Logic for Computer Scientists 2019/20. Final exam.

1. (12 points) Fix a signature Σ consisting of one unary function symbol f and one constant c. Consider the class $\mathcal{C}$ of structures $\mathbb{A} = (A, f^{\mathbb{A}}, c^{\mathbb{A}})$ with signature Σ such that

$$\{(f^{\mathbb{A}})^n(c^{\mathbb{A}}) \mid n \in \mathbb{N}\} = A,$$

where $(f^{\mathbb{A}})^n$ denotes n consecutive applications of $f^{\mathbb{A}}$.

   (a) (6 points) Show that $\mathcal{C}$ is not axiomatisable by a set of sentences of first-order logic by using the Skolem-Löwenheim theorem.

   (b) (6 points) Show that $\mathcal{C}$ is not axiomatisable by a set of sentences of first-order logic by using the compactness theorem.

2. (12 points) Let Σ be a signature consisting of a single unary function symbol f. Let $\mathbb{A} = (A, f^{\mathbb{A}})$ be a fixed structure over signature Σ where A is countably infinite and $f^{\mathbb{A}} : A \to A$ is a bijection such that $\{(f^{\mathbb{A}})^n(a) \mid n \in \mathbb{N}\}$ is finite for every $a \in A$; $(f^{\mathbb{A}})^n$ denotes n consecutive applications of $f^{\mathbb{A}}$. Find a set of sentences of second-order logic Γ s.t., for every structure $\mathbb{B} = (B, f^{\mathbb{B}})$ over signature Σ,

$$\mathbb{B} \models \Gamma \quad \text{if, and only if,} \quad \mathbb{B} \text{ is isomorphic to } \mathbb{A}.$$

3. (16 points) Fix a signature Σ consisting of a unary function f. Consider the class $\mathcal{C}$ of structures $\mathbb{A} = (A, f^{\mathbb{A}})$ over signature Σ such that A is countably infinite and $f^{\mathbb{A}} : A \to A$ is a bijection such that for every $a \in A$ and for every $n \in \mathbb{N}, (f^{\mathbb{A}})^n(a) \neq a$, where $(f^{\mathbb{A}})^n$ denotes n consecutive applications of $f^{\mathbb{A}}$.

   (a) (12 points) Provide an algorithm transforming every first-order formula $\varphi(x_1, \ldots, x_n)$ over signature Σ with free variables $x_1, \ldots, x_n$ into a quantifier-free formula $\psi(x_1, \ldots, x_n)$ with free variables included in $x_1, \ldots, x_n$ such that,

$$\text{for every } \mathbb{A} \in \mathcal{C}, \mathbb{A} \models \forall x_1, \ldots, x_n.(\varphi \leftrightarrow \psi).$$

   (b) (2 points) Let $\mathrm{Th}(\mathcal{C}) = \{\varphi \text{ sentence over } \Sigma \mid \text{for all } \mathbb{A} \in \mathcal{C}, \mathbb{A} \models \varphi\}$. Is $\mathrm{Th}(\mathcal{C})$ a decidable theory?

   (c) (2 points) Is $\mathrm{Th}(\mathcal{C})$ syntactically complete, i.e., for every sentence φ over Σ, either $\varphi \in \mathrm{Th}(\mathcal{C})$ or $\neg\varphi \in \mathrm{Th}(\mathcal{C})$?

Instructions: Write your solutions to each problem in English/Polish. Submit your solution by committing it in readable PDF or image format to GitHub by **1pm sharp**. (The deadline is strict since GitHub will automatically collect the solution by this time.) This can be achieved by scanning a clear picture of physical paper sheets containing your solution. A PDF obtained by a document processing system such as Word/LaTeX is accepted too, but be warned that it may take extra time.

You are allowed to use notes during the exam. You are not allowed to communicate with each other during the exam; we rely on your maturity to abide by this rule.

IMPORTANT: Write your **full name** and ***indeks number*** (UW email) in readable handwriting on the first page of the submitted solution. Submitted solutions not satisfying this requirement will not be accepted. Insert the same information in the file `name.txt`, or at least the *indeks number*.