantage of using homogeneous coordinates.*

- All transformations can be expressed as matrix operations.
- An arbitrary number of affine and projective mappings, applied one after the other, can be combined in one single matrix.

III. *Given two homogeneous points $\boldsymbol{p_1} = [4\ 3\ 2\ 1]^T$ and $\boldsymbol{p_2} = [1\ 2\ 3\ 4]^T$, compute the Euclidian displacement vector $\boldsymbol{d} \in R^3$ from $\boldsymbol{p_1}$ to $\boldsymbol{p_2}$.*

To compare $\mathbf{p_1}$ and $\mathbf{p_2}$ in R^3, we must first make them homogeneous. This is done by simply dividing the first three components by the fourth component of the point vectors. The corresponding vectors in R^3 are $\widetilde{\mathbf{p_1}} = [4\ 3\ 2]^\mathrm{T}$ and $\widetilde{\mathbf{p_2}} = (\frac{1}{4})[1\ 2\ 3]^\mathrm{T}$ and we therefore have $\mathbf{d} = \widetilde{\mathbf{p_2}} - \widetilde{\mathbf{p_1}} \quad = -\frac{1}{4}[15\ 10\ 5]^\mathrm{T}$ as the displacement vector.

b) **Homogenous Transformations**

I. *Decompose the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 & 47 \\ 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*into a linear transformation **L** and a translation **T**, so that **A** = **LT**. (Note: not **TL**!)*

$$\mathbf{L} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -47 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

II. *Please describe in words what the transformations **L** and **T** represent.*

L is a 90° rotation around the z axis, and **T** is a translation of $[11 - 47\ 0]^T$.

III. *In which order are **L** and **T** processed, if applied to a point in 3D?*

The point is translated by **T** and then rotated by **L**.

IV. *Let* $\boldsymbol{n}$ *be the normal of a plane H in "Hessian-Normalform". Thus, for all points* $\boldsymbol{p} \in H$, *we have* $\boldsymbol{p}^T \boldsymbol{n} = 0$. *The linear transformation A given above maps all* $\boldsymbol{p} \in H$ *to points* $\boldsymbol{p}'$ *in a second plane H' (*$\boldsymbol{p}' = \boldsymbol{A}\boldsymbol{p} \in H'$*). Compute a matrix* $\boldsymbol{B}$ *that maps* $\boldsymbol{n}$ *to the normal* $\boldsymbol{n'}$ *of the plane H' such that* $\boldsymbol{n}' = \boldsymbol{B}\boldsymbol{n}$. *(Tip: Compute* $\boldsymbol{B}$ *using the decomposition* $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{T}$ *.)*

We have $\mathbf{n}'^{\mathbf{T}}\mathbf{p}' = \mathbf{n}'^{\mathbf{T}}\mathbf{A}\mathbf{p} = \mathbf{n}'^{\mathbf{T}}\mathbf{L}\mathbf{T}\mathbf{p} = (\mathbf{T}^{\mathbf{T}}\mathbf{L}^{\mathbf{T}}\mathbf{n}')^{\mathbf{T}}\mathbf{p} = \mathbf{0}$. Comparing this with $\mathbf{n}^{\mathrm{T}}\mathbf{p} = 0$ and noting that these are true for all $\mathbf{p}$, we see that $\mathbf{n} = \mathbf{T}^{\mathbf{T}}\mathbf{L}^{\mathbf{T}}\mathbf{n}'$ and thus $\mathbf{n}' = (\mathbf{L}^{\mathbf{T}})^{-\mathbf{1}}(\mathbf{T}^{\mathbf{T}})^{-\mathbf{1}}\mathbf{n} = \mathbf{L}(\mathbf{T}^{\mathbf{T}})^{-\mathbf{1}}\mathbf{n}$ since $\mathbf{L}$ is a unitary matrix. Thus $\mathbf{B} = \mathbf{L}(\mathbf{T}^{\mathbf{T}})^{-\mathbf{1}} =$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -11 & 47 & 0 & 1 \end{bmatrix}$$

c) **Quaternions**

I. *Assemble the unit quaternion* $\boldsymbol{q}$ *= c + xi + yj + zk which describes a rotation of* 60^o *around the rotation axis* $[3\ 0\ 4]^T$. *Simplify the quaternion as much as possible.*

The rotation of an arbitrary 3D-point $\mathbf{p}$ by the rotation angle φ around an axis $\mathbf{n}$ is described by the quaternion operation $\mathbf{R_q}(\mathbf{p}) = \mathbf{q}\mathbf{p}\bar{\mathbf{q}}$, where point $\mathbf{p}$ is represented by a pure quaternion and $\mathbf{q}$ is a unit quaternion. This unit quaternion $\mathbf{q}$ can be written as $\mathbf{q} = \mathbf{cos}\frac{\boldsymbol{\varphi}}{\mathbf{2}} + \mathbf{sin}\frac{\boldsymbol{\varphi}}{\mathbf{2}}\mathbf{n}$. Thus if we write $\mathbf{n} = \frac{[3\ 0\ 4]^T}{\|[3\ 0\ 4]^T\|} = [0.6\ 0\ 0.8]$, this implies that we have $\mathbf{q} = \cos 30^o + \sin 30^o\, \mathbf{n} = \frac{\sqrt{3}}{2} + 0.3\mathrm{i} + 0.4\mathrm{k}$**.**

II. *Directly compute the quaternion of the inverse rotation from* $\boldsymbol{q}$.

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{\|\mathbf{q}\|^2} = \bar{\mathbf{q}} = \frac{\sqrt{3}}{2} - 0.3\mathrm{i} - 0.4\mathrm{k}$$

III. *Which elementary quaternion operation have you used in II?*

Possible answers: Inversion, inversion of the unit quaternion, conjugation.

IV. *Given the quaternion* $\boldsymbol{r} = [c\ x\ y\ z]^T = [0\ 1\ 0\ 0]^T$. *What rotation does* $\boldsymbol{r}$ *correspond to?*

It corresponds to a rotation of 180° around the x-axis.

V. *Please specify the rotation matrix that corresponds to* $\boldsymbol{r}$.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$