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Sergei Buyalo
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Elements of Asymptotic Geometry

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To Tania and Cornelia

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Preface

Asymptotic geometry is the study of metric spaces from a large scale point of view, where the local geometry does not come into play. An important class of spaces to be studied are the hyperbolic spaces (in the sense of Gromov), for which it turns out that the asymptotic geometry is almost completely encoded in the boundary at infinity.

The basic example of these spaces is the classical hyperbolic space H^n. A main feature of this classical space is the deep relation between the geometry of H^n and the Möbius geometry of its boundary $\partial_\infty \mathrm{H}^n$. For example the isometries of H^n correspond to Möbius transformations of $\partial_\infty \mathrm{H}^n$. The classical space itself has different realizations, but there are natural isomorphisms between these models, which induce Möbius transformations between the boundaries at infinity.

Mikhael Gromov realized that the essential asymptotic properties of H^n can be encoded in a simple condition for quadruples of points. A metric space X is called (Gromov) hyperbolic, if there exists some $\delta \geq 0$ such that every quadruple $Q = \{x, y, z, w\} \subset X$ satisfies the following inequality only involving the six distances between the four points:

$$|xz| + |yu| \leq \max\{|xy| + |zu|, |xu| + |yz|\} + 2\delta.$$

It is a remarkable fact that this inequality suffices to build up a theory of general hyperbolic spaces, which is very similar to the classical theory of the classical hyperbolic space but which allows much more flexibility and can be applied to many situations.

We develop the basics of this theory of general hyperbolic spaces in the first eight chapters of the book. In our account we stress the analogy between a Gromov hyperbolic space X and the classical hyperbolic space H^n. We describe the boundary at infinity $\partial_\infty X$ in different realizations as a bounded and as an unbounded metric space in analogy to the unit disc and the upper half-space model of H^n. We introduce a *quasi-Möbius structure* on $\partial_\infty X$ and discuss in detail the relation between the morphisms of X and the quasi-Möbius transformations of the boundary.

In the second part of the book we focus on several aspects of the asymptotic geometry of arbitrary metric spaces. It turns out that the simple philosophy to study "a boundary at infinity" does not work in this general situation.

Instead we introduce various dimension type asymptotic invariants and give several interesting applications in particular for embedding and non-embedding results.

In this book we only discuss a few elements of asymptotic geometry and our viewpoint is in no way exhaustive. For example, our book has little intersection with the recent book of John Roe [Ro] about the same subject and can be considered as a complement. Almost all of the results in this book are in the literature, but the presentation and some of the proofs are new.

This book grew out of lectures which we gave at the Steklov Institute in St. Petersburg and the University of Zürich. We want to thank the audience of these lectures, in particular Kathrin Haltiner, Alina Rull and Deborah Ruoss, for their questions and suggestions.

In particular we want to thank our friends and colleagues Yuri Burago, Thomas Foertsch, Urs Lang, Nina Lebedeva and Kolya Kosovskii for many interesting discussions about asymptotic geometry.

Chapter 1

Hyperbolic geodesic spaces

Here we recall basic notions related to metric spaces, define hyperbolic geodesic metric spaces and prove the fundamental theorem about the stability of geodesics in hyperbolic spaces.

1.1 Geodesic metric spaces

A *metric* on a set X is a function $d: X \times X \to \mathbb{R}$ which

(1) is positive: $d(x, x') \geq 0$ for every $x, x' \in X$ and $d(x, x') = 0$ if and only if $x = x'$;

(2) is symmetric: $d(x, x') = d(x', x)$ for every $x, x' \in X$;

(3) satisfies the triangle inequality: $d(x, x'') \leq d(x, x') + d(x', x'')$ for every $x, x', x'' \in X$.

Given a metric d, the value $d(x, x')$ is called *distance* between the points x, x'. We often use the notation $|xx'|$ for the distance between x, x' in a given metric space X, and λX for the metric space obtained from X by multiplying all distances by the factor $\lambda > 0$.

A map $f: X \to Y$ between metric spaces is said to be *isometric* if it preserves the distances, i.e. $|f(x)f(x')| = |xx'|$ for each $x, x' \in X$. Clearly, every isometric map is injective. If f is in addition surjective, it is called an *isometry*.

A *geodesic* in a metric space X is any isometric map $\gamma: I \to X$, where $I \subset \mathbb{R}$ is an interval (open, closed or half-open, bounded or unbounded). The image $\gamma(I)$ of such a map is also called a geodesic. A metric space X is said to be *geodesic* if any two points in X can be connected by a geodesic. We use the notation xx' for a geodesic in X between x, x', calling it a *segment* (even in the case when there are possibly several such segments).

Remark. In many theories where the local geometry plays an essential role as e.g. in Riemannian geometry, a geodesic means a curve $\gamma: I \to X$ which is only locally isometric, while on large scales the length of a segment might be larger than the distance between its end points. However, we always consider geodesics in the sense of the definition above.

1.2 Hyperbolic geodesic spaces

A *triangle* xyz in a geodesic space X is the union of segments xy, yz, zx, called the *sides*, connecting pairwise its *vertices* $x, y, z \in X$. More generally an *n-gon* $x_1 \dots x_n$ in X is the union of segments $x_1x_2, \dots, x_nx_1$.

The property of a geodesic space to be hyperbolic is defined in terms of triangles and the Gromov product, which is a useful notion in many circumstances.

1.2.1 Gromov product

Let X be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ put $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. The number $(x|x')_o$ is nonnegative by the triangle inequality, and it is called the *Gromov product* of x, x' with respect to o. Geometrically, the product can be interpreted as follows.

Lemma 1.2.1. *Let X be a geodesic space and xyz a triangle in X. There is a unique collection of points $u \in yz$, $v \in xz$, $w \in xy$ such that $|xv| = |xw|$, $|yu| = |yw|$, $|zv| = |zu|$.*

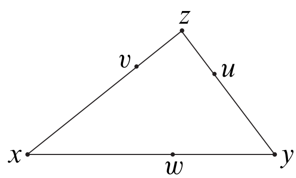


Figure 1.1. Gromov product and equiradial points.

Proof. The equation system

$$\begin{aligned} a + b \phantom{{}+c} &= |xy| \\ a \phantom{{}+b} + c &= |xz| \\ b + c &= |yz| \end{aligned}$$

has a unique solution and a, b, c are nonnegative by the triangle inequality. Then the points u, v, w are uniquely determined by the conditions $|xv| = a$, $|yw| = b$, $|zu| = c$. $\square$

The points $u \in yz$, $v \in xz$, $w \in xy$ are called *equiradial* points. Note that

$$a = \frac{1}{2}(|xy| + |xz| - |yz|) = (y|z)_x$$

and similarly $b = (x|z)_y$, $c = (x|y)_z$.

For example, if a triangle $xyz \subset X$ is a *tripod*, i.e. the union $wx \cup wy \cup wz$ with only one common point $w \in X$, then $(y|z)_x = |xw|$.

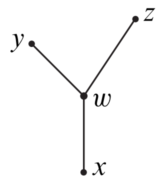


Figure 1.2. Tripod.

Definition 1.2.2. A geodesic metric space is called *δ-hyperbolic*, $\delta \geq 0$, if for any triangle $xyz \subset X$ the following holds: If $y' \in xy$, $z' \in xz$ are points with $|xy'| = |xz'| \leq (y|z)_x$, then $|y'z'| \leq \delta$.

Roughly speaking, in a δ-hyperbolic geodesic space X two sides xy and xz of any triangle xyz coming out of the common vertex x run together within the distance δ up to the moment $(y|z)_x$ and after that they start to diverge with almost maximal possible speed. This point of view becomes effective at distances large compared to δ.

The space is (*Gromov*) *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. The constant δ is called a *hyperbolicity constant* for X. Clearly, in a δ-hyperbolic space any side of any triangle lies in the δ-neighborhood of the two other sides. This is the case $k = 1$ of the following lemma.

Lemma 1.2.3. *Let $x_1 \dots x_n$ be an n-gon with $n \leq 2^k + 1$ for some $k \in \mathbb{N}$, then every side is contained in the $k\delta$-neighborhood of the union of the other sides.*

Proof. We show that a point $x \in x_n x_1$ has distance $\leq k\delta$ from $x_1x_2 \cup \dots \cup x_{n-1}x_n$. Choose the midpoint x_m with $m = [n/2] + 1$ where [ ] is the integer part and consider the triangle $x_1 x_m x_n$. By δ-hyperbolicity there exists $y \in x_1x_m \cup x_m x_n$ with $|xy| \leq \delta$. In the case $y \in x_1 x_m$ (resp. $y \in x_m x_n$) the induction hypothesis for the polygon $x_1 \dots x_m$ (resp. $x_m \dots x_n$) implies that y has distance $\leq (k-1)\delta$ from $x_1x_2 \cup \dots \cup x_{m-1}x_m$ (resp. $x_m x_{m+1} \cup \dots \cup x_{n-1}x_n$). The claim follows. □

Exercise 1.2.4. Show that if any side of any triangle in a geodesic space X lies in the δ-neighborhood of the union of the two other sides for some fixed $\delta \geq 0$, then X is hyperbolic (Rips' definition of hyperbolicity). Estimate the hyperbolicity constant for X.

Example 1.2.5. A *metric tree* is a geodesic space in which every triangle is a tripod (possibly degenerate). Clearly, every metric tree is a 0-hyperbolic space.

1.3 Stability of geodesics

In this section we show that geodesics in hyperbolic spaces are stable. This means that if we enlarge the class of geodesics to the larger class of quasi-geodesics, then still each quasi-geodesic stays in uniformly bounded distance to a geodesic. To make this concept precise we need the concept of quasi-isometric maps.

1.3.1 Quasi-isometric maps

The notion of a quasi-isometric map is a rough version of a bilipschitz map; recall that a map $f\colon X \to Y$ between metric spaces is *bilipschitz* if

$$\tfrac{1}{a}|xx'| \le |f(x)f(x')| \le a|xx'|$$

for some $a \ge 1$ and all $x, x' \in X$ (in this definition, we do not require that $f(X) = Y$).

A subset $A \subset Y$ in a metric space Y is called a *net* if the distances of all points $y \in Y$ to A are uniformly bounded.

A map $f\colon X \to Y$ between metric spaces is said to be *quasi-isometric* if there are $a \ge 1$, $b \ge 0$ such that

$$\tfrac{1}{a}|xx'| - b \le |f(x)f(x')| \le a|xx'| + b$$

for all $x, x' \in X$. In other words, a map is quasi-isometric if it is bilipschitz on large scales.

If, in addition, the image $f(X)$ is a net in Y, then f is called a *quasi-isometry*, and the spaces X and Y are called *quasi-isometric*. We also say that f is (a,b)-quasi-isometric and call a, b the *quasi-isometricity constants*.

A *quasi-geodesic* in X is a quasi-isometric map $\gamma\colon I \to X$ where $I \subset \mathbb{R}$ is an interval.

For general metric spaces a quasi-geodesic can be far from a geodesic. Consider, for example, in the Euclidean plane the spiral $\gamma\colon (1,\infty) \to \mathbb{R}^2$, $\gamma(t) = t(\cos(\ln t), \sin(\ln t))$.

Since $|\gamma(t)| = t$ and $|\gamma'(t)| = \sqrt{2}$ for all $t > 1$, we easily see

$$\tfrac{1}{\sqrt{2}}|\gamma(t)\gamma(s)| \le |t-s| \le |\gamma(t)\gamma(s)|$$

which implies that γ is a quasi-geodesic. This curve is in no way close to any geodesic. In hyperbolic geodesic spaces the situation is completely different. We will show that in a geodesic hyperbolic space every quasi-geodesic will stay in uniform bounded distance to a honest geodesic.

To start our argument we first show that, roughly speaking, in order to avoid a ball in a hyperbolic space one needs to go an exponentially long path.

We use the notation $B_r(x)$ for the open ball of radius r centered at x in a metric space X, $B_r(x) = \{x' \in X : |xx'| < r\}$. Furthermore, $\overline{B}_r(x)$ is the closed ball $\{x' \in X : |xx'| \le r\}$.

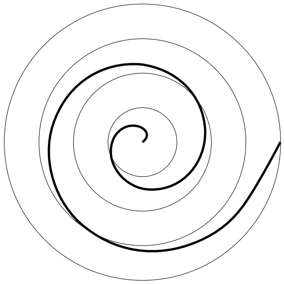


Figure 1.3. The spiral γ on logarithmic scale.

By an a-path, $a > 0$, in a metric space we mean a finite or infinite sequence of points $\{x_i\}$ with $|x_i x_{i+1}| \le a$ for each i.

Lemma 1.3.1. *Assume that an a-path $f : \{1, \dots, N\} \to X$ in a geodesic δ-hyperbolic space, $\delta > 0$, lies outside of the ball $B_r(x)$ centered at some point $x \in f(1)f(N)$. Then*

$$N \ge c \cdot 2^{r/\delta}$$

for some constant $c > 0$ depending only on a and δ.

Proof. Let k be the smallest integer with $N \le 2^k + 1$ (then $N \ge 2^{k-1}$). By Lemma 1.2.3 there exists a point $y \in f(j)f(j+1)$ for some $j \in \{1, \dots, N-1\}$ such that $|xy| \le k\delta$. Note that $|xy| \ge r - a/2$, and hence $k \ge r/\delta - a/(2\delta)$. Thus $N \ge 2^{k-1} \ge c \cdot 2^{r/\delta}$ with $c = 2^{-(a/2\delta+1)}$. $\square$

We are now able to prove the stability of quasigeodesics.

Theorem 1.3.2 (Stability of geodesics). *Let X be a δ-hyperbolic geodesic space and $a \ge 1$, $b \ge 0$. There exists $H = H(a, b, \delta) > 0$ such that for every $N \in \mathbb{N}$ the image* $\mathrm{im}(f)$ *of every (a, b)-quasi-isometric map $f : \{1, \dots, N\} \to X$ lies in the H-neighborhood of any geodesic $c : [0, l] \to X$ with $c(0) = f(1)$, $c(l) = f(N)$, and vice versa, c lies in the H-neighborhood of* $\mathrm{im}(f)$.

Proof. We first show that c lies in the h-neighborhood of $\mathrm{im}(f)$, where $h = h(a, b, \delta) > 0$ depends only on a, b and δ. Note that f is an $(a+b)$-path in X. Choose h maximal with the property that $\mathrm{im}(f)$ lies outside the ball $B_h(x)$ for some $x \in c$.

Take $y \in c(0)x$, $y' \in xc(l)$ with $|yx| = |xy'| = 2h$ (if the distance between x and one of the ends of c is less than $2h$, we take as y or y' the corresponding end). There are $i, i' \in A = \{1, \dots, N\}$ with $|f(i)y|$, $|f(i')y'| \le h$ and the segments $yf(i)$, $y'f(i')$ lie outside the ball $B_h(x)$. By taking appropriate points on these segments together with $f(i), \dots, f(i')$, we find an $(a+b)$-path between y and y' outside $B_h(x)$ which contains $K \le |i - i'| + 3 + \frac{2h}{a+b}$ points.

By quasi-isometricity of f, we have

$$|i-i'| \le a(|f(i)f(i')|+b) \le 6ah+ab.$$

On the other hand, $K \ge c \cdot 2^{h/\delta}$ by Lemma 1.3.1 where $c = c(a,b,\delta)$. These estimates together give an effective upper bound $h(a,b,\delta)$ for the radius h.

To complete the proof, consider a maximal sub-interval $\{j,\dots,j'\} \subset A$ such that $f(\{j,\dots,j'\})$ lies outside the h-neighborhood of c, $h = h(a,b,\delta)$. Since c is contained in the h-neighborhood of $\mathrm{im}(f)$, there are $i \in \{1,\dots,j\}$, $i' \in \{j',\dots,N\}$ and $z \in c$ so that $|zf(i)|, |zf(i')| \le h$. Then $|f(i)f(i')| \le 2h$, and $|i-i'| \le 2ah + ab$ by quasi-isometricity of f. Hence, $\mathrm{im}(f)$ is contained in the H-neighborhood of c, where $H = h + a(2ah+ab)+b$, $H = H(a,b,\delta)$. □

Exercise 1.3.3. Derive the following consequences of Theorem 1.3.2.

Corollary 1.3.4. *Let X be hyperbolic geodesic space. Then there is no quasi-isometric map $f : \mathbb{R}^2 \to X$.*

(Hint: Assuming that such a map exists, consider images of larger and larger equilateral triangles to obtain a contradiction using the stability of geodesics in X).

Corollary 1.3.5. *If a geodesic space X is quasi-isometric to a hyperbolic geodesic space Y, then X is also hyperbolic.*

(Hint: Take any triangle in X and compare it with its image in Y to conclude using stability of geodesics in Y that the triangle satisfies a δ-hyperbolicity condition).

1.4 Supplementary results and remarks

1.4.1 The real hyperbolic space H^n

The real hyperbolic space H^n is a simply connected, complete Riemannian manifold of dimension $n \ge 2$ having the constant sectional curvature -1. Various models of H^n are discussed in the appendix. This is the basic example of Gromov hyperbolic spaces.

Exercise 1.4.1. Using the parallelism angle formula (see Appendix, Lemma A.3.2), show that the space H^n is δ-hyperbolic with $\delta < \ln 3 = 1.0986\dots$. Actually, $\delta = 2\ln\tau = 0.9624\dots$ where τ is the golden ratio, $\tau^2 = \tau + 1$.

1.4.2 Gromov hyperbolic groups

An important class of hyperbolic spaces is the class of Gromov hyperbolic groups which are defined as follows.

Let G be a finitely generated group and $S \subset G$ a finite set generating G. We assume that S does not contain the unit element of G and is symmetric, i.e., $g \in S$ if and only if $g^{-1} \in S$. The *Cayley graph* of (G, S) is a graph $\Gamma = \Gamma(G, S)$ with the vertex set G, and vertices $g, g' \in G$ are connected by an edge if and only if $g^{-1}g' \in S$. The Cayley graph Γ carries the path metric d_S for which every edge has length one. Such a metric when viewed on G is called a *word metric*. Clearly, Γ is a geodesic space.

A finitely generated group G is said to be *word hyperbolic* or *Gromov hyperbolic* if its Cayley graph $\Gamma(G, S)$ is a hyperbolic space for some generating system S.

Exercise 1.4.2. Show (using Corollary 1.3.5) that the property of a finitely generated group G to be hyperbolic is independent of the choice of a generating system S.

1.4.3 CAT(−1)-spaces

Let xyz be a geodesic triangle in a geodesic metric space X. A *comparison triangle*

$$\tilde{x}\tilde{y}\tilde{z} \subset \mathrm{H}^2$$

is a triangle with the same side-lengths. *Comparison points* on the sides are obtained as follows. Let u be a point on one of the sides, say $u \in xy$. Then the comparison point $\tilde{u}$ is the unique point on the segment $\tilde{x}\tilde{y}$ with $|\tilde{u}\tilde{x}| = |ux|$ and $|\tilde{u}\tilde{y}| = |uy|$.

A complete geodesic space X is a CAT(-1)-*space* if for each triangle $xyz \subset X$ and each $u \in xy$, $v \in xz$, it holds that $|uv| \le |\tilde{u}\tilde{v}|$, where $\tilde{u} \in \tilde{x}\tilde{y}$, $\tilde{v} \in \tilde{x}\tilde{z}$ are comparison points on the sides of $\tilde{x}\tilde{y}\tilde{z} \subset \mathrm{H}^2$.

That is, any triangle in X is thinner than its comparison triangle in H^2. Thus by definition, every CAT(-1)-space is δ-hyperbolic with $\delta \le \delta_{\mathrm{H}^2}$.

The class of CAT(-1)-spaces is very large. Recall that a *Hadamard manifold* is a complete simply connected Riemannian manifold with nonpositive sectional curvatures. Every Hadamard manifold with sectional curvatures $K \le -1$ is a CAT(-1)-space. Furthermore, any metric tree is a CAT(κ)-space for each $\kappa < 0$, in particular, it is CAT(-1). The class of CAT(-1)-spaces also includes various hyperbolic buildings. On the other hand, there are compact nonpositively curved (in Alexandrov sense) 2-polyhedra with word hyperbolic fundamental group that admit no metric with CAT(-1) universal covering; see e.g. [BB].

Taking comparison triangles in $\mathbb{R}^2$, one similarly obtains the important class of CAT(0) or *Hadamard spaces*, i.e. complete geodesic spaces with triangles thinner than the Euclidean comparison triangles. In any Hadamard space X, all points $x, x' \in X$ are connected by a unique geodesic segment.

Bibliographical note. The stability of geodesics was discovered in the twenties of the last century by M. Morse, [Mo1], [Mo2]. There are several approaches to its proof. The proof presented in Section 1.3 is very close to Gromov's proof, [Gr1]; see also [BrH].

Chapter 2

The boundary at infinity

We start this chapter with a discussion of further properties of the Gromov product with the aim of deriving the δ-inequality for hyperbolic geodesic spaces. This allows us to extend the notion of hyperbolicity to metric spaces which are not necessarily geodesic. An important point of this discussion is the Tetrahedron Lemma, which has various applications throughout the book.

Next we define the boundary at infinity for any hyperbolic space and discuss various structures attached to it: Gromov product, quasi-metrics, visual metrics and topology. We also establish local self-similarity of the boundary at infinity of cocompact hyperbolic spaces.

2.1 δ-inequality and hyperbolic spaces

The Gromov product is monotone in the following sense.

Lemma 2.1.1. *Assume that $y' \in xy$ and $z' \in xz$ in a geodesic space X. Then $(y'|z')_x \le (y|z)_x$.*

Proof. Since

$$|xz| = |xz'| + |z'z|,$$
$$|y'z| \le |y'z'| + |z'z|,$$

we have $|xz| - |y'z| \ge |xz'| - |y'z'|$, and hence $(y'|z')_x \le (y'|z)_x$. Similarly $(y'|z)_x \le (y|z)_x$. □

Proposition 2.1.2. *If a geodesic space X is δ-hyperbolic, then*

$$(x|y)_o \ge \min\{(x|z)_o,\ (z|y)_o\} - \delta$$

for any base point $o \in X$ and any $x, y, z \in X$.

Proof. Put $t_0 = \min\{(x|z)_o, (y|z)_o\}$ and assume that $x' \in ox$, $y' \in oy$ and $z' \in oz$ satisfy $|ox'| = |oy'| = |oz'| = t_0$. Then $|x'z'|, |y'z'| \le \delta$, thus $|x'y'| \le 2\delta$. On the other hand, by Lemma 2.1.1,

$$(x|y)_o \ge (x'|y')_o = t_0 - \tfrac{1}{2}|x'y'| \ge t_0 - \delta.$$ □

The inequality from Proposition 2.1.2 is called *δ-inequality.* This inequality is characteristic for the property of a space to be hyperbolic.

Proposition 2.1.3. *Assume that a geodesic space X satisfies the δ-inequality for every base point o and every $x, y, z \in X$. Then X is 4δ-hyperbolic.*

Proof. Assume that points $x' \in ox$, $y' \in oy$ of a triangle $oxy \subset X$ satisfy the condition $|ox'| = |oy'| = t \le (x|y)_o$. It suffices to show that then $|x'y'| \le 4\delta$. By the δ-inequality we have

$$\begin{aligned}(x'|y')_o &\ge \min\{(x'|y)_o, t\} - \delta \\ &\ge \min\{\min\{(x|y)_o, t\} - \delta, t\} - \delta \\ &= t - 2\delta,\end{aligned}$$

hence $|x'y'| = 2t - 2(x'|y')_o \le 4\delta$. □

Finally, we show that the δ-inequality for some base point implies the 2δ-inequality for any other base point. The following terminology is useful. A *δ-triple* is a triple of real numbers a, b, c with the property that the two smallest of these numbers differ by at most δ. To rephrase the δ-inequality we can say that the numbers $(x|y)_o$, $(x|z)_o$, $(y|z)_o$ form a δ-triple.

It is also convenient to write $a \doteq b$ up to an error $\le c$ or $a \doteq_c b$ instead of $|a - b| \le c$.

The following important result, which has many applications in the sequel, is called Tetrahedron Lemma.

Lemma 2.1.4. *Let d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34} be six numbers such that the four triples $A_1 = (d_{23}, d_{24}, d_{34})$, $A_2 = (d_{13}, d_{14}, d_{34})$, $A_3 = (d_{12}, d_{14}, d_{24})$ and $A_4 = (d_{12}, d_{13}, d_{23})$ are δ-triples. Then*

$$B = (d_{12} + d_{34}, d_{13} + d_{24}, d_{14} + d_{23})$$

is a 2δ-triple.

Proof. Without loss of generality, we can assume that d_{34} is maximal among the listed numbers. Then $d_{13} \doteq d_{14}$ up to an error $\le \delta$ since A_2 is a δ-triple, and $d_{23} \doteq d_{24}$ up to an error $\le \delta$ since A_1 is a δ-triple. Adding these approximate equalities, we obtain that $d_{13} + d_{24} \doteq d_{23} + d_{14}$ up to an error $\le 2\delta$. Since d_{34} is maximal, this means, if we assume that B is not a 2δ-triple, that $d_{12} < \min\{d_{13}, d_{14}, d_{23}, d_{24}\} - 2\delta$. But this contradicts the fact that A_3 and A_4 are δ-triples. Thus B is a 2δ-triple. □

Lemma 2.1.5. *Assume that a metric space X satisfies the δ-inequality for a base point o. Then for any other base point $x \in X$, the 2δ-inequality is fulfilled.*

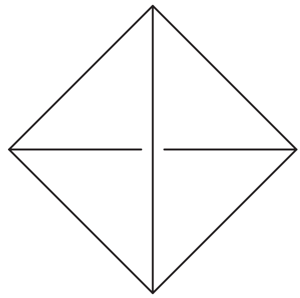


Figure 2.1. Tetrahedron Lemma.

Proof. Note that the expression

$$A = (t|y)_o + (x|z)_o - \min\{(t|z)_o + (y|x)_o, (x|t)_o + (y|z)_o\}$$

does not depend on the base point o. Choosing x as the base point, we see $A = (t|y)_x - \min\{(t|z)_x, (z|y)_x\}$. Thus, we have to prove $A \geq -2\delta$. From the δ-inequality for the base point o, it follows that the six numbers $(t|x)_o$, $(t|y)_o$, $(t|z)_o$, $(x|y)_o$, $(x|z)_o$, $(y|z)_o$ satisfy the condition of the Tetrahedron Lemma, which implies $A \geq -2\delta$. $\square$

Now we extend the notion of hyperbolicity to metric spaces which are not necessarily geodesic.

Definition 2.1.6. A metric space X is (*Gromov*) *hyperbolic* if it satisfies the δ-inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

or, what is the same, the triple $((x|y)_o, (x|z)_o, (y|z)_o)$ is a δ-triple for some $\delta \geq 0$, for every base point $o \in X$ and all $x, y, z \in X$.

For geodesic spaces this notion is equivalent to our initial definition by Propositions 2.1.2, 2.1.3. From now on, when speaking about a δ-hyperbolic space X we mean Definition 1.2.2 if X is geodesic, and Definition 2.1.6 otherwise. The same holds for hyperbolicity constants. This causes no ambiguity because of Proposition 2.1.2.

Remark 2.1.7. By Lemma 2.1.5, to prove that a space X is hyperbolic, it suffices to check that the δ-inequality holds for some $\delta \geq 0$, some base point $o \in X$ and all $x, y, z \in X$. We shall often use this remark.

2.2 The boundary at infinity of hyperbolic spaces

There are several possibilities to define the boundary at infinity of a hyperbolic space, ranging from the most geometric one, geodesic boundary, see Section 2.4.2, to the most analytic one, called Higson corona, which is not discussed in this book. We choose the original Gromov definition, since it is well adapted to the basic property of hyperbolic geodesic spaces that quasi-isometric maps have a natural extension to boundary maps, and the definition appeals to the geometric intuition.

Let X be a hyperbolic space and $o \in X$ a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity* if

$$\lim_{i,j\to\infty}(x_i|x_j)_o = \infty.$$

This property is independent of the choice of o since

$$|(x|x')_o - (x|x')_{o'}| \le |oo'|$$

for any $x, x', o, o' \in X$. Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty}(x_i|x_i')_o = \infty.$$

Using the δ-inequality, we easily see that this defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined to be the set of equivalence classes of sequences converging to infinity.

Remark 2.2.1. If $\{x_i\}$ is a sequence converging to infinity and $\{x_i'\}$ a sequence equivalent to $\{x_i\}$ in the sense that $\lim(x_i|x_i')_o = \infty$, then $\{x_i'\}$ converges to infinity itself. This easily follows from the δ-inequality.

Now we introduce natural metric structures on the boundary at infinity of a Gromov hyperbolic space X. This is done in three steps. In a first step, we extend the Gromov product to the boundary at infinity. More precisely, we define for a base point $o \in X$ and points $\xi, \eta \in \partial_\infty X$ the product $(\xi|\eta)_o$. In a second step, we define the map $\rho\colon \partial_\infty X \times \partial_\infty X \to [0,\infty)$ by $\rho(\xi,\eta) = a^{-(\xi|\eta)_o}$, where $a > 1$ is some parameter. The map ρ turns out to be a quasi-metric. In a third step, we apply a standard procedure to obtain from ρ a metric for parameters $a > 1$, a small enough.

2.2.1 Gromov product on the boundary

Fix a base point $o \in X$. For points $\xi, \xi' \in \partial_\infty X$, we define their Gromov product by

$$(\xi|\xi')_o = \inf \liminf_{i\to\infty}(x_i|x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0,\infty]$, that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$, and that

$|(\xi|\xi')_o - (\xi|\xi')_{o'}| \le |oo'|$ for any $o, o' \in X$. Furthermore, we obtain the following properties.

Lemma 2.2.2. *Let $o \in X$, let X satisfy the δ-inequality for o, and let $\xi, \xi', \xi'' \in \partial_\infty X$.*

(1) *For arbitrary sequences $\{y_i\} \in \xi$, $\{y_i'\} \in \xi'$, we have*

$$(\xi|\xi')_o \le \liminf_{i\to\infty} (y_i|y_i')_o \le \limsup_{i\to\infty} (y_i|y_i')_o \le (\xi|\xi')_o + 2\delta.$$

(2) $(\xi|\xi')_o$, $(\xi'|\xi'')_o$, $(\xi|\xi'')_o$ *is a δ-triple.*

Proof. (1) We only need to show that $\limsup_{i\to\infty} (y_i|y_i')_o \le (\xi|\xi')_o + 2\delta$. We can assume that $\xi \ne \xi'$. Applying the standard diagonal procedure, we find sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$ with $\lim (x_i|x_i')_o = (\xi|\xi')_o$. Let $\{y_i\} \in \xi$, $\{y_i'\} \in \xi'$. For i sufficiently large, we have $(x_i|x_i')_o \doteq (y_i|x_i')_o$ up to an error $\le \delta$ since $(x_i|x_i')_o$, $(y_i|x_i')_o$, $(x_i|y_i)_o$ is a δ-triple, $(x_i|y_i)_o \to \infty$, and two other members are bounded due to the assumption $\xi \ne \xi'$. In the same way we see that $(y_i|x_i')_o \doteq (y_i|y_i')_o$ up to an error $\le \delta$ for i large enough. Thus $(x_i|x_i')_o \doteq (y_i|y_i')_o$ up to an error $\le 2\delta$, which implies the claim.

(2) Without loss of generality, we have to show

$$(\xi|\xi'')_o \ge \min\{(\xi|\xi')_o, (\xi'|\xi'')_o\} - \delta.$$

Choose $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$, $\{x_i''\} \in \xi''$ such that $\lim (x_i|x_i'')_o = (\xi|\xi'')_o$. Then

$$(\xi|\xi'')_o \ge \limsup_{i\to\infty} \min\{(x_i|x_i')_o, (x_i'|x_i'')_o\} - \delta \ge \min\{(\xi|\xi')_o, (\xi'|\xi'')_o\} - \delta. \qquad \square$$

Similarly, the Gromov product

$$(x|\xi)_o = \inf \liminf_{i\to\infty} (x|x_i)_o$$

is defined for any $x \in X$, $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$, and the δ-inequality holds for any three points from $X \cup \partial_\infty X$.

2.2.2 Quasi-metric on the boundary

A *quasi-metric space* is a set Z with a function $\rho\colon Z \times Z \to \mathbb{R}$ which satisfies the conditions

(1) $\rho(z, z') \ge 0$ for every $z, z' \in Z$, and $\rho(z, z') = 0$ if and only if $z = z'$;
(2) $\rho(z, z') = \rho(z', z)$ for every $z, z' \in Z$;
(3) $\rho(z, z'') \le K \max\{\rho(z, z'), \rho(z', z'')\}$ for every $z, z', z'' \in Z$ and some fixed $K \ge 1$.

The function ρ is then called a *quasi-metric*, or more specifically, a K-quasi-metric. The property (3) is a generalized version of the *ultra-metric* triangle inequality which is the case $K = 1$.

Remark 2.2.3. If (Z, d) is a metric space, then d is a K-quasi-metric for $K = 2$. In general the p-th power d^p of the distance d is not a metric on Z for $p > 1$. But d^p is still a 2^p-quasi-metric.

Coming back to the Gromov hyperbolic space X, we fix $a > 1$ and consider the function $\rho\colon \partial_\infty X \times \partial_\infty X \to \mathbb{R}$, $\rho(\xi, \xi') = a^{-(\xi|\xi')_o}$. Then ρ is a K-quasi-metric on $\partial_\infty X$ with $K = a^\delta$: the properties (1), (2) are obvious, and (3) immediately follows from Lemma 2.2.2 (2).

Remark 2.2.4. The quasi-metric ρ defined on $\partial_\infty X$ depends on the base point $o \in X$ and the chosen parameter $a > 1$. If we emphasize this dependence, we write $\rho_{o,a}$. Let $o, o' \in X$. Since $|(\xi|\xi')_o - (\xi|\xi')_{o'}| \le |oo'|$ we compute

$$c^{-1} \le \frac{\rho_{o,a}(\xi, \xi')}{\rho_{o',a}(\xi, \xi')} \le c$$

where $c = a^{|oo'|}$. If $a, a' > 1$ are different parameters then we have

$$\rho_{o,a'} = \rho_{o,a}^{\alpha}$$

where $\alpha = \frac{\ln a'}{\ln a}$.

There is a standard procedure to construct a metric from a quasi-metric. Let (Z, ρ) be a quasi-metric space. We are interested in obtaining a metric on Z. Since the only problem is the triangle inequality, the following approach is natural. Define a map $d\colon Z \times Z \to \mathbb{R}$, $d(z, z') = \inf \sum_i \rho(z_i, z_{i+1})$, where the infimum is taken over all sequences $z = z_0, \dots, z_{k+1} = z'$ in Z. By definition, d is then symmetric and satisfies the triangle inequality. We call this construction of d the *chain construction*. The problem with the chain construction is that $d(z, z')$ could be 0 for different points z, z' and Axiom (1) would no longer be satisfied for (Z, d).

Lemma 2.2.5. *Let ρ be a K-quasi-metric on a set Z with $K \le 2$. Then the chain construction applied to ρ yields a metric d with $\frac{1}{2K}\rho \le d \le \rho$.*

Proof. Clearly, d is nonnegative, symmetric, satisfies the triangle inequality and $d \le \rho$. We prove by induction over the length of sequences $\sigma = \{z = z_0, \dots, z_{k+1} = z'\}$, $|\sigma| = k + 2$, that

$$\rho(z, z') \le \sum(\sigma) := K\Big(\rho(z_0, z_1) + 2\sum_{1}^{k-1} \rho(z_i, z_{i+1}) + \rho(z_k, z_{k+1})\Big). \tag{2.1}$$

For $|\sigma| = 3$, this follows from the triangle inequality (3) for ρ. Assume that (2.1) holds true for all sequences of length $|\sigma| \le k+1$, and suppose that $|\sigma| = k+2$.

Given $p \in \{1, \dots, k-1\}$, we let $\sigma_p' = \{z_0, \dots, z_{p+1}\}$, $\sigma_p'' = \{z_p, \dots, z_{k+1}\}$, and note that $\sum(\sigma) = \sum(\sigma_p') + \sum(\sigma_p'')$.

Because $\rho(z,z') \le K \max\{\rho(z,z_p), \rho(z_p,z')\}$, there is a maximal $p \in \{0, \dots, k\}$ with $\rho(z,z') \le K\rho(z_p,z')$. Then $\rho(z,z') \le K\rho(z,z_{p+1})$.

Assume now that $\rho(z,z') > \sum(\sigma)$. Then, in particular, $\rho(z,z') > K\rho(z,z_1)$ and $\rho(z,z') > K\rho(z_k,z')$. It follows that $p \in \{1, \dots, k-1\}$ and thus by the inductive assumption

$$\rho(z,z_{p+1}) + \rho(z_p,z') \le \sum(\sigma_p') + \sum(\sigma_p'') = \sum(\sigma) < \rho(z,z').$$

On the other hand,

$$\rho(z,z') \le K \min\{\rho(z,z_{p+1}), \rho(z_p,z')\} \le \rho(z,z_{p+1}) + \rho(z_p,z')$$

because $K \le 2$. This is a contradiction. Now it follows from (2.1) that $\rho \le 2Kd$. Hence d is a metric as required. $\square$

Proposition 2.2.6. *Let ρ be a K-quasi-metric on a set Z. Then there exists $\varepsilon_0 > 0$ only depending on K such that ρ^ε is bilipschitz equivalent to a metric for each $0 < \varepsilon \le \varepsilon_0$. More precisely, there exists a metric d_ε on Z such that*

$$\tfrac{1}{2K^\varepsilon}\rho^\varepsilon(z,z') \le d_\varepsilon(z,z') \le \rho^\varepsilon(z,z')$$

for all $z, z' \in Z$.

Proof. ρ^ε is a K^ε-quasi-metric for every $\varepsilon > 0$. If $K^\varepsilon \le 2$ then the chain construction applied to ρ^ε yields a required metric d_ε by Lemma 2.2.5. $\square$

2.2.3 Visual metrics at infinity

We now apply this construction to the quasi-metric ρ on $\partial_\infty X$. A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual* if there are $o \in X$, $a > 1$ and positive constants c_1, c_2 such that

$$c_1 a^{-(\xi|\xi')_o} \le d(\xi,\xi') \le c_2 a^{-(\xi|\xi')_o}$$

for all $\xi, \xi' \in \partial_\infty X$. In this case, we say that d is a *visual metric* with respect to the base point o and the parameter a. The inequalities above are called the *visual inequalities*.

Applying Proposition 2.2.6 we see:

Theorem 2.2.7. *Let X be a hyperbolic space. Then for any $o \in X$, there is $a_0 > 1$ such that for every $a \in (1, a_0]$ there exists a metric d on $\partial_\infty X$ which is visual with respect to o and a.* $\square$

Now we consider what happens if the base point is changed.

Proposition 2.2.8. *Visual metrics d, d' on $\partial_\infty X$ with respect to the same parameter $a > 1$ and base points o, o' respectively are bilipschitz equivalent,*

$$c^{-1} \le \frac{d'}{d} \le c$$

for some constant $c \ge 1$.

Proof. This immediately follows from the visual inequalities for d, d' and from the fact that $|(\xi|\xi')_o - (\xi|\xi')_{o'}| \le |oo'|$ for all $\xi, \xi' \in \partial_\infty X$ (see Remark 2.2.4). □

Next we consider the effect of the parameter change.

Proposition 2.2.9. *Visual metrics d, d' on $\partial_\infty X$ with respect to the same base point o and parameters a, $a' > 1$ respectively are Hölder equivalent, namely, there is a constant $c \ge 1$ such that*

$$\tfrac{1}{c} d^\alpha(\xi, \xi') \le d'(\xi, \xi') \le c d^\alpha(\xi, \xi')$$

for all $\xi, \xi' \in \partial_\infty X$, where $\alpha = \frac{\ln a'}{\ln a}$.

Proof. This immediately follows from the visual inequalities for the metrics d, d' and from the fact that $a' = a^\alpha$ (see Remark 2.2.4). □

We define the topology on the boundary at infinity $\partial_\infty X$ for a hyperbolic space X as the metric topology for some visual metric on $\partial_\infty X$. It follows from Propositions 2.2.8 and 2.2.9 that this topology is independent of the choice of a visual metric.

Exercise 2.2.10. Let X be a hyperbolic space. Show that $\partial_\infty X$ is bounded and complete for any visual metric on $\partial_\infty X$.

2.3 Local self-similarity of the boundary

Hyperbolic groups and more general cobounded hyperbolic spaces have a remarkable and useful property: their boundary at infinity are locally self-similar.

A map $f : Z \to Z'$ between metric spaces is called *homothetic* with coefficient R if

$$|f(z) f(z')| = R|zz'|$$

for all $z, z' \in Z$. Here we need a more flexible property.

Let $\lambda \ge 1$ and $R > 0$ be given. A map $f : Z \to Z'$ between metric spaces is *λ-quasi-homothetic with coefficient* R if for all $z, z' \in Z$, we have

$$R|zz'|/\lambda \le |f(z) f(z')| \le \lambda R|zz'|.$$

Note that f is also λ'-quasi-homothetic with coefficient R for every $\lambda' \geq \lambda$.

This property can be regarded as a perturbation of the property to be homothetic, and the coefficient λ describes the perturbation. We apply this notion usually to a family of quasi-homothetic maps with fixed λ when the coefficients R go to infinity.

A metric space Z is *locally similar* to (subsets of) a metric space Y if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \leq \frac{1}{R}$ there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R. If a metric space Z is locally similar to itself then we say that Z is *locally self-similar*.

Example 2.3.1. The standard ternary Cantor set X is locally self-similar. One can take $\lambda = 3$ in this case. Indeed, given $R > 3$ and $A \subset X$ with $\operatorname{diam} A \leq 1/R$, there is $k \in \mathbb{N}$ with $3^k < R \leq 3^{k+1}$. Then $\operatorname{diam} A < 1/3^k$. Hence, A is contained in the k-th step interval which it intersects. This interval is 3^k-homothetic to $[0, 1]$ and thus it is λ-quasi-homothetic to $[0, 1]$ with coefficient R.

The basic example of locally self-similar spaces is the boundary at infinity of a hyperbolic group. We consider a more general situation. A metric space X is *cobounded* if there is a bounded subset $A \subset X$ such that the orbit of A under the isometry group of X covers X.

A metric space X is *proper* if every closed ball $\overline{B}_r(x) \subset X$ is compact.

Theorem 2.3.2. *The boundary at infinity $\partial_\infty X$ of every cobounded, hyperbolic, proper, geodesic space X is locally self-similar with respect to any visual metric.*

For the proof we need the following

Lemma 2.3.3. *Let o, g, x', x'' be points of a metric space X such that the Gromov products $(x'|g)_o$, $(x''|g)_o \geq |og| - \sigma$ for some $\sigma \geq 0$. Then*

$$(x'|x'')_o \leq (x'|x'')_g + |og| \leq (x'|x'')_o + 2\sigma.$$

Proof. The left-hand inequality immediately follows from the triangle inequality: since $|ox'| \leq |og| + |gx'|$ and $|ox''| \leq |og| + |gx''|$, we have $(x'|x'')_o \leq (x'|x'')_g + |og|$.

Next we note that $(x'|o)_g = |og| - (x'|g)_o \leq \sigma$. This yields $|x'o| = |og| + |gx'| - 2(x'|o)_g \geq |og| + |gx'| - 2\sigma$ and similarly $|x''o| \geq |og| + |gx''| - 2\sigma$. Now the right-hand inequality follows. $\square$

Proof of Theorem 2.3.2. We can assume that the geodesic space X is δ-hyperbolic, $\delta \geq 0$, and that a visual metric d on $\partial_\infty X$ satisfies

$$c^{-1} a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c a^{-(\xi|\xi')_o}$$

for some base point $o \in X$, some constants $c \geq 1$, $a > 1$ and all $\xi, \xi' \in \partial_\infty X$. Note that then $\operatorname{diam} \partial_\infty X \leq c$.

There is $\rho > 0$ such that the orbit of the ball $B_\rho(o)$ under the isometry group of X covers X. Now we put $\lambda = c^3 a^{\rho+4\delta}$. Fix $R > 1$ and consider $A \subset \partial_\infty X$ with $\operatorname{diam} A \le 1/R$. For each $\xi, \xi' \in A$, we have

$$(\xi|\xi')_o \ge \log_a(R/c).$$

We fix $\xi \in A$. Since X is proper, there is a geodesic ray $o\xi \subset X$ representing ξ (see Exercise 2.4.3). We take $g \in o\xi$ with $a^{|og|} = R/c$. Then using the δ-inequality, we obtain for every $\xi' \in A$

$$(\xi'|g)_o \ge \min\{(\xi'|\xi)_o, (\xi|g)_o\} - \delta = |og| - \delta$$

because $(\xi|g)_o = |og|$.

For arbitrary $\xi', \xi'' \in A$, consider sequences $\{x_i'\} \in \xi'$, $\{x_i''\} \in \xi''$ such that $(x_i'|x_i'')_g \to (\xi'|\xi'')_g$. We can assume without loss of generality that $(x_i'|g)_o$, $(x_i''|g)_o \ge |og| - \delta$ because possible errors in these estimates disappear while taking the limit; see below.

Applying Lemma 2.3.3 to the points $o, g, x_i', x_i'' \in X$ and $\sigma = \delta$, we obtain

$$(x_i'|x_i'')_o - |og| \le (x_i'|x_i'')_g \le (x_i'|x_i'')_o - |og| + 2\delta.$$

Passing to the limit, this yields

$$(\xi'|\xi'')_o - |og| \le (\xi'|\xi'')_g \le (\xi'|\xi'')_o - |og| + 4\delta.$$

There is an isometry $f : X \to X$ with $|of(g)| \le \rho$. Then

$$(\xi'|\xi'')_g - \rho \le (f(\xi')|f(\xi''))_o \le (\xi'|\xi'')_g + \rho$$

because the Gromov products with respect to different points differ one from another at most by the distance between the points. The last two double inequalities give

$$(\xi'|\xi'')_o - |og| - \rho \le (f(\xi')|f(\xi''))_o \le (\xi'|\xi'')_o - |og| + \rho + 4\delta,$$

and therefore

$$c^{-3} a^{-(\rho+4\delta)} R d(\xi', \xi'') \le d(f(\xi'), f(\xi'')) \le c a^\rho R d(\xi', \xi'').$$

This shows that $f : A \to \partial_\infty X$ is λ-quasi-homothetic with coefficient R and hence $\partial_\infty X$ is locally self-similar. $\square$

We say that a metric space Z is *doubling* if there is a constant $N \in \mathbb{N}$ such that for every $r > 0$ every ball in Z of radius $2r$ can be covered by N balls of radius r.

If the property above holds for all sufficiently small $r > 0$ only, then we say that Z is *doubling at small scales*. Clearly, if a compact space is doubling at small scales then it is doubling.

Lemma 2.3.4. *Assume that a metric space Z is locally similar to (subsets of) a compact metric space Y. Then Z is doubling at small scales.*

Proof. There is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with diam $A \leq 1/R$ there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R.

We fix a positive $\rho \leq 1/(4\lambda)$. Since Y is compact, there is $N \in \mathbb{N}$ such that any subset $B \subset Y$ can be covered by at most N balls of radius ρ centered at points of B. Take $r > 0$ small enough so that $R = \lambda\rho/r$ satisfies the assumption above. Then for any ball $B_{2r} \subset Z$, we have

$$\operatorname{diam} B_{2r} \leq 4r \leq 1/R,$$

and thus there is a λ-quasi-homothetic map $f : B_{2r} \to Y$ with coefficient R. The image $f(B_{2r})$ is covered by at most N balls of radius ρ centered at points of $f(B_{2r})$. The preimage under f of any such ball is contained in a ball of radius $\leq \lambda\rho/R = r$ centered at a point in B_{2r}. Hence, B_{2r} is covered by at most N balls of radius r, and Z is doubling at small scales. $\square$

Example 2.3.5. The space H^n, $n \geq 2$, is locally similar to a compact subspace, e.g. to any closed ball of radius 1. However, H^n is by no means doubling.

Exercise 2.3.6. Let X be a proper geodesic hyperbolic space. Show that $\partial_\infty X$ is compact.

Corollary 2.3.7. *Assume that a hyperbolic space X satisfies the condition of Theorem 2.3.2, e.g., X is a hyperbolic group. Then $\partial_\infty X$ is doubling with respect to any visual metric.* $\square$

2.4 Supplementary results and remarks

2.4.1 A quadruple condition for hyperbolicity

Given a quadruple $Q = (x, y, z, u)$ of points in a metric space X with fixed base point o, we form the triple $A = A(Q) = ((x|y)_o + (z|u)_o, (x|z)_o + (y|u)_o, (x|u)_o + (y|z)_o)$ as in the Tetrahedron Lemma and call it the *cross-difference triple* of Q. We define the *small cross-difference* of Q, $\operatorname{scd}(Q)$, as the distance between the two smaller entries of the cross-difference triple $A(Q)$.

Proposition 2.4.1. *The metric space X is δ-hyperbolic, $\delta \geq 0$, if and only if* $\operatorname{scd}(Q) \leq \delta$ *for every quadruple $Q \subset X$.*

Proof. The condition $\operatorname{scd}(Q) \leq \delta$ is a reformulation of the property of $A(Q)$ to be a δ-triple. Note that this property is independent of the choice of o and take as o any point of Q. $\square$

Explicitly written, the condition for $A(Q)$ to be a δ-triple for $Q = (x, y, z, u)$ is the inequality

$$|xz| + |yu| \le \max\{|xy| + |zu|, |xu| + |yz|\} + 2\delta.$$

This formulation is more symmetric than the δ-inequality and has a geometric interpretation in the spirit of the Tetrahedron Lemma. Consider Q as an abstract tetrahedron. Adding the length of opposite edges of Q, we obtain three numbers which we can order as $a \le b \le c$. Then the inequality says $c - b \le 2\delta$.

2.4.2 Geodesic boundary

Two geodesic rays $\gamma, \gamma' \colon [a, \infty) \to X$ in a geodesic space X are called *asymptotic* if $|\gamma(t)\gamma'(t)| \le C < \infty$ for some constant C and all $t \ge a$. To be asymptotic is an equivalence relation on the set of the rays in X, and the set of classes of asymptotic rays is sometimes called the *geodesic boundary* of X, $\partial^g X$.

In a geodesic hyperbolic space, asymptotic rays are at a uniformly bounded distance from each other. Moreover, we have

Lemma 2.4.2. *Let X be a geodesic δ-hyperbolic space. Assume that for some constant $C > 0$, geodesic rays γ, γ' in X with common vertex o contain points $\gamma(t)$, $\gamma'(t')$ with $|\gamma(t)\gamma'(t')| \le C$ for arbitrarily large t, t'. Then $|\gamma(\tau)\gamma'(\tau)| \le \delta$ for all $\tau \ge 0$; in particular, the rays γ, γ' are asymptotic.*

Proof. We have

$$(\gamma(t)|\gamma'(t'))_o = \tfrac{1}{2}(t + t' - |\gamma(t)\gamma'(t')|) \ge \min\{t, t'\} - C/2.$$

Thus for $\tau \le \min\{t, t'\} - C/2$ we have $|\gamma(\tau)\gamma'(\tau)| \le \delta$ by δ-hyperbolicity. Since t, t' can be chosen arbitrarily large, this inequality holds for all $\tau \ge 0$. □

If a geodesic space X is Gromov hyperbolic, then obviously $\partial^g X \subset \partial_\infty X$. In general, there is no reason that the geodesic boundary of a hyperbolic geodesic space coincides with the boundary at infinity. However, there are several important cases when $\partial^g X = \partial_\infty X$.

Exercise 2.4.3. Show that if X is a proper hyperbolic geodesic space, then $\partial^g X = \partial_\infty X$.

Another important case when $\partial^g X = \partial_\infty X$ is described in Chapter 6; see Proposition 6.4.3.

2.4.3 Meaning of the function $\rho_{o,e}(\xi_1, \xi_2) = e^{-(\xi_1|\xi_2)_o}$ for H^n

For the unit ball model of the hyperbolic space H^n, $n \geq 2$ (see Appendix, Sections A.2 and A.5), the quasi-metric $\rho_{o,e}\colon S^{n-1} \times S^{n-1} \to \mathbb{R}$, $\rho_{o,e}(\xi_1, \xi_2) = e^{-(\xi_1|\xi_2)_o}$, where the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is identified with $\partial_\infty \mathrm{H}^n$ and o is the center of the ball, has a clear geometric interpretation: This function coincides with half of the chordal metric,

$$e^{-(\xi_1|\xi_2)_o} = \tfrac{1}{2}|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in S^{n-1}$. This immediately follows from the next lemma which also implies that the angle metric on S^{n-1} is a visual metric with respect to the center o and the parameter $a = e$.

Lemma 2.4.4. *For every $\xi_1, \xi_2 \in \partial_\infty \mathrm{H}^n = S^{n-1}$ we have*

$$e^{-(\xi_1|\xi_2)_o} = \sin(\theta/2),$$

where $\theta = \measuredangle_o(\xi_1, \xi_2)$.

Proof. For the geodesic rays $\gamma_i\colon [0, \infty) \to \mathrm{H}^n$ from o to ξ_i, $i = 1, 2$, we obviously have

$$e^{-(\xi_1|\xi_2)_o} = \lim_{t\to\infty} (e^{h_t} e^{-2t})^{1/2},$$

where $h_t = d(\gamma_1(t), \gamma_2(t))$ is the distance in H^n. From the hyperbolic law of cosine

$$\cosh(h_t) = \cosh^2(t) - \sinh^2(t) \cos\theta$$

and the trigonometric formula $1 - \cos\theta = 2\sin^2(\theta/2)$, we easily obtain

$$e^{h_t} \sim e^{2t} \sin^2(\theta/2)$$

as $t \to \infty$. Hence the claim. $\square$

2.4.4 The chain construction

Lemma 2.2.5 and the idea of its proof is due to A. H. Frink, [Fr]. It provides a better constant than contemporary simpler arguments; see e.g. [He, Chapter 14]. Moreover, the condition of that lemma cannot be improved according to the following result:

Example 2.4.5 ([Sch]). For every $\varepsilon > 0$, there exists a $(2 + \varepsilon)$-quasi-metric space (Z, ρ) such that the chain construction applied to ρ yields only a pseudo-metric d with $d(z, z') = 0$ for some distinct $z, z' \in Z$.

Bibliographical note. It is proven in [Bou] that the function $\rho_o(\xi_1, \xi_2) = e^{-(\xi_1|\xi_2)_o}$ is a metric on the boundary at infinity, $\xi_1, \xi_2 \in \partial_\infty X$, of any CAT$(-1)$-space X for every $o \in X$ (the only nontrivial point is to prove the triangle inequality). For further references we call this metric the *Bourdon metric*. Bourdon metrics associated with different $o, o' \in X$ are conformal to each other, and any isometry of X induces a conformal transformation of $(\partial_\infty X, \rho_o)$ [Bou].

Local self-similarity of the boundary at infinity of hyperbolic groups and more general cocompact hyperbolic spaces is certainly well known to experts in the field. Explicitly, it is established in [BL] from where basic results of Section 2.3 are taken.

Chapter 3

Busemann functions on hyperbolic spaces

The boundary at infinity of a hyperbolic space X was defined in the previous chapter using a fixed basepoint $o \in X$. Though the boundary is independent of the choice of o, the situation is different if we try to choose a basepoint at infinity. In this chapter we develop appropriate tools, the most important of which are Busemann functions. We also introduce and study properties of Gromov products and visual metrics based at infinity.

3.1 Busemann functions

The notion of a Busemann function is very useful in many areas. Intuitively, a Busemann function on a metric space X is the distance function on X from a point ω at infinity. Since literally this makes no sense, one needs to normalize it subtracting the distance from ω to a fixed reference point o: one takes the difference $b_{z,o}(x) = |zx| - |zo|$ for $z \in X$ and looks at the limit as $z \to \omega$. Since $|b_{z,o}(x)|$ is bounded by $|xo|$ by the triangle inequality, there is a good chance to get a well-defined function if one specifies how the limit $\lim_{z\to\omega} b_{z,o}(x)$ is to be understood. Moreover the cost of normalizing is that one associates with ω a class of distance functions which differ from each other by a constant. This class has no canonical representative: for example, the function $b\colon \mathrm{H}^2 \to \mathbb{R}$, defined in the upper half-plane model as $b(u, v) = c - \ln v$, is a Busemann function based at $\omega = \infty$ for each constant $c \in \mathbb{R}$.

In the case when X is Gromov hyperbolic, we avoid the problem how to define the limit $\lim_{z\to\omega} b_{z,o}(x)$ by using the already defined Gromov product $(\omega|x)_o$ for any $x, o \in X$, $\omega \in \partial_\infty X$. Geometrically, in the case when X is geodesic and ω is a point of the geodesic boundary, $\omega \in \partial^g X$, the difference $|\omega x| - |\omega o|$ makes sense as $|z_o x| - |z_x o|$, where $z_o \in x\omega$, $z_x \in o\omega$ are equiradial points of an infinite triangle $xo\omega \subset X$. Using equalities $|z_o x| = |z_\omega x| = (\omega|o)_x$ and $|z_x o| = |z_\omega o| = (\omega|x)_o$, where $z_\omega \in xo$ is the equiradial point, we arrive at the formula $b_{\omega,o}(x) = |z_\omega x| - |z_\omega o| = (\omega|o)_x - (\omega|x)_o$.

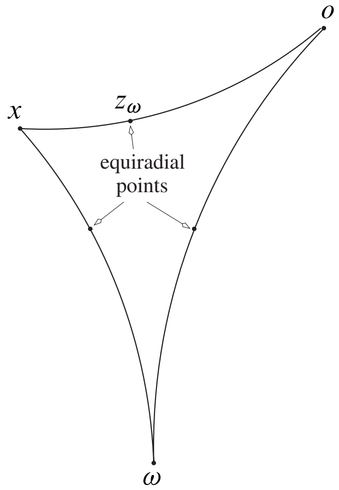


Figure 3.1. Defining a Busemann function.

All of this motivates the following considerations. Let X be a δ-hyperbolic space. For every $\omega \in \partial_\infty X$, we have a well-defined function

$$b_\omega \colon X \times X \to \mathbb{R}, \quad b_\omega(x, y) = (\omega|y)_x - (\omega|x)_y.$$

The function b_ω is obviously skew symmetric in its variables. Busemann functions based at ω associated with different reference points o differ by a constant. We show that this property holds for b_ω up to an error depending only on the hyperbolicity constant of X.

We extend our agreement about $\doteq$ as follows. For two sequences a_i, b_i, we write

$$a_i \doteq b_i \text{ up to an error } \le c \quad \text{or} \quad a_i \doteq_c b_i$$

if $\limsup_{i\to\infty} |a_i - b_i| \le c$.

Lemma 3.1.1. *For every $\omega \in \partial_\infty X$, every sequence $\{z_i\} \in \omega$ and every $x, o \in X$, we have*

$$b_\omega(x, o) \doteq (z_i|o)_x - (z_i|x)_o$$

up to an error $\le 2\delta$.

Proof. This follows from an adapted version of Lemma 2.2.2. □

Lemma 3.1.2. *For every $\omega \in \partial_\infty X$, $x, o, o' \in X$, we have*

$$b_\omega(x, o) - b_\omega(x, o') \doteq b_\omega(o', o)$$

up to an error $\le 6\delta$.

Proof. We start with the identity

$$(z|o)_x - (z|x)_o - (z|o')_x + (z|x)_{o'} = (z|o)_{o'} - (z|o')_o$$

which is straightforward to check for every quadruple of points $z, x, o, o' \in X$. When z is replaced by ω, this identity transforms into the (approximate) equality we have to prove. Taking a sequence $\{z_i\} \in \omega$ and replacing z in the identity by z_i, we apply Lemma 3.1.1 to complete the proof. $\square$

We want the set $\mathcal{B}(\omega)$ of Busemann functions based at $\omega \in \partial_\infty X$ to have the following properties:

- $\mathcal{B}(\omega)$ contains all functions $b_{\omega,o}(\cdot) = b_\omega(\cdot, o)$, $o \in X$;
- for every $b \in \mathcal{B}(\omega)$ and every constant $c \in \mathbb{R}$, the function $b + c$ is in $\mathcal{B}(\omega)$;
- any two functions $b, b' \in \mathcal{B}(\omega)$ differ from each other by a constant up to an error $\le \sigma$ with σ depending only on the hyperbolicity constant δ.

We extend the set of canonical functions $b_{\omega,o}$ to achieve an additional flexibility which is sometimes helpful; see Example 3.1.4 and the proof of Propositions 3.1.5, 3.2.3. However, this inevitably leads to an artificial choice of the error bound σ.

So consider the set $\mathcal{B}(\omega) \subset \mathbb{R}^X$ which consists of all functions $b \colon X \to \mathbb{R}$ for each of which there are $o \in X$ and a constant $c \in \mathbb{R}$ with $b \doteq b_{\omega,o} + c$ up to an error $\le 2\delta$. Then, by Lemma 3.1.2, any two functions $b, b' \in \mathcal{B}(\omega)$ differ from each other by a constant up to an error $\le 10\delta$.

Definition 3.1.3. Any function from $\mathcal{B}(\omega)$ is called a *Busemann function* based at $\omega \in \partial_\infty X$.

Example 3.1.4. For every $o \in X$, the function $\beta_{\omega,o}(x) = |ox| - 2(\omega|x)_o$ is a Busemann function based at ω, $\beta_{\omega,o} \in \mathcal{B}(\omega)$. Moreover,

$$b_{\omega,o}(x) \le \beta_{\omega,o}(x) \le b_{\omega,o}(x) + 2\delta$$

for every $x \in X$.

Proof. Given $x \in X$, there is a sequence $\{z_i\} \in \omega$ with $(\omega|x)_o = \lim (z_i|x)_o$. Using the identity

$$(z|o)_x + (z|x)_o = |xo| \tag{3.1}$$

which holds for every $o, x, z \in X$, we obtain $\lim (z_i|o)_x = |xo| - (\omega|x)_o$. Thus

$$(\omega|o)_x \le |xo| - (\omega|x)_o \le (\omega|o)_x + 2\delta$$

by Lemma 2.2.2, and the claim follows. $\square$

Proposition 3.1.5. *Every Busemann function $b \colon X \to \mathbb{R}$ based at $\omega \in \partial_\infty X$ has the following properties*:

(1) *b is roughly 1-Lipschitz, that is $|b(x)-b(x')| \le |xx'|+10\delta$ for every $x, x' \in X$;*

(2) *$b(y_i) \to +\infty$ for every $\{y_i\} \in \xi \in \partial_\infty X$, $\xi \ne \omega$;*

(3) *assume that $b(x_i) \to -\infty$ for a sequence $\{x_i\} \subset X$; then $\{x_i\} \in \omega$.*

Proof. (1) We obviously have $|b_\omega(x, x')| \le (\omega|x')_x + (\omega|x)_{x'} \le |xx'|$ for every $x, x' \in X$. Because b_ω is skew symmetric, we can apply Lemma 3.1.2 to its first variable to obtain

$$|b_{\omega,o}(x) - b_{\omega,o}(x')| \le |b_\omega(x, x')| + 6\delta \le |xx'| + 6\delta$$

for every $o \in X$. Now the claim follows easily from the definition of Busemann functions.

(2) We can assume that $b = \beta_{\omega,o}$ for some $o \in X$, $b(y) = |oy| - 2(\omega|y)_o$; see Example 3.1.4. The product $(\omega|y_i)_o$ is uniformly bounded, since $\xi \ne \omega$. Thus $b(y_i) \to +\infty$ together with $|oy_i|$ because $\{y_i\}$ converges to infinity.

(3) We can assume that $b = b_{\omega,o}$. Since $b_\omega(x_i, o) = (\omega|o)_{x_i} - (\omega|x_i)_o \to -\infty$, we have $(\omega|x_i)_o \to \infty$. Hence $\{x_i\} \in \omega$. $\square$

Remark 3.1.6. The converse to (2) and to (3) is not true: it is possible that $b(x_i) \to +\infty$ for $b \in \mathcal{B}(\omega)$ and a sequence $\{x_i\} \in \omega$. For example, the sequence $x_i = (i, \exp(-i)) \in \mathrm{H}^2$ in the upper half-plane model converges to infinity and $\{x_i\} \in \infty$, while $b(x_i) = i \to +\infty$ for the Busemann function $b(u, v) = -\ln v$ based at ∞.

3.2 Gromov products based at infinity

Let X be a δ-hyperbolic space, $\omega \in \partial_\infty X$. Busemann functions allow to define a Gromov product based at ω. We first define it on X.

3.2.1 Gromov products on X based at infinity

We fix a Busemann function $b \in \mathcal{B}(\omega)$ and for $x, y \in X$, we define their Gromov product based at b by

$$(x|y)_b = \tfrac{1}{2}(b(x) + b(y) - |xy|).$$

Note that contrary to the standard case $(x|y)_o$ with $o \in X$, the product $(x|y)_b$ might be negative. It immediately follows from the definition of Busemann functions that for different $b, b' \in \mathcal{B}(\omega)$ there is a constant $c \in \mathbb{R}$ so that

$$(x|y)_b - (x|y)_{b'} \doteq c \tag{3.2}$$

up to an error $\le 10\delta$ for every $x, y \in X$. In other words, the choice of $b \in \mathcal{B}(\omega)$ does not play an essential role.

Example 3.2.1. For the function $b = \beta_{\omega,o} \in \mathcal{B}(\omega)$, see Example 3.1.4, we have

$$(x|y)_b = (x|y)_o - (\omega|x)_o - (\omega|y)_o$$

for every $x, y \in X$.

Our next goal is to show that the Gromov product based at any Busemann function $b \in \mathcal{B}(\omega)$ satisfies the σ-inequality with $\sigma \geq 0$ depending only on the hyperbolicity constant δ of X. We begin with

Lemma 3.2.2. *Assume that numbers $a, b, c \in \mathbb{R}$ form a δ-triple, and $a' \doteq a$, $b' \doteq b$, $c' \doteq c$ up to an error $\leq \sigma$. Then the numbers a', b', c' form a $(\delta + 2\sigma)$-triple.*

Proof. We can assume that $a \geq \max\{b, c\}$. Then $|b - c| \leq \delta$. Thus $|b' - c'| \leq \delta + 2\sigma$ because $|b' - c'|$ differs from $|b - c|$ by at most 2σ. This implies the claim in the case $a' \geq \min\{b', c'\}$.

Suppose now that $a' < \min\{b', c'\}$. Since

$$a' \geq a - \sigma \geq \max\{b, c\} - \sigma \geq \min\{b', c'\} - 2\sigma,$$

the claim follows. $\square$

Proposition 3.2.3. *For every $x, y, z \in X$, the numbers $(x|y)_b$, $(x|z)_b$, $(y|z)_b$ form a 2δ-triple for every function $b = \beta_{\omega,o} \in \mathcal{B}(\omega)$, and a 22δ-triple for an arbitrary $b \in \mathcal{B}(\omega)$.*

Proof. Six numbers $(x|y)_o$, $(y|z)_o$, $(x|z)_o$, $(\omega|x)_o$, $(\omega|y)_o$, $(\omega|z)_o$ satisfy the condition of the Tetrahedron Lemma (Lemma 2.1.4) which implies that the numbers $(x|y)_o + (\omega|z)_o$, $(y|z)_o + (\omega|x)_o$, $(x|z)_o + (\omega|y)_o$ form a 2δ-triple. We let $a = (\omega|x)_o + (\omega|y)_o + (\omega|z)_o$ and first assume that $b = \beta_{\omega,o}$. Then using Example 3.2.1, we obtain

$$\begin{aligned}
(x|y)_b + a &= (x|y)_o + (\omega|z)_o, \\
(y|z)_b + a &= (y|z)_o + (\omega|x)_o, \\
(x|z)_b + a &= (x|z)_o + (\omega|y)_o.
\end{aligned}$$

Therefore, the numbers on the left-hand side form a 2δ-triple. Hence, the numbers $(x|y)_b$, $(y|z)_b$, $(x|z)_b$ form a 2δ-triple for $b = \beta_{\omega,o}$. In the general case, we apply the approximate equality $(x|y)_{b'} \doteq (x|y)_b + c$ up to an error $\leq 10\delta$, where the constant c is independent of $x, y \in X$, and Lemma 3.2.2. $\square$

3.2.2 Gromov products on $X \cup \partial_\infty X$ based at infinity

There are several possibilities to introduce a Gromov product based at infinity on $X \cup \partial_\infty X$. The simplest one, which allows to avoid any further limit procedure, is motivated by the Gromov product on X based at the Busemann function $b = \beta_{\omega,o}$. We denote $Z = X \cup \partial_\infty X$, and for every pair $(\xi,\eta) \in Z \times Z$ which is distinct from (ω,ω), we put

$$(\xi|\eta)_{\omega,o} := (\xi|\eta)_o - (\omega|\xi)_o - (\omega|\eta)_o.$$

The so defined Gromov product takes values in $[-\infty,+\infty]$, $(\xi|\eta)_{\omega,o} = +\infty$ if and only if $\xi = \eta \in \partial_\infty X \setminus \omega$, and $(\xi|\eta)_{\omega,o} = -\infty$ if and only if one of the factors equals ω. Certainly, $(x|y)_{\omega,o} = (x|y)_b$ for every $x, y \in X$.

Just as in Proposition 3.2.3, we see that for every $\xi,\eta,\zeta \in Z$ distinct from ω, the numbers $(\xi|\eta)_{\omega,o}$, $(\xi|\zeta)_{\omega,o}$, $(\eta|\zeta)_{\omega,o}$ form a 2δ-triple.

It is unpleasant that this Gromov product depends not only on $\omega \in \partial_\infty X$ but also on the reference point $o \in X$, which is conceptually not right. The standard way to eliminate this dependence is to consider a family of Gromov products which depends then only on ω. We proceed similarly to Section 2.2.1. For every Busemann function $b \in \mathcal{B}(\omega)$ and $(\xi,\eta) \in Z \times Z \setminus (\omega,\omega)$, we define the Gromov product based at b by

$$(\xi|\eta)_b = \inf \liminf_{i\to\infty} (x_i|y_i)_b,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{y_i\} \in \eta$ (here we assume that the sequence is constant for points in X). It follows from equation (3.2) that for different $b, b' \in \mathcal{B}(\omega)$, there is a constant $c \in \mathbb{R}$ so that

$$(\xi|\eta)_b - (\xi|\eta)_{b'} \doteq c \tag{3.3}$$

up to an error $\leq 10\delta$ for every $\xi,\eta \in Z \setminus \omega$. As an exercise, one can check that for the Busemann function $b = b_{\omega,o} \in \mathcal{B}(\omega)$ the approximate equality

$$(\xi|\eta)_b \doteq (\xi|\eta)_{\omega,o} \tag{3.4}$$

holds up to an error $\leq 2\delta$. Furthermore, exactly as in Lemma 2.2.2, we obtain

Lemma 3.2.4. *Assume that the σ-inequality holds in X for the Gromov product based at $b \in \mathcal{B}(\omega)$, e.g., $\sigma = 2\delta$ for $b = b_{\omega,o}$ and $\sigma = 22\delta$ for an arbitrary $b \in \mathcal{B}(\omega)$. Let $\xi,\eta,\zeta \in Z \setminus \omega$.*

(1) *For arbitrary sequences $\{x_i\} \in \xi$, $\{y_i\} \in \eta$, we have*

$$(\xi|\eta)_b \leq \liminf_{i\to\infty} (x_i|y_i)_b \leq \limsup_{i\to\infty} (x_i|y_i)_b \leq (\xi|\eta)_b + 2\sigma;$$

(2) *$(\xi|\eta)_b$, $(\eta|\zeta)_b$, $(\xi|\zeta)_b$ is a σ-triple.* ∎

3.3 Visual metrics based at infinity

We recapitulate and slightly generalize the notion of a quasi-metric space.

Definition 3.3.1. Let Z be a set, $Z_\infty \subset Z$ be a subset, $\Omega = Z_\infty \times Z_\infty$. A *quasi-metric* on Z with infinitely remote set Z_∞ is any function $\rho \colon Z \times Z \setminus \Omega \to [0, \infty]$ with the following properties:

(1) $\rho(a, b) = 0$ if and only if $a = b$;
(2) $\rho(a, b) = \rho(b, a)$;
(3) there is $K \geq 1$ with $\rho(a, b) \leq K \max\{\rho(a, c), \rho(c, b)\}$ for all $a, b, c \in Z$ for which all members of the inequality are defined;
(4) $\rho(a, b) < \infty$ if and only if $a, b \in Z \setminus Z_\infty$.

In this case, we also say that ρ is a K-quasi-metric, and (Z, ρ) is a *quasi-metric* space or shortly *Q-metric* space. This definition is a small modification of the definition in Chapter 2, Section 2.2.2. We allow the existence of an *infinitely remote* set $Z_\infty \subset Z$ such that $\rho(a, \xi) = \infty$ for all $a \in Z \setminus Z_\infty$, $\xi \in Z_\infty$. However, $\rho(\xi, \eta)$ is not defined for $\xi, \eta \in Z_\infty$.

In what follows, we always assume that the infinitely remote set contains at most one point, $|Z_\infty| \leq 1$. In this case, the generalized ultra-metric triangle inequality (3) is fulfilled for all distinct $a, b, c \in Z$. For example, if (Z, ρ) is a Q-metric space with $\rho(a, b) < \infty$ for all $a, b \in Z$, then $\overline{Z} = Z \cup \{\infty\}$ (with the obvious extension of ρ by $\rho(a, \infty) = \infty$ for all $a \in Z$) is also a Q-metric space. For another interesting example see Section 3.4.1.

The proof of Proposition 2.2.6 runs verbatim for modified Q-metrics, and we obtain

Proposition 3.3.2. *Let ρ be a K-quasi-metric on a set Z with infinitely remote set Z_∞. Then there exists $\varepsilon_0 > 0$ only depending on K such that ρ^ε is bilipschitz equivalent to a metric on $Z \setminus Z_\infty$ for each $0 < \varepsilon \leq \varepsilon_0$.* □

Coming back to our δ-hyperbolic space X, we fix $\omega \in \partial_\infty X$, a Busemann function $b \in \mathcal{B}(\omega)$ and for a parameter $a > 1$ we define the function

$$\rho_b \colon (\partial_\infty X)^2 \setminus (\omega, \omega) \to [0, \infty], \quad \rho_b(\xi, \eta) = a^{-(\xi, \eta)_b}.$$

Then, similarly to the considerations in Section 2.2.2, ρ_b is a K-quasi-metric on $Z = \partial_\infty X$ with the infinitely remote set $Z_\infty = \{\omega\}$ and with $K = a^\sigma$, where the σ-inequality holds in $\partial_\infty X \setminus \{\omega\}$ for the Gromov product based at b; see Lemma 3.2.4. Furthermore, $\rho_b(\xi, \eta) = \infty$ if and only if one of the points ξ, η coincides with ω.

A metric d on $\partial_\infty X \setminus \{\omega\}$ is called visual if it is Lipschitz equivalent to ρ_b for some Busemann function $b \in \mathcal{B}(\omega)$ and some parameter $a > 1$:

$$c_1 a^{-(\xi|\eta)_b} \leq d(\xi, \eta) \leq c_2 a^{-(\xi|\eta)_b}.$$

In this case we say that d is visual with respect to the Busemann function b and the parameter a. Because the Gromov products on $\partial_\infty X \setminus \{\omega\}$ based at different Busemann functions $b, b' \in \mathcal{B}(\omega)$ differ from each other by a constant up to an error $\leq 10\delta$, see equation (3.3), the property of a metric to be visual is independent of the choice of b. Applying Proposition 3.3.2 we see

Proposition 3.3.3. *Let X be a hyperbolic space, $\omega \in \partial_\infty X$. Then for any Busemann function $b \in \mathcal{B}(\omega)$, there is $a_0 > 1$ such that for every $a \in (1, a_0]$ there exists a visual metric d on $\partial_\infty X \setminus \{\omega\}$ with respect to b and a.* □

Example 3.3.4. For the upper half-space model of H^{n+1}, $\partial_\infty \mathrm{H}^{n+1} = \mathbb{R}^n \cup \{\infty\}$, the Euclidean metric on $\mathbb{R}^n$ is visual with respect to the Busemann function $b \in \mathcal{B}(\infty)$, $b(u, v) = -\ln v$ for $(u, v) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$, and the parameter $a = e$.

Contrary to the case of visual metrics based at a point in X, the boundary at infinity of a hyperbolic space X equipped with a visual metric based at a Busemann function $b \in \mathcal{B}(\omega)$ is unbounded with infinitely remote point $\omega \in \partial_\infty X$. This is similar to the upper half-space model of H^n.

Similarly to Propositions 2.2.8 and 2.2.9, we obtain that visual metrics with respect to one and the same parameter and different Busemann functions $b, b' \in \mathcal{B}(\omega)$ are bilipschitz to each other, and the parameter change leads to Hölder equivalent visual metrics. Much more interesting is the effect which occurs when we change the point ω. We study this effect in the next chapter.

3.4 Supplementary results and remarks

3.4.1 The boundary at infinity with respect to $\omega \in \partial_\infty X$

Similarly to Section 2.2, we say that a sequence $\{x_i\} \subset X$ *converges to infinity with respect to* $\omega \in \partial_\infty X$ if

$$\lim_{i,j\to\infty} (x_i|x_j)_b = +\infty$$

for some and hence for any Busemann function $b \in \mathcal{B}(\omega)$. Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity with respect to ω are *equivalent* if

$$\lim_{i\to\infty} (x_i|x_i')_b = +\infty$$

for some and hence for every Busemann function $b \in \mathcal{B}(\omega)$. Using Proposition 3.2.3, we easily see that this defines an equivalence relation for sequences in X converging to infinity with respect to ω. The *boundary at infinity* $\partial_\omega X$ *of X with respect to* ω is defined as the set of equivalence classes of sequences converging to infinity with respect to ω.

Proposition 3.4.1. *For every $\omega \in \partial_\infty X$ there are canonical inclusions*

$$\partial_\infty X \setminus \{\omega\} \to \partial_\omega X \to \partial_\infty X,$$

whose composition coincides with the inclusion $\partial_\infty X \setminus \{\omega\} \subset \partial_\infty X$.

Proof. Fix a reference point $o \in X$ and recall that for the Busemann function $b = \beta_{\omega,o} \in \mathcal{B}(\omega)$, we have

$$(x|y)_b = (x|y)_o - (\omega|x)_o - (\omega|y)_o$$

for every $x, y \in X$; see Example 3.2.1. Also recall that the Gromov product based at $o \in X$ is nonnegative. Now if $(x_i|x_j)_b \to +\infty$ for a sequence $\{x_i\} \subset X$, then $(x_i|x_j)_o \to \infty$, i.e., any sequence which converges to infinity with respect to ω converges to infinity in the standard sense. Similarly, two sequences equivalent to each other with respect to ω are equivalent to each other in the standard sense. This defines a map $f : \partial_\omega X \to \partial_\infty X$.

Vice versa, fix $\xi \in \partial_\infty X \setminus \{\omega\}$ and consider a sequence $\{x_i\} \in \xi$. Then $(x_i|x_j)_o \to \infty$, while $(\omega|x_i)_o$, $(\omega|x_j)_o$ are uniformly bounded. Thus $(x_i|x_j)_b \to +\infty$, and $\{x_i\}$ converges to infinity with respect to ω. Similarly, any other $\{y_i\} \in \xi$ is equivalent to $\{x_i\}$ with respect to ω. This defines a map $g : \partial_\infty X \setminus \{\omega\} \to \partial_\omega X$. Obviously, the composition $f \circ g$ coincides with inclusion $\partial_\infty X \setminus \{\omega\} \subset \partial_\infty X$.

It remains to check that the map f is injective. Assume to the contrary that there is a sequence $\{x_i\} \subset X$ with $(x_i|x_j)_b \to \infty$ and $(\omega|x_i)_o \to \infty$. Then it follows from

$$(x_i|x_j)_b = (x_i|x_j)_o - (\omega|x_i)_o - (\omega|x_j)_o \to \infty \tag{3.5}$$

and from the δ-inequality that $|(\omega|x_i)_o - (\omega|x_j)_o| \le \delta$ and therefore, for the right-hand side a_{ij} of equation (3.5), we obtain

$$a_{ij} \le (x_i|x_j)_o - 2(\omega|x_i)_o + \delta$$

for all sufficiently large i, j. We fix such an i and look at the limit as $j \to \infty$. This yields

$$\limsup_{j\to\infty} a_{ij} \le -(x_i|\omega)_o + 3\delta \to -\infty,$$

as $i \to \infty$. This contradicts the assumption $(x_i|x_j)_b \to \infty$. □

3.4.2 Boundary continuous hyperbolic spaces

Let X be a Gromov hyperbolic space which satisfies the δ-inequality for some $\delta \ge 0$. We have proven many estimates in terms of δ. In case $\delta = 0$ these inequalities become equalities. But there are also other spaces, where we have sometimes exact equalities. This holds in particular for the classical hyperbolic space.

We call a Gromov hyperbolic space X *boundary continuous* if the Gromov product extends continuously onto the boundary at infinity in the following way: if $\{x_i\}, \{y_i\}$ are sequences in X converging to points $\bar{x}, \bar{y}$ in X or in $\partial_\infty X$, then $(x_i|y_i)_o \to (\bar{x}|\bar{y})_o$ for every $o \in X$.

Gromov hyperbolic spaces satisfying the δ-inequality with $\delta = 0$ are boundary continuous. The classical hyperbolic space H^n is boundary continuous. More generally, we have

Proposition 3.4.2. *Every proper* $\mathrm{CAT}(-1)$*-space* X *is a boundary continuous Gromov hyperbolic space.*

Proof. We fix $o \in X$ and consider distinct $\xi, \xi' \in \partial_\infty X$ (the case when one of the points ξ, ξ' lies in X is even easier). Observe that there are geodesic rays $o\xi, o\xi' \subset X$ (see Exercise 2.4.3) which are uniquely determined because the space is $\mathrm{CAT}(-1)$. We use the notation $\xi = \xi(t)$ for the unit speed parametrization of $o\xi$, $\xi(0) = o$.

By monotonicity of the Gromov product (see Lemma 2.1.1), there exists a limit

$$a = \lim_{t\to\infty} (\xi(t)|\xi'(t))_o.$$

Note that $a < \infty$ because ξ, ξ' are distinct, and that $a \geq (\xi|\xi')_o$ by the definition of $(\xi|\xi')_o$. We shall prove that

$$a = \lim_{i\to\infty} (x_i|x_i')_o$$

for every $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$.

Since $(x_i|\xi)_o \to \infty$, we easily see that the segments ox_i converge to the ray $o\xi$ as $i \to \infty$ in the compact-open topology, that is, for every $\varepsilon > 0$, $R > 1$ the segment ox_i runs within the ε-neighborhood of $o\xi$ during the time R for all sufficiently large i.

To show our claim, we first prove that $\liminf_i (x_i|x_i') \geq a$. We fix $\varepsilon > 0$ and take t large enough so that $(\xi(t)|\xi'(t))_o \geq a - \varepsilon$. Next, we fix $x_i(t) \in ox_i$ with $|ox_i(t)| = |o\xi(t)| = t$. Then $|x_i(t)\xi(t)| \leq \varepsilon$ for all sufficiently large i. Hence the lengths of the segments $\xi(t)\xi'(t)$ and $x_i(t)x_i'(t)$ differ from each other by at most 2ε. Therefore, the Gromov products $(\xi(t)|\xi'(t))_o$ and $(x_i(t)|x_i'(t))_o$ differ from each other by at most ε. Using monotonicity of the Gromov product, we obtain

$$(x_i|x_i')_o \geq (x_i(t)|x_i'(t))_o \geq a - 2\varepsilon,$$

thus $\liminf_i (x_i|x_i') \geq a$. This shows in particular that $a = (\xi|\xi')_o$.

To obtain the estimate from above, $\limsup_i (x_i|x_i') \leq a$, we need the following fact: there exists a geodesic $\gamma \subset X$ with end points ξ, ξ' at infinity. (To prove this, we note that each geodesic segment $\xi(t)\xi'(t)$, $t \geq 0$, intersects a fixed ball $B_R(o)$ with $R \geq 2(\xi|\xi')_o$. Using that X is proper, we easily find γ as a sublimit of $\xi(t)\xi'(t)$ as $t \to \infty$. We leave details to the reader.)

We fix $p \in \gamma$. Then the geodesic segments px_i, px_i' converge to the subrays $p\xi$, $p\xi'$ of γ respectively in the compact-open topology as $i \to \infty$. It follows that given

$\varepsilon > 0$, the angle $\measuredangle_p(x_i, x_i') \ge \pi - \varepsilon$ for all sufficiently large i. Using comparison with H^2, we see that every point of $x_i x_i'$ is ε-close to the union $px_i \cup px_i'$.

By the CAT(-1) property, the distances $\mathrm{dist}(\xi(t), \gamma)$, $\mathrm{dist}(\xi'(t), \gamma)$ become arbitrarily small as $t \to \infty$. Thus taking t large enough, we conclude that $\mathrm{dist}(\xi(t), x_i x_i')$, $\mathrm{dist}(\xi(t), ox_i) \le \varepsilon$ and $\mathrm{dist}(\xi'(t), x_i x_i')$, $\mathrm{dist}(\xi'(t), ox_i') \le \varepsilon$ for all sufficiently large i. It follows that

$$(x_i|x_i')_o \le (\xi(t)|\xi'(t))_o + \varepsilon$$

for all sufficiently large i and hence $\limsup_i (x_i|x_i') \le a$. $\square$

For a boundary continuous hyperbolic space X, the notion of a Busemann function is simplified: the set of Busemann functions $\mathcal{B}(\omega) \subset \mathbb{R}^X$ for $\omega \in \partial_\infty X$ consists of all functions $b\colon X \to \mathbb{R}$ for each of which there are $o \in X$ and a constant $c \in \mathbb{R}$ with $b = b_{\omega,o} + c$. The approximate equality of Lemma 3.1.2 becomes the precise equality and thus two Busemann functions $b, b' \in \mathcal{B}(\omega)$ differ from each other by a constant. In particular, $b_{\omega,o} = \beta_{\omega,o}$.

Now Gromov products on X based at $\omega \in \partial_\infty X$ differ from each other by a constant, $(x|y)_b - (x|y)_{b'} = c$ for different $b, b' \in \mathcal{B}(\omega)$, some constant c and all $x, y \in X$. Furthermore, we have the following refined version of Proposition 3.2.3.

Proposition 3.4.3. *Let X be a boundary continuous δ-hyperbolic space. Then for every $x, y, z \in X$, $\omega \in \partial_\infty X$, the numbers $(x|y)_b$, $(x|z)_b$, $(y|z)_b$ form a δ-triple for every Busemann function $b \in \mathcal{B}(\omega)$.*

Proof. In view of the equality $(\omega|x)_o = \lim_{i\to\infty}(w_i|x)_o$ for every $\{w_i\} \in \omega$, $x \in X$, the numbers $(x|y)_o + (\omega|z)_o$, $(y|z)_o + (\omega|x)_o$, $(x|z)_o + (\omega|y)_o$ form a δ-triple because the difference between any two of them is independent of o (compare the proof of Proposition 2.4.1), and we can take $o = z$ for example.

The rest of the proof runs exactly as in the proof of Proposition 3.2.3. $\square$

The Gromov product on $Z = X \cup \partial_\infty X$ based at $\omega \in \partial_\infty X$ is also simplified. Given $\{x_i\} \in \xi$, $\{y_i\} \in \eta \in \partial_\infty X \setminus \{\omega\}$, we have

$$\begin{aligned}
(\xi|\eta)_{\omega,o} &= (\xi|\eta)_o - (\omega|\xi)_o - (\omega|\eta)_o \\
&= \lim_{i\to\infty} [(x_i|y_i)_o - (\omega|x_i)_o - (\omega|y_i)_o] \\
&= \lim_{i\to\infty} (x_i|y_i)_b = (\xi|\eta)_b,
\end{aligned}$$

where $b = \beta_{\omega,o} \in \mathcal{B}(\omega)$. Then for any other Busemann function $b' \in \mathcal{B}(\omega)$, the limit $\lim_{i\to\infty}(x_i|y_i)_{b'} = \lim_{i\to\infty}(x_i|y_i)_b + c$ exists, and

$$(\xi|\eta)_{b'} = \lim_{i\to\infty}(x_i|y_i)_{b'} = (\xi|\eta)_b + c.$$

In particular, Gromov products on Z based at $\omega \in \partial_\infty X$ differ from each other by a constant: $(\xi|\eta)_b - (\xi|\eta)_{b'} = c$ for different $b, b' \in \mathcal{B}(\omega)$, some constant c and all $\xi, \eta \in Z$.

Finally, we note that for every $\xi, \eta, \zeta \in Z$ distinct from ω and every $b \in \mathcal{B}(\omega)$, the numbers $(\xi|\eta)_b$, $(\eta|\zeta)_b$, $(\xi|\zeta)_b$ form a δ-triple by Proposition 3.4.3 because $(\xi|\eta)_b = \lim_{i\to\infty}(x_i|y_i)_b$ for $\{x_i\} \in \xi$, $\{y_i\} \in \eta$.

Bibliographical note. A result close to Proposition 3.3.3 is obtained in [Ha] for negatively pinched Hadamard manifolds, where a family of functions on $\partial_\infty X \setminus \{\omega\}$, defined via horospherical distances, is considered instead of the Gromov product based at a Busemann function $b \in \mathcal{B}(\omega)$.

It is proven in [FS2] that the function $\rho_b(\xi_1, \xi_2) = e^{-(\xi_1|\xi_2)_b}$ is a metric on the boundary at infinity, $\xi_1, \xi_2 \in \partial_\infty X \setminus \{\omega\}$, of any CAT$(-1)$-space X for every $\omega \in \partial_\infty X$ and every Busemann function $b \in \mathcal{B}(\omega)$. The proof is based on the properties of a Bourdon metric and the Ptolemy inequality.

Chapter 4

Morphisms of hyperbolic spaces

What is a natural class of morphisms between hyperbolic spaces? By natural we mean morphisms inducing maps of the boundaries at infinity in a way compatible with composition. There are different points of view on this question. The commonly accepted one is to consider quasi-isometric maps as natural, due to geometrical reasons.

First of all, the universal covering of a reasonable compact metric space (by reasonable we mean spaces like manifolds and polyhedra for which the covering theory holds) is quasi-isometric to its fundamental group with a word metric which for a finitely generated group is only defined up to bilipschitz transformations related to changes of generating sets. Therefore, interesting geometric invariants should be quasi-isometry invariants.

Secondly, the classical duality between the isometries of H^{n+1} and the Möbius transformations of the boundary sphere $S^n = \partial_\infty \mathrm{H}^{n+1}$ extends to a much more general class of hyperbolic spaces. A well-known theorem of Efremovich–Tihomirova [ET] based on the stability of geodesics (which was discovered much earlier by M. Morse) says that any quasi-isometry of H^{n+1} has an extension to the boundary sphere S^n. The argument works for quasi-isometric maps between any geodesic hyperbolic spaces, and with some additional effort one shows that the induced map between the boundaries at infinity is quasi-Möbius.

However, the problem is that the condition for spaces to be geodesic is too restrictive, and without it, a quasi-isometric map between hyperbolic spaces has in general no extension to the boundaries at infinity. One can develop an approach which replaces geodesics by quasi-geodesics and therefore allows to recover the extension property of quasi-isometric maps between hyperbolic spaces which are more general than geodesic ones; see e.g. [BoS], [V2].

Instead, we narrow the class of quasi-isometric maps by putting the stronger condition that a map should have a bilipschitz type control over *cross-differences*, and we call such a map *power quasi-isometric*. A power quasi-isometric map between any metric spaces is automatically quasi-isometric, and in the case of arbitrary hyperbolic spaces it naturally induces a map between their boundaries at infinity (see Corollary 4.3.3 below), which is automatically (power) quasi-Möbius (see Proposition 5.2.10).

From our point of view, power quasi-isometric maps constitute a natural class of morphisms between hyperbolic spaces. This is supported by the (nontrivial) fact that in the case of geodesic hyperbolic spaces any quasi-isometric map is power quasi-isometric (see Theorem 4.4.1), and in that way, we recover the extension property of quasi-isometric maps. Moreover, we show in Chapter 7 that any hyperbolic space (with the mild natural restriction to be visual) is roughly isometric to a subspace of a geodesic hyperbolic space with the same boundary at infinity, and hence there is no need for quasification of geodesics.

4.1 Morphisms of metric spaces and hyperbolicity

We consider in this section metric spaces X and X' and maps $f\colon X \to X'$. We classify maps f by considering the way these maps disturb the distances between points. We describe this by *control functions*. Given functions $\rho_1, \rho_2\colon [0,\infty) \to \mathbb{R}$, we are considering the class of functions f such that for all $x, y \in X$ we have

$$\rho_1(|xy|) \le |x'y'| \le \rho_2(|xy|),$$

where $x' = f(x)$ and $y' = f(y)$ are the images of x, y respectively. In this way, one can define highly rigid classes of maps, e.g. isometric maps for $\rho_1 = \rho_2 = \mathrm{id}$. On the other hand one can define much wider classes of maps. A map f is called *coarse* if there are control functions ρ_1, ρ_2 for f with the property that $\lim_{t\to\infty} \rho_i(t) = \infty$.

For example, quasi-isometric or Q-isometric maps, introduced in Chapter 1, are characterized by affine control functions, $\rho_1(t) = \frac{1}{c}t - d$, $\rho_2(t) = ct + d$, $c \ge 1$, $d \ge 0$.

In Chapter 7, roughly homothetic (or R-homothetic) and roughly isometric (or R-isometric) maps, described by control functions $\rho_1(t) = ct - d$, $\rho_2(t) = ct + d$, $d \ge 0$ and $c > 0$, $c = 1$ respectively, play an important role.

These characterizations of functions involve just the distance $|xy|$ between two points and how this distance is disturbed by the map f. For the study of hyperbolic spaces it is necessary to consider more complicated expressions which involve the distances between three and four points. This is not surprising since hyperbolicity itself is a condition on quadruples of points. These expressions can be defined in a multiplicative or in an additive way. Well known and intensively studied is the cross-ratio, which involves the distances between four points and which is written in a multiplicative form.

Given $(x, y, z, u) \in X^4$ the *classical cross-ratio* is given by

$$[x, y, z, u] = \frac{|xz||yu|}{|xy||zu|}.$$

The additive version of the classical cross-ratio is the *classical cross-difference*

$$\langle x, y, z, u\rangle = \tfrac{1}{2}(|xz| + |yu| - |xy| - |zu|).$$

The correspondence between the multiplicative and the additive version of the expression is more subtle than expected at first glance. A trivial calculation shows that

$$\langle x, y, z, u\rangle = -(x|z)_o - (y|u)_o + (x|y)_o + (z|u)_o,$$

where $o \in X$ is any chosen base point. One should think of $-(x|y)_o$ to be the additive counterpart to the multiplicative $|xy|$ (this explains the factor $1/2$). This corresponds to the fact that the quasi-metric at the boundary at infinity of a hyperbolic space is given by the exponential $a^{-(x|y)_o}$.

It is easy to express hyperbolicity in terms of the classical cross-difference. Recall that by Proposition 2.4.1, a metric space X is hyperbolic if and only if the cross-difference triple $A(Q)$ of every quadruple $Q = (x, y, z, u) \subset X$ is a δ-triple for some $\delta \geq 0$. Explicitly written, this condition is the inequality

$$|xz| + |yu| \leq \max\{|xy| + |zu|, |xu| + |yz|\} + 2\delta.$$

In turn, it can be rewritten via the classical cross-difference as

$$\min\{\langle x, y, z, u\rangle, \langle x, u, z, y\rangle\} \leq \delta.$$

One can now define classes of maps, which disturb the cross-difference in a similar way as above by using control functions,

$$\rho_1(\langle x, y, z, u\rangle) \leq \langle x', y', z', u'\rangle \leq \rho_2(\langle x, y, z, u\rangle).$$

We introduce a class of maps between metric spaces called (strongly) PQ-isometric maps. ‘PQ’ stands for ‘power quasi’ because every PQ-isometric map is quasi-isometric, and in general this is stronger than the property to be quasi-isometric.

Definition 4.1.1. We say that a map $f : X \to X'$ between metric spaces is *strongly PQ-isometric* if there are constants $c \geq 1$, $d \geq 0$ such that for all quadruples $(x, y, z, u) \in X^4$ with $\langle x, y, z, u\rangle \geq 0$,

$$\tfrac{1}{c}\langle x, y, z, u\rangle - d \leq \langle x', y', z', u'\rangle \leq c\langle x, y, z, u\rangle + d.$$

Remarks. (1) The term ‘PQ-isometric’ is reserved for a weaker condition which is introduced and used in the subsequent sections.

(2) We can also estimate $\langle x', y', z', u'\rangle$ in the case that this expression is negative. We use the antisymmetry in the second and third entry, i.e. the fact that $\langle x, y, z, u\rangle = -\langle x, z, y, u\rangle$. We then obtain for $\langle x, y, z, u\rangle < 0$ that

$$c\langle x, y, z, u\rangle - d \leq \langle x', y', z', u'\rangle \leq \tfrac{1}{c}\langle x, y, z, u\rangle + d.$$

Thus it is possible to write the condition to be a strongly PQ-isometric map as

$$-\theta(-\langle x, y, z, u\rangle) \leq \langle x', y', z', u'\rangle \leq \theta(\langle x, y, z, u\rangle)$$

where (x, y, z, u) is now an arbitrary quadruple and $\theta : \mathbb{R} \to \mathbb{R}$ is the control function $\theta(t) = \max\{ct, \tfrac{1}{c}t\} + d$.

One easily checks that the composition of strongly PQ-isometric maps is strongly PQ-isometric. It follows directly from the characterization of hyperbolic spaces by the quadruple condition of hyperbolicity that hyperbolicity is a strongly PQ-isometry invariant. More precisely we have the following.

Proposition 4.1.2. *Assume that a metric space X' is hyperbolic and that a map $f\colon X \to X'$ is strongly PQ-isometric. Then X is also hyperbolic. Furthermore, the image of a hyperbolic space under a strongly PQ-isometric map is hyperbolic.*

Proof. There are $c \geq 1$, $d \geq 0$ so that for every quadruple $Q = (x,y,z,u) \subset X$, we have

$$-\theta(-\langle x,y,z,u\rangle) \leq \langle x',y',z',u'\rangle \leq \theta(\langle x,y,z,u\rangle)$$

and

$$-\theta(-\langle x,u,z,y\rangle) \leq \langle x',u',z',y'\rangle \leq \theta(\langle x,u,z,y\rangle)$$

where $\theta(t) = \max\{ct, \frac{1}{c}t\} + d$. Then

$$\min\{\langle x',y',z',u'\rangle, \langle x',u',z',y'\rangle\} \leq \delta$$

for some $\delta \geq 0$ independent of $Q' = (x',y',z',u') = f(Q)$ because X' is hyperbolic. This implies that

$$\min\{\langle x,y,z,u\rangle, \langle x,u,z,y\rangle\} \leq c(d+\delta)$$

since we can assume that both $\langle x,y,z,u\rangle$, $\langle x,u,z,y\rangle$ are nonnegative. Therefore, X is hyperbolic. A similar argument proves the second assertion of the proposition. □

Remark 4.1.3. (1) The proof shows that f preserves hyperbolicity if f only coarsely preserves the cross-difference in the sense that there are functions $\rho_1, \rho_2\colon [0,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty} \rho_i(t) = \infty$ such that

$$\rho_1(\langle x,y,z,u\rangle) \leq \langle x',y',z',u'\rangle \leq \rho_2(\langle x,y,z,u\rangle)$$

for all $x,y,z,u \in X$ with $\langle x,y,z,u\rangle \geq 0$.

(2) In contrast it is not true in general that hyperbolicity is a quasi-isometry invariant. There are examples of quasi-isometric spaces X, X' such that one of them is hyperbolic, but the other is not. Consider the subset $X = \{0\} \cup \{a_i, b_i, c_i \in \mathbb{R}^2 : i \in \mathbb{N}\} \subset \mathbb{R}^2$, where $a_i = (10^i, 0)$, $b_i = (0, 10^i)$ and $c_i = (10^i, 10^i)$. Then, considering the quadruples $(0, a_i, b_i, c_i)$, one easily shows that X is not hyperbolic. However the map $f\colon X \to \mathbb{R}$, defined by $f(0) = 0$, $f(a_i) = 10^i$, $f(b_i) = 2 \cdot 10^i$ and $f(c_i) = 3 \cdot 10^i$ is a Q-isometric map whose image is (as a subset of $\mathbb{R}$) hyperbolic.

As another and simpler example, consider $X = \{(x,y) \in \mathbb{R}^2 : y = |x|\}$ with the metric induced from $\mathbb{R}^2$, $X' = \mathbb{R}$. Then the projection $f\colon X \to X'$, $f(x,y) = x$, is bilipschitz, X' is 0-hyperbolic, but X is not hyperbolic.

4.2 Cross-difference triples and cross-differences

Given a quadruple $Q = (x, y, z, u)$ of points in a metric space X, we have defined the classical cross-difference $\langle x, y, z, u\rangle$ of these points. However, there are drawbacks of this expression: it depends on a chosen order of the quadruple Q; as a consequence, there are six versions of the definition, and most of them (if not all) can be found in literature. Furthermore, it turns out that in a hyperbolic space X, essentially only one of those six cross-differences contains a significant geometric information encoded in the unordered Q – another one is obtained by reversing the sign and four others are inessential.

In this chapter, we define a cross-difference in another way which does not depend on the ordering of the underlying four points. For hyperbolic spaces this new cross-difference contains essentially the same information as the classical one, but is often easier to handle.

In the previous section, we have already observed the following. Given a quadruple $Q = (x, y, z, u)$ in a metric space X, the expression $(x|y)_o + (z|u)_o - (x|z)_o - (y|u)_o$ is independent of the choice of a base point $o \in X$, actually, it coincides with the classical cross-difference $\langle x, y, z, u\rangle$. This expression has an interpretation in the spirit of the Tetrahedron Lemma. Consider the quadruple Q as an abstract tetrahedron with vertices x, y, z, u. Every edge of Q is labelled by the Gromov product of its vertices with respect to o. If we fix two different pairs of opposite edges of Q, then the expression above is the difference of the sums of the labels attached to the appropriate edges.

To eliminate an ordering of the quadruple points in this expression, we proceed as follows. For a quadruple $Q = (x, y, z, u)$ of points in a metric space X, we form the cross-difference triple $A = A(Q) = ((x|y)_o + (z|u)_o, (x|z)_o + (y|u)_o, (x|u)_o + (y|z)_o)$ as in the Tetrahedron Lemma. Every element a from A corresponds to a unique pair of opposite edges of Q.

From the previous discussion, we obviously have the following fact which plays a fundamental role in the sequel.

Theorem 4.2.1. *Given a quadruple Q of points in a metric space X, the cross-difference triples $A_o(Q)$ and $A_{o'}(Q)$ with respect to different base points $o, o' \in X$ differ from each other by the same constant in each of their entries.* $\square$

Every classical cross-difference of the unordered Q is a difference of two distinct members of A, and all together, we have six such differences, three nonnegative and three opposite to them. However, in a δ-hyperbolic space, A is a δ-triple, which means that only one out of the six differences, namely, the maximal one, contains a geometrically significant information. Hence, the definition of a strongly PQ-isometric map between hyperbolic spaces puts a lot of excessive conditions which are satisfied automatically. This is the reason for the following consideration.

We define the *cross-difference* of Q,

$$\mathrm{cd}(Q) = \max_{a,a' \in A} (a - a').$$

As above, the cross-difference is independent of the choice of a base point $o \in X$ and it is an invariant of the unordered Q.

Geometrically, the best way to understand the cross-difference in a hyperbolic space is to consider the case when X is a tree.

Example 4.2.2. If Q is a quadruple of points in a metric tree X, then $\mathrm{cd}(Q)$ is the maximal distance between opposite pairs of edges of the (degenerate) tetrahedron in X spanned by Q.

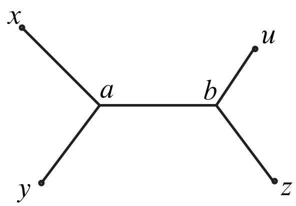


Figure 4.1. $\mathrm{cd}(Q) = |ab| = \mathrm{dist}(xy, zu)$.

In Section 2.4.1 we already defined the small cross-difference of Q, $\mathrm{scd}(Q)$, as the distance between the two smaller entries of the cross-difference triple $A(Q)$. By definition, both entries are nonnegative, $\mathrm{cd}(Q) \geq \mathrm{scd}(Q) \geq 0$ for every quadruple $Q \subset X$. Obviously, $A(Q)$ is, up to a constant, uniquely determined by the two numbers $\mathrm{cd}(Q)$ and $\mathrm{scd}(Q)$.

Lemma 4.2.3. *Let $f : X \to Y$ be a strongly (c, d)-PQ-isometric map. Then*

$$\tfrac{1}{c}\,\mathrm{cd}(Q) - d \leq \mathrm{cd}(Q') \leq c\,\mathrm{cd}(Q) + d$$

and

$$\tfrac{1}{c}\,\mathrm{scd}(Q) - d \leq \mathrm{scd}(Q') \leq c\,\mathrm{scd}(Q) + d$$

for every quadruple $Q \subset X$ with $Q' = f(Q)$.

Proof. Consider a quadruple $Q = (x, y, z, u)$ of points in X. We can assume that $\mathrm{cd}(Q) = \langle x, y, z, u\rangle \geq 0$. Then

$$\mathrm{cd}(Q') \geq \langle x', y', z', u'\rangle \geq \tfrac{1}{c}\langle x, y, z, u\rangle - d = \tfrac{1}{c}\,\mathrm{cd}(Q) - d.$$

In order to prove the estimate from above for $\mathrm{cd}(Q')$, we can assume that $\mathrm{cd}(Q') = \langle x', y', z', u'\rangle \geq 0$. In the case $\langle x, y, z, u\rangle \geq 0$ we obtain

$$\mathrm{cd}(Q') \leq c\langle x, y, z, u\rangle + d \leq c\,\mathrm{cd}(Q) + d.$$

In the case $\langle x, y, z, u\rangle < 0$ the estimate is trivial

$$\mathrm{cd}(Q') \le \frac{1}{c}\langle x, y, z, u\rangle + d < d \le c\,\mathrm{cd}(Q) + d.$$

Similar argument proves the estimates for small cross-differences. We leave this as an exercise to the reader. □

Since hyperbolicity is equivalent to the condition '$\mathrm{scd}(Q)$ is uniformly bounded for all quadruples Q', it is now evident that strongly PQ-isometric maps preserve hyperbolicity, compare Proposition 4.1.2.

4.3 PQ-isometric maps

In Section 4.1, we already introduced the notion of strongly PQ-isometric maps. We now take another look at this class of maps using the new notion of cross-difference. Maps between metric spaces are PQ-isometric if they perturb the cross-differences of quadruples in a bilipschitz manner. This is a weaker condition than to be strongly PQ-isometric. However, every PQ-isometric map is quasi-isometric, and this is in general stronger than the property of being quasi-isometric. Moreover, every PQ-isometric map between hyperbolic spaces naturally induces a map between their boundaries at infinity, which is not true necessarily for quasi-isometric maps between hyperbolic spaces.

Definition 4.3.1. We say that a map $f : X \to X'$ between metric spaces is *PQ-isometric* if there are constants $c \ge 1$, $d \ge 0$ such that

$$\tfrac{1}{c}\,\mathrm{cd}(Q) - d \le \mathrm{cd}(Q') \le c\,\mathrm{cd}(Q) + d$$

for every quadruple Q of points in X, where $Q' = f(Q)$. In this case we say that f is (c, d)-PQ-isometric.

One easily checks that the composition of PQ-isometric maps is PQ-isometric. It follows from Lemma 4.2.3 that every strongly PQ-isometric map is PQ-isometric. Furthermore, we have

Proposition 4.3.2. *Let $f : X \to X'$ be a (c, d)-PQ-isometric map. Then f is (c, d)-quasi-isometric and moreover*

$$\tfrac{1}{c}(x|y)_o - d \le (x'|y')_{o'} \le c(x|y)_o + d \tag{4.1}$$

for every $x, y, o \in X$, where 'prime' stands for the image under f. In particular, any $(1, 0)$-PQ-isometric map is isometric.

Proof. Taking the quadruple $Q = (x, x, o, o)$, we obtain that its cross-difference triple is $A = (|xo|, 0, 0)$. Thus

$$\tfrac{1}{c}|xo| - d \le |x'o'| \le c|xo| + d$$

for every $x, o \in X$, i.e., f is (c, d)-quasi-isometric. Similarly, the cross-difference triple of the quadruple $Q = (x, y, o, o)$ is $A = ((x|y)_o, 0, 0)$, and we obtain the inequalities (4.1). □

Corollary 4.3.3. *Every PQ-isometric map $f : X \to Y$ between hyperbolic spaces naturally induces a map $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$ between their boundaries at infinity.*

Proof. If $\xi \in \partial_\infty X$ and a sequence $\{x_i\} \in \xi$, then it follows from Proposition 4.3.2 that the sequence $\{f(x_i)\}$ converges to infinity. If $\{y_i\} \in \xi$ is another sequence, then similarly $\{f(x_i)\}$ and $\{f(y_i)\}$ are equivalent. Letting $\partial_\infty f(\xi)$ be the class $\eta \in \partial_\infty Y$ of $\{f(x_i)\}$, we obtain a well-defined map $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$. The map $\partial_\infty f$ is natural in the sense that $\partial_\infty(g \circ f) = \partial_\infty g \circ \partial_\infty f$ for PQ-isometric $g : Y \to Z$, where Z is a hyperbolic space. □

In Section 5.2.1 we give more information about the induced map $\partial_\infty f$; see Proposition 5.2.10.

4.4 Quasi-isometric maps of hyperbolic geodesic spaces

It is not clear whether a PQ-isometric map between general hyperbolic spaces is strongly PQ-isometric (see, however, Remark 4.5.6). But in the case of hyperbolic geodesic spaces we have a much stronger property.

Theorem 4.4.1. *Let $f : X \to X'$ be a (c, b)-quasi-isometric map of hyperbolic geodesic spaces. Then there is a constant $d \ge 0$ depending only on c, b and the hyperbolicity constants δ, δ' of X, X' such that f is strongly (c, d)-PQ-isometric and in particular (c, d)-PQ-isometric.*

As a consequence we obtain

Corollary 4.4.2. *Let $f : X \to X'$ be a map between hyperbolic geodesic spaces. The following are equivalent*:

(a) *f is quasi-isometric*;

(b) *f is PQ-isometric*;

(c) *f is strongly PQ-isometric.* □

The proof of Theorem 4.4.1 is based on the stability of geodesics in hyperbolic geodesic spaces. We first study the deviation from equiradial points. Let xyz be a geodesic triangle in a δ-hyperbolic geodesic space X. Then for the equiradial points

$u_0 \in yz$, $v_0 \in xz$, $w_0 \in xy$ (see Lemma 1.2.1), we have $|u_0v_0|$, $|v_0w_0|$, $|u_0w_0| \le \delta$ by the definition of δ-hyperbolicity. We show that the converse is true in the following sense.

Lemma 4.4.3. *Let xyz be a geodesic triangle with equiradial points $u_0 \in yz$, $v_0 \in xz$, $w_0 \in xy$ in a δ-hyperbolic geodesic space X. Then for points $u \in yz$, $v \in xz$, $w \in xy$ with $|uv|, |vw|, |uw| \le h$, we have $|uu_0|, |vv_0|, |ww_0| \le 2(\delta + h)$.*

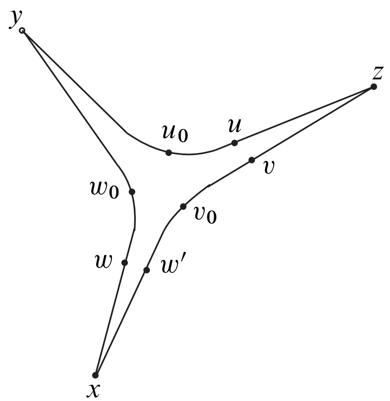


Figure 4.2. Estimating distances to equiradial points.

Proof. Without loss of generality, we can assume that $w \in xw_0$ and $v \in zv_0$. Then for $w' \in xv_0$ with $|xw'| = |xw|$ we have

$$|ww_0| = |w'v_0| \le |w'v| \le |w'w| + |wv| \le \delta + h.$$

Thus $|vv_0| \le |vw'| \le \delta + h$, and $|uu_0| \le |uw| + |ww_0| + |w_0u_0| \le 2(\delta + h)$. Hence the claim. $\square$

Proof of Theorem 4.4.1. Fix $o, x, y, z \in X$, and put $o' = f(o)$, $x' = f(x)$, $y' = f(y), z' = f(z)$. Taking o as a base point, we have $\langle x, y, z, o\rangle = (x|y)_o - (x|z)_o = s$ and similarly $\langle x', y', z', o'\rangle = (x'|y')_{o'} - (x'|z')_{o'} = s'$. It suffices to check that

$$\min\{cs, s/c\} - d \le s' \le \max\{cs, s/c\} + d$$

for an appropriate $d \ge 0$.

We first show that $|f(u)u'| \le d$ for the equiradial points $u \in xy$, $u' \in x'y'$ of the triangles oxy, $o'x'y'$ respectively.

We take the other equiradial points $v \in ox$, $w \in oy$ and consider their images $f(v)$, $f(w)$. Then $|f(u)f(v)|, |f(v)f(w)|, |f(u)f(v)| \le c\delta + b$. By the stability of geodesics, there are $\bar{u}' \in x'y'$, $\bar{v}' \in o'x'$, $\bar{w}' \in o'y'$ with $|\bar{u}'f(u)|$, $|\bar{v}'f(v)|$,

$|\bar{w}'f(w)| \le H$, where $H = H(c, b, \delta')$. Thus $|\bar{u}'\bar{v}'|$, $|\bar{v}'\bar{w}'|$, $|\bar{u}'\bar{w}'| \le h$, where $h = c\delta + b + 2H$. By Lemma 4.4.3, $|\bar{u}'u'| \le 2(\delta' + h)$ and hence $|f(u)u'| \le d = 2(\delta' + h) + H$.

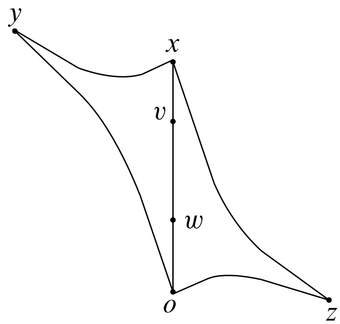


Figure 4.3. Estimating the difference of Gromov products.

Consider the equiradial points $v, w \in ox$ for the triangles oxy, oxz respectively, and the equiradial points $v', w' \in o'x'$ for the triangles $o'x'y'$, $o'x'z'$ respectively. Then $|s| = |vw|$ and $|s'| = |v'w'|$. The estimates $|f(v)v'|, |f(w)w'| \le d$ imply

$$\frac{1}{c}|s| - (b + 2d) \le |s'| \le c|s| + (b + 2d). \tag{4.2}$$

To obtain the estimate for s' itself we assume first that $s = |xv| - |xw| \ge 0$. If $s' \ge 0$ then estimate (4.2) gives the desired estimate for s and s'. Assume now that $s' < 0$. We still have $s' \le |s'| \le cs + (b + 2d)$. By the stability of geodesics, $f(w)$ lies within distance $\le H$ from a geodesic $o'f(v)$, which in particular implies $|o'f(w)| \le |o'f(v)| + H$ and hence $s' = |o'v'| - |o'w'| \ge -(2d + H)$. It follows that

$$s' = |s'| - 2|s'| \ge \frac{1}{c}s - (b + 2d) - 2(2d + H).$$

The case $s < 0$ is similar. □

4.5 Supplementary results and remarks

4.5.1 Cross-difference in X based at infinity

Assume that X is a δ-hyperbolic space, and let Q be a quadruple of points in X. Given $\omega \in \partial_\infty X$, for any Busemann function $b \in \mathcal{B}(\omega)$, we have the cross-difference triple $A = A_b(Q)$ based at b which is defined as above replacing the base point o by the Busemann function b. By the same argument as for Theorem 4.2.1 we obtain that two cross-difference triples $A_b(Q)$, $A_{b'}(Q)$, based either in X or at Busemann functions, differ from each other by a constant. In particular, the cross-difference based at b,

$$\mathrm{cd}_b(Q) = \max_{a, a' \in A}(a - a'),$$

also depends neither on $b \in \mathcal{B}(\omega)$ nor on $\omega \in \partial_\infty X$,

$$\mathrm{cd}_b(Q) = \mathrm{cd}(Q).$$

4.5.2 Cross-differences at infinity

There are several possibilities to extend the cross-difference to quadruples of points at infinity. We always assume that such quadruples consist of distinct points to ensure that cross-differences are well defined and finite.

We first consider the case when a base point $o \in X$ is fixed. Given a quadruple $Q = (\alpha, \beta, \gamma, \zeta)$ of distinct points in $\partial_\infty X$, we form the cross-difference triple $A = ((\alpha|\beta)_o + (\gamma|\zeta)_o, (\alpha|\gamma)_o + (\beta|\zeta)_o, (\alpha|\zeta)_o + (\beta|\gamma)_o)$ as above and define the cross-difference

$$\mathrm{cd}_o(Q) = \max_{a,a' \in A}(a - a').$$

This cross-difference may depend on the choice of a base point. For another base point $o' \in X$ the cross-differences $\mathrm{cd}_o(Q)$ and $\mathrm{cd}_{o'}(Q)$ differ from each other by at most 10δ. This follows from Lemma 2.2.2 (1).

Next we fix $\omega \in \partial_\infty X$, $o \in X$ and for a quadruple Q of distinct points α, β, γ, $\zeta \in \partial_\infty X$, we form the cross-difference triple $A = ((\alpha|\beta)_{\omega,o} + (\gamma|\zeta)_{\omega,o}, (\alpha|\gamma)_{\omega,o} + (\beta|\zeta)_{\omega,o}, (\alpha|\zeta)_{\omega,o} + (\beta|\gamma)_{\omega,o})$ with respect to the Gromov product $(\alpha|\beta)_{\omega,o} = (\alpha|\beta)_o - (\omega|\alpha)_o - (\omega|\beta)_o$; see Section 3.2.2. Then the cross-difference

$$\mathrm{cd}_{\omega,o}(Q) = \max_{a,a' \in A}(a - a')$$

is independent of $\omega \in \partial_\infty X$, $\mathrm{cd}_{\omega,o}(Q) = \mathrm{cd}_o(Q)$. For different reference points $o, o' \in X$, the corresponding cross-differences differ from each other by at most 10δ as above. We do not exclude the case when one of the quadruple points coincides with ω, for example, $\zeta = \omega$. In this case, all members of the triple A contain a summand $-\infty$, and because the cross-difference depends only on differences of the members of A, we simply cancel out these $(-\infty)$-summands. So we have

$$\mathrm{cd}_{\omega,o}(Q) = \max_{a,a' \in A}(a - a'),$$

where $A = ((\alpha|\beta)_{\omega,o}, (\alpha|\gamma)_{\omega,o}, (\beta|\gamma)_{\omega,o})$. In this case, the cross-difference becomes the ordinary difference.

Alternatively, fix $\omega \in \partial_\infty X$ and a Busemann function $b \in \mathcal{B}(\omega)$. Given a quadruple Q of distinct points $\alpha, \beta, \gamma, \zeta \in \partial_\infty X$, we form the cross-difference triple $A = ((\alpha|\beta)_b + (\gamma|\zeta)_b, (\alpha|\gamma)_b + (\beta|\zeta)_b, (\alpha|\zeta)_b + (\beta|\gamma)_b)$ and define the cross-difference based at b of the quadruple by

$$\mathrm{cd}_b(Q) = \max_{a,a' \in A}(a - a').$$

It follows from Chapter 3, equation (3.4) that for $b = b_{\omega,o} \in \mathcal{B}(\omega)$ the approximate equality

$$\mathrm{cd}_b(Q) \doteq \mathrm{cd}_{\omega,o}(Q)$$

holds up to an error $\le 8\delta$ for each quadruple Q of distinct points in $\partial_\infty X$.

Proposition 4.5.1. *Let X be a δ-hyperbolic space. For every quadruple Q of distinct points in $\partial_\infty X$, the cross-difference triples based at points $o \in X$ or at Busemann functions $b \in \mathcal{B}(\omega)$, where $\omega \in \partial_\infty X$, differ from each other by a constant up to a uniform error $\le c\delta$ for some $c > 0$. In particular, the corresponding cross-differences differ from each other by at most $c\delta$ for every quadruple of distinct points in $\partial_\infty X$.*

Proof. The proof is straightforward using Theorem 4.2.1, Lemma 2.2.2 and Lemma 3.2.4. We leave details as an exercise to the reader. □

In the case X is boundary continuous, all cross-differences at infinity defined above coincide.

Example 4.5.2. Assume that X is a tree. Then for every quadruple Q of distinct points in $\partial_\infty X$, the cross-difference $\mathrm{cd}(Q)$ is equal to the maximal distance between infinite geodesics in X representing opposite edges of the (abstract) tetrahedron with the quadruple vertices, cf. Example 4.2.2.

4.5.3 Cross-pairs

Let Q be a quadruple of points in a metric space X, and let $A = A(Q)$ be its cross-difference triple (with respect to a base point $o \in X$). Every member $a \in A$ corresponds to a pair of opposite edges of Q considered as an abstract tetrahedron. Such a pair is called an (additive) *cross-pair* of Q if a is maximal (this is independent of the base point o). This notion is motivated by the obvious fact that for any $a, a' \in A$ with $\mathrm{cd}(Q) = a - a'$, the element a defines a cross-pair of Q.

In general, even a strongly PQ-isometric map may not preserve cross-pairs. Here is an example.

Example 4.5.3. Consider a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is the identity outside of a ball of radius $\varepsilon \in (0,1)$ around the origin but moves the origin, $f(0,0) = (\varepsilon/2, 0)$. That is, f is a perturbation of the identity map. The identity map is certainly strongly PQ-isometric. Since the property to be strongly PQ-isometric is a coarse property, f is also strongly PQ-isometric.

Now take a large D, consider the quadruple $Q_0 = (o, x, y, z)$ with $o = (0,0)$, $x = (D, 0)$, $y = (0, D)$, $z = (D, D)$, and define Q to be the deformed quadruple, where we move the vertex x to $x' = x + (\eta, 0)$ with $\eta \ll \varepsilon$. One easily computes that the cross-difference, $\mathrm{cd}(Q_0) = (\sqrt{2} - 1)D$, is large; however, there are two cross-pairs of Q_0, namely, (xz, oy) and (yz, ox). The perturbation removes this degeneration, and there is only one cross-pair of Q, namely $(x'z, oy)$. The cross-pair of $f(Q)$ is also unique; however, it is now $(f(y)f(z), f(x')f(o))$ because $\eta \ll \varepsilon$. Therefore, f does not preserve cross-pairs.

This phenomenon is related to the fact that $\mathbb{R}^2$ is not hyperbolic. In the case the target space is hyperbolic, we have a remarkable addition to Lemma 4.2.3.

Proposition 4.5.4. *Assume that a map $f: X \to X'$ is strongly (c,d)-PQ-isometric and the space X' is δ'-hyperbolic (we do not require that X is hyperbolic). Then f is a PQ-isometric map preserving cross-pairs of quadruples $Q \subset X$ with* $\mathrm{cd}(Q) > c(d+\delta')$.

Proof. It follows from Lemma 4.2.3 that f is PQ-isometric. Consider a quadruple $Q = (x,y,z,u) \subset X$.

Assume that $\mathrm{cd}(Q) > c(d+\delta')$ and $\mathrm{cd}(Q) = \langle x,y,z,u\rangle$. Then

$$\langle x',y',z',u'\rangle \geq \tfrac{1}{c}\,\mathrm{cd}(Q) - d > \delta'.$$

Therefore, $\mathrm{cd}(Q') = \langle x',y',z',u'\rangle$ because the cross-difference triple A' of $Q' = f(Q)$ is a δ'-triple. We conclude that f preserves cross-pairs of Q. □

Combining this with Theorem 4.4.1, we obtain

Corollary 4.5.5. *Every quasi-isometric map $f: X \to X'$ between hyperbolic geodesic spaces is PQ-isometric preserving cross-pairs of quadruples $Q \subset X$ with quantitatively large cross-difference* $\mathrm{cd}(Q)$. □

Remark 4.5.6. Vice versa, one can show that any PQ-isometric map $f: X \to X'$ between hyperbolic spaces (not necessarily geodesic), which preserves cross-pairs of quadruples $Q \subset X$ with sufficiently large $\mathrm{cd}(Q)$, is strongly PQ-isometric. We leave this as an exercise to the reader.

Bibliographical note. It follows from Theorem 4.4.1 and Corollary 4.3.3 that every quasi-isometric map $f: X \to Y$ between hyperbolic geodesic spaces naturally induces a map between their boundaries at infinity. This extension property was discovered by V. A. Efremovich and E. S. Tihomirova in the case $X = Y = \mathrm{H}^n$; see [ET].

Our approach to the extension property of quasi-isometric maps of hyperbolic geodesic spaces via strongly PQ-isometric maps, Theorem 4.4.1, is similar to that from [BoS, Proposition 5.5] and [V2, Theorem 3.21], where, however, neither strongly PQ-isometric, nor PQ-isometric maps are explicitly introduced.

The second example of Remark 4.1.3 (2) is taken from [V2, Remark 3.19].

Chapter 5

Quasi-Möbius and quasi-symmetric maps

The goal of this chapter is to study properties of maps between boundaries at infinity of hyperbolic spaces induced by PQ-isometric maps between the spaces themselves. In other words, we generalize the classical result that every isometry of H^n induces a Möbius map of $\partial_\infty \mathrm{H}^n$ to arbitrary hyperbolic spaces.

5.1 Cross-ratios

A cross-ratio is the multiplicative version of a cross-difference. Let X be a δ-hyperbolic space. We fix $a > 1$ and define the cross-ratios $\mathrm{cr}_o(Q) = a^{-\mathrm{cd}_o(Q)}$ for a base point $o \in X$ and $\mathrm{cr}_b(Q) = a^{-\mathrm{cd}_b(Q)}$ for a Busemann function $b \in \mathcal{B}(\omega)$, $\omega \in \partial_\infty X$, where Q is a quadruple of distinct points in $\partial_\infty X$. These cross-ratios may depend on the choice of o or b respectively. However, by Proposition 4.5.1, such a dependence is completely controlled by the hyperbolicity constant δ.

More generally, assume that (Z, ρ) is a quasi-metric space with infinitely remote set $Z_\infty \subset Z$, $|Z_\infty| \le 1$.

Definition 5.1.1. Given a quadruple Q of distinct points $a, b, c, d \in Z$, we call the triple

$$M = (\rho(a,b)\rho(c,d), \rho(a,c)\rho(b,d), \rho(a,d)\rho(b,c)),$$

formed by the products of distances attached to pairs of opposite edges of Q, the *cross-ratio triple* of Q, and define the *cross-ratio*

$$\mathrm{cr}_\rho(Q) = \min_{m,m' \in M} m/m'.$$

Speaking about the cross-ratio of a quadruple Q of points in a quasi-metric space (Z, ρ) we always mean this notion with respect to ρ and usually omit ρ from its notation, $\mathrm{cr}(Q) = \mathrm{cr}_\rho(Q)$.

In the case when one of the quadruple points is at infinity, every member of M contains an infinite factor. In this case we cancel out every such factor, e.g., if $d \in Z_\infty$ then we put $M = (\rho(a,b), \rho(a,c), \rho(b,c))$ and define the cross-ratio $\mathrm{cr}(Q)$ as above. The cross-ratio then becomes the ordinary ratio.

The cross-ratio triple M from Definition 5.1.1 possesses a remarkable property which is the multiplicative version of the property to be a δ-triple. We say that a triple

$M = (a, b, c)$ of positive reals is a *multiplicative K-triple*, where $K \geq 1$, if the two largest members of M, say a and b, coincide up to a multiplicative error $\leq K$,

$$\frac{1}{K} \leq \frac{a}{b} \leq K.$$

As a shorthand for this, we use the notation $a \asymp b$ up to a multiplicative error $\leq K$, or $a \asymp_K b$.

For example, given distinct points a, b, c in a K-quasi-metric space (Z, ρ), the triple $M = (\rho(a, b), \rho(b, c), \rho(a, c))$ is a multiplicative K-triple. This follows from the K-ultra-metric triangle inequality for ρ. In particular, if ρ is a metric, then M is a multiplicative 2-triple, and if ρ is an ultra-metric, then M is a multiplicative 1-triple.

There is a multiplicative analog of the Tetrahedron Lemma.

Lemma 5.1.2. *Assume that ρ is a K-quasi-metric on Z, $K \geq 1$. Then for every quadruple Q of distinct points of Z, the cross-ratio triple M of Q is a multiplicative K^2-triple.*

Proof. If one of the quadruple points is in Z_∞, then M is a multiplicative K-triple (after we cancel out the infinite factors). Thus, we assume the quadruple contains no point at infinity.

The numbers $(a|b) = -\ln \rho(a, b)$, $a, b \in Z$, satisfy the δ-inequality with $\delta = \ln K$. Applying the Tetrahedron Lemma, we see that $A = -\ln M$ is a 2δ-triple. Hence, M is a multiplicative K^2-triple. $\square$

5.2 Quasi-Möbius and quasi-symmetric maps

In classical hyperbolic geometry, a Möbius map is a composition of finitely many inversions of the extended Euclidean space $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$, $\widehat{\mathbb{R}}^n = \partial_\infty \mathrm{H}^{n+1}$. Such a map is characterized by the property to preserve the classical cross-ratio; see Theorems A.7.1 and A.7.2 in the appendix.

We extend the notion of a Möbius map as follows.

5.2.1 Quasi-Möbius and PQ-Möbius maps

Let Q be a quadruple of distinct points in a quasi-metric space Z and let $M = M(Q)$ be its cross-ratio triple. Recall that every member $m \in M$ corresponds to a pair of opposite edges of Q. A (multiplicative) *cross-pair* of Q is the pair of opposite edges of Q selected by a minimal element of M. We use the notation $\mathrm{cp}(Q)$ for a cross-pair of Q even in the case that the cross-pair is not unique.

Remark 5.2.1. In Chapter 4 we introduced the notion of an additive cross-pair; see Section 4.5.3. It is always clear from the context whether we speak about the additive or the multiplicative cross-pair. If we consider quadruples of points in a (hyperbolic) space, we take the additive viewpoint. If we consider quadruples in the boundary at infinity, then we take the multiplicative viewpoint.

By Lemma 5.1.2, there is $K \geq 1$ such that $M(Q)$ is a multiplicative K^2-triple for every quadruple Q of distinct points in Z. Thus if $\mathrm{cr}(Q) < K^{-2}$ then the cross-pair of Q is uniquely determined.

Definition 5.2.2. An injective map $f\colon (Z,\rho) \to (Z',\rho')$ between quasi-metric spaces is said to be *quasi-Möbius* if there is a homeomorphism $\theta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{1}{\theta(1/\mathrm{cr}(Q))} \leq \mathrm{cr}(f(Q)) \leq \theta(\mathrm{cr}(Q))$$

for every quadruple Q of distinct points in Z.

In this case, we say that f is θ-quasi-Möbius, the function θ is called the *control function* of f.

Definition 5.2.3. The map f is called *strictly quasi-Möbius* if in addition f *eventually preserves cross-pairs*, that is, there is a constant $h \in (0,1)$ such that $\mathrm{cp}(f(Q)) = f(\mathrm{cp}(Q))$ for every quadruple $Q \subset Z$ of distinct points with $\mathrm{cr}(Q) \leq h$.

In this case, we say that the constant h is a *threshold constant*.

Remark 5.2.4. The cross-pair condition in Definition 5.2.3 is motivated by the fact that the boundary map of every strongly PQ-isometric map between hyperbolic spaces satisfies it; see Proposition 5.2.10. Furthermore, this condition is crucial for the proof of the extension theorems in Chapter 7, proving that every quasi-symmetric or quasi-Möbius map between uniformly perfect metric spaces is the boundary map of a quasi-isometric map between appropriate hyperbolic geodesic spaces.

Remark 5.2.5. Consider the following example. Take a four-point space $Z = \{x,y,z,u\}$ with distances $|xy| = |zu| = l$ and $|xz| = |xu| = |yz| = |yu| = L$ for some l, L with $L > l > 0$. This defines a metric on Z. Next, take $Z' = \{x',y',z',u'\}$ with distances $|x'z'| = |y'u'| = l$, and all other distances equal to L. Now define $f\colon Z \to Z'$ by $f(x) = x'$, $f(y) = y'$, $f(z) = z'$, $f(u) = u'$. Then $\mathrm{cr}(Q') = \mathrm{cr}(Q) = (l/L)^2$ for the unique quadruple $Q = Z$ of distinct points in Z, $Q' = f(Q)$, while $\mathrm{cp}(Q) = (xy, zu)$ and $\mathrm{cp}(Q') = (x'z', y'u')$. That is, f does not preserve the (unique) cross-pair of Q, see Figure 5.1. Nevertheless f is strictly quasi-Möbius (by a purely logical reason).

Moreover, we prove in Section 5.3.3 that any quasi-Möbius map of a uniformly perfect space is strictly quasi-Möbius; see Proposition 5.3.7.

Exercise 5.2.6. Show that every quasi-Möbius map $f\colon (Z,\rho) \to (Z',\rho')$ is continuous.

Clearly, the inverse to a (strictly) quasi-Möbius map is (strictly) quasi-Möbius and the composition of (strictly) quasi-Möbius maps is (strictly) quasi-Möbius.

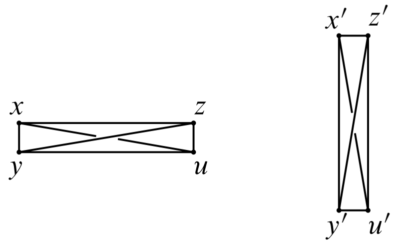


Figure 5.1. The cross-pair is not preserved.

Definition 5.2.7. For the control function $\theta(t) = q\max\{t^p, t^{\frac{1}{p}}\}$ with $p, q \geq 1$, a (strictly) quasi-Möbius map is called (*strictly*) *power quasi-Möbius* or (*strictly*) *PQ-Möbius*. That is, an injective map $f\colon (Z,\rho) \to (Z',\rho')$ is (strictly) PQ-Möbius if

$$\frac{1}{q}\,\mathrm{cr}(Q)^p \leq \mathrm{cr}(f(Q)) \leq q\,\mathrm{cr}(Q)^{\frac{1}{p}}$$

for every quadruple Q of distinct points in Z (and if f eventually preserves cross-pairs). In this case, we say that f is (strictly) (p,q)-PQ-Möbius.

This definition is motivated by the fact that any (strongly) PQ-isometric map of hyperbolic spaces induces a (strictly) PQ-Möbius map of their boundaries at infinity; see Proposition 5.2.10.

Let X be a hyperbolic space and let b be either a point in X or a Busemann function in $\mathcal{B}(\omega)$, where $\omega \in \partial_\infty X$. Recall that to a parameter $a > 1$, we associate a quasi-metric ρ_b on $\partial_\infty X$, $\rho_b(\alpha,\beta) = a^{-(\alpha|\beta)_b}$, based at b. We already know that quasi-metrics ρ_b, $\rho_{b'}$ based at different points $b, b' \in X$ are bilipschitz to each other, see Proposition 2.2.8. Now we are able to describe the effect which occurs in general when we change the base of ρ_b.

Proposition 5.2.8. *Let X be a δ-hyperbolic space. There exists a constant $q \geq 1$ which depends only on δ such that for any two quasi-metrics ρ_b, $\rho_{b'}$ on $\partial_\infty X$ with one and the same parameter $a > 1$, based in X or at infinity, the identity map* $\mathrm{id}\colon (\partial_\infty X, \rho_b) \to (\partial_\infty X, \rho_{b'})$ *is strictly $(1,q)$-PQ-Möbius.*

Proof. By Proposition 4.5.1, we have $\mathrm{cd}_b(Q) \doteq \mathrm{cd}_{b'}(Q)$ up to an error $\leq c\delta$ for every quadruple Q of distinct points in $\partial_\infty X$. Thus

$$\tfrac{1}{q}\,\mathrm{cr}_b(Q) \leq \mathrm{cr}_{b'}(Q) \leq q\,\mathrm{cr}_b(Q),$$

where $q = a^{c\delta}$. Furthermore, it also follows from Proposition 4.5.1 that id eventually preserves cross-pairs. □

Corollary 5.2.9. *Under the conditions of Proposition 5.2.8, the identity map* $\mathrm{id}\colon (\partial_\infty X, d) \to (\partial_\infty X, d')$ *is strictly $(1,q)$-PQ-Möbius for any visual metrics d, d' on $\partial_\infty X$, where the constant $q \geq 1$ depends on d, d'.* □

Proposition 5.2.10. *For every (strongly) PQ-isometric map $f: X \to X'$ between hyperbolic spaces, the induced map $\partial_\infty f: \partial_\infty X \to \partial_\infty X'$ is (strictly) PQ-Möbius quantitatively with respect to any visual metrics on $\partial_\infty X$, $\partial_\infty X'$ with base points in the corresponding spaces or in their boundaries at infinity.*

Proof. Fix $o \in X$ or a Busemann function b based at $\omega \in \partial_\infty X$. For the respective Gromov products, we use for brevity the same subscript ω. Let $Q = (\alpha, \beta, \gamma, \zeta)$ be a quadruple of distinct points in $\partial_\infty X$, $\{x_i\} \in \alpha$, $\{y_i\} \in \beta$, $\{z_i\} \in \gamma$, $\{u_i\} \in \zeta$, $Q_i = (x_i, y_i, z_i, u_i)$. Using Lemmas 2.2.2 and 3.2.4, we obtain

$$\mathrm{cd}_\omega(Q) - 4\delta \le \liminf_i \mathrm{cd}(Q_i) \le \limsup_i \mathrm{cd}(Q_i) \le \mathrm{cd}_\omega(Q) + 4\delta$$

for some $\delta \ge 0$ depending on the hyperbolicity constant of X (because the cross-difference of Q_i is independent of a base point, we omit the subscript ω from the notations). Similar estimates hold for images Q', Q'_i of the quadruples. By Proposition 4.5.4, f eventually preserves cross-pairs (for cross-difference triples) if it is strongly PQ-isometric. This implies the claim. We leave details to the reader. □

5.2.2 Quasi-symmetric and power quasi-symmetric maps

Definition 5.2.11. A map $f: X \to Y$ between metric spaces is called *quasi-symmetric* if it is not constant and if there is a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that from $|xa| \le t|xb|$ it follows that $|f(x)f(a)| \le \eta(t)|f(x)f(b)|$ for any a, b, $x \in X$ and all $t \ge 0$. In this case, we say that f is *η-quasi-symmetric*. The function η is called the *control function* of f.

Exercise 5.2.12. Show that any quasi-symmetric map $f: X \to Y$ is injective and continuous.

Lemma 5.2.13. *If $f: X \to Y$ is η-quasi-symmetric, then $f^{-1}: f(X) \to X$ is η'-quasi-symmetric, where $\eta'(t) = 1/\eta^{-1}(t^{-1})$ for $t > 0$. Moreover, if $f: X \to Y$ and $g: Y \to Z$ are η_f- and η_g-quasi-symmetric respectively, then $g \circ f: X \to Z$ is $(\eta_g \circ \eta_f)$-quasi-symmetric.*

Proof. Assume that $|xa| \ge s|xb|$. Then $|xb| \le (1/s)|xa|$, thus $|x'b'| \le \eta(1/s)|x'a'|$ and $|x'a'| \ge (1/\eta(1/s))|x'b'|$. Now if $|x'a'| < (1/\eta(1/s))|x'b'|$, then $|xa| < s|xb|$, since the opposite inequality would contradict the previous computation. We put $t = 1/\eta(1/s)$. Then $1/s = \eta^{-1}(1/t)$, and $s = 1/\eta^{-1}(1/t)$. By continuity, the inequality $|x'a'| \le t|x'b'|$ implies $|xa| \le \eta'(t)|xb|$, where $\eta'(t) = 1/\eta^{-1}(1/t)$. The last assertion is obvious. □

A quasi-symmetric map is said to be *power quasi-symmetric*, or PQ-symmetric, if its control function is of the form

$$\eta(t) = q \max\{t^p, t^{1/p}\}$$

for some $p, q \ge 1$.

Exercise 5.2.14. Show that for every strongly PQ-isometric map $f: X \to X'$ between hyperbolic spaces, the induced map $\partial_\infty f: \partial_\infty X \to \partial_\infty X'$ is PQ-symmetric quantitatively with respect to any visual metrics on $\partial_\infty X$, $\partial_\infty X'$ with base points in the corresponding spaces.

Quasi-symmetric and quasi-Möbius maps are closely related to each other, namely, every quasi-symmetric map is strictly quasi-Möbius (under the mild additional condition on spaces to be uniformly perfect) and every strictly quasi-Möbius map is quasi-symmetric if it preserves the infinitely remote set. In general, these classes of maps are different because quasi-symmetric maps preserve the property of subsets to be bounded, which is definitely not the case for quasi-Möbius maps.

The described relation is not at all trivial, and we start with the easier implication; see the proposition below. The inverse implication is proven as a consequence of the extension theorems in Chapter 7; see Corollary 7.4.2. A remarkable fact is that quasi-symmetric and quasi-Möbius maps with general control functions turn out to be (for uniformly perfect spaces) power quasi-symmetric and power quasi-Möbius respectively. This is also a consequence of the mentioned extension theorems; see Corollaries 7.4.2 and 7.4.3.

Proposition 5.2.15. *Let $f: Z \to Z'$ be a strictly quasi-Möbius map which preserves the infinitely remote set, $f(Z_\infty) = Z'_\infty$ (we assume that Z_∞ is not empty). Then f is quasi-symmetric quantitatively, in particular, any strictly PQ-Möbius f is PQ-symmetric if it preserves the infinitely remote set.*

Proof. The condition for f to be η-quasi-symmetric can be written as

$$\frac{1}{\eta(1/s)} \le s' \le \eta(s)$$

for all distinct $x, a, b \in Z \setminus Z_\infty$, where $s = \rho(x,a)/\rho(x,b)$, and the sign 'prime' stands for the image under f as usual. One easily checks that it suffices to consider the case $s \ge 1$. The cross-ratio triple of the quadruple $Q = (x, a, b, \omega)$ is

$$M = (\rho(x,a), \rho(x,b), \rho(a,b)),$$

where ω is the infinitely remote point of Z.

Let $h \in (0,1)$ be a threshold constant of f. First, consider the case $\operatorname{cr}(Q) \ge h$. Then

$$s' \le 1/\operatorname{cr}(Q') \le \theta(1/\operatorname{cr}(Q)) \le \theta(1/h) \le \theta(s/h^2),$$

where we used the inequalities $s \ge 1, h < 1$. On the other hand, $s \le 1/\operatorname{cr}(Q) \le 1/h$. Thus

$$s' \ge \operatorname{cr}(Q') \ge \frac{1}{\theta(1/\operatorname{cr}(Q))} \ge \frac{1}{\theta(1/h)} \ge \frac{1}{\theta(1/sh^2)}.$$

Therefore, $1/\eta_1(1/s) \le s' \le \eta_1(s)$ for $\eta_1(s) = \theta(s/h^2)$.

Now we consider the case $\operatorname{cr}(Q) < h$. We assume that Z and Z' are K- and K'-quasi-metric spaces respectively. Suppose that $\rho(x,b)$ is minimal in M, $\rho(x,b) \le \rho(x,a), \rho(a,b)$. Then $\operatorname{cr}(Q) \le 1/s \le K \operatorname{cr}(Q)$ (the right-hand inequality holds because M is a multiplicative K-triple). Furthermore, f preserves the cross-pair of Q, which means that $\rho'(x',b')$ is minimal in the triple M'. Thus $\operatorname{cr}(Q') \le 1/s' \le K' \operatorname{cr}(Q')$ because M' is a multiplicative K'-triple. We obtain

$$s' \le 1/\operatorname{cr}(Q') \le \theta(1/\operatorname{cr}(Q)) \le \theta(Ks) \le \eta_2(s)$$

and

$$s' \ge \frac{1}{K' \operatorname{cr}(Q')} \ge \frac{1}{K'\theta(\operatorname{cr}(Q))} \ge \frac{1}{K'\theta(1/s)} \ge \frac{1}{\eta_2(1/s)}$$

for $\eta_2(s) = K'\theta(Ks)$.

It remains to consider the case that $\rho(a,b)$ is minimal in M. Then $\rho'(a',b')$ is minimal in M', and we have $1 \le s \le K$, $1/K' \le s' \le K'$. It follows that $s' \le K'\theta(s)/\theta(1) \le \eta_3(s)$ and

$$s' \ge \frac{1}{K'} \cdot \frac{\theta(K/s)}{\theta(K/s)} \ge \frac{\theta(1)}{K'\theta(K/s)} = \frac{1}{\eta_3(1/s)}$$

for $\eta_3(s) = K'\theta(Ks)/\theta(1)$. Putting $\eta = \max\{\eta_1, \eta_2, \eta_3\}$, we obtain that f is η-quasi-symmetric. $\square$

Remark 5.2.16. This proposition does not cover the case that the spaces Z, Z' are bounded, even if we artificially extend the Q-metrics ρ, ρ' to $Z \cup \{\infty\}$, $Z' \cup \{\infty\}$ respectively and put $f(\infty) = \infty$. The problem is that there is no obvious reason for the extended f to be still quasi-Möbius. We solve this problem in Chapter 7 for the case that ρ, ρ' are uniformly perfect metrics.

The converse, that any quasi-symmetric homeomorphism between uniformly perfect metric spaces is strictly quasi-Möbius, is shown in Chapter 7, Corollary 7.4.2.

Theorem 5.2.17. *Let $f: X \to Y$ be a quasi-isometric map of hyperbolic geodesic spaces. Then f naturally induces a well-defined map $\partial_\infty f: \partial_\infty X \to \partial_\infty Y$ of their boundaries at infinity which is*

(1) *strictly PQ-Möbius with respect to any visual metrics on $\partial_\infty X$, $\partial_\infty Y$ with base points in X, Y or in $\partial_\infty X$, $\partial_\infty Y$ respectively,*

and

(2) *PQ-symmetric with respect to any visual metrics with base points in X, Y or with base points $\omega \in \partial_\infty X$, $\partial_\infty f(\omega) \in \partial_\infty Y$ respectively.*

Proof. By Theorem 4.4.1, f is strongly PQ-isometric (quantitatively). Now using Proposition 5.2.10, we obtain (1). The first part of (2) follows from Exercise 5.2.14. The second part of (2) follows from Proposition 5.2.15 because in this case the strictly PQ-Möbius $\partial_\infty f$ preserves the infinitely remote points by the assumption. $\square$

Corollary 5.2.18. *The hyperbolic spaces* H^n, H^m*,* $n, m \geq 2$*, are not quasi-isometric for* $n \neq m$*.*

Proof. Indeed, any quasi-isometry between H^n and H^m would induce by Theorem 5.2.17 a homeomorphism between their boundary spheres S^{n-1} and S^{m-1} which is impossible for $n \neq m$. □

5.3 Supplementary results and remarks

5.3.1 Möbius structure on a quasi-metric space

An injective map $f : Z \to Z'$ between quasi-metric spaces is called *Möbius* if it preserves the classical cross-ratio.

Exercise 5.3.1. Show that any injective map $f : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ which preserves the cross-ratio, that is, $\mathrm{cr}(f(Q)) = \mathrm{cr}(Q)$ for every quadruple of distinct points $Q \subset \widehat{\mathbb{R}}^n$, is Möbius, cf. Theorem A.7.2.

Clearly, the inverse to a Möbius map is Möbius and the composition of Möbius maps is Möbius. The *Möbius structure* on a quasi-metric space (Z, ρ) is the set of all quasi-metrics on Z which are Möbius equivalent to ρ.

As an example consider a boundary continuous hyperbolic space, see Section 3.4.2. Proposition 5.2.8 is then refined as follows.

Proposition 5.3.2. *Let* X *be a boundary continuous hyperbolic space. Then for two quasi-metrics* ρ_b*,* $\rho_{b'}$ *on* $\partial_\infty X$ *with one and the same parameter* $a > 1$*, based in* X *or at infinity, the identity map* $\mathrm{id}: (\partial_\infty X, \rho_b) \to (\partial_\infty X, \rho_{b'})$ *is Möbius; in particular,* ρ_b*,* $\rho_{b'}$ *are Möbius equivalent to each other.*

Proof. For a boundary continuous hyperbolic space, one can take $c = 0$ in Proposition 4.5.1, that is, for every quadruple Q of distinct points in $\partial_\infty X$, the cross-difference triples based at points $o \in X$ or at Busemann functions $b \in \mathcal{B}(\omega)$, where $\omega \in \partial_\infty X$, differ from each other by the same constant in each of their entries. This follows from Theorem 4.2.1, Lemmas 2.2.2, 3.2.4 and from properties of boundary continuous hyperbolic spaces. Thus corresponding cross-differences coincide with each other irrespectively of their bases. Therefore, the identity map $\mathrm{id}: (\partial_\infty X, \rho_b) \to (\partial_\infty X, \rho_{b'})$ is Möbius. □

Remark 5.3.3. In the case that X is a CAT(-1)-space, it follows from the results [Bou] and [FS2] that any quasi-metric $\rho_b(\alpha, \beta) = e^{-(\alpha|\beta)_b}$ on $\partial_\infty X$ based either at $b \in X$ or at a Busemann function $b \in \mathcal{B}(\omega)$ with $\omega \in \partial_\infty X$ is actually a metric, i.e., it satisfies the triangle inequality. By Proposition 3.4.2, every proper CAT(-1)-space X is boundary continuous. Hence, there is a Möbius structure on $\partial_\infty X$ whose members are honest metrics. This should be compared to [Bou] where it is proven that any ρ_b, $\rho_{b'}$ as above with $b, b' \in X$ are conformal to each other.

Assume that (Z,ρ) is a K-quasi-metric space with infinitely remote set Z_∞ with $|Z_\infty| \le 1$. The most important example of a quasi-metric which is Möbius to ρ is the *inversion* ρ' of ρ with radius $r > 0$ centered at $o \in Z$, $o \in Z \setminus Z_\infty$. We put

$$\rho'(a,b) = \frac{r^2\rho(a,b)}{\rho(o,a)\rho(o,b)}$$

for every $a,b \in Z$, $(a,b) \ne (o,o)$, and $\rho'(\omega,\omega) = 0$. In the case $b = \omega$, this means $\rho'(a,\omega) = \rho'(\omega,a) = r^2/\rho(o,a)$, and in the case $b = o$, we have $\rho'(a,o) = \rho'(o,a) = \infty$, i.e., the infinitely remote set for ρ' is $Z'_\infty = \{o\}$. In the case $Z = \widehat{\mathbb{R}}^n$, we have $\rho'(a,b) = \rho(\varphi(a),\varphi(b))$, where $\varphi\colon \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ is the inversion with respect to the sphere of radius r centered at $o \in \mathbb{R}^n$; see Appendix, Section A.6.1. This justifies our terminology. Furthermore, the inversion operation is involutive in the sense that the inversion ρ'' of ρ' with the same radius r centered at ω coincides with ρ, $\rho'' = \rho$.

Example 5.3.4. For a hyperbolic space X, we fix $o \in X$, $\omega \in \partial_\infty X$ and consider the Gromov product $(\xi,\eta)_{\omega,o} = (\xi|\eta)_o - (\omega|\xi)_o - (\omega|\eta)_o$ on $\partial_\infty X$, see Section 3.2.2. Then the (unbounded) quasi-metric $\rho_{\omega,o} = a^{-(\cdot|\cdot)_{\omega,o}}$, $a > 1$, on $\partial_\infty X$ is the inversion of the bounded quasi-metric $\rho = a^{-(\cdot|\cdot)_o}$ on $\partial_\infty X$ centered at ω with radius $r = 1$,

$$\rho_{\omega,o}(\xi,\eta) = \frac{\rho(\xi,\eta)}{\rho(\xi,\omega)\rho(\eta,\omega)} \quad \text{for each } \xi,\eta \in \partial_\infty X.$$

We generalize the notion of the inversion of ρ as follows. We call a function $\lambda\colon Z \to [0,\infty]$ *admissible* if $Z_\infty = \lambda^{-1}(\infty)$ and if there is $r > 0$ such that $\rho(a,b) \le K'\max\{r\lambda(a), r\lambda(b)\}$ and $r\lambda(a) \le K'\max\{\rho(a,b), r\lambda(b)\}$, i.e., $(\rho(a,b), r\lambda(a), r\lambda(b))$ is a multiplicative K'-triple for some $K' \ge K$ and all distinct $a,b \in Z$. This is a projective invariant, i.e., if λ is admissible then $s\lambda$ is admissible for every $s > 0$. We call r a *coefficient* of λ.

Any admissible λ assumes the value 0 at most once. Furthermore, if $\lambda(o) = 0$ for some $o \in Z$ then clearly $r\lambda \asymp_{K'} \rho(o,\cdot)$ for some $r > 0$, $K' \ge K$.

Example 5.3.5. The following functions are admissible:

(1) $\lambda(a) = \rho(o,a)$ for some $o \in Z \setminus Z_\infty$. Here, $r = 1$ and $K' = K$.

(2) $\lambda(a) = \max\{1, \rho(o,a)\}$ for some $o \in Z \setminus Z_\infty$. Here again $r = 1$ and $K' = K$.

(3) $\lambda(a) = (1 + \rho^2(o,a))^{1/2}$ for some $o \in Z \setminus Z_\infty$. In the case that ρ is a metric, the l_2-product metric $\hat\rho$ on $Z \times \mathbb{R}$, $\hat\rho^2((a,s),(b,t)) = \rho^2(a,b) + (s-t)^2$ is a 2-quasi-metric, and $\lambda(a) = \hat\rho((a,0),(o,1))$ is admissible with $r = 1$, $K' = K = 2$. In the general case, i.e., for an arbitrary K-quasi-metric ρ we only have that λ is admissible with $r = 1$ and $K' = 2K$. Indeed, this function equals at most two times the function of (2).

In the last two examples, λ is uniformly separated from 0 on Z, $\lambda(a) \ge 1$ for all $a \in Z$.

Given an admissible function λ with coefficient $r > 0$, we put $Z'_\infty = \{\lambda^{-1}(0)\}$ and define λ-*inversion* $\rho_\lambda\colon (Z \times Z) \setminus (Z'_\infty \times Z'_\infty) \to [0, \infty]$ of ρ by

$$\rho_\lambda(a, b) = \frac{\rho(a, b)}{\lambda(a)\lambda(b)},$$

where we set $\rho_\lambda(a, \omega) = \rho_\lambda(\omega, a) = r/\lambda(a)$ for $\omega \in Z_\infty$, in particular, $\rho_\lambda(\omega, \omega) = 0$. Furthermore, for $o \in Z'_\infty$, we have $\rho_\lambda(a, o) = \infty$ for every $a \in Z \setminus Z'_\infty$, i.e., Z'_∞ is the infinitely remote set for ρ_λ.

In the case that λ assumes the value 0, we clearly have $\rho_\lambda \asymp_{K'^2} \rho'$, where ρ' is the inversion of ρ with radius r^2 centered at o, $\lambda(o) = 0$.

Proposition 5.3.6. *Let (Z, ρ) be a K-quasi-metric space with infinitely remote set $Z_\infty = \{\omega\}$, $|Z_\infty| \le 1$. Then for every admissible function λ on Z, the λ-inversion ρ_λ of ρ is a ${K'}^2$-quasi-metric on Z with infinitely remote set $Z'_\infty = \{\lambda^{-1}(0)\}$ and some $K' \ge K$, which is Möbius equivalent to ρ. Furthermore, in the case $\lambda(z) \ge \lambda_0 > 0$ for every $z \in Z$, the space (Z, ρ_λ) is bounded,* $\operatorname{diam}(Z, \rho_\lambda) \le rK'/\lambda_0$, *where r is a coefficient of λ.*

Proof. The function ρ_λ obviously satisfies conditions (1), (2), (4) of Definition 3.3.1 with infinitely remote set Z'_∞. Let $Q = (a, b, c, d)$ be a quadruple of distinct points in Z. If $\omega \notin Q$, then the cross-ratio triple M_λ of Q with respect to ρ_λ is proportional to the cross-ratio triple M of Q with respect to ρ. Indeed $\lambda(a)\lambda(b)\lambda(c)\lambda(d)M_\lambda = M$. If $\omega \in Q$, e.g. $d = \omega$, then

$$M_\lambda = \frac{r}{\lambda(a)\lambda(b)\lambda(c)}(\rho(a, b), \rho(a, c), \rho(b, c)) = \frac{r}{\lambda(a)\lambda(b)\lambda(c)}M.$$

It follows that the identity map $\mathrm{id}\colon (Z, \rho) \to (Z, \rho_\lambda)$ is Möbius.

Now we check condition 3.3.1 (3). If λ assumes the value 0 then $Z'_\infty = \{\lambda^{-1}(0)\}$ consists of one point which we denote by o, $\lambda(o) = 0$, and $\rho_\lambda \asymp_{K'^2} \rho'$, where ρ' is an inversion of ρ centered at o. Thus it suffices to check (3) for the inversion ρ' of ρ.

Let a, b, $c \in Z$ be distinct points. If one of them coincides with o, then (3) is obvious for ρ' with any $K' \ge 1$. Thus we assume that a, b, c are different from o. Then the triple

$$M' = (\rho'(a, b), \rho'(a, c), \rho'(b, c))$$

is proportional to the cross-ratio triple M (with respect to ρ) of the quadruple $Q = (a, b, c, o)$,

$$M = (\rho(a, b)\rho(c, o), \rho(a, c)\rho(b, o), \rho(a, o)\rho(b, c)).$$

It follows from Lemma 5.1.2 that M is a multiplicative K^2-triple, i.e., condition (3) is fulfilled for ρ' with $K' = K^2$.

Otherwise, if λ does not assume the value 0, consider the space $\widehat{Z}$ which is the disjoint union of Z and a point o, $\widehat{Z} = Z \cup \{o\}$. Since λ is admissible with the coefficient $r > 0$, $(\rho(a,b), r\lambda(a), r\lambda(b))$ is a multiplicative K'-triple, $K' \geq K$, for all distinct $a, b \in Z$. We define the function $\hat{\rho}\colon (\widehat{Z} \times \widehat{Z}) \setminus (\omega, \omega) \to [0, \infty]$ as follows: $\hat{\rho}(o, o) = 0$, $\hat{\rho}(o, a) = \hat{\rho}(a, o) = r\lambda(a)$ for every $a \in Z$, and $\hat{\rho}$ restricted to $(Z \times Z) \setminus (\omega, \omega)$ coincides with ρ. Then $\hat{\rho}$ is a K'-quasi-metric on $\widehat{Z}$ with infinitely remote set Z_∞. Hence, by Lemma 5.1.2, the cross-ratio triple $\widehat{M}$ of any quadruple $Q = (a, b, c, o)$ of distinct points in $\widehat{Z}$ is a multiplicative K'^2-triple with respect to $\hat{\rho}$.

Now for distinct $a, b, c \in Z$, we consider the triple

$$M = (\rho_\lambda(a,b), \rho_\lambda(a,c), \rho_\lambda(b,c)).$$

If one of the points is ω, e.g. $c = \omega$, then

$$M = \frac{1}{\lambda(a)\lambda(b)}(\rho(a,b), r\lambda(b), r\lambda(a))$$

is a multiplicative K'-triple. Otherwise, M is proportional to $\widehat{M}$,

$$\lambda(a)\lambda(b)\lambda(c)M = (\lambda(c)\rho(a,b), \lambda(b)\rho(a,c), \lambda(a)\rho(b,c)) = \widehat{M},$$

and thus M is a multiplicative K'^2-triple. Therefore, condition (3) of Definition 3.3.1 holds for ρ_λ, and ρ_λ is a K'^2-quasi-metric in Z. Finally, if $\lambda \geq \lambda_0$ then $\rho_\lambda(a,b) \leq rK'/\lambda_0$ for every $a, b \in Z$, because λ is admissible. Hence, the space (Z, ρ_λ) is bounded. $\square$

5.3.2 Stereographic projections

As illustration, we look at the classical stereographic projection $\varphi\colon S^n \to \widehat{\mathbb{R}}^n$,

$$\varphi(x) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{for } x = (x_1, \ldots, x_{n+1}).$$

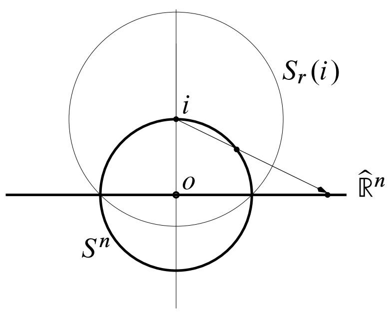


Figure 5.2. The stereographic projection as an inversion.

The inversion $\widehat{\varphi}\colon \widehat{\mathbb{R}}^{n+1} \to \widehat{\mathbb{R}}^{n+1}$ of the extended $\widehat{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ with respect to the sphere $S_r(i) \subset \mathbb{R}^{n+1}$, $i = (0, \dots, 0, 1)$, $r = \sqrt{2}$ (see Section A.6.1), restricted to the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$, coincides with the stereographic projection, $\widehat{\varphi}|S^n = \varphi$. Thus φ as well as its inverse $\pi\colon \widehat{\mathbb{R}}^n \to S^n$ are Möbius maps.

We put $o = (0, \dots, 0) \in \mathbb{R}^{n+1}$ and denote by ρ the standard metric on $\mathbb{R}^{n+1}$, $\rho(x, y) = |x - y|$, canonically extended to $\widehat{\mathbb{R}}^{n+1}$. We use the same notation ρ for the induced metric on $S^n \subset \mathbb{R}^{n+1}$, and for the induced metric on $\widehat{\mathbb{R}}^n = \{x_{n+1} = 0\} \cup \{\infty\} \subset \widehat{\mathbb{R}}^{n+1}$.

Consider the following two spaces: $(S^n, \frac{1}{2}\rho)$ and $(\widehat{\mathbb{R}}^n, \rho)$ (the choice $\frac{1}{2}\rho$ instead of ρ for S^n is motivated by the fact that $\frac{1}{2}\rho$ is a visual metric for the unit disc model of H^{n-1} with parameter $a = e$, see Section 2.4.3). The function $\lambda\colon S^n \to \mathbb{R}$, $2\lambda(x) = \rho(x, i)$, is admissible for $(S^n, \frac{1}{2}\rho)$, see Example 5.3.5 (1), and the function $\bar{\lambda}\colon \widehat{\mathbb{R}}^n \to \mathbb{R}$, $\bar{\lambda}(x) = (1 + \rho^2(x, o))^{1/2}$, is admissible for $(\widehat{\mathbb{R}}^n, \rho)$, see Example 5.3.5 (3).

For the pull back metric $\widehat{\varphi}^*\rho$ on $\widehat{\mathbb{R}}^{n+1}$, $\widehat{\varphi}^*\rho(x, y) = \rho(\widehat{\varphi}(x), \widehat{\varphi}(y))$, we have (cf. proof of Theorem A.7.1, p. 189)

$$\widehat{\varphi}^*\rho(x, y) = \frac{r^2\rho(x, y)}{\rho(x, i)\rho(y, i)}$$

for each $x, y \in \mathbb{R}^{n+1} \setminus \{i\}$. Thus $\varphi^*\rho = (\frac{1}{2}\rho)_\lambda$ for the pull back metric $\varphi^*\rho$ on S^n. Since $\widehat{\varphi}$ is involutive, $\widehat{\varphi}^2 = \mathrm{id}$, we also obtain

$$\pi^*\left(\frac{1}{2}\rho(x, y)\right) = \frac{\rho(x, y)}{\rho(x, i)\rho(y, i)} = \frac{\rho(x, y)}{(1 + \rho^2(x, o))^{1/2}(1 + \rho^2(y, o))^{1/2}}$$

for each $x, y \in \mathbb{R}^n$. Then $\pi^*(\frac{1}{2}\rho) = \rho_{\bar{\lambda}}$ for the pull back metric $\pi^*(\frac{1}{2}\rho)$ which is bounded on $\widehat{\mathbb{R}}^n$.

In the general situation, if (Z, ρ) is a quasi-metric space with at least two points, then using Proposition 5.3.6 and admissible functions as in Example 5.3.5, we easily see that there are bounded and unbounded Möbius equivalent quasi-metrics on Z, cf. Example 5.3.4.

5.3.3 Möbius maps of uniformly perfect spaces

A (quasi) metric space Z is said to be *uniformly perfect* if there is a constant $\mu \in (0, 1)$ so that for every $x \in Z$ and every $r > 0$, for which the set $Z \setminus B_r(x)$ is nonempty, we have $B_r(x) \setminus B_{\mu r}(x) \neq \emptyset$.

Proposition 5.3.7. *If a quasi-metric space Z is uniformly perfect, then any quasi-Möbius map $f\colon Z \to Z'$ is strictly quasi-Möbius.*

Before we prove this proposition, we need some additional results about quadruples of points in a quasi-metric space. Let Z be a K-quasi-metric space, and

$Q = (x, y, z, w)$ a quadruple of four distinct points in Z. We use the following notation: Let l be the length of a smallest side and L the length of a largest side of Q. Note that $\operatorname{cr}(Q) \geq l^2/L^2$. Assume that $|xy| = l$.

Lemma 5.3.8. *Assume that $M \geq K$ and that $\operatorname{cr}(Q) < \frac{1}{M^{16}}$. Then the four distances $|xz|$, $|xw|$, $|yz|$, $|yw|$ are all $> M^4 l$. The cross-pair is the pair (xy, zw).*

Proof. Assume that one of the four distances, say $|xz|$ is $\leq M^4 l$. By the quasi-metric property $|yz| \leq M^5 l$. Hence we have

$$l \leq |xy|,\ |xz|,\ |yz| \leq M^5 l.$$

Since $\operatorname{cr}(Q) \geq l^2/L^2$ we have $L \geq M^8 l$. This implies that in any triangle containing the point $w \in Q$, the two sides adjacent to w are the two largest sides. Using the K- and hence the M-quasi-metric property again, this implies that

$$L/M \leq |wx|,\ |wy|,\ |wz| \leq L,$$

and hence every entry of the cross-ratio triple is bounded from below by lL/M and above by $M^5 lL$. Thus $\operatorname{cr}(Q) \geq \frac{1}{M^6}$, a contradiction to the assumption.

It remains to prove that (xy, zw) is the cross-pair.

Therefore we prove $|xy| \cdot |zw| < |zx| \cdot |yw|$. We can assume (w.l.o.g) that $|zx| \geq |yw|$. The quasi-metric inequality applied to the triangle xyw gives $|xw| \leq M|yw| \leq M|xz|$. Thus

$$|zw| \leq M \max\{|xw|, |zx|\} \leq M^2|xz|$$

and therefore

$$|xy| \cdot |zw| \leq lM^2|xz| \leq lM^4|xz| < |yw| \cdot |xz|,$$

since $|yw| > M^4 l$ by the first part of the proof. Similarly we see that $|xy| \cdot |zw| < |zy| \cdot |xw|$. $\square$

Lemma 5.3.9. *Assume that there exists $M \geq K$ with $\operatorname{cr}(Q) \geq \frac{1}{M^3}$. Then there exists $z \in Q \setminus \{x, y\}$ with $l \leq |xy|,\ |yz|,\ |xz| \leq M^5 l$.*

Proof. If not, then the four distances $|xz|$, $|xw|$, $|yz|$, $|yw|$ are all $> M^4 l$. Note that by the quasi-metric property at least one of these distances, say $|xz|$, is $\geq L/M$. We now have $|xz| \cdot |yw| > M^3 Ll$, while $|xy| \cdot |zw| \leq Ll$ in contradiction to the assumption. $\square$

The following is an easy consequence of the definition of uniform perfectness.

Lemma 5.3.10. *Let Z be uniformly perfect. Then there exists a constant C such that for any given differing points $x, y \in Z$ there exists $v \in Z$ with*

$$a \leq |xy|,\ |xv|,\ |vy| \leq Ca,$$

for some $a > 0$. $\square$

Proof of Proposition 5.3.7. Assume that Z is uniformly perfect and that $f: Z \to Z'$ is quasi-Möbius.

Firstly, choose M large enough such that $M \geq \max\{K^4, K'^4, C\}$, where C is the constant of Lemma 5.3.10. Secondly, choose $M' \geq M$ large enough such that the following holds: if $Q \subset Z$ is a quadruple with $\mathrm{cr}(Q) \geq \frac{1}{M^4}$, then $\mathrm{cr}(Q') \geq \frac{1}{M'^3}$.

Finally choose h small enough such that (a) $h < \frac{1}{M^{16}}$ and (b) $\mathrm{cr}(Q) < h$ implies $\mathrm{cr}(Q') < \frac{1}{M'^{68}}$.

Now assume that $Q = (x, y, z, w)$ is a quadruple with $\mathrm{cr}(Q) < h$, and that xy is a shortest side. We show that f preserves the cross-pair. Since $\mathrm{cr}(Q) < \frac{1}{M^{16}}$ we see by Lemma 5.3.8 that the cross-pair of Q is (xy, zw). Thus we have to show that the cross-pair of Q' is $(x'y', z'w')$.

Let us assume that this is not the case. Let l' be the length of the shortest side in Q'. Then (since by our choices $\mathrm{cr}(Q') \leq \frac{1}{M'^{68}}$) by Lemma 5.3.8, neither $x'y'$ nor $z'w'$ is a shortest side. We can thus (without loss of generality) assume that $y'w'$ is a shortest side and hence $|y'w'| = l'$. By Lemma 5.3.8 (applied to Q') and the fact that $\mathrm{cr}(Q') \leq \frac{1}{M'^{68}}$ we see that the four lengths $|x'y'|$, $|x'w'|$, $|z'y'|$ and $|z'w'|$ are all $> M'^{17}l'$.

Now we choose according to Lemma 5.3.10 a point $v \in Z$ with $a \leq |xv|$, $|xy|$, $|vy| \leq Ma$, and consider the two quadruples $Q_z = (z, x, v, y)$ and $Q_w = (w, x, v, y)$. Let us take a closer look at Q_z. Note that by Lemma 5.3.8 (applied to Q), we have $|zx|, |zy| > M^4|xy|$ and hence also $|zv| > M^3|xy|$. Thus the smallest side is among the sides xv, vy, yx. Thus the smallest side has an adjacent side whose length is at most M times larger. By Lemma 5.3.8 (applied now to Q_z with constant $M^{1/4} \geq K$) this implies, that $\mathrm{cr}(Q_z) \geq \frac{1}{M^4}$. In the same way we see that $\mathrm{cr}(Q_w) \geq \frac{1}{M^4}$.

By the choices of our constants $\mathrm{cr}(Q'_z), \mathrm{cr}(Q'_w) \geq \frac{1}{M'^3}$.

Let l'_w be the length of the smallest side of Q'_w, in particular $l'_w \leq l' = |y'w'|$. Recall $|x'w'|, |x'y'| > M'^{17}l' > M'^5 l'_w$. This implies that x' cannot be among the three distinguished points from Lemma 5.3.9 (applied to Q'_w). Thus Lemma 5.3.9 implies that $l'_w \leq |v'y'|, |v'w'|, |y'w'| \leq M'^5 l'_w$. In particular $|v'y'| \leq M'^5 l'$.

Now consider the quadruple Q_z and its image Q'_z. Let l'_z be the length of a smallest side of this tetrahedron. By the above $l'_z \leq |v'y'| \leq M'^5 l'$. Since $|z'y'| > M'^{17}l'$ we obtain by the quasi-metric inequality that $|z'v'| > M'^{16}l'$. In the same way we have $|x'y'| > M'^{17}l'$ and obtain also that $|x'v'| > M'^{16}l'$. This implies for all four distances

$$|z'y'|, |z'v'|, |x'y'|, |x'v'| > M'^{16}l' \geq M'^5 l'_z.$$

This is in contradiction to Lemma 5.3.9 applied to Q'_z. □

Bibliographical note. Quasi-symmetric maps in metric spaces were introduced in [TV]. Quasi-Möbius maps were introduced in [V1] in a different form via classical cross-ratio. Due to symmetry properties of the classical cross-ratio, it suffices to require

$$[x', y', z', u'] \le \theta([x, y, z, u])$$

for every quadruple of distinct points x, y, z, u. As an exercise to the reader, we suggest to check that this definition is equivalent to Definition 5.2.3.

5.4 Summary

A number of different types of morphisms between hyperbolic spaces and their boundaries at infinity are introduced in this and former chapters and various relations between them are discussed. To help the reader to get a general picture, we state here the basic definitions and results in a more systematic way.

5.4.1 Tuples of points and associated quantities

For a metric space X, we have the distance $|xy|$ between two points. For a triple $x, y, z \in X$, it is useful to consider the *ordinary difference*

$$\langle x, y, z\rangle_o = (x|y)_o - (x|z)_o,$$

which depends on a base point $o \in X$. In the case that x, y, z are distinct, we also have the *ordinary ratio*,

$$[x, y, z] = \frac{|xz|}{|xy|},$$

which is the multiplicative version of the ordinary difference and typically useful for studying the boundary maps of hyperbolic spaces.

In the context of hyperbolic spaces it is most important to deal with quadruples of points and their cross-differences. Given $Q = (x, y, z, u)$, we have the classical cross-difference

$$\langle x, y, z, u\rangle = \tfrac{1}{2}(|xz| + |yu| - |xy| - |zu|),$$

which can also be expressed via Gromov products

$$\langle x, y, z, u\rangle = -(x|z)_o - (y|u)_o + (x|y)_o + (z|u)_o$$

this time independently of the base point o.

The best way to understand how these expressions are formed is to think of Q as an abstract tetrahedron, to label its edges by distances or Gromov products of vertices, and to take the difference of two sums corresponding to pairs of opposite edges. This leads to the notion of the cross-difference triple of Q, which is formed by three sums corresponding to the three pairs of opposite edges:

$$A = ((x|y)_o + (z|u)_o, (x|z)_o + (y|u)_o, (x|u)_o + (y|z)_o).$$

It is remarkable that A is always a δ-triple if X is δ-hyperbolic. This follows from Theorem 4.2.1: take $o = u$ for example. Thus out of six possible classical cross-differences, which are associated with *unordered* Q, only one, namely a maximal one, has a geometrically significant meaning, and we call it the cross-difference of Q,

$$\operatorname{cd}(Q) = \max_{a,a' \in A} (a - a').$$

To keep track of the corresponding pair, we have introduced the notion of the cross-pair of Q, $\operatorname{cp}(Q)$, which is a pair of opposite edges of Q with maximal $a \in A$.

Given a quasi-metric space (Z, ρ), we associate the classical cross-ratio

$$[a, b, c, d] = \frac{\rho(a, c)\rho(b, d)}{\rho(a, b)\rho(c, d)}$$

to every quadruple $Q = (a, b, c, d) \subset Z$ of distinct points (in the case one of the points is infinitely remote, we cancel out the factors containing this point and the cross-ratio becomes an ordinary ratio, e.g., $[a, b, c, \infty] = \frac{\rho(a,c)}{\rho(a,b)}$). The cross-ratio is a multiplicative version of the cross-difference and the ordinary ratio is that of the ordinary difference. These multiplicative versions naturally occur on the boundary at infinity of a hyperbolic space.

Since the triple $(\rho(a, b), \rho(b, c), \rho(a, c))$ is a multiplicative K-triple for a K-quasi-metric ρ, the cross-ratio triple

$$M = (\rho(a, b)\rho(c, d), \rho(a, c)\rho(b, d), \rho(a, d)\rho(b, c)),$$

formed by the products of distances attached to pairs of opposite edges of a quadruple $Q \subset Z$, is a multiplicative K^2-triple by the multiplicative version of the Tetrahedron Lemma (Lemma 5.1.2). Thus again only one out of six classical cross-ratios associated with unordered Q, a minimal one, has a geometrically significant meaning, and we call it the cross-ratio of Q,

$$\operatorname{cr}(Q) = \min_{m,m' \in M} m/m'.$$

As in the additive case, to keep track of a corresponding pair, we call the pair of opposite edges of Q selected by a minimal element of M the cross-pair of Q, $\operatorname{cp}(Q)$.

An essential distinction from the additive case is that the cross-ratio $\operatorname{cr}(Q)$ together with the cross-pair $\operatorname{cp}(Q)$ contains information basically equivalent to that encoded in the six classical cross-ratios of an unordered Q for any quasi-metric space, while for cross-differences this is only true under the assumption that the space is hyperbolic.

5.4.2 Control functions for additive quantities

Now using various control functions applied to the distance and to the (classical) cross-difference, we obtain two families of classes of maps $f : X \to X'$ between metric spaces labelled by Q and PQ respectively:

(Q) $\rho_1(|xy|) \le |x'y'| \le \rho_2(|xy|)$ for every $x, y \in X$;

(PQ) $\rho_1(\langle x, y, z, u\rangle) \le \langle x', y', z', u'\rangle \le \rho_2(\langle x, y, z, u\rangle)$ for every $x, y, z, u \in X$

(as usual, the sign 'prime' stands for the image under the map). Then $\mathrm{PQ} \subset \mathrm{Q}$ because $|xy| = \langle x, x, y, y\rangle$ for every $x, y \in X$. Note that introducing similar conditions for the ordinary difference, we obtain nothing new because $\langle x, y, z\rangle_o = \langle x, y, z, o\rangle$.

Why do we stick to affine control functions? An obvious reason is that these are the simplest ones in the coarse category. Deeper and mathematically more supported is the reason discussed in Chapter 7: quasi-Möbius maps with arbitrary control functions between the boundaries at infinity of hyperbolic spaces are automatically subordinated to very special control functions, namely power functions $\theta(t) = q \max\{t^p, t^{1/p}\}$, which correspond to affine control functions on the level of the spaces themselves. (This kind of rigidity holds under the assumption that the boundaries are uniformly perfect, however, this assumption is not too restrictive.) Maps with PQ affine control functions, we call (strongly) PQ-isometric.

Together with general affine control functions $\rho_1(t) = \frac{1}{c}t - d$, $\rho_2(t) = ct + d$, $c \ge 1$, $d \ge 0$, it is useful to consider a homothety or similarity type control $\rho_1(t) = ct - d$, $\rho_2(t) = ct + d$ with $c > 0$, in particular, an isometry type control with $c = 1$. We discuss the last two classes in more detail and give important applications in Chapter 7. Note that for the homothety type control, the two families coincide, $\mathrm{Q} = \mathrm{PQ}$. Thus, we further discuss general affine control functions.

For classical PQ affine classes, one should use a piecewise affine control, that is $\rho_1(t) = \min\{t/c, ct\} - d$, $\rho_2(t) = \max\{ct, t/c\} + d$, because classical cross-differences take also negative values.

For maps between hyperbolic geodesic spaces, we have the coincidence $\mathrm{Q} = \mathrm{PQ}$ of the two affine control classes, which in a sense completes the picture on the level of spaces.

For general hyperbolic spaces, a quasi-isometric map between them may not induce any reasonable map between their boundaries at infinity, while any map in PQ affine classes automatically induces the boundary map. Moreover, the induced map is subordinated to strong control functions applied to the ordinary ratios and/or to the cross-ratios. In other words, we have a duality for general hyperbolic spaces which generalizes the classical duality

$$\text{isometries of } \mathrm{H}^{n+1} \longleftrightarrow \text{Möbius maps of } S^n = \partial_\infty \mathrm{H}^{n+1}.$$

More precisely, we recall that the canonical quasi-metrics on $\partial_\infty X$ for a hyperbolic X, $\rho(\xi, \xi') = a^{-(\xi|\xi')_o}$, depend on two parameters, $a > 0$ and $o \in X$. Changing the base point o results in a bilipschitz transformation of ρ, and changing the constant a results in taking a power of ρ; see Remark 2.2.4. However, replacing o by a boundary point $\omega \in \partial_\infty X$ already leads to a Möbius transformation, see Example 5.3.4 and Proposition 5.2.8. This should always be taken into account while discussing the duality.

5.4.3 Control functions for multiplicative quantities and duality

Using various control functions applied to the ordinary ratio and to the cross-ratio, we obtain two families of classes of maps $f:(Z,\rho)\to(Z',\rho')$ between quasi-metric spaces labelled by QS and QM, respectively:

(QS) there is a control function $\eta\colon[0,\infty)\to[0,\infty)$, which is a homeomorphism such that
$$[x',y',z']\le\eta([x,y,z])$$
for all triples of distinct $x,y,z\in Z$;

(QM) there is a control function $\theta\colon[0,\infty)\to[0,\infty)$, which is a homeomorphism such that
$$\frac{1}{\theta(1/\operatorname{cr}(Q))}\le\operatorname{cr}(Q')\le\theta(\operatorname{cr}(Q))$$
for all quadruples of distinct points $Q\subset Z$; moreover, if $\operatorname{cr}(Q)<h$ for some $h\in(0,1)$ then $\operatorname{cp}(Q')=\operatorname{cp}(Q)'$ for the cross-pairs.

We prove in Chapter 7 that QS $\subset$ QM under the assumption that the spaces are uniformly perfect, see Corollary 7.4.2. On the other hand, any quasi-Möbius map (QM-map) preserving infinitely remote points is quasi-symmetric (QS-map), see Proposition 5.2.15, and any quasi-Möbius map between uniformly perfect bounded spaces is quasi-symmetric, see Corollary 7.3.14.

In general, these families are distinct, QS $\neq$ QM, because any quasi-symmetric map transforms bounded sets into bounded sets, which is not the case for quasi-Möbius maps.

We further classify these families by choosing special control functions. The most important are of power type, $\eta(t)=\theta(t)=q\max\{t^p,t^{1/p}\}$ with $p,q\ge 1$. The reason is that the power type control functions correspond via duality to general affine type control functions for additive quantities. Moreover, it turns out that there is basically no other class inside of QS or QM different from the power control type. This is proven in Chapter 7 under the assumption that the spaces are uniformly perfect; see Corollaries 7.4.2 and 7.4.3. QS-maps or QM-maps subordinated to the power type control are called PQ-symmetric or PQ-Möbius, respectively.

Now the most general analog of the classical duality for hyperbolic spaces X is as follows:

$$\text{PQ-isometries of } X \longleftrightarrow \text{PQ-Möbius maps of } \partial_\infty X.$$

The right arrow, i.e., every PQ-isometric map between hyperbolic spaces induces a PQ-Möbius map of their boundaries at infinity, is explained in this chapter, see Proposition 5.2.10. The left arrow holds under the additional assumptions that X is geodesic and that $\partial_\infty X$ is uniformly perfect, and this is explained in Chapter 7.

One can further refine the classes of PQ-symmetric and PQ-Möbius maps by taking control functions $\eta(t)=\theta(t)=qt^p$ with $p>0$, $q\ge 1$. It follows from the discussion in Section 5.3.2 that if (Z, ρ) is a quasi-metric space, then there are always quasi-metrics ρ_1 and ρ_2 on Z, which are Möbius equivalent to ρ such that ρ_1 is bounded and ρ_2 is unbounded.

The most important example with $p \neq 1$ is a *snow-flake transformation* $\rho \mapsto \rho^p$ of a quasi-metric, which occurs e.g. as the dual to roughly homothetic maps of a hyperbolic space into itself. Snow-flake transformations play an important role in the Assouad embedding theorem; see Chapter 8.

Chapter 6

Hyperbolic approximation of metric spaces

A *hyperbolic cone* is a hyperbolic space with prescribed boundary at infinity. More precisely, a hyperbolic cone X over a metric space Z is a hyperbolic space, usually geodesic, whose boundary at infinity is identified with Z, $\partial_\infty X = Z$, and the metric of Z coincides with a visual metric on $\partial_\infty X$. In this chapter we introduce a special kind of hyperbolic cones called *hyperbolic approximations*. The construction of a hyperbolic approximation is simple and transparent, and it has many applications. The main advantage of the construction is that it directly includes the combinatorics of coverings by balls of the space in the geometry of a hyperbolic approximation.

6.1 Construction

A subset V of a metric space Z is called *a-separated*, $a > 0$, if $d(v, v') \geq a$ for any distinct $v, v' \in V$. Note that if V is maximal with this property, then the union $\bigcup_{v \in V} B_a(v)$ covers Z.

A *hyperbolic approximation* of a metric space Z is a graph X which is defined as follows. We fix a positive $r \leq 1/6$ which is called the *parameter* of X. For every $k \in \mathbb{Z}$, let $V_k \subset Z$ be a maximal r^k-separated set. We associate with every $v \in V_k$ the ball $B(v) \subset Z$ of radius $r(v) := 2r^k$ centered at v. We consider the disjoint union $V = \bigcup_{k \in \mathbb{Z}} V_k$, or better the set of balls $B(v)$, $v \in V$, as the vertex set of a graph X. Vertices $v, v' \in V$ are connected by an edge if and only if they either belong to the same level, V_k, and the closed balls $\overline{B}(v)$, $\overline{B}(v')$ intersect, $\overline{B}(v) \cap \overline{B}(v') \neq \emptyset$, or they lie on neighboring levels V_k, V_{k+1} and the ball of the upper level, V_{k+1}, is contained in the ball of the lower level, V_k.

An edge $vv' \subset X$ is called *horizontal,* if its vertices belong to the same level, $v, v' \in V_k$ for some $k \in \mathbb{Z}$. Other edges are called *radial*. We consider the path metric on X for which every edge has length 1. We denote by $|vv'|$ the distance between points $v, v' \in V$ in X, and by $d(v, v')$ the distance between them in Z. The level function $\ell\colon V \to \mathbb{Z}$ is defined by $\ell(v) = k$ for $v \in V_k$.

We often use the following:

Remark 6.1.1. For every $z \in Z$ and every $k \in \mathbb{Z}$, there is a vertex $v \in V_k$ with $d(z, v) < r^k$. This follows from the fact that V_k is a maximal r^k-separated set in Z.

6.2 Geodesics in a hyperbolic approximation

Here we study the behavior of geodesics in X. First, we note that any (finite or infinite) sequence $\{v_k\} \subset V$ such that $v_k v_{k+1}$ is a radial edge for every k and the level function ℓ is monotone along $\{v_k\}$, is the vertex sequence of a geodesic in X. Such a geodesic is called *radial*.

Lemma 6.2.1. *For every $v \in V$ there is a vertex $w \in V$ with $\ell(w) = \ell(v) - 1$ connected with any $v' \in V$, $\ell(v') = \ell(v)$, $|vv'| \leq 1$, by a radial edge. Furthermore $d(v, w) \leq r^k$ where $k = \ell(w)$.*

We call the vertex w a *central ancestor* of v. In general, a central ancestor of v may not be unique.

Proof. By Remark 6.1.1, there is a vertex $w \in V_k$ for which the distance in Z between v and w is at most r^k, $d(v, w) \leq r^k$. Thus for every vertex $v' \in V_{k+1}$ adjacent to v in X we have

$$d(v', w) \leq d(v', v) + d(v, w) < 4r^{k+1} + r^k.$$

For each $z \in B(v')$ we have

$$d(z, w) \leq d(z, v') + d(v', w) < 6r^{k+1} + r^k \leq 2r^k,$$

since $r \leq 1/6$. Hence $B(v') \subset B(w)$, and wv' is a radial edge. □

Lemma 6.2.2. *For every $v, v' \in V$ there exists $w \in V$ with $\ell(w) \leq \ell(v), \ell(v')$ such that v, v' can be connected to w by radial geodesics. In particular, the space X is geodesic.*

Proof. Let $\ell(v) = k$ and $\ell(v') = k'$. Choose $m < \min\{k, k'\}$ small enough such that $d(v, v') \leq r^{m+1}$. Applying Lemma 6.2.1, we find radial geodesics $\gamma = v_k v_{k-1} \dots v_m$ and $\gamma' = v'_{k'} v'_{k'-1} \dots v'_m$ in X connecting $v = v_k$ and $v' = v'_{k'}$ respectively with m-th level. It follows from the definition of radial edges that $v \in B(u)$, $v' \in B(u')$ for every vertex $u \in \gamma$, $u' \in \gamma'$. Then

$$d(v', v_m) \leq d(v', v) + d(v, v_{m+1}) + d(v_{m+1}, v_m) \leq 3r^{m+1} + r^m \leq 2r^m$$

since $r \leq 1/6$. Thus $v' \in \bar{B}(v_m) \cap \bar{B}(v'_m)$, and the vertices v_m, v'_m are connected by a horizontal edge. Applying Lemma 6.2.1 once again, we find $w \in V_{m-1}$ connected with v_m, v'_m by radial edges. Therefore v, v' are connected to w by radial geodesics, and X is connected. This implies that X is geodesic because distances between vertices take integer values. □

Lemma 6.2.3. *Assume that $|vv'| \le 1$ for vertices v, v' of one and the same level, $\ell(v) = \ell(v')$. Then $|ww'| \le 1$ for any vertices w, w' adjacent to v, v' respectively and sitting one level below.*

Proof. The balls $\bar{B}(w)$, $\bar{B}(w')$ intersect since they contain the intersecting balls $\bar{B}(v)$, $\bar{B}(v')$ respectively. □

From Lemma 6.2.3, we immediately obtain:

Corollary 6.2.4. *For any two radial geodesics γ, $\gamma' \subset X$ with common ends, the distance in X between vertices of γ and γ' of the same level is at most* 1. □

Lemma 6.2.5. *Any two vertices $v, v' \in V$ can be joined by a geodesic $\gamma = v_0 \dots v_{n+1}$ such that $\ell(v_i) < \max\{\ell(v_{i-1}), \ell(v_{i+1})\}$ for all $1 \le i \le n$.*

Proof. Let $n = |vv'| - 1$. Consider a geodesic $\gamma = v_0 \dots v_{n+1}$ from $v_0 = v$ to $v_{n+1} = v'$ such that $\sigma(\gamma) = \sum_{i=1}^{n} \ell(v_i)$ is minimal. We claim that γ has the desired properties. Let $1 \le i \le n$, and let $k = \ell(v_i)$. Consider the sequence $(\ell(v_{i-1}), \ell(v_i), \ell(v_{i+1}))$. There are nine combinatorial possibilities for this sequence. To prove the result, it remains to show that the sequences $(k-1, k, k-1)$, (k, k, k), $(k-1, k, k)$ and $(k, k, k-1)$ cannot occur.

If the sequence is $(k-1, k, k-1)$, then $|v_{i-1}v_{i+1}| \le 1$ by Lemma 6.2.3, in contradiction to the fact that γ is a geodesic. In the case (k, k, k) Lemma 6.2.1 implies the existence of $w \in V_{k-1}$ with $|v_{i-1}w| \le 1$ and $|v_{i+1}w| \le 1$. Replacing the string $v_{i-1}v_i v_{i+1}$ by $v_{i-1}wv_{i+1}$ we obtain a new geodesic γ' between v, v' with $\sigma(\gamma') < \sigma(\gamma)$ in contradiction to the choice of γ. The two last cases are symmetric and we consider only the case $(k-1, k, k)$. Choose similar as above $w \in V_{k-1}$ with $|v_{i+1}w| \le 1$. Then $|v_{i-1}w| \le 1$ by Lemma 6.2.3. Again $v_{i-1}wv_{i+1}$ defines a geodesic with smaller σ. □

From this we easily obtain the following

Lemma 6.2.6. *Any vertices $v, v' \in V$ can be connected in X by a geodesic which contains at most one horizontal edge. If there is such an edge, then it lies on the lowest level of the geodesic.* □

The following corollary is useful in many circumstances.

Corollary 6.2.7. *Assume that for some $v, v' \in V$ the balls $B(v)$, $B(v')$ intersect. Then $|vv'| \le |\ell(v) - \ell(v')| + 1$.*

Proof. We can assume that $\ell(v) \ge \ell(v')$. For every vertex $w \in V$ of a radial geodesic descending from v we have $B(v) \subset B(w)$; in particular, if $\ell(w) = \ell(v')$ then $|wv'| \le 1$. It follows that v' is the lowest vertex of a geodesic $v'v \subset X$ as in Lemma 6.2.6, hence the claim. □

We use the following terminology. Let $V' \subset V$ be a subset. A point $u \in V$ is called a *cone point* for V' if $\ell(u) \le \inf_{v \in V'} \ell(v)$ and every $v \in V'$ is connected to u by a radial geodesic. A cone point of maximal level is called a *branch point* of V'. By Lemma 6.2.2, for any two points $v, v' \in V$ there is a cone point. Thus every finite V' possesses a cone point and hence a branch point.

Corollary 6.2.8. *Let $v, v' \in V$ and let w be a branch point for $\{v, v'\}$. Then $(v|v')_w \in \{0, \frac{1}{2}\}$, in particular $|vv'| \ge |vw| + |wv'| - 1$.*

Proof. Let u be any cone point of $\{v, v'\}$. Then $|vu| = \ell(v) - \ell(u)$ and $|v'u| = \ell(v') - \ell(u)$ and hence

$$2(v|v')_u = \ell(v) + \ell(v') - 2\ell(u) - |vv'|.$$

In particular for different branch points w_1, w_2 of $\{v, v'\}$ the corresponding Gromov products coincide since $\ell(w_1) = \ell(w_2)$. Therefore it suffices to construct a branch point with this property. By Lemma 6.2.6 there is a geodesic between v and v' having at most one horizontal edge. If there are no horizontal edges, we pick $w \in V$, which is the lowest level vertex of that geodesic. Clearly w is a branch point, and $(v|v')_w = 0$. If there is one horizontal edge, then by Lemma 6.2.1 there is $w \in V$ such that its level is one less than that of the vertices of the horizontal edge, and w is connected by radial edges with these vertices. Thus w is a cone point with $(v|v')_w = \frac{1}{2}$ and therefore also a branch point. □

From Lemma 6.2.6 and Corollaries 6.2.4, 6.2.8 it is clear that the behavior of geodesics in X is similar to that in a tree (one should bear in mind that for every $k \in \mathbb{Z}$ the subgraph $X_k \subset X$ spanned by V_k plays the role of a (horo)sphere in X). Thus very likely X is hyperbolic.

Proposition 6.2.9. *Let $v, v', v'' \in V$ and let w, w', w'' be branch points for the pairs of vertices $\{v', v''\}$, $\{v, v''\}$ and $\{v, v'\}$ respectively. Let u be a cone point of $\{w, w', w''\}$. Then*

$$(v|v')_u \ge \min\{(v|v'')_u, (v'|v'')_u\} - \delta$$

with $\delta = 3/2$.

Proof. We will show that the numbers $(v|v')_u$, $(v|v'')_u$ and $(v'|v'')_u$ form a δ-triple with $\delta = 3/2$. Note that $(v|v')_u = (v|v')_{w''} + |uw''|$ and corresponding equations hold for the other Gromov products. Since the terms $(v|v')_{w''}$ are bounded by $1/2$ due to Corollary 6.2.8, it remains to show that the numbers $|uw|$, $|uw'|$, $|uw''|$ form a δ-triple with $\delta = 1$. We assume without loss of generality that $|uw| \le |uw'| \le |uw''|$, and put $\sigma := |uw'| - |uw|$. It suffices to show that $\sigma \le 1$.

We pick vertices w_1 and w_1' on the radial geodesic uw'', for which $|uw_1| = |uw|$ and $|uw_1'| = |uw'|$. We also pick w_2 on the radial geodesic uw' with $|uw_2| = |uw|$. Then $\sigma = |w_1 w_1'| = |w' w_2|$.

The concatenation of the geodesics uw' and $w'v$ is a radial geodesic from u to v. Also the concatenation of uw'' and $w''v$ is a (maybe different) radial geodesic from u to v. It follows from Corollary 6.2.4 that $|w'w'_1| \le 1$.

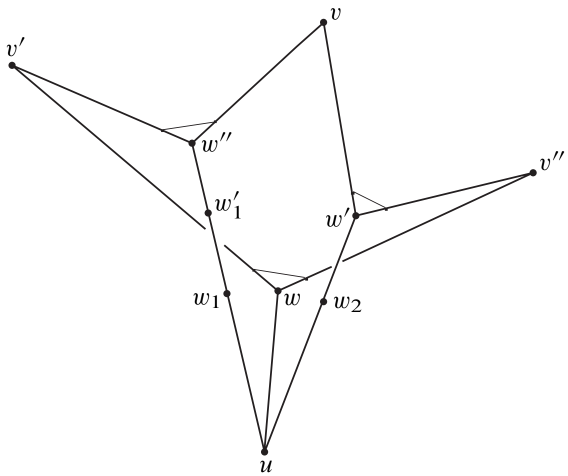


Figure 6.1. δ-hyperbolicity.

The broken geodesic $v'w''w'_1w'v''$ has length

$$L = |v'w'_1| + |w'_1w'| + |w'v''| \le |v'w'_1| + 1 + |w'v''|.$$

Furthermore we have

$$|v'w| = |v'w_1| = |v'w'_1| + \sigma$$

and

$$|v''w| = |v''w_2| = |v''w'| + \sigma.$$

Putting everything together we estimate

$$\begin{aligned} |v'v''| \le L &\le |v'w'_1| + 1 + |w'v''| \\ &= |v'w| + |wv''| + 1 - 2\sigma \\ &\le |v'v''| + 2 - 2\sigma, \end{aligned}$$

where the last inequality comes from Corollary 6.2.8. Thus $\sigma \le 1$. □

Proposition 6.2.10. *A hyperbolic approximation of any metric space is a geodesic 2δ-hyperbolic space with $2\delta = 3$.* □

Proof. Choose some base point $x \in V$. We show that X satisfies the 2δ-inequality for the base point x. Let t, y and z be arbitrary points in V. (We use the notation as in Lemma 2.1.5 since we use similar arguments.) Choose a branch point for every of the six pairs $\{t,x\}$, $\{t,y\}$, $\{t,z\}$, $\{x,y\}$, $\{x,z\}$, $\{y,z\}$, and choose a cone point u for the set of branch points. Proposition 6.2.9 implies now that the six numbers $(t|x)_u$, $(t|y)_u$, $(t|z)_u$, $(x,y)_u$, $(x|z)_u$, $(y|z)_u$ satisfy the condition of the Tetrahedron Lemma 2.1.4 with $\delta = 3/2$ which implies $A \geq -2\delta$ for

$$A = (t|y) + (x|z) - \min\{(t|z) + (y|x), (x|t) + (y|z)\}$$

(recall that this expression is independent of a base point for the Gromov products involved). Since $A = (t|y)_x - \min\{(t|z)_x, (z|y)_x\}$ (compare the proof of Lemma 2.1.4), we obtain the result. $\square$

6.3 The boundary at infinity of a hyperbolic approximation

Let X be a hyperbolic approximation with parameter $r \leq 1/6$ of a metric space Z. The main result of this section is that the metric d of Z is a visual metric for X under a natural identification $\partial_\infty X = Z \cup \{\infty\}$. More precisely, we have the following result.

Theorem 6.3.1. *Given a complete metric space Z, its hyperbolic approximation X is a Gromov hyperbolic geodesic space, and there is a canonical identification $\partial_\infty X = Z \cup \{\infty\}$ such that the metric of Z is a visual metric on $\partial_\infty X \setminus \{\omega\}$ with respect to some and hence any Busemann function $b \in \mathcal{B}(\omega)$ and to the parameter $a = 1/r$, where $\omega \in \partial_\infty X$ corresponds to the infinitely remote point ∞.*

For the proof we need several lemmas.

Lemma 6.3.2. *For every $z \in Z$, there is a sequence $\{\gamma_k(z) \in V_k\}$ such that $d(z, \gamma_k(z)) < r^k$, in particular $z \in B(\gamma_k(z))$, and $\gamma_k(z)\gamma_{k+1}(z)$ is a radial edge for every $k \in \mathbb{Z}$.*

Proof. By Remark 6.1.1, there is $v_k \in V_k$ with $d(z, v_k) < r^k$. Then $z \in B(v_k)$, and for every $z' \in B(v_{k+1})$ we have

$$d(z', v_k) \leq d(z', v_{k+1}) + d(v_{k+1}, z) + d(z, v_k) < 2r^{k+1} + r^{k+1} + r^k \leq 2r^k,$$

because $r \leq 1/6$. Thus $B(v_{k+1}) \subset B(v_k)$, and $v_k v_{k+1}$ is a radial edge. It remains to define $\gamma_k(z) := v_k$. $\square$

Recall that every point $\xi \in \partial_\infty X$ is represented by a sequence $\{x_k\} \subset X$ which converges to infinity, see Section 2.2.

Corollary 6.3.3. *For every $z \in Z$, $\gamma(z) = \dots \gamma_{-1}(z)\gamma_0(z)\gamma_1(z)\dots$ is a bi-infinite geodesic line in X. Its 'negative' tails $\{\gamma_k(z) : k < 0\}$ define one and the same point $\omega \in \partial_\infty X$ for all $z \in Z$.*

Proof. Clearly, the 'positive' and 'negative' tails of $\gamma(z)$ converge to infinity. If z, $z' \in Z$ then the balls of radius $2r^k$ associated to the vertices $\gamma_k(z)$, $\gamma_k(z')$ intersect, $B(\gamma_k(z)) \cap B(\gamma_k(z')) \neq \emptyset$, for all k with $d(z,z') < r^k$, thus $|\gamma_k(z)\gamma_k(z')| \leq 1$. This shows that the sequences $\{\gamma_k(z) : k < 0\}$ define the same point $\omega \in \partial_\infty X$ for all $z \in Z$. □

This point $\omega \in \partial_\infty X$ will correspond to the point ∞ under the identification $\partial_\infty X = Z \cup \{\infty\}$.

Now we fix a base point $o \in V_0$ which is also considered as a point of Z. The geodesic $\gamma(o)$ with $\gamma_0(o) = o$ will serve as a reference geodesic.

Lemma 6.3.4. *Let $v \in V$ and let u be a branch point of the pair $\{o, v\}$. Then $|\gamma_k(o)u| \leq 1$, where $k = \ell(u)$, $\gamma_{k-1}(o)$ is a cone point of $\{o, v\}$ and $(v|\omega)_o \doteq |ou|$ up to an error $\leq 3\delta$, where $\delta = 3/2$.*

Proof. Since $o \in B(u) \cap B(\gamma_k(o))$, we have $|\gamma_k(o)u| \leq 1$. Since

$$d(u, \gamma_{k-1}(o)) \leq d(u,o) + d(o, \gamma_{k-1}(o)) \leq 2r^k + r^{k-1},$$

we have $B(u) \subset B(\gamma_{k-1}(o))$ which implies that $\gamma_{k-1}(o)$ is a cone point of $\{o, v\}$.

Let $n < k$. It follows from the above that $|\gamma_n(o)v| = |\gamma_n(o)u| + |uv|$, $|\gamma_n(o)o| = |\gamma_n(o)u| + |uo|$ and $|ov| \leq |ou| + |uv| \leq |ov| + 1$. This implies $(\gamma_n(o)|v)_o \doteq |ou|$ up to an error $\leq 1/2 = \delta/3$. Since $\lim_{n\to-\infty}(\gamma_n(o)|v)_o \doteq (\omega|v)_o$ up to error 2δ we obtain the result. □

Lemma 6.3.5. *Every sequence $\{v_n\} \subset V$ that converges to infinity in X and represents a point from $\partial_\infty X$ different from ω has a limit in Z. For any other sequence $\{v'_n\} \subset V$ converging to infinity and equivalent to $\{v_n\}$ we have $\lim_n v'_n = \lim_n v_n$.*

Proof. Note that if $w \in V$ is a branch point for $v, v' \in V$, then both balls $B(v)$, $B(v')$ are contained in the ball $B(w)$ by the definition of radial edges. Thus $d(v, v') < 2r(w)$, where recall $r(w)$ is the radius of $B(w)$.

The assumptions imply that $(v_n|\omega)_o$ is bounded, hence by Lemma 6.3.4 also $k = \inf\{\ell(v_n)\}$ is bounded. Then this lemma implies that $\gamma_{k-1}(o)$ is a cone point of the set $\{v_n\}$.

It follows from this and from $(v_n|v_m)_o \to \infty$ that the level of branch points $w_{n,m}$ for v_n, v_m increases to infinity and thus $r(w_{n,m}) \to 0$ as $n, m \to \infty$. Therefore, $\{v_n\}$ is a Cauchy sequence if considered as a sequence in Z. Thus there is a limit $\lim_n v_n \in Z$ because Z is complete. A similar argument shows that if $\{v'_n\}$ is equivalent to $\{v_n\}$ then $\lim_n v'_n = \lim_n v_n$. □

Lemma 6.3.6. *Assume that a sequence $\{v_n\} \subset V$ converges to infinity in X and represents a point $\xi \in \partial_\infty X \setminus \{\omega\}$. Let $z = \lim_n v_n$ (see Lemma 6.3.5). Then any 'positive tail' of the geodesic $\gamma(z)$ represents the same point ξ, $\{\gamma_k(z)\}_{k\in\mathbb{N}} \in \xi$.*

Proof. It follows from the condition that for every $k \in \mathbb{Z}$, a tail of $\{v_n\}$ is contained in $B(\gamma_k(z))$. Passing to a subsequence, we can assume that $v_k \in B(\gamma_k(z))$ and $\ell(v_k) \geq k$ for all $k \in \mathbb{N}$.

Then $B(v_k) \cap B(\gamma_k(z)) \neq \emptyset$ for all $k \in \mathbb{N}$. By Corollary 6.2.7, there is a geodesic $\gamma_k(z)v_k \subset X$ which is either radial or the level of its unique horizontal edge is k. Then

$$(\gamma_k(z)|v_k)_{\gamma_0(z)} \geq k - 1 \to \infty,$$

as $k \to \infty$. This shows that the sequences $\{\gamma_k(z)\}$ and $\{v_k\}$, $k \in \mathbb{N}$, are equivalent. $\square$

Lemma 6.3.7. *The map $\psi\colon \partial_\infty X \to Z \cup \{\infty\}$ that associates to every $\xi \in \partial_\infty X \setminus \omega$ the limit point $z = \psi(\xi)$ of a sequence $\{v_n\} \in \xi$, considered as the sequence in Z, and $\psi(\omega) = \infty$, is a well-defined bijection.*

Proof. The map ψ is well defined by Lemma 6.3.5. We define the map $\eta\colon Z \cup \{\infty\} \to \partial_\infty X$ letting $\eta(\infty) = \omega$ and $\eta(z) \in \partial_\infty X \setminus \omega$ for $z \in Z$ be the class of the 'positive' tail of the geodesic $\gamma(z)$; see Corollary 6.3.3. This class is well defined, since if $\gamma'(z)$ is another geodesic with $z \in B(\gamma'_k(z))$, then $|\gamma_k(z)\gamma'_k(z)| \leq 1$ for all $k \in \mathbb{Z}$, and the sequences $\{\gamma_k(z)\}$, $\{\gamma'_k(z)\}$, $k \geq 0$, are equivalent. Clearly, $\lim_k \gamma_k(z) = z$ in Z, thus η is a right inverse to ψ, $\psi \circ \eta = \mathrm{id}$. By Lemma 6.3.6, $\eta \circ \psi = \mathrm{id}$, i.e., η and ψ are mutually inverse bijections. $\square$

From now on we identify $\partial_\infty X$ with $Z \cup \{\infty\}$ using the bijection ψ.

Proof of Theorem 6.3.1. To complete the proof of Theorem 6.3.1, we show that the metric d of $Z = \partial_\infty X \setminus \{\omega\}$ is visual with respect to any Busemann function $b \in \mathcal{B}(\omega)$ and the parameter $a = 1/r$. Because the functions ρ_b are bilipschitz to each other for all Busemann functions $b \in \mathcal{B}(\omega)$, it suffices to consider the case $b = b_{\omega,o}$; see Definition 3.1.3.

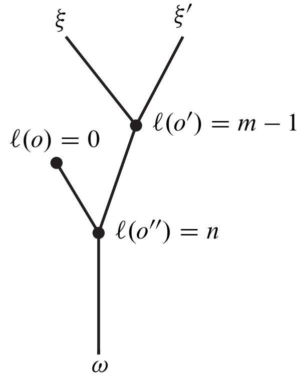
Fix distinct $\xi, \xi' \in Z$ and consider corresponding bi-infinite geodesics $\gamma(\xi)$, $\gamma(\xi')$ in X. For brevity, we use the notations $v_k = \gamma_k(\xi)$, $v'_k = \gamma_k(\xi')$, $k \in \mathbb{Z}$. Let $m = \max\{k \in \mathbb{Z} : |v_k v'_k| \leq 1\}$. One easily estimates that

$$r^{m+1} \leq d(\xi, \xi') \leq 6r^m \leq r^{m-1}. \tag{6.1}$$

By Lemma 6.2.1, there is a cone point o' for $\{v_m, v'_m\}$ of the level $\ell(o') = m - 1$. The idea is to compute the Gromov products $(v_k|v'_k)_b$ as $k \to \infty$ via $(v_k|v'_k)_{b'}$ for the Busemann function $b' = b_{\omega,o'}$ using the approximate equality

$$b_{\omega,o}(v) \doteq b_{\omega,o'}(v) + b_\omega(o', o) \tag{6.2}$$

up to an error $\leq 12\delta$ for all $v \in V$; see Lemma 3.1.2 (and take into account that X is 2δ-hyperbolic).



The point o' is obviously the branch point for the pairs $\{o', v_k\}$, $\{o', v'_k\}$ for all $k \ge m$, thus by Lemma 6.3.4, we obtain $|(\omega|v_k)_{o'}|$, $|(\omega|v'_k)_{o'}| \le 3\delta$. Furthermore, one easily computes $(\omega|o')_{v_k}, (\omega|o')_{v'_k} \doteq_{2\delta} k - m + 1$. Using $b'(v_k) = (\omega|o')_{v_k} - (\omega|v_k)_{o'}$, we obtain $b'(v_k), b'(v'_k) \doteq_{5\delta} k - m + 1$. Since $|v_k v'_k| \doteq 2(k - m + 1)$ up to an error $\le 1 \le \delta$, we have $|(v_k|v'_k)_{b'}| \le 6\delta$.

Let o'' be a branch point for the pair $\{o, o'\}$. Then $n = \ell(o'') \le \ell(o) = 0$, and by Lemma 6.3.4 we have

$$(\omega|o)_{o'} \doteq_{3\delta} |o'o''| = m - n, \ (\omega|o')_o \doteq_{3\delta} |oo''| = -n$$

and therefore $b_\omega(o', o) = (\omega|o)_{o'} - (\omega|o')_o \doteq m$ up to an error $\le 6\delta$.

Using equation (6.2), we obtain

$$(v_k|v'_k)_b = \tfrac{1}{2}(b(v_k) + b(v'_k) - |v_k v'_k|) \doteq_{12\delta} (v_k|v'_k)_{b'} + b_\omega(o', o),$$

thus $(v_k|v'_k)_b \doteq m$ up to an error $\le 24\delta$ for all $k \ge m$. Together with equation (6.1), this implies the existence of constants $c_1, c_2 > 0$, depending only on r and δ, with

$$c_1 a^{-(\xi|\xi')_b} \le d(\xi, \xi') \le c_2 a^{-(\xi|\xi')_b}$$

for all $\xi, \xi' \in \partial_\infty X \setminus \{\omega\}$. □

6.4 Supplementary results and remarks

6.4.1 Hyperbolic approximation of bounded spaces

Assume now that the metric space Z is bounded and nontrivial, i.e. $\operatorname{diam} Z < \infty$, and contains at least two points. Then the largest integer k with $\operatorname{diam} Z < r^k$ exists, and we denote it by $k_0 = k_0(\operatorname{diam} Z, r)$. Observe that if $r < \min\{\operatorname{diam} Z, 1/\operatorname{diam} Z\}$ then $k_0 = 0$ (the case $\operatorname{diam} Z < 1$) or $k_0 = -1$ (the case $\operatorname{diam} Z \ge 1$).

Note that for every $k \le k_0$ the vertex set V_k consists of one point, and therefore contains no essential information about Z. Thus we modify the graph X by putting $V_k = \emptyset$ for every $k < k_0$, and call the modified graph the *truncated hyperbolic approximation* of Z. Clearly, all properties of the hyperbolic approximations discussed above hold as well for the truncated hyperbolic approximations.

In particular, the level function ℓ has the unique minimum o, $\ell(o) = k_0$, and the point o is naturally considered as the base point of X. Furthermore, o is obviously a cone point for V. The bounded version of Theorem 6.3.1 is the following:

Theorem 6.4.1. *Let X be a truncated hyperbolic approximation of a complete, bounded metric space Z. Then there is the canonical identification $\partial_\infty X = Z$ under which the metric d of Z is a visual metric on $\partial_\infty X$ with respect to the base point o of X and the parameter $a = 1/r$.*

Proof. Let $\widehat{X}$ be the standard (non truncated) hyperbolic approximation of Z, from which X is obtained by truncation. We fix an auxiliary vertex $\bar{o}$ of $\widehat{X}$ with level zero, $\ell(\bar{o}) = 0$, and consider the Busemann function $b = b_{\omega,\bar{o}} \in \mathcal{B}(\omega)$. Now we only have to find a relation between the Gromov products $(v|v')_b$ and $(v|v')_o$ on the vertex set V of X.

Since $b_{\omega,o}(v) = (\omega|o)_v - (\omega|v)_o = |ov|$ for every $v \in V$ and $b_\omega(o,\bar{o}) = (\omega|\bar{o})_o - (\omega|o)_{\bar{o}} = -k_0$, we obtain from equation (6.2)

$$b(v) \doteq_{12\delta} b_{\omega,o}(v) + b_\omega(o,\bar{o}) = |ov| - k_0.$$

Thus $(v|v')_b \doteq (v|v')_o - k_0$ up to an error $\le 12\delta$ for all $v, v' \in V$, which shows via Theorem 6.3.1 that

$$c_1 a^{-(\xi|\xi')_o} \le d(\xi,\xi') \le c_2 a^{-(\xi|\xi')_o}$$

for some positive constants c_1, c_2 depending only on r, δ and $k_0 = k_0(\operatorname{diam} Z, r)$. $\square$

Exercise 6.4.2. Show that the infinitely remote point $\infty \in \partial_\infty X$ of a hyperbolic approximation of a metric space Z is isolated in $\partial_\infty X$ if and only if Z is bounded.

6.4.2 Geodesic boundary of a hyperbolic approximation

For the hyperbolic approximation of a complete bounded metric space (which is not necessarily compact), we have the following

Proposition 6.4.3. *Let X be a truncated hyperbolic approximation of a complete bounded space Z. Then $\partial^g X = \partial_\infty X$, and for any two radial rays $\gamma = o \dots v_k \dots$, $\gamma' = o \dots v_k' \dots$ ($v_k, v_k' \in V_k$) in X, representing the same point $\xi \in \partial_\infty X$, we have $|v_k v_k'| \le 1$ for all $k \ge k_0$.*

Proof. Recall that $\partial^g X \subset \partial_\infty X$ for any hyperbolic geodesic space. For the truncated hyperbolic approximation X, the equality $\partial^g X = \partial_\infty X$ follows from Lemma 6.3.6. For the rays γ, γ', we have $\lim_k v_k = z = \lim_k v_k'$ and $z \in \overline{B}(v_k) \cap \overline{B}(v_k')$ for every $k \ge k_0$. Hence the claim. $\square$

6.4.3 Hyperbolic approximation of a compact metric space

Exercise 6.4.4. Show that a hyperbolic approximation X is proper if and only if $\partial_\infty X$ is compact.

6.4.4 The hyperbolic cone of a bounded metric space

There are several constructions of a hyperbolic cone over a bounded space besides a hyperbolic approximation. Here we discuss one of them, which is useful in some circumstances.

Let Z be a bounded metric space. Assuming that $\operatorname{diam} Z > 0$ we put $\mu = \pi/\operatorname{diam} Z$ and note that $\mu|zz'| \in [0,\pi]$ for every z, $z' \in Z$. The *hyperbolic cone* $\mathrm{Co}(Z)$ over Z is the space $Z \times [0,\infty)/Z \times \{0\}$ with metric defined as follows. Given $x = (z,t)$, $x' = (z',t') \in \mathrm{Co}(Z)$ we consider a triangle $\bar{o}\bar{x}\bar{x}' \subset \mathrm{H}^2$ with $|\bar{o}\bar{x}| = t$, $|\bar{o}\bar{x}'| = t'$ and the angle $\measuredangle_{\bar{o}}(\bar{x},\bar{x}') = \mu|zz'|$. Now we put $|xx'| := |\bar{x}\bar{x}'|$. The point $o = Z \times \{0\} \in \mathrm{Co}(Z)$ is called the *vertex* of $\mathrm{Co}(Z)$. In the case Z is isometric to the unit standard sphere $S^{n-1} \subset \mathbb{R}^n$ (with induced intrinsic metric) the cone $\mathrm{Co}(Z)$ is isometric to H^n. In the general case, we have

Proposition 6.4.5. *Let Z be a bounded metric space. Then the hyperbolic cone $Y = \mathrm{Co}(Z)$ is a hyperbolic space which satisfies the δ-inequality with respect to the vertex o with $\delta = \delta(\mathrm{H}^2)$. Furthermore, there is a canonical inclusion $Z \subset \partial_\infty Y$, and the metric of Z is visual with respect to the base point o and the parameter $a = e$.*

Proof. Assume that $(y|y')_o \le (y|y'')_o \le (y''|y')_o$ for $y, y', y'' \in Y$. We have to show that $(y|y'')_o \le (y|y')_o + \delta$. To this end, consider triangles $\bar{o}\bar{y}\bar{y}''$ and $\bar{o}\bar{y}''\bar{y}'$ in H^2 with common side $\bar{o}\bar{y}''$ separating them such that $|\bar{o}\bar{y}| = |oy|$, $|\bar{o}\bar{y}'| = |oy'|$, $|\bar{o}\bar{y}''| = |oy''|$, and $|\bar{y}\bar{y}''| = |yy''|$, $|\bar{y}''\bar{y}'| = |y''y'|$. Then $|yy'| \le |\bar{y}\bar{y}'|$ by the triangle inequality in Z. It follows that $(\bar{y}|\bar{y}'')_{\bar{o}} = (y|y'')_o$, $(\bar{y}''|\bar{y}')_{\bar{o}} = (y''|y')_o$ and $(\bar{y}|\bar{y}')_{\bar{o}} \le (y|y')_o$. Therefore, $(y|y'')_o - (y|y')_o \le (\bar{y}|\bar{y}'')_{\bar{o}} - (\bar{y}|\bar{y}')_{\bar{o}} \le \delta$ since the δ-inequality holds for H^2 (see Exercise 1.4.1).

For every $z \in Z$ the ray $\{z\} \times [0,\infty) \subset Y$ represents a point from $\partial_\infty Y$ which we identify with z. This yields the inclusion $Z \subset \partial_\infty Y$.

It remains to check that the metric of Z is visual. Given $z, z' \in Z$, consider the geodesic rays $\gamma(t) = (z,t)$, $\gamma'(t) = (z',t)$ in $\mathrm{Co}(Z)$. Then $\gamma \in z$, $\gamma' \in z'$ viewed as points at infinity, and for $(\gamma|\gamma')_o = \lim_{t\to\infty}(\gamma(t)|\gamma'(t))_o$ we have (cf. Lemma 2.2.2)

$$(z|z')_o \le (\gamma|\gamma')_o \le (z|z')_o + 2\delta.$$

For comparison geodesic rays $\bar{\gamma}$, $\bar{\gamma}' \subset \mathrm{H}^2$ with common vertex $\bar{o}$ and

$$\measuredangle_{\bar{o}}(\bar{\gamma},\bar{\gamma}') = \mu|zz'|$$

(recall $\mu = \pi/\operatorname{diam} Z$) we have $(\bar{\gamma}|\bar{\gamma}')_{\bar{o}} = (\gamma|\gamma')_o$ and $(\bar{\gamma}|\bar{\gamma}')_{\bar{o}} \le d \le (\bar{\gamma}|\bar{\gamma}')_{\bar{o}} + \delta$, where $d = \operatorname{dist}(\bar{o}, \bar{z}\bar{z}')$ and $\bar{z}\bar{z}' \subset \mathrm{H}^2$ is the infinite geodesic with the end points at infinity $\bar{z} = \bar{\gamma}(\infty)$, $\bar{z}' = \bar{\gamma}'(\infty)$. By the parallelism angle formula (see Appendix, Lemma A.3.2), we have $\tan \frac{\mu|zz'|}{4} = e^{-d}$, therefore, we conclude that

$$e^{-3\delta} e^{-(z|z')_o} \le \tan \frac{\mu|zz'|}{4} \le e^{-(z|z')_o}$$

for every $z, z' \in Z$. The function $s \mapsto \frac{1}{s} \tan \frac{\mu s}{4}$ is uniformly bounded and separated from 0 on $[0, \operatorname{diam} Z]$. It follows that the metric on $Z \subset \partial_\infty Y$ is visual with respect to the vertex $o \in Y$ and the parameter $a = e$. $\square$

A disadvantage of this construction is that the space $\mathrm{Co}(Z)$ is not necessarily geodesic even if Z is complete. However, $\mathrm{Co}(Z)$ is roughly similar to any hyperbolic approximation of Z, see the next chapter.

Exercise 6.4.6. Show that if Z is bounded and complete then the boundary at infinity of $\mathrm{Co}(Z)$ coincides with Z.

Bibliographical note. The construction of a hyperbolic approximation of a metric space is a further development of constructions in [El] for compact subspaces of a Euclidean space and in [BP] for arbitrary compact metric spaces.

Our construction differs from that of [BP] by the definition of radial edges and radii of balls, which provides some technical advantages. Theorem 6.4.1 is similar to [BP], Proposition 2.1.

Chapter 7

Extension theorems

In this chapter we prove three extension results, each saying that given a map with certain properties between the boundaries at infinity of hyperbolic spaces, there is a map in an appropriate class between the spaces themselves which induces the given map of the boundaries. The simplest case is if the boundary map is bilipschitz (with respect to visual metrics). Then the extension map is roughly homothetic. This case is most important in view of a number of applications. In the other two cases, the extension map is quasi-isometric while the boundary map is quasi-symmetric or quasi-Möbius. These results show that all asymptotic properties of a hyperbolic space are encoded in its boundary at infinity.

7.1 Extension theorem for bilipschitz maps

Here we discuss refined versions of quasi-isometric maps. A map $f\colon X \to X'$ between metric spaces is said to be *roughly homothetic* if $|f(x)f(x')| \doteq a|xx'|$ up to an error $\le b$ for some constants $a > 0$, $b \ge 0$, and for all $x, x' \in X$ (recall our agreement to write $A \doteq B$ up to an error $\le C$ instead of $|A-B| \le C$, see Chapter 2). If in addition the image $f(X)$ is a net in X', then f is called a *rough similarity*. In the case $a = 1$, the map f is called *roughly isometric* and a *rough isometry* respectively. If there is a rough similarity (isometry) between X and X', then the spaces X and X' are called roughly similar (isometric) to each other, and these relations are obviously equivalence relations.

A hyperbolic space Y is said to be *visual* if for some base point $o \in Y$ there is a positive constant D such that for every $y \in Y$ there is $\xi \in \partial_\infty Y$ with $|oy| \le (y|\xi)_o + D$ (one easily sees that this property is independent of the choice of o). For hyperbolic geodesic spaces this property is a rough version of the property that every segment $oy \subset Y$ can be extended to a geodesic ray beyond the end point y.

Exercise 7.1.1. Show that the property to be visual is a quasi-isometry invariant of hyperbolic geodesic spaces. Show that if $\partial_\infty Y$ consists of one point then a visual hyperbolic Y is roughly isometric to a subspace of a ray.

Theorem 7.1.2. *Let X be a visual and X' a geodesic hyperbolic space. Assume that there is a bilipschitz embedding $f\colon (\partial_\infty X, d) \to (\partial_\infty X', d')$ where d, d' are visual* *metrics with respect to base points $o \in X$, $o' \in X'$ and the same parameter a. Then there exists a roughly isometric map $F\colon X \to X'$ such that $f = \partial_\infty F$.*

In the following arguments, all Gromov products are taken with respect to the base points $o \in X$, $o' \in X'$ respectively. To simplify the notation, we omit the base point and denote $|x| = |ox|$, $|x'| = |o'x'|$ for $x \in X$, $x' \in X'$ respectively.

The idea of the proof is easily explained in the case that every point in X and X' lies on rays emanating from the origin. In this case for $x \in X$ let $\xi \in \partial_\infty X$ such that $x \in o\xi$. Let $\xi' = f(\xi)$ and choose a ray $o'\xi' \subset X'$. Then define $x' = F(x) \in o'\xi'$ to be the point with $|x'| = |x|$. The bilipschitz property of f implies that for $x_1, x_2 \in X$ and for the corresponding points $\xi_1, \xi_2 \in \partial_\infty X$ the equality

$$(\xi_1'|\xi_2') \doteq (\xi_1|\xi_2)$$

holds up to a uniformly bounded error. Using this one can check that F is roughly isometric. Under the more general assumptions of the theorem we have to modify the argument.

Lemma 7.1.3. *Let X be a Gromov hyperbolic space satisfying the δ-inequality with respect to the base point $o \in X$. Assume that $|x_i| \le (x_i|z_i) + D$ for some $D \ge 0$, $x_i \in X$, $z_i \in X \cup \partial_\infty X$, $i = 1, 2$. Then*

$$(x_1|x_2) \doteq \min\{|x_1|, (z_1|z_2), |x_2|\}$$

up to an error $\le D + 2\delta$.

Proof. Applying the δ-inequality twice and using the condition $(x_i|z_i) \ge |x_i| - D$, $i = 1, 2$, we obtain

$$\begin{aligned}(z_1|z_2) &\ge \min\{(z_1|x_1), (x_1|x_2), (x_2|z_2)\} - 2\delta \\ &\ge \min\{|x_1|, (x_1|x_2), |x_2|\} - (D + 2\delta) = (x_1|x_2) - (D + 2\delta),\end{aligned}$$

where the last equality follows from $(x_1|x_2) \le \min\{|x_1|, |x_2|\}$. Thus

$$\min\{|x_1|, (z_1|z_2), |x_2|\} \ge (x_1|x_2) - (D + 2\delta).$$

Similarly we have $(x_1|x_2) \ge \min\{|x_1|, (z_1|z_2), |x_2|\} - (D + 2\delta)$. Hence $(x_1|x_2) \doteq \min\{|x_1|, (z_1|z_2), |x_2|\}$ up to an error $\le D + 2\delta$. □

Proof of Theorem 7.1.2. By the assumption there exists $D > 0$ such that for every point $x \in X$ there is a point $\xi = \xi(x) \in \partial_\infty X$ with $(x|\xi) \ge |x| - D$. Choose such a ξ and let $\xi' = f(\xi) \in \partial_\infty X'$. Choose a point $z' \in X'$ with $(z'|\xi') \ge |x|$, in particular, $|z'| \ge |x|$. Let $o'z'$ be a geodesic from o' to z'. Then we define $x' = F(x) \in o'z'$ to be the point with $|x'| = |x|$. Note that $F(o) = o'$.

Since X, X' satisfy the δ-, δ'-inequalities and d, d' are visual metrics with respect to the base points $o \in X$, $o' \in X'$ and with respect to the same parameter a, we see that

$$c_1 a^{-(\xi_1|\xi_2)} \le d(\xi_1, \xi_2) \le c_2 a^{-(\xi_1|\xi_2)}$$

for every $\xi_1, \xi_2 \in \partial_\infty X$, and

$$c_1' a^{-(\xi_1'|\xi_2')} \le d'(\xi_1', \xi_2') \le c_2' a^{-(\xi_1'|\xi_2')}$$

for every $\xi_1', \xi_2' \in \partial_\infty X'$. Using that

$$\frac{1}{\Lambda} \le \frac{d'(f(\xi_1), f(\xi_2))}{d(\xi_1, \xi_2)} \le \Lambda$$

for some $\Lambda \ge 1$, we obtain

$$(f(\xi_1)|f(\xi_2)) \doteq (\xi_1|\xi_2)$$

up to an error $\le c$ for all $\xi_1, \xi_2 \in \partial_\infty X$, where the constant c depends only on a, Λ and c_i, c_i', $i = 1, 2$.

Now given $x_i \in X$ consider $\xi_i = \xi(x_i) \in \partial_\infty X$, $z_i' \in X'$ with $(z_i'|f(\xi_i)) \ge |x_i|$ and $x_i' = F(x_i) \in o'z_i'$, $i = 1, 2$. By Lemma 7.1.3 we have

$$(x_1|x_2) \doteq \min\{|x_1|, (\xi_1|\xi_2), |x_2|\}$$

up to an error $\le D + 2\delta$. Since $|x_i| = |x_i'| = (x_i'|z_i')$, we obtain

$$(x_i'|f(\xi_i)) \ge \min\{(x_i'|z_i'), (z_i'|f(\xi_i))\} - \delta' = |x_i'| - \delta'.$$

Then again by Lemma 7.1.3 we have $(x_1'|x_2') \doteq \min\{|x_1'|, (f(\xi_1)|f(\xi_2)), |x_2'|\}$ up to an error $3\delta'$. This implies $(x_1'|x_2') \doteq (x_1|x_2)$ and hence $|x_1'x_2'| \doteq |x_1x_2|$ up to an error $\le c + D + 2\delta + 3\delta'$.

This shows that F is roughly isometric, and it easily follows from the definition that $\partial_\infty F = f$. □

Corollary 7.1.4. *Under the same conditions except that d' is now defined with respect to a different parameter a', the map F is roughly homothetic.*

Proof. Replace X' by the homothetic $\lambda X'$, where $\lambda = \ln a / \ln a'$. This allows us to replace the parameter a' of d' by $a'^{\lambda} = a$, while the metric d' on $\partial_\infty X' = \partial_\infty \lambda X'$ remains unchanged. Then the result follows from Theorem 7.1.2. □

Corollary 7.1.5. *Every visual hyperbolic space X is roughly similar to a subspace of a hyperbolic geodesic space X' with the same boundary at infinity, $\partial_\infty X' = \partial_\infty X$.*

Proof. Apply Corollary 7.1.4 to a hyperbolic approximation X' of $\partial_\infty X$ and the identity map $\mathrm{id}\colon \partial_\infty X \to \partial_\infty X'$. □

We say that hyperbolic spaces X, X' have *bilipschitz equivalent boundaries at infinity* if for some visual metrics d on $\partial_\infty X$, d' on $\partial_\infty X'$ the metric spaces $(\partial_\infty X, d)$ and $(\partial_\infty X', d')$ are bilipschitz equivalent.

Corollary 7.1.6. *Visual hyperbolic geodesic spaces X and X' with bilipschitz equivalent boundaries at infinity are roughly similar to each other. In particular, every visual hyperbolic geodesic space is roughly similar to any hyperbolic approximation of its boundary at infinity, and any two hyperbolic approximations of a complete bounded metric space Z are roughly similar to each other.*

Proof. Apply Corollary 7.1.4 to the spaces X, X' and then exchange their roles. For the last two assertions note that every hyperbolic approximation is a visual hyperbolic space. □

7.2 Extension theorem for quasi-symmetric maps

Recall that a metric space X is said to be *uniformly perfect* if there is a constant $\mu \in (0,1)$ so that for every $x \in X$ and every $r > 0$ we have $B_r(x) \setminus B_{\mu r}(x) \neq \emptyset$ unless $X = B_r(x)$.

Recall that sometimes we use the notation $a \asymp b$ up to a multiplicative error $\le c$ or $a \asymp_c b$ instead of

$$\frac{1}{c} \le \frac{a}{b} \le c$$

especially in situations when the error bound c is uniformly bounded over the range of variables a, b.

Theorem 7.2.1. *For every quasi-symmetric homeomorphism $f : Z \to Z'$ of uniformly perfect, complete metric spaces, there is a quasi-isometry of their hyperbolic approximations $F : X \to X'$ which induces f, $\partial_\infty F(z) = f(z)$ for every $z \in Z$.*

Remark 7.2.2. In the case Z and therefore Z' are bounded we consider *truncated* hyperbolic approximations in the theorem above.

Proof. It suffices to define F as a map $F : V \to V'$ of the corresponding vertex sets. Recall that X is a graph with the vertex set V whose edges have length 1. The vertices from $V = \bigcup_k V_k$ are the balls $B(v)$, $v \in V_k$, of radius $r(v) = 2r^k$, $r \le 1/6$, and their centers V_k form a maximal r^k-separated set in Z. Furthermore, we assume (without loss of generality) that the hyperbolic approximation X' of Z' is defined with the same parameter r as X. We also use notation $a = 1/r$.

For every $v \in V$ there is a vertex $v' = F(v) \in V'$ of highest level for which the ball $B(F(v))$ contains $f(B(v)) \subset Z'$. This defines a map $F : V \to V'$, $v \mapsto v'$.

The basic idea is to show that the distance $|v'w'|$ is uniformly bounded for any neighboring $v, w \in V$. From the qualitative point of view this is almost obvious.

Assuming that this is not the case, we find a neighboring $v, w \in V$ with arbitrarily large distance $|v'w'|$. Since the balls $B(v)$, $B(w)$ and hence their images under f intersect, this means that the level difference $|\ell(v') - \ell(w')|$ is arbitrarily large; see Corollary 6.2.7. Therefore the ratio of radii $r(v')/r(w')$ and hence the ratio $\operatorname{diam} f(B(v))/\operatorname{diam} f(B(w))$ is arbitrarily large (assuming that $\ell(v') \le \ell(w')$). Since the balls $B(v)$ and $B(w)$ have comparable diameters due to the uniform perfection condition, this is certainly incompatible with the condition that f preserves the ratio of distances with common point (up to a uniformly bounded multiplicative error).

Now F is Lipschitz because the space X is geodesic. By the same reason a similarly defined map $G\colon V' \to V$ is Lipschitz, and similar arguments show that both compositions $G \circ F$ and $F \circ G$ are at a finite distance from the corresponding identities.

We give details for the reader interested in a quantitative proof. We assume that the space Z is μ-uniformly perfect, and that f is η-quasi-symmetric.

Given neighboring $v, w \in V$, $|vw| \le 1$, we estimate the distance $|v'w'|$ as follows. The definition of F implies that the balls $B(v')$ and $B(w') \subset X'$ intersect. Thus $|v'w'| \le |\ell(v') - \ell(w')| + 1$ by Corollary 6.2.7, and we have to estimate the difference of levels $|\ell(v') - \ell(w')|$. To this end, it suffices to show that $r(v') \asymp r(w')$ up to a uniformly bounded multiplicative error.

We do this in two steps. First, we show that $r(v')$ is comparable with $\operatorname{diam} f(B(v))$, $r(v') \asymp \operatorname{diam} f(B(v))$ quantitatively. Clearly, $\operatorname{diam} f(B(v)) \le 2r(v')$. For the opposite estimate we choose $k \in \mathbb{Z}$ with $r^{k+1} \le \operatorname{diam} f(B(v)) < r^k$. There is $u' \in V'_k$ with $d(u', f(v)) \le r^k$ and hence $f(B(v)) \subset B(u')$. It follows

$$r(v') \le r(u') = 2r^k = 2ar^{k+1} \le 2a \operatorname{diam} f(B(v)). \tag{7.1}$$

Second, we check that the diameters $\operatorname{diam} f(B(v))$, $\operatorname{diam} f(B(w))$ are comparable with each other. At this point we use both the uniform perfection property of X and the quasi-symmetry property of f. It follows from $|vw| \le 1$ that $|\ell(v) - \ell(w)| \le 1$ and $\overline{B}(v) \cap \overline{B}(w) \ne \emptyset$. Thus $r(w) \le ar(v)$ and $d(v, w) \le ar(v)$. For any $z \in B(w)$, we have $d(z, v) \le d(z, w) + d(w, v) \le r(w) + ar(v) \le 2ar(v)$. On the other hand, we pick $\hat{v} \in B(v)$ with $d(v, \hat{v}) \ge \mu' r(v)$ which is going to play the role of a reference point,

$$d(z, v) \le \frac{2a}{\mu'} d(\hat{v}, v)$$

for every $z \in B(w)$ (the existence of $\hat{v}$ follows from the μ-perfection condition if $B(v) \ne Z$, otherwise we take $\hat{v} \in B(v)$ with $d(v, \hat{v}) \ge \frac{1}{4} \operatorname{diam} B(v)$ and note that $r(v) \le r(o) \le 2a \operatorname{diam} Z$ by definition of the base vertex $o \in V_{k_0}$; now $\mu' = \min\{\mu, 1/8a\}$). Hence $d(f(z), f(v)) \le \eta(c) d(f(\hat{v}), f(v))$ for $c = 2a/\mu'$, which implies

$$\operatorname{diam} f(B(w)) \le 2\eta(c) d(f(\hat{v}), f(v)) \le 2\eta(c) \operatorname{diam} f(B(v)). \tag{7.2}$$

This completes the proof that the distance $|v'w'|$ is uniformly bounded and thus F is Lipschitz.

Consider the inverse homeomorphism $g = f^{-1}: Z' \to Z$. It also induces a map $G: V' \to V, v' \mapsto v''$. Our next goal is to show that the composition $G \circ F: V \to V$ is at a finite distance from the identity.

Given $v \in V$, we have $B(v') \supset f(B(v))$ and thus $B(v'') \supset g(B(v')) \supset B(v)$; in particular, the balls $B(v)$ and $B(v'')$ intersect. Thus as above to estimate the distance $|vv''|$ it suffices to show that $r(v) \asymp r(v'')$ up to a uniformly bounded multiplicative error.

The inclusion above implies $r(v) \le 2r(v'')$. For the opposite estimate we note that as above $r(v'')$ is comparable with $\operatorname{diam} g(B(v'))$, and by (7.1) we have

$$d(z', f(v)) \le 2r(v') \le 4a \operatorname{diam} f(B(v))$$

for every $z' \in B(v')$. Applying (7.2) with $w = v$ we see that $d(f(\hat{v}), f(v)) \ge \operatorname{diam} f(B(v))/2\eta(c)$, hence $d(z', f(v)) \le td(f(\hat{v}), f(v))$ with $t \le 8a\eta(c)$. Thus $d(g(z'), v) \le \eta'(t)d(\hat{v}, v) \le \eta'(t)r(v)$ for every $z' \in B(v')$ and therefore $\operatorname{diam} g(B(v')) \le 2\eta'(t)r(v)$, where η' is the control function of g. This concludes the proof that $G \circ F$ is at a finite distance from the identity, and hence that F is a quasi-isometry.

It remains to check that F induces the initial homeomorphism $f: Z \to Z'$ extended to the infinitely remote points by $f(\infty) = \infty$ in the case Z, Z' are unbounded (recall that $\partial_\infty X = Z \cup \{\infty\}$ and $\partial_\infty X' = Z' \cup \{\infty\}$ in that case). We consider only the unbounded case, the bounded case is even simpler.

By what we have already proven, $F: X \to X'$ is a quasi-isometry of hyperbolic spaces. Hence, by Theorem 5.2.17, F induces a homeomorphism $\partial_\infty F: \partial_\infty X \to \partial_\infty X'$.

Every point $z \in Z$ represents a point of $\partial_\infty X$ different from ∞. By identification $Z \cup \{\infty\} = \partial_\infty X$, we have $v_n \to z$ in Z for every sequence $\{v_n\} \in z$. We choose $\{v_n\}$ in a way that $B(v_{n+1}) \subset B(v_n)$ for every $n \ge 0$, see Lemma 6.3.2. The sequence $\{v_n' = F(v_n)\} \subset V'$ converges to infinity and thus defines a point $z' \in \partial_\infty X' = Z' \cup \{\infty\}$, $z' = \partial_\infty F(z)$. Because $f(B(v_n)) \subset B(F(v_0))$ for all $n \ge 0$, z' is distinct from ∞, $z' \ne \infty$. In particular, the levels $\ell(n)$ of v_n' tend to infinity as $n \to \infty$. Furthermore $d(f(v_n), v_n') \le 2r_{\ell(n)}$ and since $f(v_n) \to f(z)$, we have $v_n' \to f(z)$ in Z'. By identification $Z' \cup \{\infty\} = \partial_\infty X'$, we have $z' = f(z)$ and thus $\partial_\infty F(z) = f(z)$ for every $z \in Z$. Hence, $\partial_\infty F = f$. □

Combining Corollary 7.1.6 and Theorem 7.2.1 we obtain

Corollary 7.2.3. *Let X, X' be visual hyperbolic geodesic spaces such that their boundaries at infinity $\partial_\infty X$, $\partial_\infty X'$ are uniformly perfect. Then any quasi-symmetry $f: \partial_\infty X \to \partial_\infty X'$ can be extended to a quasi-isometry $F: X \to X'$.* □

7.3 Extension theorem for quasi-Möbius maps

It is convenient to use the following agreement. Let Z be a metric space. Using the notation $Z \cup \{\omega\}$, we assume that $\{\omega\} = \emptyset$, i.e. $Z \cup \{\omega\} = Z$, if Z is bounded, and $\omega = \infty$ is an infinitely remote point, $\operatorname{dist}(z, \omega) = \infty$ for every $z \in Z$, otherwise.

Theorem 7.3.1. *Let Z, Z' be complete, uniformly perfect metric spaces, and let $f\colon Z \cup \{\omega\} \to Z' \cup \{\omega'\}$ be a quasi-Möbius bijection. Then there is a quasi-isometry $F\colon X \to X'$ between hyperbolic approximations X, X' of Z, Z' respectively, which induces f, $\partial_\infty F = f$ (here as usual we use the canonical identifications $\partial_\infty X = Z \cup \{\omega\}$, $\partial_\infty X' = Z' \cup \{\omega'\}$ and for bounded spaces, we consider truncated hyperbolic approximations).*

This theorem is obtained as a combination of the quasi-symmetry extension theorem and the following inversion extension theorem.

7.3.1 Extension theorem for inversions

By $\bar{Z}$ we denote the metric completion of a metric space Z. Let Z' be an unbounded metric space. One can think of the map f from the following theorem as a generalized inversion centered at z_0.

Theorem 7.3.2. *Let $f : Z \setminus z_0 \to Z'$ be a quasi-Möbius map with control function $\theta(t) = qt^p$, $p > 0$, $q \geq 1$, and with dense image, $\overline{f(Z \setminus z_0)} = \bar{Z}'$, so that $f(z) \to \infty$ as $z \to z_0$ for $z_0 \in Z$ and Z is a uniformly perfect metric space.*

Then for every $0 < r \leq 1/6$, there is a rough similarity $F\colon X \to X'$ with coefficient p between hyperbolic approximations of Z, Z' respectively both with parameter r such that $\partial_\infty F(z) = f(z)$ for every $z \in Z \setminus z_0$, $\partial_\infty F(z_0) = \omega'$, where we use the identification of $\partial_\infty X \setminus \omega$ with the metric completion $\bar{Z}$ of Z, $\partial_\infty X \setminus \omega = \bar{Z}$, and the identification $\partial_\infty X' \setminus \omega' = \bar{Z}'$, and where we assume that X is truncated in the case Z is bounded.

Corollary 7.3.3. *Under the conditions of Theorem 7.3.2, f extends to a θ-quasi-Möbius homeomorphism $\bar{f}\colon \bar{Z} \cup \{\omega\} \to \bar{Z}' \cup \{\infty\}$ with $\bar{f}(z_0) = \infty$, where $\{\omega\} = \emptyset$ if Z is bounded, and $\omega = \infty$ otherwise.* □

Our main tool used in the proof is triples of points, called representative, associated with vertices of X which are balls $B(v) \subset Z$, $v \in V$. We say that a triple of points $T \subset B_s(z)$ is λ-*representative* for the ball $B_s(z) \subset Z$, $\lambda \in (0, 1)$, if

(1) T is λs-separated, i.e. $d(t, t') \geq \lambda s$ for each distinct $t, t' \in T$;
(2) $\operatorname{cr}(Q) \geq \lambda$ for the quadruple $Q = (T, z_0)$.

We first establish the existence of λ-representative triples.

Lemma 7.3.4. *If the metric space Z is μ-uniformly perfect, then every ball $B_s(z) \subset Z$ with nonempty complement, $Z \setminus B_s(z) \neq \emptyset$, contains a λ-representative triple for any positive $\lambda \leq \mu^3/16$.*

Proof. We put $\kappa = \mu/4$ and first consider the case $d(z_0, z) \geq \kappa s$. There are $x, y \in Z$ with $\mu\kappa s/2 \leq d(z,x) \leq \kappa s/2$ and $\mu^2\kappa s/4 \leq d(z,y) \leq \mu\kappa s/4$. Clearly, the triple $T = (x,y,z)$ is contained in $B_s(z)$ and it is λs-separated. Note that $\operatorname{diam} T \leq (\mu+2)\kappa s/4$ and $\operatorname{dist}(z_0,T) \geq d(z_0,z) - \kappa s/2 \geq \kappa s/2$. Thus $\operatorname{diam} T/\operatorname{dist}(z_0,T) \leq (\mu+2)/2$ and using $m := \max_{t\in T} d(z_0,t) \leq \operatorname{dist}(z_0,T) + \operatorname{diam} T$, we obtain $\operatorname{dist}(z_0,T)/m \geq 2/(\mu+4)$. It follows

$$\operatorname{cr}(Q) \geq \frac{\operatorname{dist}(z_0,T)}{m}\frac{\lambda s}{\operatorname{diam} T} \geq \lambda,$$

that is, the triple T is λ-representative for the ball $B_s(z)$.

Now consider the case $d(z_0,z) < \kappa s$. By uniform perfection, there is $x \in Z$ with $\kappa s \leq d(z_0,x) < s/4$. Then $d(z,x) \leq d(z,z_0) + d(z_0,x) < (\mu+1)s/4$ and therefore $B_\rho(x) \subset B_s(z)$ for $\rho = s - (\mu+1)s/4 = (3-\mu)s/4$. Since the upper estimate $(\mu+2)\kappa s/4$ for the diameter of the triple T constructed above is $< \rho$, applying the same argument to $B_\rho(x)$ and z_0, we find a λ-representative triple for the ball $B_s(z)$ also in this case. □

Assume that Z is μ-uniformly perfect, and that for every vertex $v \in V$ of the hyperbolic approximation X a λ-representative triple $T_v \subset B(v)$ is fixed with $\lambda = \mu^3/16$ according to Lemma 7.3.4. In a sense, the triple T_v replaces or represents the ball $B(v)$, considered as a vertex of X, and we use the construction of the extension F similar to that in the proof of Theorem 7.2.1.

The distance between vertices of X can be expressed via special quadruples which are determined using representative triples, see equation (7.3) below. On the other hand, we have a good control over the cross-ratio of quadruples under the map f, see Lemma 7.3.9, which leads directly to desired properties of the extension we construct.

The technical reason why pairs of points cannot be used instead of triples for this purpose is explained in the case $\ell(v) \leq \ell(w) + 1$ in the proof of Proposition 7.3.5.

Given $v, v' \in V$, a quadruple of distinct points $Q \subset T_v \cup T_{v'}$ is said to be *admissible for the triples* T_v, $T_{v'}$ if it contains pairs of points from both T_v and $T_{v'}$ which form a cross-pair for Q (in multiplicative setting with respect to the metric d of Z). Furthermore, we put $a = 1/r$.

In the following proposition, we express the distance in X between distinct vertices via cross-ratios of admissible quadruples for the corresponding representative triples, which yields therefore a key ingredient of the proof of Theorem 7.3.2.

Proposition 7.3.5. *There are constants $\Lambda_0 > 1$, $\lambda_0 \in (0,1)$ depending only on a and λ (one can take $a^{\Lambda_0} > \max\{4a^6, 36a^3/\lambda^2\}$ and $\lambda_0 = \lambda^2 a^{-(\Lambda_0+1)}/4$) such that for given vertices v, $v' \in V$, the following holds:*

(1) *if* $|vv'| \geq \Lambda_0$ *then there is an admissible quadruple* Q *for the triples* T_v*,* $T_{v'}$ *and*

$$a^{-|vv'|} \asymp \min_Q \operatorname{cr}(Q) \text{ up to a multiplicative error at most } \sigma_0, \tag{7.3}$$

where the minimum is taken over all admissible quadruples Q *for* T_v*,* $T_{v'}$ *and* σ_0 *depends only on* a *and* λ *(one can take* $\sigma_0 = \max\{4a^6, (6a)^2/\lambda^2\}$*);*

(2) *if there is an admissible quadruple* Q *for* T_v*,* $T_{v'}$ *with* $\operatorname{cr}(Q) \leq \lambda_0$ *then* $|vv'| \geq \Lambda_0$.

Proof. We fix a branch point $w \in V$ for the pair $\{v, v'\}$ and note that $|vw|+|wv'|-1 \leq |vv'| \leq |vw| + |wv'|$ while $|vw| + |wv'| = \ell(v) + \ell(v') - 2\ell(w)$. Therefore, $|vv'| \leq \ell(v) + \ell(v') - 2\ell(w) \leq |vv'| + 1$. Using that $2a^{-\ell(v)} = r(v)$ for every $v \in V$ (recall that $r(v)$ is the radius of the ball $B(v)$), we obtain the following estimates which are several times used in the proof:

$$a^{-(|vv'|+1)} \leq \frac{r(v)r(v')}{r(w)^2} \leq a^{-|vv'|}. \tag{7.4}$$

Furthermore, since the triples T_v, $T_{v'}$ both are contained in $B(w)$, we have $d(t,t') \leq 2r(w)$ for any $t \in T_v$, $t' \in T_{v'}$. This yields

$$\operatorname{cr}(Q) \geq \lambda^2 r(v)r(v')/(2r(w))^2 \geq \lambda^2 a^{-|vv'|}/4a \tag{7.5}$$

for any quadruple Q admissible for T_v, $T_{v'}$ and hence

$$a^{-|vv'|} \leq \sigma_0 \operatorname{cr}(Q). \tag{7.6}$$

In what follows, we assume without loss of generality that $\ell(v) \leq \ell(v')$. First, consider the case $\ell(v) \geq \ell(w) + 2$. We show that then

$$\operatorname{dist}(B(v), B(v')) \geq a^{-3} r(w).$$

Assume to the contrary that $\operatorname{dist}(\tilde{v}, B(v')) < a^{-3} r(w)$ for some $\tilde{v} \in B(v)$. Since the vertex set V_k is an r^k-net in Z for every k, there is a vertex $u \in V$ of the level $\ell(u) = \ell(w) + 2$ such that $\operatorname{dist}(u, \tilde{v}) \leq r(u)/2$. By our assumption,

$$\operatorname{dist}(u, B(v')) \leq d(u, \tilde{v}) + \operatorname{dist}(\tilde{v}, B(v')) < r(u)/2 + a^{-3} r(w) \leq r(u)$$

because $r(u) = a^{-2} r(w) \geq 2a^{-3} r(w)$. Thus the ball $B(u)$ intersects both balls $B(v)$ and $B(v')$. Using Corollary 6.2.8, we obtain

$$|vv'| \leq |vu| + |uv'| \leq \ell(v) + \ell(v') - 2\ell(u) + 2 = \ell(v) + \ell(v') - 2\ell(w) - 2 < |vv'|,$$

a contradiction.

Under assumption $|vv'| \geq \Lambda_0$, we show that every quadruple $Q \subset T_v \cup T_{v'}$ formed by any distinct $t_1, t_2 \in T_v$ and distinct $t_1', t_2' \in T_{v'}$ is admissible. To this end, consider the cross-ratio triple M of Q,

$$M = (d(t_1,t_2)d(t_1',t_2'), d(t_1,t_1')d(t_2,t_2'), d(t_1,t_2')d(t_2,t_1')).$$

For its first member, we have $d(t_1,t_2)d(t_1',t_2') \leq 4r(v)r(v')$, while for any other $m \in M$ the estimate $m \geq \operatorname{dist}(B(v), B(v'))^2 \geq a^{-6}r(w)^2$ holds. We have $4r(v)r(v') < a^{-6}r(w)^2$ because $2a^{-\ell(v)} = r(v)$ and $\ell(v) + \ell(v') - 2\ell(w) \geq |vv'| \geq \Lambda_0$. Hence, the cross-pair of Q is determined by the first member of M, which means that Q is admissible. Furthermore, by (7.4),

$$\operatorname{cr}(Q) \leq 4r(v)r(v')/\operatorname{dist}(B(v), B(v'))^2 \leq 4r(v)r(v')/(a^{-3}r(w))^2 \leq 4a^6a^{-|vv'|}$$

and together with (7.6), we obtain (7.3).

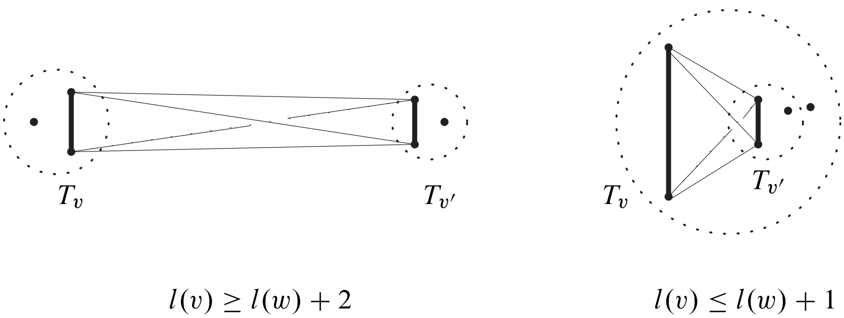


Figure 7.1. Admissible quadruples (cross-pairs are in bold).

Now consider the case $\ell(v) \leq \ell(w) + 1$. Then $r(v) \geq r(w)/a$ and thus T_v is $\lambda r(w)/a$-separated. Furthermore, $|vw| \leq 1$ and hence $|vv'| \leq |vw| + |wv'| \leq 1 + \ell(v') - \ell(w)$. The condition $|vv'| \geq \Lambda_0$ implies $\ell(v') \geq \ell(w) + \Lambda_0 - 1$ and therefore $r(v') \leq a^{1-\Lambda_0}r(w)$. Thus $\operatorname{diam} T_{v'} \leq 2r(v') \leq \lambda r(w)/3a$. Now if $\operatorname{dist}(t_0, T_{v'}) \leq \lambda r(w)/3a$ for some $t_0 \in T_v$, then for any other $t \in T_v$ we have

$$\operatorname{dist}(t, T_{v'}) \geq \operatorname{dist}(t, t_0) - \operatorname{dist}(t_0, T_{v'}) - \operatorname{diam} T_{v'} \geq \lambda r(w)/3a.$$

Thus in any case, there are distinct $t_1, t_2 \in T_v$ with $\operatorname{dist}(t_i, T_{v'}) \geq \lambda r(w)/3a$, $i = 1, 2$.

We show that every quadruple Q formed by t_1, t_2 and any distinct $t_1', t_2' \in T_{v'}$ is admissible. Indeed, we obtain as above that $d(t_1,t_2)d(t_1',t_2') \leq 4r(v)r(v')$ while $m \geq (\lambda/3a)^2 r(w)^2$ for any other member m of the cross-ratio triple M of Q. We have chosen Λ_0 in a way that $4r(v)r(v') < (\lambda/3a)^2 r(w)^2$, thus Q is admissible with

$$\operatorname{cr}(Q) \leq 4(3a/\lambda)^2 \frac{r(v)r(v')}{r(w)^2} \leq 4(3a/\lambda)^2 a^{-|vv'|}.$$

Together with (7.6), this implies (7.3), and (1) is proved.

Finally, assume that there is an admissible quadruple Q for T_v, $T_{v'}$ with $\mathrm{cr}(Q) \le \lambda_0$. Using (7.5), we obtain $\lambda_0 \ge \lambda^2 a^{-(|vv'|+1)}/4$ and thus $|vv'| \ge \log_a(\lambda^2/4\lambda_0) - 1 \ge \Lambda_0$. Hence, (2) holds. $\square$

Remark 7.3.6. In the proof above, we only used that triples T_v, $v \in V$, are $\lambda r(v)$-separated and did not use the condition $\mathrm{cr}(T_v, z_0) \ge \lambda$. This last condition will be used when we consider what happens with triples T_v while applying the map f to them.

We define a map $F: V \to V'$ by taking for every $v \in V$ a vertex $v' = F(v) \in V'$ of highest level for which the ball $B(v') \subset Z'$ contains the triple $f(T_v)$. Here as usual, V' is the vertex set of the graph X'. The map F is well defined (up to the distance error ≤ 1) because the singular point $z_0 \in Z$ is a member of no triple T_v, $v \in V$. One of the reasons why we change the definition of F comparing with that of Theorem 7.2.1 is that for vertices $v \in V$ with $z_0 \in B(v)$ the image $f(B(v))$ is unbounded.

In the next step of the proof of Theorem 7.3.2 we show that F is roughly homothetic. For a triple of distinct points $T \subset Z$, we let $r(T)$ be the minimal ratio of their distances,

$$r(T) = \min \frac{d(t,t'')}{d(t,t')},$$

where the minimum is taken over all permutations of the points of T.

Lemma 7.3.7. *Under the conditions of Theorem 7.3.2, assume that a triple $T \subset B_s(z)$ is λ-representative. Then $r(f(T)) \ge \lambda_1$, where $\lambda_1 \in (0,1)$ depends only on λ, p and q (one can take $\lambda_1 = \lambda^p/q$).*

Proof. Consider the quadruple $Q = (T, z_0) \subset Z$. The condition $f(z) \to \infty$ as $z \to z_0$ implies $r(f(T)) = \mathrm{cr}(f(Q))$. Since f is θ-quasi-Möbius, $\mathrm{cr}(f(Q)) \ge \mathrm{cr}(Q)^p/q$. On the other hand, $\mathrm{cr}(Q) \ge \lambda$ by the assumption on T. Hence, the claim. $\square$

Lemma 7.3.8. *For every $v \in V$, the triple $T'_v = f(T_v) \subset B(v')$ is $\lambda' r(v')$-separated, $v' = F(v)$, with $\lambda' \in (0,1)$ depending only on a, λ, p, q (one can take $\lambda' = \lambda_1/2a$, see Lemma 7.3.7).*

Proof. There is $k \in \mathbb{Z}$ with $r^{k+1} \le \operatorname{diam} T'_v < r^k$ and $w \in V'$ with $\ell(w) = k$ and $\operatorname{dist}(w, T'_v) \le r^k$. Then $d(w,t) \le \operatorname{dist}(w, T'_v) + \operatorname{diam} T'_v < r(w)$ for every $t \in T'_v$, i.e. $T'_v \subset B(w)$. It follows from the definition of F that $\ell(v') \ge \ell(w) = k$. Thus for $s = r(v')$, we have $s \le 2ar^{k+1} \le 2a \operatorname{diam} T'_v$. Since the minimal ratio $r(T'_v) \ge \lambda_1$ by Lemma 7.3.7 and thus the triple T'_v is $(\lambda_1 \operatorname{diam} T'_v)$-separated, we see that T'_v is $\lambda' s$-separated with $\lambda' = \lambda_1/2a$. $\square$

Since the space Z is uniformly perfect, the map f is strictly quasi-Möbius by Proposition 5.3.7. In particular, there is a threshold constant $h \in (0,1)$ such that $\mathrm{cp}(f(Q)) = f(\mathrm{cp}(Q))$ for every quadruple $Q \subset Z$ of distinct points with $\mathrm{cr}(Q) \le h$. Moreover, f^{-1} is strictly quasi-Möbius as well, and we easily see that any $h' \in (0,1)$ with $h' \le 1/\theta(1/h) = h^p/q$ is a threshold constant for f^{-1}.

Lemma 7.3.9. *There is $\lambda_2 \in (0,1)$ depending only on h, p and q (one can take $\lambda_2 \le h$ and $q\lambda_2^p \le h^p/q$) such that if $v, v' \in V$ satisfy the condition*

$$m = \min_Q \mathrm{cr}(Q) \le \lambda_2,$$

where the minimum is taken over all admissible quadruples for the triples T_v, $T_{v'}$, then

$$m' = \min_{Q'} \mathrm{cr}(Q') \asymp_q m^p,$$

where the minimum is taken over all admissible quadruples for the triples $f(T_v)$, $f(T_{v'})$.

Proof. We fix admissible quadruples Q for T_v, $T_{v'}$ and Q' for $f(T_v)$, $f(T_{v'})$ with $\mathrm{cr}(Q) = m$ and $\mathrm{cr}(Q') = m'$ respectively. Since $\lambda_2 \le h$, the map f preserves the cross-pair of Q, $f(\mathrm{cp}(Q)) = \mathrm{cp}(f(Q))$. Thus $f(Q)$ is admissible for $f(T_v)$, $f(T_{v'})$ with

$$m' \le \mathrm{cr}(f(Q)) \asymp_q \mathrm{cr}(Q)^p = m^p,$$

in particular, $\mathrm{cr}(Q') \le \mathrm{cr}(f(Q)) \le q\lambda_2^p$. Since $q\lambda_2^p \le h^p/q$ which is a threshold constant for f^{-1}, the quadruple $Q'' \subset T_v \cup T_{v'}$ with $f(Q'') = Q'$ is admissible for T_v, $T_{v'}$. We have

$$m' = \mathrm{cr}(Q') \asymp_q \mathrm{cr}(Q'')^p \ge m^p,$$

hence, the claim. $\square$

We denote by Λ_0', λ_0', σ_0' the constants of Proposition 7.3.5 obtained from Λ_0, λ_0, σ_0 respectively by replacing the separation constant λ by the constant λ' from Lemma 7.3.8.

Proposition 7.3.10. *There is $\Lambda > 1$ depending only on a, h, p, q, λ such that for every $v, v' \in V$ with $|vv'| \ge \Lambda$ the equality*

$$|F(v)F(v')| \doteq p|vv'|$$

holds up to an error $\le \sigma$ (one can take $\Lambda \ge \Lambda_0$ so that $\sigma_0 a^{-\Lambda} \le \lambda_2$, $q\sigma_0^p a^{-p\Lambda} \le \lambda_0'$ and $\sigma = \log_a(q\sigma_0'\sigma_0^p)$).

Proof. Since $|vv'| \ge \Lambda \ge \Lambda_0$, we have by Proposition 7.3.5 (1)

$$m = \min_Q \mathrm{cr}(Q) \asymp a^{-|vv'|}$$

up to a multiplicative error $\le \sigma_0$, where the minimum is taken over all admissible quadruples Q for the triples T_v, $T_{v'}$.

We have $m \le \sigma_0 a^{-\Lambda} \le \lambda_2$ by the choice of Λ, hence by Lemma 7.3.9

$$m' = \min_{Q'} \operatorname{cr}(Q') \asymp_q m^p,$$

where the minimum is taken over all admissible quadruples for the triples $f(T_v)$, $f(T_{v'})$.

Since $m^p \asymp a^{-p|vv'|}$ up to a multiplicative error $\le \sigma_0^p$, we obtain $m' \asymp a^{-p|vv'|}$ up to a multiplicative error $\le q\sigma_0^p$. Therefore, $m' \le q\sigma_0^p a^{-p\Lambda} \le \lambda_0'$ by the choice of Λ, and $|F(v)F(v')| \ge \Lambda_0'$ by Proposition 7.3.5 (2) applied for $F(v)$, $F(v')$ (see Remark 7.3.6). Then again by Proposition 7.3.5 (1) we have

$$a^{-|F(v)F(v')|} \asymp m' \text{ up to a multiplicative error at most } \sigma_0',$$

hence $|F(v)F(v')| \doteq_\sigma p|vv'|$ with $\sigma = \log_a(q\sigma_0'\sigma_0^p)$. □

By Proposition 7.3.10 the map $F\colon X \to X'$ is roughly homothetic and thus it induces a map $\partial_\infty F\colon \partial_\infty X \to \partial_\infty X'$.

Corollary 7.3.11. *We have $\partial_\infty F(z) = f(z)$ for every $z \in Z \setminus z_0$ and $\partial_\infty F\colon \partial_\infty X \to \partial_\infty X'$ is a bijection with $\partial_\infty F(z_0) = \omega'$, where $\omega' \in \partial_\infty X'$ corresponds to the infinitely remote point ∞ under the identification $\partial_\infty X' = \overline{Z}' \cup \{\infty\}$.*

Proof. The argument is actually the same as at the end of the proof of Theorem 7.2.1. Namely, given $z \in Z \setminus z_0$, the point z_0 misses a closed ball $\overline{B}$ in Z containing z. Then $f(B) \subset Z'$ is bounded. Choosing a sequence of vertices $\{v_n\} \in z$ with $B(v_{n+1}) \subset B(v_n) \subset B$ for every $n \ge 0$, we find that the sequence $\{v_n' = F(v_n)\} \subset V'$ determines a point $z' = \partial_\infty F(z) \in \overline{Z}' \cup \{\infty\}$. Note that for the corresponding representative triples, we have $T_{v_n} \to z$ and $f(T_{v_n}) \to f(z)$ as $n \to \infty$, because f is continuous. Since $f(T_{v_n}) \subset f(B)$ for all $n \ge 0$, we see that $z' \ne \infty$ and the levels $\ell(n)$ of v_n' tend to infinity as $n \to \infty$. Thus $f(T_{v_n}) \to z'$ which implies $z' = f(z) \in Z'$, i.e., $\partial_\infty F(z) = f(z)$ for every $z \in Z \setminus z_0$.

For every $z' \in \overline{Z}'$, there is a Cauchy sequence $z_n' \in f(Z \setminus z_0)$ converging to z', $z_n' \to z'$. Then the singular point $z_0 \in Z$ cannot be an accumulation point for $z_n = f^{-1}(z_n')$, since otherwise $z_n' \to \infty$. Using that f controls the cross-ratios, we easily see that $\{z_n\}$ is Cauchy and thus $z' = \partial_\infty F(z)$ for some $z \in \overline{Z} \setminus z_0$. This means $\partial_\infty F(z_0) \notin \overline{Z}'$, thus $\partial_\infty F(z_0) = \omega'$.

Finally, the metric completion $\overline{f(Z \setminus z_0)}$ is contained in the image $\partial_\infty F(\partial_\infty X)$ because the last one is complete and contains $f(Z \setminus z_0)$. Since $\overline{f(Z \setminus z_0)} = \overline{Z}' = \partial_\infty X' \setminus \omega'$, we see that $\partial_\infty X'$ coincides with the image of $\partial_\infty F$ and thus $\partial_\infty F$ is bijective. □

To complete the proof of Theorem 7.3.2, it remains to show that $F(X)$ is a net in X'. This follows from a general fact:

Lemma 7.3.12. *Let $F: X \to X'$ be a quasi-isometric map between hyperbolic approximations with bijective induced map $\partial_\infty F: \partial_\infty X \to \partial_\infty X'$. Then F is a quasi-isometry, i.e., $F(X)$ is a net in X'.*

Proof. Given $v' \in V'$, consider a bi-infinite radial geodesic $\gamma' \subset X'$ with $\gamma'(0) = v'$. Then for its ends $\xi', \omega' \in \partial_\infty X'$, there are distinct $\xi, \eta \in \partial_\infty X$ with $\partial_\infty F(\xi) = \xi'$, $\partial_\infty F(\eta) = \omega'$. Consider a bi-infinite geodesic $\gamma \subset X$ with end points ξ, η (the existence of γ is obvious in the case ξ or η coincides with $\omega \in \partial_\infty X$; otherwise, there is a branch point $w \in V$ for ξ, η, which yields γ). Now by stability of geodesics, $\operatorname{dist}(v', F(\gamma))$ is bounded by a constant depending only on F. □

Now for the proof of Theorem 7.3.1, we use as the basic tool the following:

Lemma 7.3.13. *Given a metric space Z and a nonisolated point $z_0 \in Z$, there is an unbounded metric space Z_0 and a quasi-Möbius map $\varphi: Z \setminus z_0 \to Z_0$ with dense image, $\overline{\varphi(Z \setminus z_0)} = \overline{Z}_0$, and with control function $\theta(t) = qt^p$ for some $p > 0$, $q \geq 1$, such that $\varphi(z) \to \infty$ as $z \to z_0$.*

Proof. Let d be the metric on Z. We first consider the inversion $\rho = d'$ of d with respect to z_0, $\rho(z, z') = d(z, z')/(d(z_0, z)d(z_0, z'))$ for every z, $z' \in Z$ distinct from z_0, see Section 5.3.1. By Proposition 5.3.6, ρ is a K-quasi-metric with $K = 4$, and (Z, ρ) is an unbounded quasi-metric space with infinitely remote point $\omega = z_0$. Moreover, the identity map $\varphi: (Z \setminus z_0, d) \to (Z, \rho)$, $\varphi(z) = z$, is Möbius with dense image, and $\varphi(z) \to \infty$ as $z \to z_0$. That is, all requirements of the lemma are satisfied for (Z, ρ), except that ρ might not be a metric.

To obtain a metric space Z_0, we take a power of ρ, $\rho' = \rho^p$ with a sufficiently small $p \in (0, 1)$. By Proposition 2.2.6, if $p \leq 1/4$, then ρ' is bilipschitz to a metric d_0. Now we take as Z_0 the metric space $(Z \setminus z_0, d_0)$. Then the identity map $\varphi: Z \setminus z_0 \to Z_0$ is quasi-Möbius with control function $\theta(t) = qt^p$ for some $q \geq 1$, and it satisfies all requirements of the lemma. □

Proof of Theorem 7.3.1. If the spaces Z, Z' both are unbounded and f preserves the infinitely remote points, $f(\omega) = \omega'$, then f is a quasi-symmetry by Proposition 5.2.15, and Theorem 7.2.1 can be applied.

Assume that Z, Z' both are bounded. We fix $z_0 \in Z$ and put $z_0' = f(z_0) \in Z'$. Applying Lemma 7.3.13, we find quasi-Möbius maps $\varphi: Z \setminus z_0 \to Z_0$, $\varphi': Z' \setminus z_0' \to Z_0'$ with dense images to which in turn Theorem 7.3.2 can be applied. In that way, we find rough similarities $\Phi: X \to X_0$, $\Phi': X' \to X_0'$ which induce φ, φ', where X_0, X_0' are hyperbolic approximations of Z_0, Z_0' respectively. Now the spaces Z_0, Z_0' are unbounded, and the composition $\psi = \varphi' \circ f \circ \varphi^{-1}: Z_0 \to Z_0'$ is quasi-Möbius preserving the infinitely remote points. Applying the first case above, we find a quasi-isometry $\Psi: X_0 \to X_0'$ which induces ψ. Therefore, the quasi-isometry $\Phi'^{-1} \circ \Psi \circ \Phi: X \to X'$ induces $\varphi'^{-1} \circ \psi \circ \varphi = f$.

The remaining cases are similar, and we leave them to the reader as an exercise. □

Corollary 7.3.14. *Every quasi-Möbius map $f: Z \to Z'$ between bounded, uniformly perfect metric spaces is power quasi-symmetric.*

Proof. One can assume that the spaces are complete and f is bijective. Then, by Theorem 7.3.1, f is the boundary value of a quasi-isometry between (truncated) hyperbolic approximations X, X' of Z, Z' respectively. Since any hyperbolic approximation is geodesic, f is power quasi-symmetric by Theorem 5.2.17 (2), because the metrics of Z, Z' being bounded are visual with respect to points inside of X, X' respectively. □

7.4 Supplementary results and remarks

7.4.1 The hyperbolic cone

It follows from the proof of Theorem 7.1.2 that the condition on the target space X' to be geodesic can be relaxed to the existence of $o' \in X'$ such that every $x' \in X'$ can be connected to o' by a geodesic segment. This is useful e.g. for the hyperbolic cone construction; see Section 6.4.4. Thus we obtain

Corollary 7.4.1. *Every visual hyperbolic space X is roughly similar to a subspace of the cone $\mathrm{Co}(\partial_\infty X)$ over its boundary at infinity taken with any visual metric. Furthermore, if X is in addition geodesic then X and $\mathrm{Co}(\partial_\infty X)$ are roughly similar to each other.* □

7.4.2 Power quasi-symmetric and quasi-Möbius embeddings

It is known that a quasi-symmetric embedding of a uniformly perfect space Z is power quasi-symmetric, see e.g. [He], Theorem 11.3. It is also known that such a map is quasi-Möbius (in classical sense and without uniform perfection assumption), [V1], Theorem 3.2. Theorem 7.2.1 combined with Theorem 5.2.17 gives another proof of these facts.

Corollary 7.4.2. *Any quasi-symmetric map $f: Z \cup \{\omega\} \to Z' \cup \{\omega'\}$, where Z is uniformly perfect, is power quasi-symmetric and power quasi-Möbius.*

Proof. We can assume without loss of generality that f is surjective, hence a homeomorphism, see Exercise 5.2.12, and that Z is complete. Then Z' is also uniformly perfect, complete, and, furthermore, f preserves the infinitely remote point, $f(\omega) = \omega'$ (in the case $\omega = \emptyset$, i.e., Z is bounded, this means that the image Z' is also bounded). By Theorem 7.2.1, f is the boundary value of a quasi-isometry between hyperbolic approximations of Z, Z'. Since the metrics of Z, Z' are visual for X, X' with appropriately chosen base points, see Theorems 6.3.1 and 6.4.1, it follows from Theorem 5.2.17 (1) that f is PQ-Möbius and from Theorem 5.2.17 (2) that f is PQ-symmetric. □

Similarly, from Theorem 7.3.1 and Theorem 5.2.17 (1), we obtain

Corollary 7.4.3. *Let Z, Z' be complete, uniformly perfect metric spaces. Then any quasi-Möbius bijection $f : Z \cup \{\omega\} \to Z' \cup \{\omega'\}$ is power quasi-Möbius.* □

Bibliographical note. There are several approaches to extension results discussed in this chapter; see [Tu], [Pa], [BoS]. The idea of our approach via hyperbolic approximations was discussed in [BP] without giving however any detail.

The important Lemma 7.1.3 is taken from [BoS], Lemma 5.1.

Chapter 8

Embedding theorems

In this chapter we prove two important embedding results which have a number of applications. The second one, the Bonk–Schramm embedding theorem (Theorem 8.2.1), is an application of the first one, the Assouad embedding theorem (Theorem 8.1.1).

8.1 Assouad embedding theorem

A metric space Z is said to be *doubling* if there is a constant $M \in \mathbb{N}$ such that every ball in Z can be covered by at most M balls of the half radius. For more about doubling spaces see Section 8.3.1.

If d is a metric on a space Z then d^p is also a metric on Z for every $p \in (0,1)$ (cf. Remark 2.2.3). The transformation $d \mapsto d^p$ is sometimes called a *snow-flake transformation*, because any curve in Z that is rectifiable with respect to d becomes nonrectifiable with respect to d^p.

Theorem 8.1.1. *Let (Z,d) be a doubling metric space. Then for every $p \in (0,1)$ there is a bilipschitz embedding $\varphi\colon (Z,d^p) \to \mathbb{R}^N$, where $N \in \mathbb{N}$ depends only on p and the doubling constant of the metric d.*

For the proof we use a hyperbolic approximation of Z. Recall that for every $k \in \mathbb{Z}$ a maximal r^k-separated set $V_k \subset Z$ is fixed, where $r \le 1/6$. The graph structure X is not used in the proof and plays only a heuristic role.

Proof. There are two important ingredients of the proof. The first one is a coloring of the vertex set $V = \bigcup_{k\in\mathbb{Z}} V_k$. Since Z is doubling we can find a finite set A with cardinality $|A|$ depending only on the doubling constant of d such that for every $k \in \mathbb{Z}$ there is a coloring $\lambda_k\colon V_k \to A$ with $\lambda_k(v) \neq \lambda_k(v')$ for any distinct $v, v' \in V_k$ with $d(v,v') \le 4r^{k-1}$ (see Proposition 8.3.3). Furthermore, we take a finite set B, whose cardinality will depend only on p and which we will specify later in the proof. We fix a periodic coloring $\mu\colon \mathbb{Z} \to B$ such that $\mu(k) = \mu(k')$ if and only if k and k' are congruent modulo $|B|$.

Now for $C = A \times B$ we define the coloring $\lambda\colon V \to C$ by $\lambda(v) = (\lambda_k(v), \mu(k))$ for every $v \in V_k$, $k \in \mathbb{Z}$. For every color $c \in C$ we put $V^c = \lambda^{-1}(c)$, $V_k^c = V^c \cap V_k$.

Given an element z in Z let us define for a vertex $v \in V$ the number $z(v) = \max\{0, 1 - d(v,z)/r(v)\} \in [0,1]$, where $r(v) = 2r^k$ for $v \in V_k$. The "decimal" decomposition of z for a color $c \in C$ is now defined as the set

$$D_c(z) = \{z(v) : v \in V^c\}.$$

Since $z(v) = 0$ for $d(v,z) \geq 2r^k$ and $4r^k < 4r^{k-1}$, there is, by the properties of the coloring λ, for every level $k \in \mathbb{Z}$ at most one nonzero component $z(v)$ of $D_c(z)$. Therefore, for every color $c \in C$ every point $z \in Z$ determines a sequence $\gamma_c(z) = \{v \in V_k^c : z(v) \neq 0,\ k \in \mathbb{Z}\}$, which can be regarded as a "geodesic" in X and which converges to z as $k \to +\infty$. Note that for different $z, z' \in Z$ the "geodesics" $\gamma_c(z)$, $\gamma_c(z')$ coincide on all sufficiently large negative levels.

The second important ingredient of the proof is the scaling of $D_c(z)$ by the factor $(2r^k)^p$ for every level $k \in Z$ in the following definition of the map $\varphi\colon Z \to \mathbb{R}^C$. We fix $z_0 \in Z$ and define φ by its coordinate functions $\varphi_c\colon Z \to \mathbb{R}$,

$$\varphi_c(z) := \sum_{v \in V^c} (z(v) - z_0(v))\, r(v)^p, \quad c \in C.$$

The reference point z_0 is introduced to guarantee that the series converges on negative levels $k < 0$. Indeed, $|z(v) - z_0(v)|\, r(v)^p \leq d(z, z_0) r(v)^{p-1}$, and since $p < 1$, the series converges on negative levels as a geometric series. On the other hand, $|z(v) - z_0(v)|\, r(v)^p \leq 2r(v)^p$, and the series also converges on positive levels as a geometric series. Clearly, for the bilipschitz property the reference point z_0 plays no role.

The idea is that the labelling and the scaling together allow to locate precisely the place in the hyperbolic approximation X where the "geodesics" $\gamma_c(z)$, $\gamma_c(z')$ start to diverge by comparing the scaled "decimal" decompositions of $z, z' \in Z$ for some color $c \in C$. This leads directly to the required bilipschitz property of the map φ with respect to the metric d^p.

Given different $z, z' \in Z$, we define their *critical level* $n = n(z,z') \in \mathbb{Z}$ by the condition

$$3r^{n+1} < d(z,z') \leq 3r^n. \tag{8.1}$$

In the following estimates we put d as a short for $d(z,z')$. For $c \in C$ and $k \in \mathbb{Z}$ we put

$$\Delta_{c,k} = \Delta_{c,k}(z,z') = \sum_{v \in V_k^c} (z(v) - z'(v)) r(v)^p.$$

The following calculations are elementary. As tool we will only use the standard estimates for the geometric series: for $0 \leq b < 1$ we have

$$\sum_{k \geq m} b^k \leq \frac{b^m}{1-b}$$

and for $1 < b$ we have

$$\sum_{k \le m} b^k \le \frac{b^{m+1}}{b-1}.$$

We will use this series for $b = r^p < 1$ respectively for $b = r^{(p-1)} > 1$. We will also use the estimate

$$r^{p(n+1)} \le 3^p r^{p(n+1)} \le d^p, \tag{8.2}$$

which follows from (8.1).

Since for every k there exists at most one $v \in V_k^c$ with $z(v) \neq 0$ and one $v' \in V_k^c$ with $z'(v') \neq 0$, we have $|\Delta_{c,k}| \le 2(2r^k)^p = 2^{p+1} r^{pk}$ and hence for all $m \ge n$ we obtain

$$\sum_{k \ge m} |\Delta_{c,k}| \le \frac{2^{p+1}}{1 - r^p} r^{pm} \le c_1(r,p) r^{p(m-n)} d^p \tag{8.3}$$

with $c_1(r,p) = \frac{2^{p+1}}{r^p(1-r^p)}$, where we used in the last inequality the estimate (8.2).

If $k \le n$ and $d(z,v)$, $d(z',v') \le 2r^k$ for $v, v' \in V_k$, then $d(v,v') \le 4r^k + d(z,z') \le 4r^k + 3r^n \le 4r^{k-1}$ and hence v and v' have different colors or $v = v'$. This shows that for $k \le n$ the set V_k^c contains at most one vertex v for which $z(v) - z'(v) \neq 0$. Thus we estimate

$$|\Delta_{c,k}| \le (2r^k)^p |z(v) - z'(v)| \le 2^{p-1} d \cdot r^{(p-1)k} \le 3 \cdot 2^{p-1} r^n r^{(p-1)k}$$

for some $v \in V_k^c$, since $2r^k |z(v) - z'(v)| \le d(z,z') = d$ and $d \le 3r^n$. We obtain therefore for $m \le n$

$$\sum_{k \le m} |\Delta_{c,k}| \le \frac{3 \cdot 2^{p-1} r^n}{r^{p-1} - 1} \cdot r^{(p-1)(m+1)} \le c_2(r,p) r^{(1-p)(n-m)} d^p, \tag{8.4}$$

where $c_2(r,p) = (3/2)^{1-p}/(r^p - r)$. Here we used again (8.2) in the last inequality.

Now φ is Lipschitz because

$$|\varphi_c(z) - \varphi_c(z')| \le \sum_{k \le n} |\Delta_{c,k}| + \sum_{k \ge n+1} |\Delta_{c,k}| \le c_3(r,p) d^p(z,z')$$

for every color c, where $c_3(r,p) = c_1(r,p) r^p + c_2(r,p)$.

To prove the bilipschitz property we consider different $z, z' \in Z$. Since the balls $B_{r_k}(v)$, $v \in V_k$, cover Z for every $k \in \mathbb{Z}$ we note that there exists a color $c \in C$ such that $d(v,z) \le r^{n+1}$ for some $v \in V_{n+1}^c$, where $n = n(z,z')$ is the critical level. We decompose

$$\varphi_c(z) - \varphi_c(z') = \sum_{k \le n} \Delta_{c,k} + \Delta_{c,n+1} + \sum_{k \ge n+2} \Delta_{c,k} \tag{8.5}$$

and show that only the middle term essentially contributes to this decomposition. From $d(v,z) \leq r^{n+1}$ we conclude that first $z(v) \geq 1/2$ and second z' is not in $B_{2r^{n+1}}(v)$ since $d(z,z') > 3r^{n+1}$. Thus $z'(v) = 0$ and we obtain

$$\Delta_{c,n+1} = (2r^{n+1})^p z(v) \geq 2^{p-1} r^{p(n+1)} \geq 2^{p-1}(r/3)^p d^p$$

by the right-hand side of estimates (8.1).

Using the periodicity of the coloring, we obtain

$$\sum_{k \leq n} |\Delta_{c,k}| = \sum_{k \leq m} |\Delta_{c,k}| \leq c_2(r,p) r^{(1-p)(|B|-1)} d^p$$

for $m = n + 1 - |B|$ by (8.4), and

$$\sum_{k \geq n+2} |\Delta_{c,k}| = \sum_{k \geq m} |\Delta_{c,k}| \leq c_1(r,p) r^{p(|B|+1)} d^p$$

for $m = n + 1 + |B|$ by (8.3). If we choose $|B|$ large enough (only depending on p and r), we see that the sum of the two boundary terms in the decomposition (8.5) is bounded by $2^{p-2}(r/3)^p d^p$ which implies the bilipschitz property of φ. $\square$

Remark. For $p = 1/2$ one can check using the expressions for $c_1(r,p)$, $c_2(r,p)$ that four periodic colors, $|B| = 4$, suffice for the mapping φ to be bilipschitz if r is chosen sufficiently small. In general, $|B| \to \infty$ as $p \to 0$ or $p \to 1$.

8.2 Bonk–Schramm embedding theorem

Theorem 8.2.1. *Let X be a visual Gromov hyperbolic geodesic space whose boundary at infinity is doubling for some visual metric. Then there exists $n \geq 2$ such that X is roughly similar to a convex subset of* H^n.

Let Z be a compact subset of $\partial_\infty \mathrm{H}^n$, $n \geq 2$. Its *convex hull* in H^n is the intersection of all convex $Y' \subset \mathrm{H}^n$ with $\partial_\infty Y' \supset Z$.

Lemma 8.2.2. *Given the convex hull Y of a compact $Z \subset \partial_\infty \mathrm{H}^n$, $n \geq 2$, containing more than one point, we have*

(1) $\partial_\infty Y = Z$;

(2) *Y is visual in* H^n.

Proof. First of all $Y \neq \emptyset$ since Z contains more than one point.

(1) It follows from the definition that $Z \subset \partial_\infty Y$. To prove the opposite inclusion, given $z \in \partial_\infty \mathrm{H}^n \setminus Z$, we consider a geodesic $\gamma\colon \mathbb{R} \to \mathrm{H}^n$ with $\gamma(\infty) = z$. Then every hyperplane $E \subset \mathrm{H}^n$ orthogonal to γ bounds a half-space $E_+ \subset \mathrm{H}^n$, whose boundary at infinity $\partial_\infty E_+$ is a neighborhood of z in $\partial_\infty \mathrm{H}^n$ and such neighborhoods

form a basis for z. Since Z is compact there is a hyperplane $E = E(t)$ orthogonal to γ at some $\gamma(t)$ with $\partial_\infty E_+ \cap Z = \emptyset$. Then for every $t' > t$ the half-space $E_-(t')$ opposite to $E_+(t')$ is a convex subset in H^n with $Z \subset \partial_\infty E_-(t')$ and $z \notin \partial_\infty E_-(t')$. Thus $z \notin \partial_\infty Y$.

(2) Fix $o \in Y$. It suffices to show that for every $y \in Y$ there is $\xi \in Z = \partial_\infty Y$ with $|oy| \le (y|\xi)_o + \delta$, where δ is the hyperbolicity constant of H^n.

Assume that this is not the case, and for some $y \in Y$ we have $|oy| > (y|\xi)_o + \delta$ for all $\xi \in Z$. We show that there is $\alpha \in (0, \pi/2)$ such that the angle of the (infinite) triangle $oy\xi$ at y is at most α, $\measuredangle_y(o,\xi) \le \alpha$, for all $\xi \in Z$. Indeed, otherwise since Z is compact there is $\xi \in Z$ with $\measuredangle_y(o,\xi) \ge \pi/2$. Let $z_0 \in oy$, $y_0 \in o\xi$, $o_0 \in y\xi$ be the equiradial points of $oy\xi$. Then $|z_0 y| \le |z_0 o_0| \le \delta$ and $|oy| = |oz_0| + |z_0 y| \le (y|\xi)_o + \delta$, which contradicts our assumption.

Now we conclude that the hyperplane $E \subset \mathrm{H}^n$, orthogonal to the segment oy at some point $y' \in oy$ close enough to y, bounds the half-space $E_+ \subset \mathrm{H}^n$ whose boundary at infinity contains Z, $\partial_\infty E_+ \supset Z$. Therefore, $y \notin Y$, a contradiction. □

Lemma 8.2.3. *Given a compact $Z \subset \mathbb{R}^n$, the Euclidean metric on Z is bilipschitz to the restriction to Z of some visual metric on $\partial_\infty \mathrm{H}^{n+1} = \mathbb{R}^n \cup \{\infty\}$, where H^{n+1} is considered in the upper half-space model.*

Proof. Let $g\colon B^{n+1} \to C^{n+1}$ be the isomorphism of the unit ball and the upper half-space models of H^{n+1}; see Appendix, Section A.3 and A.5. Then g^{-1} restricted to any compact $Z \subset \mathbb{R}^n = \partial_\infty C^{n+1}$ is bilipschitz with respect to the Euclidean metric on $\mathbb{R}^n$ and the spherical metric on $S^n = \partial_\infty B^{n+1}$ (the bilipschitz constant depends on Z).

On the other hand, by Section 2.4.3, the spherical distance on S^n is a visual metric on H^{n+1} with respect to the base point o and the parameter $a = e$. □

Proof of Theorem 8.2.1. We can assume that X is unbounded. Then since it is visual, $\partial_\infty X \ne \emptyset$. If $\partial_\infty X$ consists of one point, then X is roughly isometric to a ray (see Exercise 7.1.1), and the claim is obvious. Thus we assume that $\partial_\infty X$ contains more than one point. By Theorem 8.1.1, there is a bilipschitz embedding $(\partial_\infty X, d^{1/2}) \to \mathbb{R}^n$ for some $n \ge 1$, where d is a visual metric on $\partial_\infty X$. Note that taking the power $p = 1/2$ of d corresponds to the p-homothety of X, pX.

Let $Z \subset \mathbb{R}^n$ be the image of the embedding. We consider $\mathbb{R}^n$ as a part of the boundary at infinity $\mathbb{R}^n \cup \{\infty\}$ of the hyperbolic space H^{n+1} (in the upper half-space model). Then the convex hull $Y \subset \mathrm{H}^{n+1}$ of Z is hyperbolic and geodesic with respect to the induced metric, and its boundary at infinity coincides with Z by Lemma 8.2.2 (1). By Lemma 8.2.2 (2), Y is visual. By Lemma 8.2.3, there is a visual metric on $\partial_\infty Y$ bilipschitz to the metric of Z induced from $\mathbb{R}^n$. Now applying Corollary 7.1.6, we see that the space $\frac{1}{2}X$ and hence X is roughly similar to Y. □

Remark 8.2.4. The same argument as above (with Corollary 7.1.6 replaced by Corollary 7.1.4) shows that any visual hyperbolic (not necessarily geodesic) space X with doubling boundary at infinity is roughly similar to a subspace of a convex subset of H^n for an appropriate $n \geq 2$.

8.3 Supplementary results and remarks

8.3.1 More about doubling spaces

An equivalent definition of the property of Z to be doubling is that there is $M \in \mathbb{N}$ such that for every $r > 0$ every ball of radius $2r$ in Z contains at most M points which are r-separated.

Exercise 8.3.1. Show that this definition is equivalent to the initial one. Show that the property of a metric space Z to be doubling is equivalent to the fact that there is a function $M\colon [1,\infty) \to \mathbb{N}$ such that for every $r > 0$ every ball of radius ρr with $\rho \geq 1$ in Z contains at most $M(\rho)$ points which are r-separated.

The property to be doubling is hereditary: if a metric space X is doubling, then every subspace $X' \subset X$ is doubling. The basic example is $\mathbb{R}^n$ and its subsets. On the other hand, any tree with uniformly bounded length of edges and degree of every vertex ≥ 3 is not doubling, as well as H^n for any $n \geq 2$ and any Hadamard manifold with sectional curvatures separated from 0. Recall that a *Hadamard manifold* X is a simply connected, complete Riemannian manifold with nonpositive sectional curvatures.

Exercise 8.3.2. Show that the property to be doubling is quasi-symmetry invariant.

The *degree of a vertex* v of a graph X is the number of edges in X adjacent to v.

Proposition 8.3.3. *A metric space Z is doubling if and only if the degree of vertices of a hyperbolic approximation X of Z is uniformly bounded.*

Proof. Assume that the degree of vertices of X is uniformly bounded, i.e., there is a constant $M \in \mathbb{N}$ such that the number of edges adjacent to any vertex of X is at most M. Consider a ball $B_{2s}(z) \subset Z$. There is $k \in \mathbb{Z}$ with $r^{k+1} < 2s \leq r^k$, where we recall $r \leq 1/6$ is the parameter of X (see Chapter 6).

Since the vertex set $V_k \subset Z$ of level k is an r^k-net in Z, there is $v \in V_k$ with $d(z,v) \leq r^k$. Then $B_{2s}(z) \subset B(v)$, where $B(v) = B_{2r^k}(v)$. For the radius $2r^{k+2}$ of every ball $B(w) \subset Z$, $w \in V_{k+2}$, we have $2r^{k+2} = 2r^{k+1}r < 4sr < s$. Furthermore, recall that the balls $B_{r^j}(w)$, $w \in V_j$, cover Z for every $j \in \mathbb{Z}$. Thus to show that Z is doubling, it suffices to estimate the number N of $w \in V_{k+2}$ with $B(w) \cap B(v) \neq \emptyset$. By Corollary 6.2.7, the distance in X between v and w is $|vw| \leq |\ell(v) - \ell(w)| + 1 = 3$. Hence $N \leq M^3$, and Z is doubling.

Conversely, assume that Z is doubling, and let $M : [1, \infty) \to \mathbb{N}$ be an appropriate control function, see Exercise 8.3.1. Consider a vertex v of the graph X, $v \in V_k$ for some $k \in \mathbb{Z}$. Every horizontal edge of X adjacent to v corresponds to a vertex $w \in V_k$ with $\overline{B}(v) \cap \overline{B}(w) \neq \emptyset$. Thus $w \in \overline{B}_{4r^k}(v) \subset Z$ and the number of horizontal edges adjacent to v is bounded above by the number of points $w \in V_k$ sitting in the ball $\overline{B}_{4r^k}(v)$. Since the set V_k is r^k-separated, the last number is at most $M(4)$. By a similar argument, the number of radial edges connecting v with upper level vertices is bounded above by $M(2/r)$.

Finally, consider radial edges connecting v with lower level vertices. Every such edge corresponds to a ball $B(w) \subset Z$ with $w \in V_{k-1}$, containing the ball $B(v)$. Hence $w \in B_{2r^{k-1}}(v)$. Since V_{k-1} is r^{k-1}-separated, we obtain that there are at most $M(2)$ radial edges connecting v with lower level vertices. □

8.3.2 Spaces with bounded geometry

Here we discuss the question how to decide whether a boundary at infinity of a hyperbolic space X is doubling by looking at the space X itself. This is important in view of the Bonk–Schramm theorem (Theorem 8.2.1).

There are a number of definitions reflecting a property we are looking for. We prefer to use the following one basically for the reason that the corresponding property is quasi-isometry invariant.

A metric space X has *bounded geometry* if there are a constant $r > 0$ and a function $M: [1, \infty) \to \mathbb{N}$ such that every ball of radius ρr, $\rho \geq 1$, in X can be covered by $M(\rho)$ balls of radius r. This is equivalent to the property that every ball of radius $\rho r'$ in X contains at most $M'(\rho)$ points which are r'-separated for some constant $r' > 0$ and a function $M': [1, \infty) \to \mathbb{N}$.

The property to have bounded geometry is, obviously, hereditary. Moreover, it is straightforward to check the following.

Lemma 8.3.4. *If a metric space X has bounded geometry and $f : X' \to X$ is quasi-isometric, then X' also has bounded geometry.* □

Consequently, the property to have bounded geometry is a quasi-isometry invariant. However, it is not at all clear how to check that a given metric space has bounded geometry. Thus we consider the following property which is typically easy to check.

A metric space X is *doubling at some scale* if there are constants $r > 0$ and $M \in \mathbb{N}$ such that every ball of radius $2r$ in X contains at most M points which are r-separated.

For example, every doubling space is doubling at some scale. Clearly, spaces with bounded geometry are doubling at some scale.

Exercise 8.3.5. Show that if a geodesic metric space X is doubling at some scale, then X has bounded geometry. This is not true in general without assumption that the space is geodesic.

Examples 8.3.6. (i) Every Hadamard manifold X with bounded sectional curvature has bounded geometry. Indeed, X is geodesic and every ball $B_r(x) \subset X$ is bilipschitz to an open Euclidean ball of radius r, where the bilipschitz constant is arbitrarily close to 1 for sufficiently small r which is separated from 0 independent of x, since the sectional curvatures are uniformly bounded. Hence, X is doubling at some scale and has bounded geometry by Exercise 8.3.5.

(ii) Every graph X with length of edges uniformly separated from zero and uniformly bounded degree of vertices is, obviously, doubling at some scale and therefore it has bounded geometry by Exercise 8.3.5. In particular, every word hyperbolic group (see Section 1.4.2) has bounded geometry and by Proposition 8.3.3, any hyperbolic approximation of a doubling metric space has bounded geometry.

The converse to the last example is also true.

Proposition 8.3.7. *Assume that a hyperbolic approximation X of a metric space Z has bounded geometry. Then Z is doubling.*

In what follows, we use our standard notations V_k for the vertex set of X of level $k \in \mathbb{Z}$. For the proof we need the following lemma.

Lemma 8.3.8. *Given $k \in \mathbb{Z}$, for every $w \in V_k$, there is a radial ray $w_0w_1\dots$ in X starting at $w_0 = w$ such that for distinct $w, w' \in V_k$ we have $|w_nw_n'| \geq 2(n-1)$ for every $n \geq 1$.*

Proof. Assume that the vertex $w_n \in V_{k+n}$, $n \geq 0$, is already defined. Then we take as w_{n+1} a vertex from V_{k+n+1} with $d(w_n, w_{n+1}) \leq r^{k+n+1}$ (recall that we use notation $d(v,w)$ for the distance in Z between vertices v, w of X, and that V_j is an r^j-net in Z for every $j \in \mathbb{Z}$). Then $B(w_{n+1}) \subset B(w_n)$ since $r \leq 1/6$, and this determines a radial ray in X.

We have $d(w,w_n) \leq \sum_{j\geq 1} r^{k+j} = r^{k+1}/(1-r)$ for every $n \geq 1$. For distinct $w, w' \in V_k$ we have $d(w,w') \geq r^k$, thus we obtain

$$d(w_n, w_n') \geq r^k - 2r^{k+1}/(1-r) = r^k(1-3r)/(1-r)$$

for every $n \geq 1$. On the other hand, let $v_j, v_j' \in V_j$ be lowest level vertices of a shortest segment w_nw_n' in X, $|v_jv_j'| \leq 1$ (see Lemma 6.2.6). Then $B(w_n) \subset B(v_j)$, $B(w_n') \subset B(v_j')$, and we obtain $d(w_n, w_n') \leq 2r^j + d(v_j, v_j') + 2r^j \leq 8r^j$. Together with the former estimate, this yields $8r^{j-k} \geq (1-3r)/(1-r)$ and thus $j - k \leq 1$ since $r \leq 1/6$. Therefore, $|w_nw_n'| \geq 2(n+k-j) \geq 2(n-1)$. □

Proof of Proposition 8.3.7. By Proposition 8.3.3, it suffices to show that the degree of vertices of X is uniformly bounded.

Let $v \in V_k$ be a vertex of X. Applying Lemma 8.3.8 to the vertices of the same level $j = k-1, k, k+1$ adjacent to v, we see that for every $n \geq 2$, there are at least

N_j points in the ball of radius $n+1$ in X centered at v which are $2(n-1)$-separated. Here N_j is the number of vertices of level j adjacent to v, so $N = N_{k-1} + N_k + N_{k+1}$ is the degree of v.

On the other hand, there is $s > 0$ and a function $M : [1, \infty) \to \mathbb{N}$ such that every ball of radius ρs in X contains at most $M(\rho)$ points which are s-separated, since X has bounded geometry. Choosing n sufficiently large with $2(n-1) \geq s$ and taking $\rho = (n+1)/s$, we obtain $N \leq 3M(\rho)$ independently of v. Hence, the claim. $\square$

Theorem 8.3.9. *Assume that a hyperbolic geodesic space X has bounded geometry. Then its boundary at infinity $\partial_\infty X$ is doubling with respect to any visual metric.*

Proof. By Theorem 7.1.2 and Corollary 7.1.4, any hyperbolic approximation of $\partial_\infty X$ is roughly similar to a subset of X and hence it has bounded geometry. Then, by Proposition 8.3.7, $\partial_\infty X$ is doubling. $\square$

Corollary 8.3.10. *The boundary at infinity of every Hadamard manifold with pinched negative sectional curvature, $-b^2 \leq K \leq -a^2 < 0$, is doubling. The boundary at infinity of every word hyperbolic group is doubling. In particular, every such space is roughly similar to a convex subset of H^n for an appropriate $n \geq 2$.* $\square$

Remark 8.3.11. The last corollary for hyperbolic groups has already been proved in Chapter 2 using local self-similarity of the boundary at infinity; see Corollary 2.3.7.

Corollary 8.3.12. *The boundary at infinity of every word hyperbolic group has finite topological dimension.* $\square$

Bibliographical note. Theorem 8.1.1 is due to P. Assouad [As2] and is optimal in the sense that in general $p = 1$ cannot be taken for the value of the parameter p: the Carnot–Caratheodory metric on S^3 which naturally occurs a boundary metric of the complex hyperbolic plane $\mathbb{C}\,\mathrm{H}^2$ is doubling, but admits no bilipschitz embedding into any Euclidean space $\mathbb{R}^N$; see [He], Chapter 12.

Our proof of Theorem 8.1.1 follows the original ideas from [As2] but technically it is somewhat different due to the explicit use of a hyperbolic approximation.

Theorem 8.1.1 is the main ingredient of the proof of Theorem 8.2.1, the Bonk–Schramm embedding result [BoS].

Theorem 8.3.9 has appeared in [BoS, Theorem 9.2] in a slightly different form. Our proof is based on different ideas and uses different techniques.

Chapter 9

Basics of dimension theory

Important tools in studying metric spaces are various coverings, and basic dimension type invariants are defined via multiplicity of coverings. In this chapter, we discuss a number of dimensions all of which are close relatives of the classical topological or covering dimension. Similarly as the topological dimension is invariant under homeomorphisms, other dimensions we consider are invariant under different types of morphisms, i.e., they are quasi-isometry, or bilipschitz, or even quasi-symmetry invariants. All those dimensions were proven to be useful for a large spectrum of problems, in particular, for embedding and nonembedding problems.

9.1 Various dimensions

Let X be a metric space. For $U,\ U' \subset X$ we denote by $\operatorname{dist}(U,U')$ the distance between U and U', $\operatorname{dist}(U,U') = \inf\{|uu'| : u \in U,\ u' \in U'\}$ where $|uu'|$ is the distance between u, u'. For $r > 0$ we denote by $B_r(U)$ the open r-neighborhood of U, $B_r(U) = \{x \in X : \operatorname{dist}(x,U) < r\}$, and by $\overline{B}_r(U)$ the closed r-neighborhood of U, $\overline{B}_r(U) = \{x \in X : \operatorname{dist}(x,U) \le r\}$. We extend the notation $\overline{B}_r(U)$ for all real r putting $B_r(U) = U$ for $r = 0$, and defining $B_r(U)$ for $r < 0$ as the complement of the closed $|r|$-neighborhood of $X \setminus U$, $B_r(U) = X \setminus \overline{B}_{|r|}(X \setminus U)$.

Given a family $\mathcal{U}$ of subsets in X we define $\operatorname{mesh}(\mathcal{U}) = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}$. The *multiplicity* of $\mathcal{U}$, $m(\mathcal{U})$, is the maximal number of members of $\mathcal{U}$ with nonempty intersection. A family $\mathcal{U}$ is called a *covering* of X if $\bigcup\{U : U \in \mathcal{U}\} = X$.

9.1.1 Topological dimension

The *topological dimension* of X is the minimal integer $\dim X = n$ such that for every $\varepsilon > 0$ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n + 1$ and $\operatorname{mesh}(\mathcal{U}) \le \varepsilon$.

In this and all cases below, if the appropriate number n does not exist, we say that the corresponding dimension of X is infinite.

The topological dimension is obviously invariant under uniformly continuous homeomorphisms, and it was extensively studied during the last century.

The definitions of other dimensions involve control over the Lebesgue number of coverings, which – in contrast to the topological dimension – makes them depending on a chosen metric in a more crucial way. This feature is highly useful for applications.

Let $\mathcal{U}$ be an open covering of a metric space X. Given $x \in X$, we let

$$L(\mathcal{U}, x) = \sup\{\operatorname{dist}(x, X \setminus U) : U \in \mathcal{U}\}$$

be the *Lebesgue number* of $\mathcal{U}$ at x and $L(\mathcal{U}) = \inf_{x \in X} L(\mathcal{U}, x)$ the Lebesgue number of $\mathcal{U}$. For every $x \in X$, the open ball $B_r(x)$ of radius $r = L(\mathcal{U})$ centered at x is contained in some member of the covering $\mathcal{U}$. Note that $L(\mathcal{U})$ might be larger than $\operatorname{mesh}(\mathcal{U})$ and even infinite, e.g., if $\mathcal{U}$ contains X as a covering element for a bounded X.

Though this definition can be applied to any covering, sometimes it yields inappropriate results, e.g. for the covering $\mathcal{U} = \{[i, i+1] : i = 0, \dots, n-1\}$ of the segment $X = [0, n]$, we have $L(\mathcal{U}) = 0$.

Remark 9.1.1. We shall often use the following obvious fact. If the Lebesgue number of an (open) covering $\mathcal{U}$ of a metric space X is positive, $r = L(\mathcal{U}) > 0$, then the family $B_{-s}(\mathcal{U}) = \{B_{-s}(U) : U \in \mathcal{U}\}$ is still a covering of X for every $s < r$.

9.1.2 Asymptotic dimension

The *asymptotic dimension* of X is the minimal integer $\operatorname{asdim} X = n$ such that for every positive d there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n+1$, $\operatorname{mesh}(\mathcal{U}) < \infty$ and $L(\mathcal{U}) \ge d$.

Clearly, $\operatorname{asdim} X = 0$ for every bounded X (take the covering with only one member, X). However, for unbounded spaces, the asymptotic dimension is highly interesting and useful. Due to the large Lebesgue numbers, the condition for coverings to be open is not essential. We easily see that $\operatorname{asdim} X$ is a quasi-isometry invariant of X.

9.1.3 Assouad–Nagata dimension

The *Assouad–Nagata dimension* of X is the minimal integer $\operatorname{ANdim} X = n$ with the following property: there exists $\delta \in (0, 1)$ such that for every positive τ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n+1$, $\operatorname{mesh}(\mathcal{U}) \le \tau$ and $L(\mathcal{U}) \ge \delta\tau$.

This dimension is obviously a bilipschitz invariant. Surprisingly, the Assouad–Nagata dimension as well as the following dimension are in fact quasi-symmetry invariants, which is not at all obvious.

The Assouad–Nagata dimension is an example of linearly controlled dimensions, which means that the Lebesgue number of coverings involved in the definition of a dimension is at least a fixed linear function of their mesh.

9.1.4 Linearly controlled metric dimension

The *linearly controlled metric dimension* or *ℓ-dimension* of a metric space X is the minimal integer $\ell\text{-}\dim X = n$ with the following property: there exists $\delta \in (0, 1)$ such that for every sufficiently small $r > 0$ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \leq n + 1$, $\operatorname{mesh}(\mathcal{U}) \leq r$ and $L(\mathcal{U}) \geq \delta r$.

9.1.5 Linearly controlled asymptotic dimension

The *linearly controlled asymptotic dimension* or asymptotic ℓ-dimension of X is the minimal integer $\ell\text{-asdim}\, X = n$ with the following property: there exists $\delta \in (0, 1)$ such that for every sufficiently large $R > 1$ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \leq n + 1$, $\operatorname{mesh}(\mathcal{U}) \leq R$ and $L(\mathcal{U}) \geq \delta R$.

Exercise 9.1.2. Check that the linearly controlled asymptotic dimension is a quasi-isometry invariant.

The distinction between the last three dimensions is that while the Assouad–Nagata dimension takes into account all scales, the ℓ-dimension reflects only features of the space at arbitrarily small scales, and the asymptotic ℓ-dimension takes into account the large scales only. We obviously have

$$\dim X \leq \ell\text{-}\dim X \leq \operatorname{ANdim} X \quad \text{and} \quad \operatorname{asdim} X \leq \ell\text{-asdim}\, X \leq \operatorname{ANdim} X,$$

and we easily see that

$$\operatorname{ANdim} X = \max\{\ell\text{-}\dim X, \ell\text{-asdim}\, X\}$$

for every metric space X, in particular, $\ell\text{-}\dim X = \operatorname{ANdim} X$ for every bounded space X.

One can continue this list indefinitely, e.g., we discuss another useful dimension, the hyperbolic dimension, in Chapter 13.

How to work with these definitions? For example, how to compute the topological dimension of the Euclidean space $\mathbb{R}^n$? Everybody knows that $\dim \mathbb{R}^n = n$, however, the argument is not at all on the surface. Even to observe the easier estimate $\dim \mathbb{R}^n \leq n$, the best way is to use an alternative definition of the topological dimension, called the colored definition.

Namely, we say that a family $\mathcal{U}$ of subsets in a metric space X is *disjoint* if its multiplicity equals 1, $m(\mathcal{U}) = 1$. A covering $\mathcal{U}$ is said to be *colored* if it is the union of $m \geq 1$ disjoint families, $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$, $|A| = m$. In this case, we also say that $\mathcal{U}$ is m-colored. Clearly, the multiplicity of an m-colored covering is at most m.

9.1.6 Colored definition of a dimension

We define $\dim_{\mathrm{col}} X$ as a minimal integer n such that for every $\varepsilon > 0$ there is an open $(n+1)$-colored covering $\mathcal{U}$ of X with $\operatorname{mesh}(\mathcal{U}) \le \varepsilon$. Similarly, there are colored definitions of every dimension from the list above.

Clearly, $\dim_{\mathrm{col}} X \ge \dim X$ and it turns out that $\dim_{\mathrm{col}} X = \dim X$ for every metric space X, see Proposition 9.3.7, and a similar equality is true for every dimension on the list.

Example 9.1.3. We show that $\dim_{\mathrm{col}} \mathbb{R}^n \le n$. As a color set, we take the cyclic group $A = \mathbb{Z}/(n+1)\mathbb{Z}$ of order $n+1$, which we identify with $A = \{0, \dots, n\}$. Now we define the family $\mathcal{U}^0$ as follows. Take the middle open subsegment $J \subset [0,1]$ of length $l \in (0,1)$ and consider the cube $J^n \subset \mathbb{R}^n$. The integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ acts on $\mathbb{R}^n$ by translations, and we put

$$\mathcal{U}^0 = \{\gamma J^n : \gamma \in \mathbb{Z}^n\}.$$

For every color $a \in A$, we take the cube $J^n_a = J^n + a\nu$, where the vector $\nu \in \mathbb{R}^n$ is of the form $\nu = (n+1)^{-1}\{1, \dots, 1\}$, and define

$$\mathcal{U}^a = \{\gamma J^n_a : \gamma \in \mathbb{Z}^n\}.$$

Clearly, every $\mathcal{U}^a$, $a \in A$, is a disjoint family in $\mathbb{R}^n$. We easily see that if l is sufficiently close to 1, then the family $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$ is a covering of $\mathbb{R}^n$ (see Figure 9.1 for the case $n = 2$). Moreover, its Lebesgue number $L(\mathcal{U}) \ge \delta_n$, where the constant $\delta_n > 0$ depends only on n.

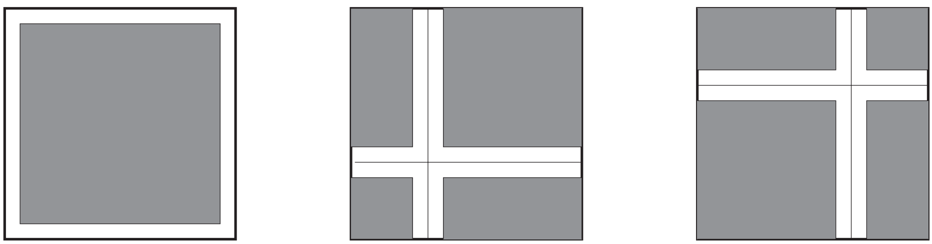


Figure 9.1. Three shifted copies of J^2 (grey) cover the torus $\mathbb{R}^2/\mathbb{Z}^2$.

Taking a homothety $h_\tau \colon \mathbb{R}^n \to \mathbb{R}^n$, $h_\tau(x) = \tau x$, with coefficient $\tau > 0$, we obtain a covering $\mathcal{U}_\tau = h_\tau(\mathcal{U})$ with $\operatorname{mesh}(\mathcal{U}_\tau) \le \tau\sqrt{n}$ and $L(\mathcal{U}_\tau) \ge \delta_n \tau$. This shows that the colored dimension of $\mathbb{R}^n$ of each type from the list above (i.e. topological, asymptotic and all three linearly controlled ones) is at most n.

We explain how to get the estimate $\dim \mathbb{R}^n \ge n$ in Section 9.8.

9.1.7 Polyhedral definition of a dimension

There is another useful characterization of dimensions via Lipschitz maps into simplicial polyhedra. For example, one can define $\dim_{\mathrm{pol}} X$ as a minimal integer n such that for every $\varepsilon > 0$ there is a Lipschitz map $f : X \to K$ into a simplicial polyhedron K of combinatorial dimension $\le n$ with $\operatorname{diam} f^{-1}(y) \le \varepsilon$ for every $y \in K$.

Similarly, there is a polyhedral definition of every dimension from the list above, and for every dimension, the different definitions are equivalent. The advantage of this is an additional flexibility in working with dimensions.

Now to proceed further, we need to introduce some standard constructions.

9.2 Constructions

9.2.1 Uniform simplicial polyhedra

Given an index set J, we let $\mathbb{R}^J$ be the Euclidean space of functions $J \to \mathbb{R}$ with finite support, i.e., $x \in \mathbb{R}^J$ if and only if only finitely many coordinates $x_j = x(j)$ are non-zero. The distance $|xx'|$ is well defined by

$$|xx'|^2 = \sum_{j \in J} (x_j - x_j')^2.$$

Let $\Delta^J \subset \mathbb{R}^J$ be the *standard simplex*, i.e., $x \in \Delta^J$ if and only if $x_j \ge 0$ for all $j \in J$ and $\sum_{j \in J} x_j = 1$. The metric of $\mathbb{R}^J$ induces a metric on Δ^J and on every subcomplex $K \subset \Delta^J$, i.e., the distance between two points in K is the distance between them in $\mathbb{R}^J$. The topology of K is the metric topology.

A metric in a simplicial polyhedron K is said to be *uniform* if K is isometric to a subcomplex of Δ^J for some index set J. Every simplex $\sigma \subset K$ is then isometric to the standard k-simplex $\Delta^k \subset \mathbb{R}^{k+1}$, $k = \dim \sigma$. So, for a finite J, $\dim \Delta^J = |J| - 1$, and this equality serves as the definition of the combinatorial dimension: the *combinatorial dimension* of a simplicial polyhedron is the maximal dimension of its simplices. Speaking about dimension of a simplicial polyhedron, we always mean the combinatorial dimension.

For every simplicial polyhedron K, there is the canonical embedding $u : K \to \Delta^J$, where J is the vertex set of K, which is affine on every simplex. Its image $K' = u(K)$ is called the *uniformization* of K, and u is the *uniformization map*.

A subpolyhedron K' of a simplicial polyhedron K is said to be *complete* if every simplex of K whose vertices are in K' is also a simplex of K'. A simplicial polyhedron is *locally finite* if every vertex is the member of only finitely many simplices. The standard simplex Δ^J is not locally finite if $|J| = \infty$.

Given a simplicial polyhedron K and a vertex $v \in K$, its *star* $\overline{\mathrm{st}}_v$ consists of all simplices of K, containing v, and the *open star* st_v is the star $\overline{\mathrm{st}}_v$ without faces opposite to v. For each vertex v of every standard simplex Δ^J, the open star st_v is open in Δ^J. Indeed, any point $x \in \mathrm{st}_v$ is characterized by the condition $x_v > 0$. Then for every $\varepsilon \in (0, x_v)$, the intersection $B_\varepsilon(x) \cap \Delta^J$ is contained in st_v because for $y \in B_\varepsilon(x)$ we have $|xy| < \varepsilon$, hence $|x_v - y_v| < \varepsilon$ and thus $y_v > 0$.

As a consequence, we obtain that the open star of every vertex of a uniform simplicial polyhedron K is open in K.

Remark 9.2.1. The open stars of the vertices cover any simplicial polyhedron K, and if $\dim K$ is finite for a uniform K then clearly the Lebesgue number of this covering is bounded from below by a positive constant depending only on $\dim K$.

9.2.2 The nerve of a covering and barycentric maps

With every covering $\mathcal{U} = \{U_j\}_{j \in J}$ of a space X, one associates a simplicial polyhedron $\mathcal{N} = \mathcal{N}(\mathcal{U})$ called the *nerve* of $\mathcal{U}$. The vertex set of $\mathcal{N}$ is identified with the set J representing the covering members, and a subset $J' \subset J$ spans a simplex if and only if all U_j with $j \in J'$ have a common point. The nerve $\mathcal{N}$ can always be considered as a subcomplex of Δ^J, $\mathcal{N} \subset \Delta^J$, and therefore as a uniform polyhedron. Furthermore, we have $\dim \mathcal{N} = m(\mathcal{U}) - 1$.

A covering $\mathcal{U}$ of a space X is said to be *locally finite* if any $x \in X$ has a neighborhood which meets only finitely many members of $\mathcal{U}$.

Let $\mathcal{U} = \{U_j\}_{j \in J}$ be a locally finite open covering of a metric space X and $\mathcal{N} = \mathcal{N}(\mathcal{U}) \subset \Delta^J$ its nerve. One defines a *barycentric map*

$$p \colon X \to \mathcal{N}$$

associated with $\mathcal{U}$ as follows. Note that the Lebesgue number at x, $L(\mathcal{U}, x)$, is positive for every $x \in X$ though it might be infinite. We fix a positive function $d \colon X \to \mathbb{R}$ with $d(x) \le L(\mathcal{U}, x)$ for every $x \in X$ called a *cut function*. Given $j \in J$, we put $q_j \colon X \to \mathbb{R}$, $q_j(x) = \min\{d(x), \mathrm{dist}(x, X \setminus U_j)\}$. Then we have $\sum_{j \in J} q_j(x) \ge d(x) > 0$, and $\sum_{j \in J} q_j(x) < \infty$ for every $x \in X$ because $\mathcal{U}$ is locally finite.

Now the map $p \colon X \to \Delta^J$ is well defined by its coordinate functions $p_j(x) = q_j(x) / \sum_{j \in J} q_j(x)$, $j \in J$, which are nothing else than the barycentric coordinates. Clearly, the image of a point lands in the nerve, $p(X) \subset \mathcal{N}$.

Note that for each vertex $v \in \mathcal{N}$, the preimage of its open star, $p^{-1}(\mathrm{st}_v) \subset X$, coincides with that member of the covering $\mathcal{U}$ associated with v.

Lemma 9.2.2. *Assume that the multiplicity of an open locally finite covering $\mathcal{U}$ is finite, $m(\mathcal{U}) = m < \infty$, and $L(\mathcal{U}) \ge d > 0$. Then there is a Lipschitz barycentric map $p \colon X \to \mathcal{N}$ with Lipschitz constant* $\mathrm{Lip}(p) \le (m+1)^2/d$.

Proof. We take the barycentric map $p \colon X \to \mathcal{N}$ determined via the constant cut function, $d(x) = d$ for every $x \in X$. Then $\sum_{j \in J} q_j(x) \ge d$ for every $x \in X$.

Furthermore, $q_j(x) \le q_j(x') + |xx'|$ and

$$\sum_{j \in J} q_j(x) \le \sum_{j \in J} q_j(x') + m|xx'|,$$

because there are at most m nonzero summands in each sum. Using this, we obtain

$$\frac{1}{\sum_{j \in J} q_j(x')} \le \frac{1}{\sum_{j \in J} q_j(x)} + \frac{m|xx'|}{d \cdot \sum_{j \in J} q_j(x)}.$$

Then abbreviating $\sigma = \sum_{j \in J} q_j(x)$, $\sigma' = \sum_{j \in J} q_j(x')$, we obtain

$$p_j(x') - p_j(x) = \frac{q_j(x') - q_j(x)}{\sigma'} + \left(\frac{1}{\sigma'} - \frac{1}{\sigma}\right) q_j(x) \le \frac{m+1}{d}|xx'|.$$

Finally, for $p = \{p_j\}_{j \in J}$ we have

$$|p(x')p(x)|^2 = \sum_{j \in J} (p_j(x') - p_j(x))^2 \le \frac{(m+1)^2(2m)}{d^2}|xx'|^2,$$

hence, $\operatorname{Lip}(p) \le (m+1)^2/d$. □

9.2.3 Barycentric subdivision

The *barycentric subdivision* of the standard simplex Δ^J is a subcomplex $\operatorname{ba}\Delta^J$ of the standard simplex $\Delta^{\mathcal{J}}$ where $\mathcal{J}$ is the set of all finite nonempty subsets of J called the *finite power set*. The subcomplex $\operatorname{ba}\Delta^J$ is defined as follows: a finite collection α of vertices of $\Delta^{\mathcal{J}}$ spans a simplex of $\operatorname{ba}\Delta^J$ if and only if for any two members of α considered as subsets in J, one of them is contained in the other.

There is the canonical bijection $\operatorname{ba}_J : \Delta^J \to \operatorname{ba}\Delta^J$ called the *barycentric subdivision map*. The inverse map ba_J^{-1} sends every vertex $a \in \mathcal{J}$ into the barycenter

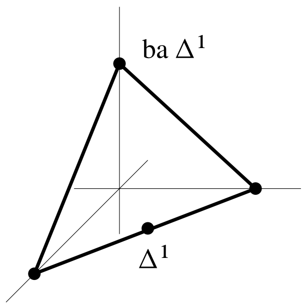


Figure 9.2. The barycentric subdivision of Δ^1.

of the corresponding face $\Delta^a \subset \Delta^J$, and it is affine on every face of $\operatorname{ba}\Delta^J$. Given a uniform polyhedron $K \subset \Delta^J$, its barycentric subdivision is by definition the subcomplex $\operatorname{ba} K = \operatorname{ba}_J(K) \subset \operatorname{ba}\Delta^J$. Therefore, $\operatorname{ba} K$ is also a uniform simplicial polyhedron. Often we consider $\operatorname{ba} K$ as the barycentric triangulation of K, i.e., we identify $\operatorname{ba} K$ with K via ba_J if there is no danger of confusion.

Lemma 9.2.3. *Assume that an index set J is finite, $|J| < \infty$. Then for each $y, y' \in \operatorname{ba}\Delta^J$, there is $y'' \in \operatorname{ba}\Delta^J$ such that the pairs y, y'' and y'', y' lie in simplices of $\operatorname{ba}\Delta^J$ and $|yy''| + |y''y'| \le C|yy'|$ for some constant depending only on $|J|$.*

Proof. There are simplices $\sigma, \sigma' \subset \operatorname{ba}\Delta^J$ with a common face, $\sigma \cap \sigma' \neq \emptyset$, containing y, y' respectively. Take $\sigma \ni y, \sigma' \ni y'$ of minimal dimension. If $\sigma \cap \sigma' = \emptyset$ then the subset I of J which is the union of the subsets representing all vertices of σ, σ' is the common vertex of the simplices spanned by σ, I and σ', I.

Now we take $y'' \in \sigma \cap \sigma'$ such that the broken geodesic path $yy''y'$ is a shortest path in $\sigma \cup \sigma'$ between y, y'. It suffices to estimate from below the angle α between the segments $y''y$ and $y''y'$ at y'' by a positive constant depending only on $|J|$. We can assume that $\sigma \neq \sigma'$, and thus $\sigma \cap \sigma'$ is a proper face of σ, σ'. Let $A \subset \mathbb{R}^{\mathcal{J}}$ be the affine subspace spanned by the face $\sigma \cap \sigma'$, and let B, $B' \subset \mathbb{R}^{\mathcal{J}}$ be the affine half subspaces bounded by A, $\partial B = A = \partial B'$, containing y, y' respectively. Then α is bounded from below by the angle between B, B', that is, by the angle between the rays with a common vertex in A, which are orthogonal to A and lie in B, B' respectively.

When y, y' run over σ, σ' respectively, the corresponding subspaces B, B' are parameterized by points of the faces $\delta \subset \sigma$, $\delta' \subset \sigma'$ opposite to $\sigma \cap \sigma'$, i.e., we can assume that $y \in \delta$, $y' \in \delta'$. The worst case occurs when we take $y \in \delta$, $y \in \delta'$ as the barycenters and $\dim \sigma \cap \sigma' = 0$. In this case, $\alpha > 0$ depends only on the distances $|y|, |y'|$ to the origin of $\mathbb{R}^{\mathcal{J}}$, which are bounded from below by a constant depending only on $|J|$. $\square$

Lemma 9.2.4. *For every uniform, simplicial, finite dimensional polyhedron $K \subset \Delta^J$, the map $\operatorname{ba}_J : K \to \operatorname{ba} K$ is bilipschitz,*

$$|xx'|/C \le |\operatorname{ba}_J(x)\operatorname{ba}_J(x')| \le C|xx'|$$

for each $x, x' \in K$, where the constant $C \ge 1$ depends only on $\dim K$.

Proof. The property is obvious for x, x' whose images $\operatorname{ba}_J(x)$, $\operatorname{ba}_J(x')$ lie in one and the same simplex of $\operatorname{ba}\Delta^J$ because the map ba_J^{-1} is affine on every face of $\operatorname{ba}\Delta^J$.

In the general case, we note that x, x' are contained in a simplex $\sigma \subset \Delta^J$ whose dimension depends only on $\dim K$, and we consider the restriction of ba_J to σ. Then for any $y, y' \in \operatorname{ba}\sigma$ according to Lemma 9.2.3, there is $y'' \in \operatorname{ba}\sigma$ such that each of the pairs y, y'' and y'', y' lie in simplices of $\operatorname{ba}\sigma$ and $|yy''| + |y''y'| \le C|yy'|$ for some constant C depending only on $\dim\sigma$. From this, we easily obtain the bilipschitz property of ba_J for x, x'. $\square$

Remark 9.2.5. One important feature of the barycentric subdivision is that the covering $\mathcal{U}$ of any finite dimensional polyhedron $K \subset \Delta^J$ by the open stars of the vertices of its barycentric subdivision $\mathrm{ba}\, K$ is open and m-colored with $m = \dim K + 1$. The color of a star $\mathrm{st}_v \subset \mathrm{ba}\, K$ for $v \in \mathcal{J}$ is $|v|$, the number of elements from J,

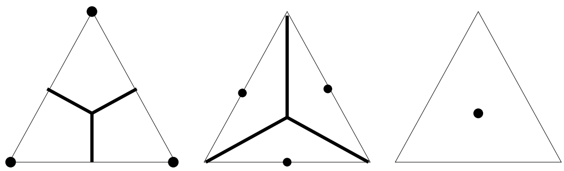


Figure 9.3. Three disjoint families of open stars cover Δ^2.

$|v| \le \dim K + 1$. Furthermore, it follows from Remark 9.2.1 and Lemma 9.2.4 that the Lebesgue number of $\mathcal{U}$ is bounded from below by a positive number depending only on $\dim K$.

9.2.4 The barycentric triangulation of a product

Here, we describe the canonical triangulation of the product of two simplicial complexes which is based on the barycentric subdivision.

We start with the case of two simplices. Let J_1, J_2 be index sets, $J = J_1 \cup J_2$ the disjoint union, and $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}$ the power sets of J_1, J_2, J respectively.

Recall that the vertex set of the simplicial complex $\mathrm{ba}\, \Delta^J$ is $\mathcal{J}$. Every member of $\mathcal{J}_1 \times \mathcal{J}_2$ being a pair with members from $\mathcal{J}_1$, $\mathcal{J}_2$ is canonically identified with a member of $\mathcal{J}$, that is, it is a vertex of $\mathrm{ba}\, \Delta^J$. By definition, the *barycentric triangulation* of $\Delta^{J_1} \times \Delta^{J_2}$ is the complete simplicial subcomplex $\Delta^{J_1} \times_s \Delta^{J_2}$ of $\mathrm{ba}\, \Delta^J$ spanned by $\mathcal{J}_1 \times \mathcal{J}_2$. This means that a finite collection α of members of $\mathcal{J}_1 \times \mathcal{J}_2$ spans a simplex of $\Delta^{J_1} \times_s \Delta^{J_2}$ if and only if for any two members of α considered as subsets in J, one of them is contained in the other, see Section 9.2.3.

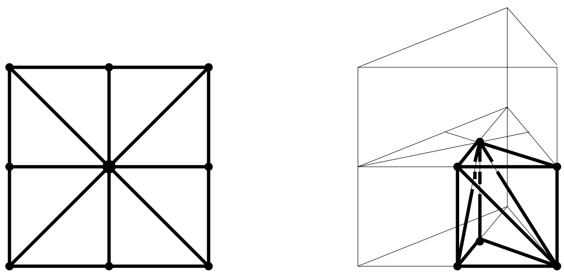


Figure 9.4. The barycentric triangulation of $\Delta^1 \times \Delta^1$ (left) and a part of the barycentric triangulation of $\Delta^2 \times \Delta^1$ (right).

There is the canonical bijection $\varphi\colon \Delta^{J_1} \times \Delta^{J_2} \to \Delta^{J_1} \times_s \Delta^{J_2}$ also called the *barycentric triangulation map*, for which the inverse map φ^{-1} sends every vertex $a = (a_1, a_2) \in \mathcal{J}_1 \times \mathcal{J}_2$ of $\Delta^{J_1} \times_s \Delta^{J_2}$ into $(b_1, b_2) \in \Delta^{J_1} \times \Delta^{J_2}$, where b_i is the barycenter of the corresponding face $\Delta^{a_i} \subset \Delta^{J_i}$, $i = 1, 2$, and it is affine on every simplex of $\Delta^{J_1} \times_s \Delta^{J_2}$.

Now let K_1, K_2 be uniform simplicial polyhedra which we identify with subpolyhedra of Δ^{J_1}, Δ^{J_2} respectively, where J_i is the vertex set of K_i, $i = 1, 2$. Then $K_1 \times K_2 \subset \Delta^{J_1} \times \Delta^{J_2}$ and $\varphi(K_1 \times K_2) \subset \Delta^{J_1} \times_s \Delta^{J_2}$ is clearly a subcomplex, where $\varphi\colon \Delta^{J_1} \times \Delta^{J_2} \to \Delta^{J_1} \times_s \Delta^{J_2}$ is the barycentric triangulation map. We define the barycentric triangulation of the product $K_1 \times K_2$ as $K_1 \times_s K_2 = \varphi(K_1 \times K_2)$.

Lemma 9.2.6. *Given finite dimensional simplicial polyhedra $K_1 \subset \Delta^{J_1}$, $K_2 \subset \Delta^{J_2}$, the barycentric triangulation map*

$$\varphi\colon K_1 \times K_2 \to K_1 \times_s K_2$$

is bilipschitz with bilipschitz constant depending only on the dimensions of K_1, K_2.

Proof. Any pair of points in either of the factors K_1, K_2 is contained in a simplex with dimension at most two times the dimension of the factor. Thus we can assume that $K_i = \Delta^{J_i}$ with finite J_i, and we are looking for an estimate depending only on both $|J_i|$, $i = 1, 2$.

Now the proof is similar to that of Lemma 9.2.4. If $y = \varphi(x)$, $y' = \varphi(x')$ lie in one and the same simplex of $K = K_1 \times_s K_2$ for some $x, x' \in K_1 \times K_2$ then the property is obvious because φ^{-1} is affine on every simplex.

Otherwise, the argument of Lemma 9.2.3 can be applied to K as well, and we find $y'' \in K$ such that the pairs y, y'' and y'', y' lie in simplices of K and $|yy''| + |y''y'| \le C|yy'|$ for some constant depending only on $|J_1| + |J_2|$. Hence, the claim. □

Remark 9.2.7. Since the open stars of the vertices of $K_1 \times_s K_2$ form a covering, their preimages under the barycentric triangulation map cover $K_1 \times K_2$. In the case that $\dim K_1, \dim K_2$ are finite, this covering is open, and it follows from Remark 9.2.1 and Lemma 9.2.6 that its Lebesgue number is bounded from below by a positive constant depending only on $\dim K_1, \dim K_2$.

We need the following:

Lemma 9.2.8. *Let K_1, K_2 be uniform simplicial polyhedra, $K = K_1 \times_s K_2$. For every vertex $v \in K$, there are vertices $v_1 \in K_1$, $v_2 \in K_2$ such that $\varphi^{-1}(\mathrm{st}_v) \subset \mathrm{st}_{v_1} \times \mathrm{st}_{v_2}$, where $\varphi\colon K_1 \times K_2 \to K$ is the barycentric triangulation map.*

Proof. Consider first the case of two simplices. Let J_1, J_2 be index sets and $\mathcal{J}_1$, $\mathcal{J}_2$ the power sets of J_1, J_2 respectively. Every vertex v of $\Delta^{J_1} \times_s \Delta^{J_2}$ is a member of $\mathcal{J}_1 \times \mathcal{J}_2$, $v = (a_1, a_2)$, and therefore, there are $v_1 \in J_1$, $v_2 \in J_2$ entering v, $v_1 \in a_1$,

$v_2 \in a_2$. Then $\varphi^{-1}(v) = (b_1, b_2)$, where b_i is the barycenter of Δ^{a_i}. Because v_i is a vertex of Δ^{a_i}, $i = 1, 2$, we have $\varphi^{-1}(\mathrm{st}\, v) \subset \mathrm{st}_{v_1} \times \mathrm{st}_{v_2}$.

In the general case, we can assume that $K_i \subset \Delta^{J_i}$, where J_i is the vertex set of K_i, $i = 1, 2$. Then the open star of any vertex in K_i, K is the part of the open star in Δ^{J_i}, $\Delta^{J_1} \times_s \Delta^{J_2}$ of the same vertex sitting in K_i, K respectively, $i = 1, 2$. The claim follows because $K = \varphi(K_1 \times K_2)$. □

9.3 P-dimensions

The dimensions listed at the beginning of the chapter have three types of equivalent definitions and possess the same basic properties like monotonicity and a product theorem. It would be awkward to discuss these things for every dimension separately. Thus we introduce the general concept of P-dimension (P stands for Property, indicating the characterizing property of an appropriate dimension) with the aim to provide a device, which allows us to present a large number of similar results in a unified way.

9.3.1 Property spaces

Recall that a *filter* $\mathcal{F}$ on a set P is a collection of subsets of P with the following properties:

(1) $\emptyset \notin \mathcal{F}$;
(2) if $A \in \mathcal{F}$ and $B \supset A$ then $B \in \mathcal{F}$;
(3) if $A, A' \in \mathcal{F}$ then $A \cap A' \in \mathcal{F}$.

Every P-dimension of metric spaces is characterized by a property space. A *property space* P associated with a dimension is a set together with a filter $\mathcal{F}$ of subsets called *characteristic*. Every point $p \in P$ represents a property of open coverings of a metric space X or a property of Lipschitz maps of X into uniform simplicial polyhedra. We write $\mathcal{U} \in p$ if a covering $\mathcal{U}$ of X has the property (represented by) p. Similarly, $f \in p$ for a Lipschitz map $f : X \to K$ into a uniform simplicial polyhedron K, if the covering $\mathcal{U} = \{f^{-1}(\mathrm{st}_v) : v \in K\}$ of X by the preimages of the open stars st_v of the vertices of K has the property p. Speaking more formally, every point $p \in P$ is a function on the set of all open coverings of X or of all Lipschitz maps of X into uniform simplicial polyhedra with values in $\{0, 1\}$. Its value $p(\mathcal{U})$ on a covering $\mathcal{U}$ equals 1 if and only if the covering $\mathcal{U}$ has the property p.

Now we give examples of property spaces.

For the topological dimension, the property space P is identified with positive reals $(0, \infty)$, and the filter $\mathcal{F}$ consists of all subsets each of which contains some subinterval $(0, t) \subset (0, \infty)$, that is, $\mathcal{F}$ is *generated* by the intervals $(0, t)$, $t > 0$. We say that a covering $\mathcal{U}$ of a metric space X has the property $t \in (0, \infty)$, $\mathcal{U} \in t$, if and only if $\mathcal{U}$ is open and $\mathrm{mesh}(\mathcal{U}) \le t$.

For the asymptotic dimension, the property space P is identified with the interval $(0,\infty)$, and the filter $\mathcal{F}$ is generated by the intervals $(t,\infty) \subset (0,\infty)$, $t > 0$. We say that a covering $\mathcal{U}$ of a metric space X has the property $t \in (0,\infty)$, $\mathcal{U} \in t$, if and only if $\mathcal{U}$ is open, $\operatorname{mesh}(\mathcal{U}) < \infty$ and $L(\mathcal{U}) \ge t$.

For the Assouad–Nagata dimension, the property space P is identified with $P = (0,\infty)\times(0,1)$, and the filter $\mathcal{F}$ is generated by the sets $(0,\infty)\times(0,\delta)$, $\delta \in (0,1)$. We say that a covering $\mathcal{U}$ of a metric space X has the property $p = (\tau,\delta) \in P$, $\mathcal{U} \in p$, if and only if $\mathcal{U}$ is open, $L(\mathcal{U}) \ge \delta\tau$ and $\operatorname{mesh}(\mathcal{U}) \le \tau$.

For the ℓ-dimension of X, the property space P is identified with $P = (0,\infty) \times (0,1)$, and the filter $\mathcal{F}$ is generated by the sets $(0,\tau)\times(0,\delta) \subset P$, $\delta \in (0,1)$, $\tau > 0$. We say that a covering $\mathcal{U}$ of a metric space X has the property $p = (\tau,\delta) \in P$, $\mathcal{U} \in p$, if and only if $\mathcal{U}$ is open, $L(\mathcal{U}) \ge \delta\tau$ and $\operatorname{mesh}(\mathcal{U}) \le \tau$.

9.3.2 Axioms of property spaces

We introduce a set of axioms which allow us to prove a number of basic properties of the P-dimension.

Let X be a metric space, $\mathcal{U}$ its locally finite open covering and $p_{\mathcal{U}}\colon X \to \mathcal{N}$ a barycentric map; see Section 9.2.2. We denote by $\operatorname{ba}(p_{\mathcal{U}})$ the covering of X by the preimages of the open stars in $\operatorname{ba}\mathcal{N}$ with respect to $p_{\mathcal{U}}$, $\operatorname{ba}(p_{\mathcal{U}}) = \{p_{\mathcal{U}}^{-1}(\operatorname{st}_v) : v \in \operatorname{ba}\mathcal{N}\}$.

Given two barycentric maps $f_i\colon X_i \to \mathcal{N}_i$ associated with open, locally finite coverings $\mathcal{U}_i$ of X_i, $f_i = p_{\mathcal{U}_i}$, $i = 1,2$, we denote by $\mathcal{U}_{f_1,f_2}$ the covering of the product $X_1 \times X_2$ by the preimages of the open stars in $K = \mathcal{N}_1 \times_s \mathcal{N}_2$ with respect to the map $\varphi \circ (f_1 \times f_2)\colon X_1 \times X_2 \to K$, where $\varphi\colon \mathcal{N}_1 \times \mathcal{N}_2 \to K$ is the barycentric triangulation map; see Section 9.2.4 and Lemma 9.2.6. Note that if $m(\mathcal{U}_i) \le n_i + 1$ then $\dim \mathcal{N}_i \le n_i$, $i = 1,2$, therefore $\dim K \le n_1+n_2$ and $m(\mathcal{U}_{f_1,f_2}) \le n_1+n_2+1$.

We assume that any property space P we consider satisfies the following axioms.

Axioms 9.3.1. (1) For every natural number m, there exists a map $\mathbf{ba}_m\colon P \to P$ such that $\mathbf{ba}_m(\mathcal{F}) \subset \mathcal{F}$ for the characteristic filter $\mathcal{F}$, and $\operatorname{ba}(p_{\mathcal{U}}) \in \mathbf{ba}_m(p)$ for all $p \in P$, every open covering $\mathcal{U} \in p$ of a metric space X with multiplicity $\le m$ and some barycentric map $p_{\mathcal{U}}\colon X \to \mathcal{N}$.

This axiom is responsible for the equivalence of the three definitions of P-dimension given below.

(2) If $X' \subset X$ and a covering $\mathcal{U}$ of X has the property $p \in P$, $\mathcal{U} \in p$, then the restriction of $\mathcal{U}$ to X' has the same property p, $\mathcal{U}|X' \in p$.

This axiom is responsible for monotonicity of the P-dimension.

(3) For every natural number m, there exists a map $\operatorname{prod}_m\colon P \times P \to P$ such that $\operatorname{prod}_m(\mathcal{F} \times \mathcal{F}) \subset \mathcal{F}$ for the characteristic filter $\mathcal{F}$, and such that $\mathcal{U}_{f_1,f_2} \in \operatorname{prod}_m(p_1,p_2)$ for every $p_i \in P$, each open covering $\mathcal{U}_i \in p_i$ of a metric space X_i with multiplicity $\le m$, and some barycentric map $f_i = p_{\mathcal{U}_i}\colon X_i \to \mathcal{N}_i$, $i = 1,2$.

This axiom is responsible for the product theorem.

Example 9.3.2. For the topological dimension, recall that $P = (0, \infty)$ and the filter $\mathcal{F}$ is generated by the intervals $(0, t)$, $t > 0$. We take the identity map as the map $\mathbf{ba}_m : P \to P$ for each $m \geq 0$. Axiom 9.3.1 (1) is satisfied because the open star of every vertex of the barycentric subdivision is contained in the open star of an appropriate vertex of the polyhedron, and the covering of the finite dimensional nerve $\mathcal{N}$ by the open stars of ba $\mathcal{N}$ is open; see Remark 9.2.5. Axiom 9.3.1 (2) is obvious. Axiom 9.3.1 (3) is also satisfied, if we define $\mathrm{prod}_m : P \times P \to P$ by $\mathrm{prod}_m(t_1, t_2) = 2 \max\{t_1, t_2\}$ for every natural m.

Indeed, given $t_i \in P$ and an open covering $\mathcal{U}_i \in t_i$ of a metric space X_i with multiplicity $\leq m$, there is a Lipschitz barycentric map $f_i = p_{\mathcal{U}_i} : X_i \to \mathcal{N}_i$, $i = 1, 2$ by Lemma 9.2.2. For any vertex $v \in K = \varphi(\mathcal{N}_1 \times \mathcal{N}_2)$, there are by Lemma 9.2.8 vertices $v_1 \in \mathcal{N}_1$, $v_2 \in \mathcal{N}_2$ with $\varphi^{-1}(\mathrm{st}_v) \subset \mathrm{st}_{v_1} \times \mathrm{st}_{v_2}$, where $\varphi : \mathcal{N}_1 \times \mathcal{N}_2 \to K$ is the barycentric triangulation map.

Then for $U = (f_1 \times f_2)^{-1} \circ \varphi^{-1}(\mathrm{st}_v)$ we have that $U \subset U_1 \times U_2$, where $U_i = f_i^{-1}(\mathrm{st}_{v_i}) \in \mathcal{U}_i$. Since $\mathrm{diam}(U_1 \times U_2) \leq 2 \max_i \mathrm{diam}\, U_i \leq 2 \max_i t_i$, and $\varphi^{-1}(\mathrm{st}_v) \subset \mathcal{N}_1 \times \mathcal{N}_2$ is open (see Remark 9.2.7), we have $\mathcal{U}_{f_1, f_2} \in \mathrm{prod}_m(t_1, t_2)$. Finally, the inclusion $\mathrm{prod}_m(\mathcal{F} \times \mathcal{F}) \subset \mathcal{F}$ follows from the definition of prod_m and properties of filters.

Example 9.3.3. For the asymptotic dimension, $P = (0, \infty)$, and the filter $\mathcal{F}$ is generated by the intervals (t, ∞), $t > 0$. Recall that a covering $\mathcal{U}$ of X has the property $t \in P$, $\mathcal{U} \in t$, if and only if $\mathcal{U}$ is open, mesh $\mathcal{U} < \infty$ and $L(\mathcal{U}) \geq t$.

By Remark 9.2.5, there is for every natural number m a lower bound $l_m \in (0, 1)$ for the Lebesgue number of the covering of any uniform polyhedron K, $\dim K + 1 \leq m$, by the open stars of ba K. We put $\lambda_m = l_m/(m + 1)^2$ and define $\mathbf{ba}_m : P \to P$ as $\mathbf{ba}_m(t) = \lambda_m t$ for every $t > 0$. Axiom 9.3.1 (1) is satisfied because, for any covering $\mathcal{U} \in t \in P$ with multiplicity $\leq m$, the covering $\mathcal{U}' = \mathrm{ba}(p_{\mathcal{U}})$ is open, $\mathrm{mesh}(\mathcal{U}') \leq \mathrm{mesh}(\mathcal{U}) < \infty$ and $L(\mathcal{U}') \geq l_m/\mathrm{Lip}(p_{\mathcal{U}}) \geq \lambda_m t$, where we used the estimate $\mathrm{Lip}(p_{\mathcal{U}}) \leq (m + 1)^2/t$ of Lemma 9.2.2. The property $\mathbf{ba}_m(\mathcal{F}) \subset \mathcal{F}$ is evident.

Axiom 9.3.1 (2) is obvious because $L(\mathcal{U}|X') \geq L(\mathcal{U})$ for every open covering $\mathcal{U}$ of X and every subspace $X' \subset X$.

For every natural number m, there is by Remark 9.2.7 a constant $c_m \in (0, 1)$ with the property: given uniform polyhedra K_1, K_2, $\dim K_i + 1 \leq m$, the Lebesgue number of the covering of $K_1 \times K_2$ by $\varphi^{-1}(\mathrm{st}_v)$, $v \in K_1 \times_s K_2$, is bounded from below by c_m, where $\varphi : K_1 \times K_2 \to K_1 \times_s K_2$ is the barycentric triangulation map. We put $\mu_m = c_m/(m + 1)^2$ and define $\mathrm{prod}_m : P \times P \to P$ by $\mathrm{prod}_m(t_1, t_2) = \mu_m \min\{t_1, t_2\}$ for every $t_1, t_2 > 0$.

Then clearly $\mathrm{prod}_m(\mathcal{F} \times \mathcal{F}) \subset \mathcal{F}$. Moreover, for every covering $\mathcal{U}_i \in t_i \in P$ with multiplicity $\leq m$, there is by Lemma 9.2.2 a barycentric map $f_i = p_{\mathcal{U}_i} : X_i \to \mathcal{N}_i$ with $\mathrm{Lip}(f_i) \leq (m + 1)^2/t_i$, $i = 1, 2$. Then the covering $\mathcal{U}_{f_1, f_2}$ of $X_1 \times X_2$ is open and

$$L(\mathcal{U}_{f_1, f_2}) \geq c_m/\mathrm{Lip}(f_1 \times f_2) \geq \mu_m \min\{t_1, t_2\},$$

because $\operatorname{Lip}(f_1 \times f_2) \le (m+1)^2 \max\{1/t_1, 1/t_2\}$. That is, $\mathcal{U}_{f_1,f_2} \in \operatorname{prod}_m(t_1, t_2)$, and Axiom 9.3.1 (3) is satisfied.

Example 9.3.4. For the ℓ-dimension, $P = (0,\infty) \times (0,1)$, and the filter $\mathcal{F}$ is generated by $(0,\tau) \times (0,\delta)$ for all $\tau > 0$, $\delta \in (0,1)$. Recall that a covering $\mathcal{U}$ of X has the property $p = (\tau,\delta) \in P$, $\mathcal{U} \in p$, if and only if $\mathcal{U}$ is open, $\operatorname{mesh}(\mathcal{U}) \le \tau$ and $L(\mathcal{U}) \ge \delta\tau$.

Given a natural number m, we define $\mathbf{ba}_m\colon P \to P$ by $\mathbf{ba}_m(\tau,\delta) = (\tau, \lambda_m\delta)$ and $\operatorname{prod}_m\colon P \times P \to P$ by

$$\operatorname{prod}_m((\tau_1,\delta_1),(\tau_2,\delta_2)) = \left(2\max\{\tau_1,\tau_2\}, \frac{\mu_m \min\{\delta_1\tau_1, \delta_2\tau_2\}}{2\max\{\tau_1,\tau_2\}}\right),$$

where the constants $\lambda_m, \mu_m \in (0,1)$ are defined as in Example 9.3.3. Combining the arguments of the two examples above, one easily checks the Axioms 9.3.1 for the ℓ-dimension. We leave details to the reader.

Exercise 9.3.5. Check the axioms for the Assouad–Nagata dimension.

Exercise 9.3.6. Describe the property space and check the axioms for the asymptotic ℓ-dimension.

In the following let X be a metric space.

9.3.3 Colored definition of the P-dimension

For every integer $n \ge 0$, the space X is represented in P by a subset $\operatorname{col} = \operatorname{col}(X,n)$ of P: namely, $p \in \operatorname{col}$ if and only if there is an $(n+1)$-colored covering $\mathcal{U}$ of X with the property p. Now the *colored* P-dimension of X is

$$\operatorname{Pdim}_{\mathrm{col}} X = \min\{n : \operatorname{col}(X,n) \text{ contains some set from } \mathcal{F}\}.$$

9.3.4 Covering definition of the P-dimension

For every integer $n \ge 0$, the space X is represented in P by a subset $\operatorname{cov} = \operatorname{cov}(X,n) \subset P$: namely, $p \in \operatorname{cov}$ if and only if there is a covering $\mathcal{U}$ of X with the multiplicity $m(\mathcal{U}) \le n+1$ and the property p. Now the *covering* P-dimension of X is

$$\operatorname{Pdim}_{\mathrm{cov}} X = \min\{n : \operatorname{cov}(X,n) \text{ contains some set from } \mathcal{F}\}.$$

For the property spaces of the topological, asymptotic, Assouad–Nagata, ℓ- and asymptotic ℓ-dimensions, this gives back the definitions of these dimensions at the beginning of the chapter.

9.3.5 Polyhedral definition of the P-dimension

For every integer $n \geq 0$, the space X is represented in P by a subset $\mathrm{pol} = \mathrm{pol}(X, n)$ of P: namely, $p \in \mathrm{pol}$ if and only if there is a Lipschitz map $f : X \to K$ having the property p, where K is a uniform simplicial complex of the (combinatorial) dimension n. Now the *polyhedral* P-dimension of X is

$$\mathrm{Pdim}_{\mathrm{pol}} X = \min\{n : \mathrm{pol}(X, n) \text{ contains some set from } \mathcal{F}\}.$$

Note that $\mathrm{col}(X, n) \subset \mathrm{cov}(X, n) \subset \mathrm{pol}(X, n)$: the first inclusion follows from the fact that the multiplicity of an m-colored covering is at most m, and the second one follows from the existence of a barycentric map associated with a covering; see Section 9.2.2. Hence,

$$\mathrm{Pdim}_{\mathrm{col}} X \geq \mathrm{Pdim}_{\mathrm{cov}} X \geq \mathrm{Pdim}_{\mathrm{pol}} X$$

for every metric space X.

Proposition 9.3.7. *For every metric space X, the dimensions* $\mathrm{Pdim}_{\mathrm{col}} X$, $\mathrm{Pdim}_{\mathrm{cov}} X$, $\mathrm{Pdim}_{\mathrm{pol}} X$ *coincide,*

$$\mathrm{Pdim}_{\mathrm{col}} X = \mathrm{Pdim}_{\mathrm{cov}} X = \mathrm{Pdim}_{\mathrm{pol}} X.$$

Proof. It suffices to check that $\mathrm{col}(X, n)$ contains a set $F' \in \mathcal{F}$ for some n if $\mathrm{pol}(X, n)$ contains a set $F \in \mathcal{F}$, because this implies the inequality $\mathrm{Pdim}_{\mathrm{pol}} X \geq \mathrm{Pdim}_{\mathrm{col}} X$ completing the proof.

Now if $p \in F \subset \mathrm{pol}(X, n)$, then there is a Lipschitz map $f : X \to K$ with $f \in p$, where K is a uniform polyhedron of dimension n. Consider the covering $\mathcal{U}$ of X by preimages of open stars of K, $\mathcal{U} = \{f^{-1}(\mathrm{st}_v) : v \in K\}$. Note that the nerve of $\mathcal{U}$ coincides with K. By Axiom 9.3.1 (1), there is a barycentric map $p_{\mathcal{U}} : X \to K$ with $\mathrm{ba}(p_{\mathcal{U}}) \in \mathbf{ba}_{n+1}(p)$, and the covering $\mathrm{ba}(p_{\mathcal{U}})$ is an $(n + 1)$-colored covering of X, see Remark 9.2.5. Thus $\mathbf{ba}_{n+1}(p) \in \mathrm{col}(X, n)$. This shows that $\mathbf{ba}_{n+1}(F) \subset \mathrm{col}(X, n)$. Finally, we note that $F' = \mathbf{ba}_{n+1}(F) \in \mathcal{F}$ again by Axiom 9.3.1 (1). □

From now on, we use the notation $\mathrm{Pdim}\, X$ for the common value of all three P-dimensions.

9.4 The monotonicity theorem

Theorem 9.4.1. *Given a property space P, for a metric space X and any subspace $X' \subset X$, we have* $\mathrm{Pdim}\, X' \leq \mathrm{Pdim}\, X$.

Proof. It follows from Axiom 9.3.1 (2), that $\mathrm{cov}(X', n) \supset \mathrm{cov}(X, n)$ for every n. Hence, the claim. □

9.5 The product theorem

Theorem 9.5.1. *For metric spaces X_1, X_2 and a given dimension of type P, we have* $\mathrm{Pdim}(X_1 \times X_2) \le \mathrm{Pdim}\, X_1 + \mathrm{Pdim}\, X_2$.

Proof. Let $(P, \mathcal{F})$ be the property space associated with the given dimension. Assume that $F_i \subset \mathrm{cov}(X_i, n_i) \subset P$ for some characteristic set $F_i \in \mathcal{F}$ and some integer n_i, $i = 1, 2$. It suffices to show that the set $\mathrm{cov}(X_1 \times X_2, n_1 + n_2) \subset P$ contains a characteristic set $F \in \mathcal{F}$, because this would imply $\mathrm{Pdim}(X_1 \times X_2) \le n_1 + n_2$ and therefore the required inequality.

It follows from the assumption that for every property $p_i \in F_i$ there is a covering $\mathcal{U}_i \in p_i$ with multiplicity $\le n_i + 1$. By Axiom 9.3.1 (3), there is a barycentric map $f_i = p_{\mathcal{U}_i} : X_i \to \mathcal{N}_i$, $i = 1, 2$, such that $\mathcal{U}_{f_1, f_2} \in \mathrm{prod}_m(p_1, p_2)$ with $m = \max\{n_1, n_2\} + 1$. Since $\mathcal{U}_{f_1, f_2}$ is a covering of $X_1 \times X_2$ with multiplicity $\le n_1 + n_2 + 1$, we obtain $\mathrm{prod}_m(p_1, p_2) \in \mathrm{cov}(X_1 \times X_2, n_1 + n_2)$. This shows that $\mathrm{prod}_m(F_1, F_2) \subset \mathrm{cov}(X_1 \times X_2, n_1 + n_2)$. Finally, $F = \mathrm{prod}_m(F_1, F_2) \in \mathcal{F}$ again by Axiom 9.3.1 (3). □

9.6 The saturation of families

Here, we discuss a powerful and flexible construction called the *saturation* of a given family $\mathcal{U}$ of subsets of a space X by another such family $\mathcal{V}$. The construction has important applications in a number of questions including estimations of various types of dimensions and embedding questions.

The saturation of $U \in \mathcal{U}$ by the family $\mathcal{V}$ is the union $U * \mathcal{V}$ of U and all members $V \in \mathcal{V}$ with $U \cap V \ne \emptyset$. Now the saturation of $\mathcal{U}$ by $\mathcal{V}$ is the family

$$\mathcal{U} * \mathcal{V} = \{U * \mathcal{V} : U \in \mathcal{U}\}.$$

Note that $\{\emptyset\} * \mathcal{V} = \emptyset$, $\mathcal{U} * \{\emptyset\} = \mathcal{U}$.

We slightly modify the notion of the Lebesgue number to adapt it to open coverings of a subset A in a metric space X. Let $\mathcal{U}$ be a family of open subsets in a metric space X which cover $A \subset X$. Then we put $L(\mathcal{U}) = \inf_{x \in A} L(\mathcal{U}, x)$, the Lebesgue number of the covering $\mathcal{U}$ of A, where as usual $L(\mathcal{U}, x) = \sup\{\mathrm{dist}(x, X \setminus U) : U \in \mathcal{U}\}$. For every $x \in A$, the open ball $B_r(x) \subset X$ of radius $r = L(\mathcal{U})$ centered at x is contained in some member of the covering $\mathcal{U}$.

Proposition 9.6.1. *Suppose that X is a metric space and A, $B \subset X$. Let*

$$\mathcal{U} = \bigcup_{c \in C} \mathcal{U}^c \quad \text{and} \quad \mathcal{V} = \bigcup_{c \in C} \mathcal{V}^c$$

be coverings of A and B, respectively, which are both open in X and m-colored with $m = |C| \ge 1$. If $\mathrm{mesh}(\mathcal{V}) \le L(\mathcal{U})/2$ *then the family $\mathcal{W} = \bigcup_{c \in C} \mathcal{W}^c$ is* *the open m-colored covering of $A \cup B$ with $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}$ and* $\operatorname{mesh}(\mathcal{W}) \leq \max\{\operatorname{mesh}(\mathcal{U}), \operatorname{mesh}(\mathcal{V})\}$ *where*

$$\mathcal{W}^c = B_{-r}(\mathcal{U}^c) * \mathcal{V}^c \cup \{V \in \mathcal{V}^c : B_{-r}(U) \cap V = \emptyset \text{ for all } U \in \mathcal{U}^c\},$$

for $r = L(\mathcal{U})/2$

Proof. We can assume that $L(\mathcal{U}) < \infty$, i.e. no member of $\mathcal{U}$ covers X, because otherwise there is nothing to prove. The family $B_{-r}(\mathcal{U}) = \{B_{-r}(U) : U \in \mathcal{U}\}$ still covers A because $r < L(\mathcal{U})$. Thus, the family $\mathcal{W}$ covers $A \cup B$. Clearly, for the Lebesgue number of $\mathcal{W}$, we have

$$L(\mathcal{W}) \geq \min\{L(B_{-r}(\mathcal{U})), L(\mathcal{V})\} \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}.$$

Since $\operatorname{diam} V \leq r$ for every $V \in \mathcal{V}$, we have $B_{-r}(U) * \mathcal{V}^c \subset U$ for every $U \in \mathcal{U}^c$, $c \in C$. Hence, $\operatorname{mesh} \mathcal{W} \leq \max\{\operatorname{mesh}(\mathcal{U}), \operatorname{mesh} \mathcal{V}\}$. Now since $\mathcal{V}^c$ is disjoint, it easily follows from the definition that $\mathcal{W}^c$ is disjoint for every $c \in C$. Hence, $\mathcal{W}$ is m-colored. □

9.7 The finite union theorem

We apply the saturation construction to prove the following theorem.

Theorem 9.7.1. *Assume that a metric space X is the union of two subsets, $X = A \cup B$. Then*

$$\operatorname{Pdim} X = \max\{\operatorname{Pdim} A, \operatorname{Pdim} B\}$$

for each Pdim *from the list:* asdim*,* ANdim*,* ℓ-dim*,* ℓ-asdim.

Proof. Because of the monotonicity theorem, it suffices to show that $\operatorname{Pdim} X \leq m = \max\{\operatorname{Pdim} A, \operatorname{Pdim} B\}$. We can assume that m is finite.

The case Pdim $=$ asdim. Given $d > 0$, there is an open $(m+1)$-colored covering $\mathcal{V}$ of B with $\operatorname{mesh}(\mathcal{V}) < \infty$ and $L(\mathcal{V}) \geq d$. For $d' \geq 2\max\{d, \operatorname{mesh}(\mathcal{V})\}$, there is an open $(m+1)$-colored covering $\mathcal{U}$ of A with $\operatorname{mesh}(\mathcal{U}) < \infty$ and $L(\mathcal{U}) \geq d'$. Then $\operatorname{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$. By Proposition 9.6.1, there is an open $(m+1)$-colored covering $\mathcal{W}$ of X with $\operatorname{mesh}(\mathcal{W}) < \infty$ and $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\} \geq d$. Hence, $\operatorname{asdim} X \leq m$.

The case Pdim $=$ ANdim: We can assume that the constants δ_A, $\delta_B \in (0,1)$ from the definition of ANdim for A, B coincide, $\delta_A = \delta_B = \delta$, taking the smaller one if necessary. Given $\tau > 0$, there is an open $(m+1)$-colored covering $\mathcal{U}$ of A with $\operatorname{mesh}(\mathcal{U}) \leq \tau$ and $L(\mathcal{U}) \geq \delta\tau$. For $\tau' = \delta\tau/2$, there is an open $(m+1)$-colored covering $\mathcal{V}$ of B with $\operatorname{mesh}(\mathcal{V}) \leq \tau'$ and $L(\mathcal{V}) \geq \delta\tau'$. Then $\operatorname{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$, and, by Proposition 9.6.1, there is an open $(m+1)$-colored covering $\mathcal{W}$ of X with $\operatorname{mesh}(\mathcal{W}) \leq \max\{\tau, \tau'\} = \tau$ and $L(\mathcal{W}) \geq \min\{\delta\tau/2, \delta\tau'\} = \frac{\delta^2}{2}\tau$. Hence, $\operatorname{ANdim} X \leq m$.

The cases Pdim $= \ell$-dim and Pdim $= \ell$-asdim are similar. □

9.8 Sperner lemma

The following nice combinatorial result definitely has a homological nature, and it can be used for elementary proofs of a number of classical topological facts like the invariance of the dimension and interior points for manifolds, the existence of fixed points of continuous maps, etc. Nowadays, these facts are usually proven by homological means. It seems, however, that the potential of the Sperner lemma is not yet completely exhausted.

Lemma 9.8.1. *Let τ be a triangulation of the standard simplex Δ^n. Assume that with every vertex $t \in \tau$ one associates a vertex $\varphi(t)$ of Δ^n such that the following holds: if $t \in \sigma \subset \Delta^n$, where σ is a face of Δ^n, then $\varphi(t) \in \sigma$.*

Then there is an n-dimensional simplex $\gamma = [t_0 \dots t_n] \subset \tau$ such that the vertices $\varphi(t_0), \dots, \varphi(t_n)$ are pairwise distinct.

Proof. Let Γ be the set of all n-dimensional simplices of the triangulation τ. A simplex γ from Γ is said to be *normal*, if the vertices of Δ^n associated with the vertices of γ are pairwise distinct. We claim that the number of the normal simplices is odd.

For $n = 0$ this is obvious. Assume that the assertion is proven for all $(n-1)$-simplices, and prove it for Δ^n. A face of dimension $n-1$ of a simplex $\gamma \in \Gamma$ is called *distinguished* if its vertices are mapped onto the vertices $e_1, \dots, e_n$ of $\Delta^n = [e_0 e_1 \dots e_n]$. Note that the number of distinguished faces of γ is 0, or 1, or 2. Indeed, if γ is normal, then the number of its distinguished faces is 1; if γ is not normal and contains a distinguished face, then for the remaining vertex $t_\gamma^0 \in \gamma$, we have $\varphi(t_\gamma^0) \in \{e_1, \dots, e_n\}$. Thus γ has another distinguished face which contains the vertex t_γ^0, and no other distinguished face apart from these two.

Let a_γ be the number of distinguished faces of γ. We put

$$a = \sum_{\gamma \in \Gamma} a_\gamma.$$

Then the number of normal simplices has the same parity as a. Thus it suffices to show that a is odd. We compute a in another way: if one of $n-1$ faces of a simplex $\gamma \in \Gamma$ lies inside of Δ^n, then it has exactly 2 adjacent simplices from that list. Thus its contribution to a is even; if such a face lies on some proper face of Δ^n different from $[e_1 \dots e_n]$, then, by the condition on φ, it cannot be distinguished, and its contribution to a is zero. The remaining case is that such a face lies in $[e_1 \dots e_n]$. Clearly, in this case its contribution to a is 0 or 1. By the inductive assumption, the number of faces with contribution 1 in a is odd. Thus a is odd. $\square$

Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard n-simplex, and let $\mathcal{U}$ be a covering of a metric space X. A map $f\colon \Delta^n \to X$ is said to be *coherent with* $\mathcal{U}$ if no element of $\mathcal{U}$

intersects the image of every $(n-1)$-face of Δ^n. A covering $\mathcal{V}$ of Δ^n is called *coherent* if the identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ is coherent with $\mathcal{V}$.

Lemma 9.8.2. *Let $\mathcal{V}$ be a coherent covering of Δ^n. Then for any triangulation τ of Δ^n, there is an n-simplex $\gamma \subset \tau$ whose vertices lie in $n+1$ different elements of the covering $\mathcal{V}$.*

Proof. Any set containing two vertices of Δ^n intersects all $(n-1)$-faces of Δ^n. Thus there are $n+1$ different elements $V_0, \dots, V_n \in \mathcal{V}$ such that every vertex of Δ^n is contained in one and only one of those elements, $e_i \in V_i$, $i = 0, \dots, n$. Assume that there is an element $V \in \mathcal{V}$ different from $V_0, \dots, V_n$. Since $\mathcal{V}$ is coherent, there is an $(n-1)$-face of Δ^n disjoint with V. Let $e_i \in \Delta^n$ be the opposite vertex, $i \in \{0, \dots, n\}$. We modify $\mathcal{V}$ by taking the union $V_i \cup V$. This does not change the coherent condition. Therefore, from the beginning we can assume that $\mathcal{V} = \{V_0, \dots, V_n\}$.

Now with every vertex $t \in \tau$, we associate a vertex $\varphi(t) = e_i \in \Delta^n$ so that $t \in V_i$. Note that any face $[e_0 \dots e_r] \subset \Delta^n$ is contained in $V_0 \cup \dots \cup V_r$, because if $i \notin \{0, \dots, r\}$ then V_i misses an $(n-1)$-face containing $[e_0 \dots e_r]$ and hence their intersection. Therefore, the condition of Lemma 9.8.1 is satisfied, and the claim follows. □

Corollary 9.8.3. *The multiplicity of every open coherent covering $\mathcal{V}$ of Δ^n is at least $n+1$, $m(\mathcal{V}) \geq n+1$.*

Proof. Since Δ^n is compact, we can assume that $\mathcal{V}$ is finite. Then the Lebesgue number $L(\mathcal{V}) = 2r > 0$, and the family $\mathcal{V}' = B_{-r}(\mathcal{V})$ still covers Δ^n. By Lemma 9.8.2, for any triangulation τ of Δ^n, there is an n-simplex $\gamma \subset \tau$ whose vertices lie in $n+1$ different elements of the covering $\mathcal{V}'$. Assuming that mesh $\tau < r$, we find a simplex $\gamma \subset \tau$ which is contained in $n+1$ different elements of the covering $\mathcal{V}$. Then $m(\mathcal{V}) \geq n+1$. □

As an application, we obtain the following theorem.

Theorem 9.8.4. *For every $n \geq 0$, we have*

$$\dim \mathbb{R}^n = \operatorname{asdim} \mathbb{R}^n = \operatorname{ANdim} \mathbb{R}^n = \ell\text{-}\dim \mathbb{R}^n = \ell\text{-asdim}\, \mathbb{R}^n = n.$$

Proof. We have already shown that all dimensions, dim, asdim, ANdim, ℓ-dim and ℓ-asdim, of $\mathbb{R}^n$, are at most n, see Example 9.1.3. Now we first show that $\dim \mathbb{R}^n \geq n$.

By the monotonicity theorem, it suffices to prove that $\dim \Delta^n \geq n$. We fix $\varepsilon > 0$ and consider an open covering $\mathcal{U}$ of Δ^n with $\operatorname{mesh}(\mathcal{U}) \leq \varepsilon$. If ε is sufficiently small, then $\mathcal{U}$ is obviously coherent. By Corollary 9.8.3, $m(\mathcal{U}) \geq n+1$ and thus $\dim \Delta^n \geq n$.

Since $\operatorname{ANdim} \geq \ell\text{-}\dim \geq \dim$, we also obtain the required estimate for the ℓ- and Assouad–Nagata dimensions.

It remains to consider the asymptotic dimension. Given any covering $\mathcal{U}$ of $\mathbb{R}^n$ with $\mathrm{mesh}(\mathcal{U}) < \infty$, we have $\mathrm{mesh}(h(\mathcal{U})) \le \lambda\,\mathrm{mesh}(\mathcal{U})$ for the covering $h(\mathcal{U})$, where $h\colon \mathbb{R}^n \to \mathbb{R}^n$, $h(x) = \lambda x$, is the homothety with coefficient $\lambda > 0$. Choosing λ sufficiently small, we produce a covering $h(\mathcal{U})$ with arbitrarily small mesh. Thus, $\dim \mathbb{R}^n \le \mathrm{asdim}\,\mathbb{R}^n \le \ell\text{-asdim}\,\mathbb{R}^n$. □

9.9 Supplementary results and remarks

9.9.1 The finite union theorem for the topological dimension

For the topological dimension, the classical Menger–Urysohn theorem states: $\dim(A \cup B) \le \dim A + \dim B + 1$ with equality in some cases. The difference to the finite union theorem for the asymptotic and the linearly controlled dimensions is rooted in the control of the Lebesgue number of coverings. However, for a compact X and closed $A, B \subset X$, one has $\dim(A \cup B) = \max\{\dim A, \dim B\}$.

9.9.2 Qualified multiplicity and separation

There is an alternative approach to definitions of dimensions where, instead of the Lebesgue number, one uses a qualified multiplicity or separation condition.

For $d > 0$, the *d-multiplicity* of a family $\mathcal{U}$ of subsets in a metric space X, $m_d(\mathcal{U})$, is the multiplicity of the family $B_d(\mathcal{U})$ obtained by taking open d-neighborhoods of the members of $\mathcal{U}$. So $m_d(\mathcal{U}) = m(B_d(\mathcal{U}))$.

Then the covering definition of the asymptotic dimension can be formulated as follows. The asymptotic dimension of a metric space X is at most n, if for every $d > 0$ there is a covering $\mathcal{U}$ of X with $\mathrm{mesh}(\mathcal{U}) < \infty$ and $m_d(\mathcal{U}) \le n + 1$.

A family $\mathcal{U}$ is called *d-disjoint* if $m(B_d(\mathcal{U})) = 1$. Then the colored definition of the asymptotic dimension can be formulated as follows. The asymptotic dimension of a metric space X is at most n if for every $d > 0$ there is an $(n+1)$-colored uniformly bounded covering $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$, $|A| = n + 1$ of X, where each family $\mathcal{U}^a$, $a \in A$, is d-disjoint.

Exercise 9.9.1. Formulate similar colored, covering and polyhedral definitions for asdim, ANdim, ℓ-dim and ℓ-asdim using qualified multiplicity and separation conditions instead of the Lebesgue number and prove their equivalence.

9.9.3 Asymptotic dimension and coarse maps

Recall that a map $f : X \to Y$ between metric spaces is coarse if there are control functions ρ_1, ρ_2 for f, that is,

$$\rho_1(|xx'|) \le |f(x)f(x')| \le \rho_2(|xx'|)$$

for all $x, x' \in X$, with the property that $\lim_{t\to\infty} \rho_i(t) = \infty$. The spaces X, Y are *coarsely equivalent* if there exist coarse maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ and $f \circ g$ are at a finite distance from the identity maps on X and on Y respectively.

Exercise 9.9.2. Show that the asymptotic dimension is a coarse invariant, i.e. it is invariant under coarse equivalence.

9.9.4 Mapping simplices into a metric space

There is another application of the Sperner lemma in the spirit of the Borsuk–Ulam theorem.

Theorem 9.9.3. *Let X be a metric space with* $\dim X \le n$. *Then for any continuous $f: \Delta^{n+1} \to X$, there is $x_0 \in X$ such that $f^{-1}(x_0)$ meets every n-face of Δ^{n+1}.*

Proof. Assume to the contrary that for every $x \in X$, $f^{-1}(x)$ misses an n-face of Δ^{n+1}. Then f is coherent with every open covering $\mathcal{U}$ of X with sufficiently small mesh. Choosing $\mathcal{U}$ with multiplicity $m(\mathcal{U}) \le n+1$, we obtain the open coherent covering $\mathcal{V} = f^{-1}(\mathcal{U})$ of Δ^{n+1} with $m(\mathcal{V}) \le n+1$. This contradicts Corollary 9.8.3. □

Recall that the boundary $\partial\Delta^{n+1}$ is homeomorphic to the sphere S^n and that by the classical Borsuk–Ulam theorem for any continuous map $f: S^n \to \mathbb{R}^n$, there is $x_0 \in \mathbb{R}^n$ such that $f^{-1}(x_0)$ contains a pair of antipodal points of S^n.

Corollary 9.9.4. *For any continuous $f: \partial\Delta^{n+1} \to \mathbb{R}^n$, there is $x_0 \in \mathbb{R}^n$ such that $f^{-1}(x_0)$ meets every n-face of $\partial\Delta^{n+1}$.*

Proof. The map f can be extended to a continuous map $\Delta^{n+1} \to \mathbb{R}^n$. □

9.9.5 An alternative approach to the product theorem

Here we briefly discuss an alternative approach to the product theorem based on the so called Kolmogorov trick and the following theorem due to Ostrand [Os1].

Theorem 9.9.5 (P. A. Ostrand). *A metric space X is of dimension $\le n$ if and only if for each open cover $\mathcal{C}$ of X and each integer $k \ge n+1$ there exist k disjoint families of open sets $\mathcal{U}_1, \dots, \mathcal{U}_k$ such that the union of any $n+1$ of them is a covering of X which refines $\mathcal{C}$.*

Here, we say that a covering $\mathcal{U}$ *refines* $\mathcal{C}$ or *is inscribed* in $\mathcal{C}$ if every member of $\mathcal{U}$ is contained in some member of $\mathcal{C}$. The existence of additional $k-(n+1)$ families can be used in the proof of the product theorem as follows. Assume that $\dim X = m$, $\dim Y = n$, and let $k = m+n+1$. Then we can find k disjoint families

$\{\mathcal{U}_i\}$ in X and k disjoint families $\{\mathcal{V}_i\}$ in Y as in Theorem 9.9.5 with arbitrarily small mesh. Then the family $\{\mathcal{U}_i \times \mathcal{V}_i\}_{i=1}^k$ covers $X \times Y$, as for any point $(x, y) \in X \times Y$, x is contained in sets from at least $k - m = n + 1$ families from $\{\mathcal{U}_i\}$ and y is contained in sets from at least $k - n = m + 1$ families from $\{\mathcal{V}_i\}$. Thus there is at least one index i such that x is covered by $\mathcal{U}_i$ and y is covered by $\mathcal{V}_i$. The covering $\{\mathcal{U}_i \times \mathcal{V}_i\}_{i=1}^k$ is disjoint and its mesh is as small as required. A similar argument works for any dimension we discussed above, see [BDLM].

9.9.6 Coarse structures and dimensions

In this book, we took a moderate approach using a metric to define various asymptotic dimensions. There is a more advanced approach based on the notion of coarse structures that does not rely on metrics. For that, we refer to [Ro], [Gra], [DH], [Dr2].

Historical note. The covering definition of the topological dimension originates from A. Lebesgue. The colored definition goes back to the paper [Os1] by P. A. Ostrand which in turn is based on Kolmogorov's idea used in the solution to Hilbert's 13th problem, [Ko], [Ar].

The notion of the asymptotic dimension is introduced in [Gr2], the notion of the Assouad–Nagata dimension is due to P. Assouad, see [As1] where it is called the Nagata dimension.

The idea of the saturation of one family of subsets by another was used in [BD1] to prove a countable union theorem for the asymptotic dimension which includes the finite union theorem.

Basic results of Section 9.8 (Lemmas 9.8.1, 9.8.2 and Corollary 9.8.3) are due E. Sperner, [Sp].

Chapter 10

Asymptotic dimension

10.1 Estimates from below

We prove two theorems which give optimal estimates from below for the asymptotic dimension of different classes of metric spaces.

Recall that a Hadamard manifold is a complete simply connected Riemannian manifold X with nonpositive sectional curvature. For every point $x \in X$, there is the exponential map $\exp_x : T_x X \to X$ from the tangent space $T_x X$ which is a noncontracting diffeomorphism, $|\exp_x v \exp_x w| \geq |vw|$ for each $v, w \in T_x X$.

Theorem 10.1.1. *For every Hadamard manifold X, we have*

$$\operatorname{asdim} X \geq \dim X.$$

Proof. The proof is based on the Sperner lemma, see Section 9.8. Let $\mathcal{U}$ be a uniformly bounded covering of X, $\operatorname{mesh}(\mathcal{U}) < \infty$. Then for every sufficiently large $R > 1$, we obtain an n-dimensional, $n = \dim X$, simplex $f : \Delta^n \to X$ coherent with $\mathcal{U}$ as follows. Consider an isometric copy of Δ^n in some tangent space $T_x X$ with the barycenter at the origin and identify Δ^n with the closed ball $\overline{B}_R \subset T_x X$ of the radius R via a radial homotopy. Composing with $\exp_x : \overline{B}_R \to X$, we obtain a map $f : \Delta^n \to X$. Since $\exp_x$ is noncontracting, no member of $\mathcal{U}$ meets all $(n-1)$-faces of $f(\Delta^n)$ if R is chosen sufficiently large compared to $\operatorname{mesh}(\mathcal{U})$. Thus f is coherent with $\mathcal{U}$.

Then $m(\mathcal{U}) \geq n + 1$ by Corollary 9.8.3. It follows that $\operatorname{asdim} X \geq n$. $\square$

For Hadamard manifolds that are also hyperbolic spaces, the following argument can be applied, which also works for arbitrary proper geodesic hyperbolic spaces. That argument is based on the well-known fact from dimension theory that $\dim(Z \times I) = \dim Z + 1$ for every compact space Z, where $I = [0, 1]$. For the proof of this fact we refer to [Dr1].

Theorem 10.1.2. *For every proper geodesic hyperbolic space X, we have*

$$\operatorname{asdim} X \geq \dim \partial_\infty X + 1.$$

Proof. We can assume that $\partial_\infty X \neq \emptyset$. By Corollary 7.1.5, the hyperbolic cone $\mathrm{Co}(Z)$ over $Z = \partial_\infty X$, taken with some visual metric, can be roughly similarly embedded in X because X is geodesic. Thus $\operatorname{asdim} X \geq \operatorname{asdim} \mathrm{Co}(Z)$, and we show that $\operatorname{asdim} \mathrm{Co}(Z) \geq \dim Z + 1$.

The *annulus* $\mathrm{An}(Z) \subset \mathrm{Co}(Z)$ consists of all $x \in \mathrm{Co}(Z)$ with $1 \leq |xo| \leq 2$, where $o \in \mathrm{Co}(Z)$ is the vertex. Clearly, $\mathrm{An}(Z)$ is homeomorphic to $Z \times [0,1]$. Since X is proper, Z is compact. Then $\dim \mathrm{An}(Z) = \dim Z + 1$.

Consider the sequence of contracting homeomorphisms $F_k\colon \mathrm{Co}(Z) \to \mathrm{Co}(Z)$ given by $F_k(z,t) = (z, \frac{1}{k}t)$, $(z,t) \in \mathrm{Co}(Z)$, $k \in \mathbb{N}$. Given a uniformly bounded covering $\mathcal{U}$ of $\mathrm{Co}(Z)$, the coverings $\mathcal{U}_k = F_k(\mathcal{U}) \cap \mathrm{An}(Z)$ of the annulus $\mathrm{An}(Z)$ have arbitrarily small mesh as $k \to \infty$. Therefore, $\operatorname{asdim} \mathrm{Co}(Z) \geq \dim \mathrm{An}(Z)$, and the estimate follows. □

Remark 10.1.3. For some important classes of metric spaces, we obtain an optimal estimate from below for the asymptotic dimension via the monotonicity theorem. For example, let X be a Euclidean building whose apartments are isometric to $\mathbb{R}^n$, $n \geq 1$. Then $\operatorname{asdim} X \geq \operatorname{asdim} \mathbb{R}^n = n$ (see Theorem 9.8.4). In particular, for every metric tree T with nonempty boundary at infinity, we have $\operatorname{asdim} T \geq 1$.

Similarly, let X be a hyperbolic building whose apartments are isometric to H^n, $n \geq 2$. Then $\operatorname{asdim} X \geq n$ because $\operatorname{asdim} \mathrm{H}^n \geq n$ by Theorem 10.1.1. Actually, in all cases discussed in this remark, the equality holds (we show this below for the case $n = 1$).

10.2 Estimates from above

To estimate the asymptotic dimension from above is both, more important and more difficult. Estimates from above have interesting applications. By a result of G. Yu [Yu], for every finitely generated group G with $\operatorname{asdim} G < \infty$ and finite classifying space, the Novikov higher signature conjecture holds. As a corollary, the Gromov–Lawson–Rosenberg conjecture, saying that there is no Riemannian metric with positive scalar curvature on closed $K(\pi,1)$-manifolds, holds when the fundamental group π has finite asymptotic dimension. Furthermore, the extension property for Lipschitz maps is closely related to finiteness of the Assouad–Nagata dimension [LS].

On the other hand, even the question whether every cocompact Hadamard manifold has finite asymptotic dimension is open. For hyperbolic spaces the situation is better understood, and we give in Chapter 12 some general optimal estimates. In part, they are based on the following estimate for trees.

Proposition 10.2.1. *The asymptotic dimension as well as the asymptotic ℓ-dimension of every metric tree T is at most* 1*,* $\operatorname{asdim} T \leq \ell\text{-}\operatorname{asdim} T \leq 1$.

Proof. Fix a base point $o \in T$ and consider Gromov products with respect to o,

$(x|y) = (x|y)_o$. Since T is 0-hyperbolic, the triple

$$((x|y), (y|z), (x|z))$$

is a 0-triple, i.e., the two smallest members coincide for every $x, y, z \in T$.

We use the notation $|x| = |ox|$ for $x \in T$. Given $R > 1$, we consider the annuli

$$A_k = \{x \in T : kR \le |x| < (k+1)R\}$$

for every integer $k \ge 0$. The relation $x \sim x'$ if and only if $(x|x') \ge (k - 1/2)R$ is an equivalence relation on A_k because T is 0-hyperbolic. If $x \sim x'$ then

$$|xx'| = |x| + |x'| - 2(x|x') < 2(k+1)R - 2(k - \tfrac{1}{2})R = 3R,$$

i.e., each equivalence class has diameter $\le 3R$. Furthermore, for nonequivalent $x, x' \in A_k$, we have

$$|xx'| = |x| + |x'| - 2(x|x') > 2kR - 2(k - \tfrac{1}{2})R = R,$$

i.e., different equivalence classes are $R/2$-disjoint, see Section 9.9.2.

The covering $\mathcal{U}$ of T by all equivalence classes has mesh$(\mathcal{U}) \le 3R$. Now the coloring of the annuli by even and odd k makes this covering 2-colored such that members of the same color are $R/2$-disjoint. Thus $\operatorname{asdim} T \le \ell\text{-}\operatorname{asdim} T \le 1$, cf. Exercise 9.9.1. $\square$

There is a general idea behind this proof that can be applied in many other cases though the estimates from above it produces are often not optimal.

Let $\{X_\alpha\}$ be a family of metric spaces. The *uniform asymptotic dimension* of $\{X_\alpha\}$ is the minimal n such that for every positive d there is $R > 1$ and an (open) $(n+1)$-colored covering $\mathcal{U}_\alpha$ of each X_α with $L(\mathcal{U}_\alpha) \ge d$ and mesh$(\mathcal{U}_\alpha) \le R$.

Lemma 10.2.2. *Let $f : X \to Y$ be a Lipschitz map between metric spaces. Suppose that* $\operatorname{asdim} Y \le n$ *and for each $R > 1$ the inverse image $f^{-1}(B_R(y))$ has asymptotic dimension $\le k$ uniformly in $y \in Y$. Then* $\operatorname{asdim} X \le (n+1)(k+1) - 1$.

Proof. Let $\lambda = \operatorname{Lip}(f)$ be the Lipschitz constant of f. Given a positive d, there is an open $(n+1)$-colored covering $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$, $|A| = n+1$, of Y with $L(\mathcal{U}) \ge \lambda d$ and mesh$(\mathcal{U}) \le R$ for some $R > 1$. Note that for the covering $\widetilde{\mathcal{U}} = f^{-1}(\mathcal{U})$ of X, we have $L(\widetilde{\mathcal{U}}) \ge L(\mathcal{U})/\lambda \ge d$.

Then there is $D > 1$ such that for each $a \in A$, the inverse image $V = f^{-1}(U)$ of every member $U \in \mathcal{U}^a$ can be covered by a $(k+1)$-colored family $\mathcal{W}_V$ of open sets with $L(\mathcal{W}_V) \ge d$ and mesh$(\mathcal{W}_V) \le D$. Here the condition $L(\mathcal{W}_V) \ge d$ means that $B_d(x) \cap V$ is contained in some member of $\mathcal{W}_V$ for every $x \in V$. Therefore, the union $\bigcup_{U \in \mathcal{U}^a} f^{-1}(U)$ is covered by the open family $\bigcup_V \mathcal{W}_V$ that is $(k+1)$-colored and uniformly bounded by D with Lebesgue number $\ge d$.

Taking the union over all $a \in A$, we obtain the open $(n+1)(k+1)$-colored covering $\mathcal{W}$ of X with mesh$(\mathcal{W}) \le D$. Since each ball $B_d \subset X$ of radius d is contained in some member $V \in \widetilde{\mathcal{U}}$, we see that $L(\mathcal{W}) \ge d$. Hence, $\operatorname{asdim} X \le (n+1)(k+1) - 1$. $\square$

As an application, we obtain:

Proposition 10.2.3. *For every $m \geq 1$, we have* $\operatorname{asdim} \mathrm{H}^{m+1} \leq 2m+1$.

Proof. Consider H^{m+1} in the upper half-space model $\mathbb{R}^{m+1}_{+} = \mathbb{R}^m \times (0, \infty)$, see Appendix. The Busemann function $b\colon \mathrm{H}^{m+1} \to \mathbb{R}$, $b(u,v) = -\ln v$ for $(u,v) \in \mathbb{R}^{n+1}_{+}$ is 1-Lipschitz and the inverse images $b^{-1}(B_R)$ of balls $B_R \subset \mathbb{R}$ are isometric to each other for every fixed radius $R > 0$. In particular, $b^{-1}(B_R)$ have one and the same asymptotic dimension uniformly. Since the asymptotic dimension is a quasi-isometry invariant, $\operatorname{asdim} b^{-1}(B_R)$ coincides with the asymptotic dimension of any horosphere $S = b^{-1}(t)$, $t > 0$. The horosphere S is isometric to $\mathbb{R}^m$ with respect to the induced intrinsic metric ρ and thus it has open, uniformly bounded, $(m+1)$-colored coverings with arbitrarily large Lebesgue number with respect to ρ, see Example 9.1.3. To apply Lemma 10.2.2, it remains to note that $|xx'| \asymp \ln \rho(x,x')$ for sufficiently separated $x, x' \in S$, see Exercise A.3.3, where $|xx'|$ is the distance in H^{m+1}. $\square$

This estimate is not optimal: we show below that $\operatorname{asdim} \mathrm{H}^2 = 2$ by constructing a quasi-isometric embedding into the product of two trees. The embedding approach into products of trees works in the general case and we use it to show that $\operatorname{asdim} \mathrm{H}^m = m$ for every $m \geq 2$, see Theorem 12.3.3. However, Proposition 10.2.3 suffices to prove that the asymptotic dimension of every hyperbolic group is finite. Moreover, we have:

Corollary 10.2.4. *Let X be a visual Gromov hyperbolic space whose boundary at infinity is doubling for some visual metric. Then* $\operatorname{asdim} X < \infty$.

Proof. Indeed, by the Bonk–Schramm embedding theorem, Theorem 8.2.1 and Remark 8.2.4, X is roughly similar to a subset of H^m for some $m \in \mathbb{N}$. Hence $\operatorname{asdim} X \leq \operatorname{asdim} \mathrm{H}^m \leq 2m-1$. $\square$

10.3 Embedding of H^2 into a product of two trees

We describe a quasi-isometric embedding of H^2 into a product of two trees using a tessellation of H^2 by regular right-angled hexagons. The trees have infinite valence at every vertex and the embedding is equivariant with respect to the action of the Coxeter group generated by reflections on the sides of a hexagon.

Consider a right-angled hexagon F in H^2. Let $x_0 \in F$ be the symmetry center, and let $A = \{a^1, a^2, a^3\}$ and $B = \{b^1, b^2, b^3\}$ denote the alternating lines that contain the sides of the hexagon. Thus, $a^i \cap a^j = \emptyset$ and $b^i \cap b^j = \emptyset$ for $i \neq j$. We denote by G the Coxeter group generated by reflections with respect to the lines a^i, b^j, $i, j = 1, 2, 3$ (we use the same notation for the reflections as for the corresponding lines). We then have the tessellation

$$\mathrm{H}^2 = \bigcup_{g \in G} g(F)$$

of the hyperbolic plane H^2 by regular right-angled hexagons.

An element $r \in G$ is called a-reflection (b-reflection) if $r = gag^{-1}$ ($r = gbg^{-1}$), $a \in A$ ($b \in B$), $g \in G$. The fixed point set of an a-reflection r is called *a-mirror* $M_r = \{x \in \mathrm{H}^2 : r(x) = x\}$. Let L_a denote the union of all a-mirrors. Let T_a be a the graph whose vertices are components of $\mathrm{H}^2 \setminus L_a$, and whose edges correspond to common boundary a-mirrors. Clearly, the graph T_a is a tree. We consider it as a metric tree assuming that the length of each edge of T_a is 1. The metric tree T_b is defined similarly.

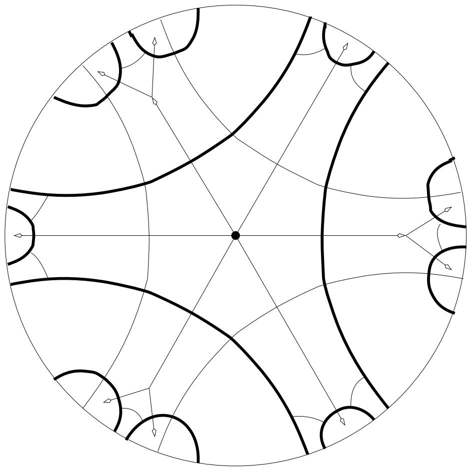


Figure 10.1. a-mirrors and the tree T_a.

Proposition 10.3.1. *There exists a quasi-isometric embedding* $\mathrm{H}^2 \to T_a \times T_b$.

Proof. To define a quasi-isometric map, it suffices to define it on a net. In our case, we take as a net the orbit $G(x_0) = \{g(x_0) : g \in G\} \subset \mathrm{H}^2$.

We define a map $f_a \colon G(x_0) \to T_a$ by the rule: $f_a(x)$ is the vertex in T_a that corresponds to the component of $x \in G(x_0)$. Similarly, we define $f_b \colon G(x_0) \to T_b$.

The geodesic segment $g(x_0)g'(x_0)$ runs inside of a tessellation hexagon during a time not longer than diam F. Furthermore, any two disjoint hexagons are separated at least by the distance l, the edge length of F. It follows that $|g(x_0)g'(x_0)| \doteq m$ up to uniformly bounded error, where m is the number of hexagons which are met by $g(x_0)g'(x_0)$. Since every geodesic segment of the form $g(x_0)g'(x_0)$ intersects any mirror at most once, we see that the product map $(f_a, f_b)\colon G(x_0) \to T_a \times T_b$ is quasi-isometric where the product of trees is considered with the l_1-metric. $\square$

It is easy to prove the following corollary by directly constructing appropriate coverings (see [Gr1], 1.E). However, for higher dimensions, this does not work properly while embeddings into products of trees are still effective.

Corollary 10.3.2. $\operatorname{asdim} \mathrm{H}^2 = \ell\text{-asdim}\, \mathrm{H}^2 = 2$.

Proof. By Theorem 10.1.1, we have $\operatorname{asdim} \mathrm{H}^2 \geq 2$. Using Proposition 10.3.1 and Proposition 10.2.1, we obtain $\ell\text{-asdim}\, \mathrm{H}^2 \leq 2$. □

10.4 Supplementary results and remarks

10.4.1 Estimates from above for the asymptotic dimension

Proposition 10.2.1 is well known, see e.g. [DJ], Proposition 4. The proof shown above was taken from [Ro], where it is generalized to the case of hyperbolic groups, avoiding the Bonk–Schramm embedding theorem, see also [Ro1].

The above Lemma 10.2.2 is [Ro], Lemma 9.16, which generalizes arguments from [BD1], Theorem 2.

In [DJ], it was shown that Coxeter groups have finite asymptotic dimension. The following optimal estimate for a finite graph of groups is obtained in [BD2].

Theorem 10.4.1. *Let π be the fundamental group of a finite graph of groups with finitely generated vertex groups G_v having* $\operatorname{asdim} G_v \leq n$ *for all vertices v. Then* $\operatorname{asdim} \pi \leq n + 1$.

This result is generalized in [Be] to the case of groups acting on complexes. The last two results are further generalized and sharpened in [BD3], where the following Hurewicz-type theorem for the asymptotic dimension is proven.

Theorem 10.4.2. *Let $f : X \to Y$ be a Lipschitz map of a geodesic metric space X to a metric space Y. Suppose that for every $R > 0$, $\{f^{-1}(B_R(y)) : y \in Y\}$ satisfies the inequality* $\operatorname{asdim} \leq n$ *uniformly in R. Then* $\operatorname{asdim} X \leq \operatorname{asdim} Y + n$.

A further progress is achieved in [BDLM], where a Hurewicz-type theorem is proven for the Assouad–Nagata dimension as well as for the linearly controlled asymptotic dimension.

Refined versions of Lemma 10.2.2 have been used in [LS] to prove the following:

1. $\operatorname{ANdim} X = n$ for any Euclidean building X of rank $n \geq 1$. The argument actually gives $\operatorname{asdim} X = \ell\text{-asdim}\, X = n$;

2. Let X be a homogeneous Hadamard manifold, i.e. a Hadamard manifold with transitive isometry group. Then $\operatorname{ANdim} X < \infty$.

Since $\operatorname{ANdim} X = \max\{\ell\text{-dim}\, X, \ell\text{-asdim}\, X\}$ and $\ell\text{-dim}\, X = \dim X$, this yields $\operatorname{asdim} X \leq \ell\text{-asdim}\, X < \infty$.

Let $M = G/K$ be a homogeneous space with a G-invariant metric, where G is a connected Lie group, and K its maximal compact subgroup. It is proven in [CG] that $\operatorname{asdim} M = \dim M$. In particular, this equality holds for each symmetric space of noncompact type.

Let S be a compact orientable surface. The *curve graph* X of S is a graph whose vertices are free homotopy classes of essential simple closed curves on S. Two distinct vertices are joined by an edge if the corresponding classes can be realized by disjoint curves. Let g be the genus of S, and p the number of the boundary components. If $3g - 3 + p > 1$ then, by a result of H. Masur and Y. Minsky [MM], the curve graph $X(S)$ is hyperbolic with respect to the intrinsic metric with length 1 edges. The graph $X = X(S)$ plays an important role in the study of the mapping class group of S, which naturally acts on X. It is proven in [BeFu] that if $3g - 3 + p > 1$ then the asymptotic dimension of X is finite, asdim $X < \infty$.

Chapter 11

Linearly controlled metric dimension: Basic properties

Let $\mathcal{U}$ be an open covering of a metric space Z with $\operatorname{mesh}(\mathcal{U}) < \infty$. We define the *capacity* of $\mathcal{U}$ by

$$\operatorname{cap}(\mathcal{U}) = \sup\{\delta : L(\mathcal{U}) \geq \delta \operatorname{mesh}(\mathcal{U})\}.$$

The basic motivation of linearly controlled dimensions is that in some circumstances we need control over the Lebesgue number $L(\mathcal{U})$ of coverings involved in the definition of a dimension, e.g., that the capacity $\operatorname{cap}(\mathcal{U})$ stays separated from zero for appropriately chosen $\mathcal{U}$'s. In general, there is no reason for that. However, if we allow coverings with larger multiplicity, we can typically gain control over $L(\mathcal{U})$.

Examples. (1) The topological dimension of the space $Z = \{0\} \cup \{1/m : m \in \mathbb{N}\} \subset [0, 1]$ is zero, $\dim Z = 0$, because it obviously admits open coverings of multiplicity 1 with arbitrarily small mesh. However, ℓ-$\dim Z > \dim Z$ by the following argument.

Given $\tau > 0$, let $\mathcal{U}$ be an open covering of Z with $m(\mathcal{U}) = 1$ and $\operatorname{mesh}(\mathcal{U}) \leq \tau$. Take the member $U \in \mathcal{U}$ containing 0 and consider the maximal $z = 1/m \in U$. Then $1/m = \operatorname{diam} U \leq \tau$, and we have

$$L(\mathcal{U}) \leq \operatorname{dist}(z, Z \setminus U) \leq \frac{1}{m-1} - \frac{1}{m} \leq \frac{\tau^2}{1-\tau}.$$

Hence, there is no way to find a multiplicity one open covering $\mathcal{U}$ of Z with $\operatorname{mesh}(\mathcal{U}) \leq \tau$, $L(\mathcal{U}) \geq \delta\tau$ for some fixed $\delta > 0$ and arbitrarily small τ. This implies ℓ-$\dim Z > 0$, and in fact, ℓ-$\dim Z = 1$ by the monotonicity theorem because ℓ-$\dim \mathbb{R} = 1$. Actually, the ℓ-dimension of a space might be arbitrarily larger than the topological dimension; see Section 11.3.1.

(2) The asymptotic dimension of the space $X = \{n^2 : n \in \mathbb{N}\}$ with metric induced from $\mathbb{R}$ is zero, $\operatorname{asdim} X = 0$, while ℓ-$\operatorname{asdim} X = 1$. We leave the proof to the reader as an exercise.

The basic result of the chapter is the existence of separated sequences of colored coverings related to the linearly controlled metric dimension, ℓ-dim, see Theorem 11.1.1. This theorem has a number of applications in the sequel. The proof of quasi-symmetry invariance of the linearly controlled dimensions, which is the main result of Section 11.2, is based on Theorem 11.1.1.

In the next chapter, we prove a general and sharp embedding result of hyperbolic spaces into the product of metric trees. This result uses the whole power of Theorem 11.1.1.

11.1 Separated sequences of colored coverings

The main feature of linearly controlled dimensions is that the coverings involved in their definitions have the Lebesgue number at the same scale as their mesh. This is the source of astonishing flexibility in manipulating with coverings, which allows to achieve many useful properties.

Theorem 11.1.1. *Let Z be a metric space with finite ℓ-dimension, $\ell\text{-}\dim Z \le n$. Then there are constants $\delta, \gamma \in (0,1)$ such that for every sufficiently small $r \in (0,1)$ there exists a sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, of $(n+1)$-colored (by a set A) open coverings of Z with the following properties*:

(i) $\operatorname{mesh} \mathcal{U}_j < r^j$ *and* $L(\mathcal{U}_j) \ge \delta r^j$ *for every* $j \in \mathbb{N}$;

(ii) *for every $a \in A$ and any different members $U \in \mathcal{U}_j^a$, $U' \in \mathcal{U}_{j'}^a$, with $j' \le j$, we have either $B_s(U) \cap U' = \emptyset$ or $B_s(U) \subset U'$ with $s = \gamma r^j$.*

The construction of $\{\mathcal{U}_j\}$ is based on the saturation construction, see Section 9.6, and consists of infinitely many steps. The idea of an elementary step can be explained as follows. Recall that a family $\mathcal{U}$ is r-disjoint, $r \ge 0$, if the multiplicity $m(B_r(\mathcal{U})) = 1$. Assume we have a separated family $\mathcal{V}$ of subsets in X, that is, any two members of $\mathcal{V}$ are either disjoint or one of them is contained in the other. Now given an r-disjoint family $\mathcal{U}$ of subsets in X with $r > \operatorname{mesh}(\mathcal{V})$, the saturated union

$$(\mathcal{U} * \mathcal{V}) \cup \mathcal{V}$$

is a separated family, where we recall $\mathcal{U} * \mathcal{V} = \{U * \mathcal{V} : U \in \mathcal{U}\}$ is the saturation of $\mathcal{U}$ by $\mathcal{V}$, i.e., $U * \mathcal{V}$ is the union of U and all members $V \in \mathcal{V}$ with $U \cap V \ne \emptyset$. This is because $V \subset B_r(U)$ whenever $U \cap V \ne \emptyset$.

We begin with the following fact used in Lemma 11.1.5.

Lemma 11.1.2. *Given $U \subset Z$, for every $0 < s < t$ we have*

$$B_{t-s}(U) \subset B_{-s}(B_t(U)).$$

Proof. If $z \notin B_{-s}(B_t(U))$ then there is $z' \in Z \setminus B_t(U)$ with $d(z,z') \le s$. For every $u \in U$, we have $d(z',u) \ge t$ and thus $d(z,u) \ge d(z',u) - d(z,z') \ge t - s$. Therefore, $z \notin B_{t-s}(U)$. □

Next we prepare a sequence of colored coverings modifying which we construct a required separated sequence.

Lemma 11.1.3. *Under the condition of Theorem 11.1.1, there are constants δ, $r_0 \in (0,1)$ such that for every $r \in (0, r_0)$, there exists a sequence of open, $(n+1)$-colored (by a color set A) coverings $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$, of Z such that* $\mathrm{mesh}(\widehat{\mathcal{U}}_j) < r^j$, *$L(\widehat{\mathcal{U}}_j) \ge \delta r^j$ and for every $a \in A$, the family $\widehat{\mathcal{U}}_j^a$ is δr^j-disjoint for every $j \in \mathbb{N}$.*

Proof. It follows from the colored definition of the ℓ-dimension that there are constants δ', $r_0 \in (0,1)$, with the following property. Given $r \in (0, r_0)$, for every $j \in \mathbb{N}$, there is an open $(n+1)$-colored covering $\mathcal{U}_j$ of Z with $\mathrm{mesh}(\mathcal{U}_j) < r^j$ and $L(\mathcal{U}_j) \ge \delta' r^j$. Fix $j \in \mathbb{N}$ and for $s = \delta' r^j/2$ consider the family $\widehat{\mathcal{U}}_j = B_{-s}(\mathcal{U}_j)$. Then $\widehat{\mathcal{U}}_j$ is an open covering of Z (see Remark 9.1.1), and we have $\mathrm{mesh}(\widehat{\mathcal{U}}_j) \le \mathrm{mesh}(\mathcal{U}_j) < r^j$, $L(\widehat{\mathcal{U}}_j) \ge \delta r^j$ with $\delta = \delta'/2$. Furthermore, for every color $a \in A$, the family $\widehat{\mathcal{U}}_j^a$ is δr^j-disjoint. □

We need a qualified version of the saturation construction. Given $s \ge 0$, $U \subset Z$ and the family $\mathcal{V}$ of subsets in Z, the s-saturation of U by $\mathcal{V}$ is the union $U *_s \mathcal{V}$ of U and all members $V \in \mathcal{V}$ with $B_s(U) \cap B_s(V) \ne \emptyset$.

Lemma 11.1.4. *Assume that a family of sets $\mathcal{V}$ is δs-disjoint for $\delta \in (0, 2/3]$, $s > 0$, and* $\mathrm{mesh}(\mathcal{V}) < 2s$. *Then the operation*

$$U \mapsto U^* = B_{\delta s}\left(B_{-4s}(U) *_{\delta s} \mathcal{V}\right)$$

does not increase any set $U \subset Z$, $U^* \subset U$, and for every $V \in \mathcal{V}$ it holds that either $B_{\delta s}(V) \cap U^* = \emptyset$ or $B_{\delta s}(V) \subset U^*$.

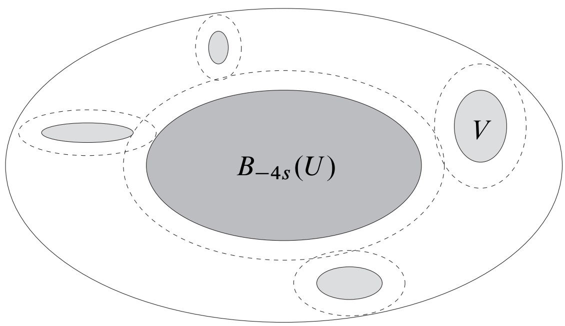


Figure 11.1. Operation $U \mapsto U^*$.

Proof. For every $V \in \mathcal{V}$, we have that diam $B_{\delta s}(V) < 2s + 2\delta s$ and therefore $\delta s + \mathrm{diam}\, B_{\delta s}(V) < 2s(1 + 3\delta/2) \le 4s$. Hence if $B_{\delta s}(V)$ intersects $B_{\delta s}(B_{-4s}(U))$ then $B_{\delta s}(B_{-4s}(U) \cup V) \subset U$, and $U^* \subset U$, in particular, in that case $B_{\delta s}(V) \subset U^*$. Otherwise, $B_{\delta s}(V)$ misses the δs-neighborhood of $B_{-4s}(U)$ as well as the one of every other member of V, and thus $B_{\delta s}(V) \cap U^* = \emptyset$. □

This sophisticated version $U \mapsto U^*$ of the saturation is chosen to provide also the following important property which allows to apply the operation inductively infinitely many times leaving invariant quantitative separation conditions.

Lemma 11.1.5. *Under the conditions of Lemma 11.1.4 assume that*

$$B_t(U_1) \subset U_2,\ t > 4s,$$

for some sets $U_1, U_2 \subset Z$. Then $B_{t'}(U_1^*) \subset U_2^*$ for $t' = t - 4s$.

Proof. By Lemma 11.1.4, we have $B_{t'}(U_1^*) \subset B_{t'}(U_1)$. By Lemma 11.1.2, $B_{t-4s}(U_1) \subset B_{-4s}(B_t(U_1))$. Finally, we have $B_{-4s}(B_t(U_1)) \subset B_{-4s}(U_2) \subset U_2^*$. □

Proof of Theorem 11.1.1. We take the sequence of the colored coverings $\widehat{\mathcal{U}}_j$, $j \in \mathbb{N}$, constructed in Lemma 11.1.3 for $r \in (0, r_0)$, and modify it to obtain a separated sequence. We further assume that $\frac{2r}{1-r} \le \delta/4 \le 1/6$.

Fix a color $a \in A$ and define $\mathcal{V}_1^a = \mathcal{U}_{1,1}^a := \widehat{\mathcal{U}}_1^a$. Then the family $\mathcal{U}_{1,1}^a$ is δr-disjoint and $\operatorname{mesh}(\mathcal{U}_{1,1}^a) < r$.

Assume that for $k \ge 1$ the family $\mathcal{V}_k^a$ is already defined and has the following properties:

(i) $\mathcal{V}_k^a = \bigcup_{j=1}^k \mathcal{U}_{j,k}^a$;

(ii) the family $\mathcal{U}_{j,k}^a$ is δr^j-disjoint and $\operatorname{mesh}(\mathcal{U}_{j,k}^a) < r^j$ for every $1 \le j \le k$;

(iii) given $1 \le j' < j \le k$, for every $U' \in \mathcal{U}_{j',k}^a$, $U \in \mathcal{U}_{j,k}^a$, we have either $B_t(U) \cap U' = \emptyset$ with $t = \delta r^j/2$ or $B_t(U) \subset U'$ with $t = \gamma_{k,j} r^j$, where $\gamma_{k,j}$ is defined recurrently by $\gamma_{j,j} = \delta/2$ and $\gamma_{k,j} = \gamma_{k-1,j} - 2r^{k-j}$ for $k > j$.

We define

$$\mathcal{V}_{k+1}^a := B_{\delta s}(B_{-4s}(\mathcal{V}_k^a) *_{\delta s} \widehat{\mathcal{U}}_{k+1}^a) \cup \widehat{\mathcal{U}}_{k+1}^a$$

with $s = r^{k+1}/2$. Then $\mathcal{V}_{k+1}^a = \bigcup_{j=1}^{k+1} \mathcal{U}_{j,k+1}^a$, where

$$\mathcal{U}_{j,k+1}^a = B_{\delta s}(B_{-4s}(\mathcal{U}_{j,k}^a) *_{\delta s} \widehat{\mathcal{U}}_{k+1}^a)$$

for $1 \le j \le k$ and $\mathcal{U}_{k+1,k+1}^a = \widehat{\mathcal{U}}_{k+1}^a$. Since the family $\widehat{\mathcal{U}}_{k+1}^a$ is $2\delta s$-disjoint and $\operatorname{mesh}(\widehat{\mathcal{U}}_{k+1}^a) < 2s$, we can apply Lemma 11.1.4, by which every $U^* \in \mathcal{U}_{j,k+1}^a$ with $1 \le j \le k$ is contained in the appropriate $U \in \mathcal{U}_{j,k}^a$, in particular, the family $\mathcal{U}_{j,k+1}^a$ is δr^j-disjoint and $\operatorname{mesh}(\mathcal{U}_{j,k+1}^a) \le \operatorname{mesh}(\mathcal{U}_{j,k}^a) < r^j$. Furthermore, for every $\widehat{U} \in \mathcal{U}_{k+1,k+1}^a$ we have either $B_{\delta s}(\widehat{U}) \cap U^* = \emptyset$ or $B_{\delta s}(\widehat{U}) \subset U^*$.

Now if $U' \in \mathcal{U}_{j',k}^a$ with $j' < j$ and $B_t(U) \cap U' = \emptyset$ with $t = \delta r^j/2$ then $B_t(U^*) \cap U'^* = \emptyset$ since $U^* \subset U$ and $U'^* \subset U'$. In the case when $B_t(U) \subset U'$ with $t = \gamma_{j,k} r^j$, we have $B_{t-2r^{k+1}}(U^*) \subset U'^*$ by Lemma 11.1.5 and $t - 2r^{k+1} =$

$\gamma_{k+1,j} r^j$ with $\gamma_{k+1,j} = \gamma_{k,j} - 2r^{k+1-j}$. Note that $\lim_{k\to\infty} \gamma_{k,j} = \delta/2 - \frac{2r}{1-r} \geq \delta/4$ for every $j > 1$.

Therefore, for every color $a \in A$, we have the sequence $\mathcal{V}_k^a$, $k \in \mathbb{N}$, of families of sets in Z with properties (i)–(iii). It follows from the definition of the $*$-operation that every member $U^* \in \mathcal{V}_{k+1}^a$ is contained in its well-defined predecessor $U \in \mathcal{V}_k^a$, moreover, $U^* \in \mathcal{U}_{j,k+1}^a$ if and only if $U \in \mathcal{U}_{j,k}^a$. In this sense, the sequence $\mathcal{V}_k^a$ is monotone, $\mathcal{V}_k^a \supset \mathcal{V}_{k+1}^a$, and we define $\mathcal{U}_j^a = \operatorname{Int}\left(\bigcap_{k\geq j} \mathcal{U}_{j,k}^a\right)$, where $\operatorname{Int} U$ is the interior of a subset $U \subset Z$, $\mathcal{U}_j = \bigcup_{a\in A} \mathcal{U}_j^a$ for every $j \in \mathbb{N}$.

We put $\hat{s}_j = \sum_{k\geq j} 2r^{k+1} = \frac{2r}{1-r} r^j$. Then $\hat{s}_j \leq \delta r^j/4 < L(\widehat{\mathcal{U}}_j)$ and $B_{-\hat{s}_j}(\widehat{\mathcal{U}}_j^a) \subset \mathcal{U}_j^a$ for every $a \in A$, $j \in \mathbb{N}$. By Remark 9.1.1, the family $\mathcal{U}_j$ is still an open $(n+1)$-colored covering of Z with $L(\mathcal{U}_j) \geq \delta r^j - \hat{s}_j \geq \delta r^j/2$. From (ii) we obtain $\operatorname{mesh}(\mathcal{U}_j) < r^j$ and the family $\mathcal{U}_j^a$ is δr^j-disjoint for every $a \in A$, $j \in \mathbb{N}$. Property (iii) implies that given $1 \leq j' < j$, for every $U' \in \mathcal{U}_{j'}^a$, $U \in \mathcal{U}_j^a$, we have either $B_t(U) \cap U' = \emptyset$ or $B_t(U) \subset U'$ with $t = \gamma r^j$, $\gamma \geq \delta/4$. □

11.2 Quasi-symmetry invariance of ℓ-dim

By the definition, the ℓ-dimension is a bilipschitz invariant. On the other hand it turns out that the ℓ-dimension perfectly fits a number of questions related to the boundary at infinity of hyperbolic spaces taken with visual metrics. Recall that visual metrics (based at interior points of a hyperbolic space) are only defined up to a quasi-symmetry transformation. Surprisingly, the ℓ-dimension is actually a quasi-symmetry invariant of uniformly perfect spaces. In that way the ℓ-dimension of the boundary at infinity of a hyperbolic space is well defined independently of the choice of a visual metric.

Theorem 11.2.1. *The ℓ-dimension is a quasi-symmetry invariant of uniformly perfect metric spaces, that is, if there is a quasi-symmetric homeomorphism $f: X \to Y$ between uniformly perfect metric spaces then* $\ell\text{-dim}\, X = \ell\text{-dim}\, Y$.

The proof is based on the existence of a separated sequence of colored coverings, Theorem 11.1.1. The idea can be explained as follows. One easily realizes that there is no reasonable control over the capacity $\operatorname{cap}(\mathcal{U})$ of a covering $\mathcal{U}$ of a metric space X under quasi-symmetry homeomorphisms, and thus there is no way for a straightforward argument. However, the *local capacity* of $\mathcal{U}$ defined by

$$\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) = \inf_{x\in X} \sup\{\delta : L(\mathcal{U}, x) \geq \delta \operatorname{mesh}(\mathcal{U}, x)\},$$

where $\operatorname{mesh}(\mathcal{U}, z) = \sup\{\operatorname{diam} U : x \in U \in \mathcal{U}\}$, has the advantage over the capacity that its positivity is preserved under quasi-symmetries quantitatively, see Lemma 11.2.2. Furthermore, if $\mathcal{U}$ is *c-balanced*, that is $\inf\{\operatorname{diam}(U) : U \in \mathcal{U}\} \geq c \cdot \operatorname{mesh}(\mathcal{U})$ with $c > 0$, then $\operatorname{cap}(\mathcal{U}) \geq c \cdot \operatorname{cap}_{\mathrm{loc}}(\mathcal{U})$, see Lemma 11.2.3. We combine these two facts to obtain the desired estimate in the following way.

Taking a separated sequence of colored coverings $\mathcal{U}_j$, $j \in \mathbb{N}$, of X, we construct out of it a covering $\mathcal{V}$ of X with local capacity uniformly separated from 0, see Lemma 11.2.6, such that the image $f(\mathcal{V})$ is balanced and has an arbitrarily small mesh, see Lemma 11.2.7. Then the capacity of the covering $f(\mathcal{V})$ of Y is positive quantitatively, which implies $\ell\text{-dim}\, Y \le \ell\text{-dim}\, X$.

Recall that a (nonconstant) map $f: X \to Y$ between metric spaces is η-quasi-symmetric with control function $\eta\colon [0,\infty) \to [0,\infty)$ if from $|xa| \le t|xb|$, it follows that $|f(x)f(a)| \le \eta(t)|f(x)f(b)|$ for any $a, b, x \in X$ and all $t \ge 0$; see Definition 5.2.11.

Lemma 11.2.2. *Let $\mathcal{U}$ be an open covering of a metric space Z with finite mesh,* $\operatorname{mesh}(\mathcal{U}) < \infty$*, let $f: Z \to Z'$ be an η-quasi-symmetry and let $\mathcal{U}' = f(\mathcal{U})$ be the image of $\mathcal{U}$. Then for the local capacities of $\mathcal{U}$ and $\mathcal{U}'$, we have*

$$\frac{1}{\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}')} \le 16\eta\left(\frac{2}{\operatorname{cap}_{\mathrm{loc}}(\mathcal{U})}\right).$$

Proof. We can assume that $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) > 0$ and that no member of $\mathcal{U}$ coincides with Z, since otherwise $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) = \operatorname{cap}_{\mathrm{loc}}(\mathcal{U}') = \infty$. We fix $z \in Z$ and consider $U \in \mathcal{U}$ for which $z \in U$ and $\operatorname{dist}(z, Z \setminus U) \ge L(\mathcal{U}, z)/2$. For $z' = f(z)$ and $U' = f(U)$ there is $a' \in Z' \setminus U'$ with $|z'a'| \le 2\operatorname{dist}(z', Z' \setminus U')$. Then $|z'a'| \le 2L(\mathcal{U}', z')$, and for $a = f^{-1}(a')$ we have $|za| \ge \operatorname{dist}(z, Z \setminus U) \ge L(\mathcal{U}, z)/2$.

Similarly, consider $V' \in \mathcal{U}'$ with $z' \in V'$ and $\operatorname{diam} V' \ge \operatorname{mesh}(\mathcal{U}', z')/2$. There is $b' \in V'$ with $|z'b'| \ge \operatorname{diam} V'/4$. Then $|z'b'| \ge \operatorname{mesh}(\mathcal{U}', z')/8$, and for $b := f^{-1}(b')$ we have $|zb| \le \operatorname{mesh}(\mathcal{U}, z)$. Therefore, we have

$$\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \le \frac{L(\mathcal{U}, z)}{\operatorname{mesh}(\mathcal{U}, z)} \le \frac{2|za|}{|zb|} \quad \text{and} \quad |zb| \le t|za|$$

with $t = 2/\operatorname{cap}_{\mathrm{loc}}(\mathcal{U})$. It follows that $|z'b'| \le \eta(t)|z'a'|$ and

$$\frac{L(\mathcal{U}', z')}{\operatorname{mesh}(\mathcal{U}', z')} \ge \frac{|z'a'|}{16|z'b'|} \ge (16\eta(t))^{-1}$$

for every $z' \in Z'$. Then $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}') \ge \big(16\eta(\frac{2}{\operatorname{cap}_{\mathrm{loc}}(\mathcal{U})})\big)^{-1}$. □

Lemma 11.2.3. *If an open covering $\mathcal{U}$ of a metric space Z is c_1-balanced and its local capacity satisfies* $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \ge c_0$*, then* $\operatorname{cap}(\mathcal{U}) \ge c_0 \cdot c_1$.

Proof. Since $\mathcal{U}$ is c_1-balanced, we have $\operatorname{mesh}(\mathcal{U}, z) \ge c_1 \cdot \operatorname{mesh}(\mathcal{U})$ for every $z \in Z$. Since $\operatorname{cap}_{\mathrm{loc}}(\mathcal{U}) \ge c_0$, we have $L(\mathcal{U}, z) \ge c_0 \cdot \operatorname{mesh}(\mathcal{U}, z)$ for every $z \in Z$. Therefore $L(\mathcal{U}) \ge c_0 c_1 \cdot \operatorname{mesh}(\mathcal{U})$. □

It suffices to prove that $\ell\text{-dim}\, Y \le \ell\text{-dim}\, X$, because then the opposite inequality is obtained be permuting X and Y. Thus we assume that $n = \ell\text{-dim}\, X < \infty$.

By Theorem 11.1.1, for some $\delta \in (0,1)$ and for every sufficiently small $r > 0$, there is a separated sequence of $(n+1)$-colored coverings $\mathcal{U}_j$ of X with $\mathrm{mesh}(\mathcal{U}_j) < r^j$ and $L(\mathcal{U}_j) \ge \delta r^j$. We take $r < \delta$. Then $\mathrm{mesh}(\mathcal{U}_{j+1}) < L(\mathcal{U}_j)$ and, hence, the covering $\mathcal{U}_{j+1}$ is *inscribed* in $\mathcal{U}_j$ for every $j \in \mathbb{N}$, that is, every member of $\mathcal{U}_{j+1}$ is contained in some member of $\mathcal{U}_j$.

Since X is uniformly perfect, there is $\mu \in (0,1)$ such that for every $x \in X$ and every $r > 0$ with $X \setminus B_r(x) \neq \emptyset$ we have $B_r(x) \setminus B_{\mu r}(x) \neq \emptyset$.

We can additionally assume that $\mathrm{diam}\, U \ge \mu L(\mathcal{U}_j)$ for every $U \in \mathcal{U}_j$, since if $\mathrm{diam}\, U < \mu L(\mathcal{U}_j)$ then U cannot coincide with the ball $B_\rho(x)$ of radius $\rho = L(\mathcal{U}_j)$ centered at any point $x \in U$ because X is μ-uniformly perfect. Thus U is contained in another member $U' \in \mathcal{U}_j$ and hence it can be deleted from $\mathcal{U}_j$ without destroying any property of the sequence. This is the only place where we use the uniform perfection condition.

We put $\mathcal{U} = \bigcup_{j \in \mathbb{N}} \mathcal{U}_j$ and for $s > 0$ consider the family $\mathcal{U}(s) = \{U \in \mathcal{U} : \mathrm{diam}\, f(U) \le s\}$.

Lemma 11.2.4. *For every $s > 0$, the family $\mathcal{U}(s)$ is a covering of X.*

Proof. We fix $x \in X$, consider an element $x' \in X$ different from x and put $y = f(x)$, $y' = f(x')$. For every $j \in \mathbb{N}$, there is $U_j \in \mathcal{U}_j$ containing x. Take $y_j \in f(U_j)$ with $\mathrm{diam}\, f(U_j) \le 4|yy_j|$ and consider $x_j = f^{-1}(y_j)$. Then $|xx_j| \le t_j|xx'|$ with $t_j \to 0$ as $j \to \infty$, since $\mathrm{diam}\, U_j \le r^j \to 0$. Therefore, $\mathrm{diam}\, f(U_j) \le 4|yy_j| \le 4\eta(t_j)|yy'| \le s$ for sufficiently large j. Hence, $U_j \in \mathcal{U}(s)$. $\square$

A family $\mathcal{V} \subset \mathcal{U}(s)$ is *maximal* if every $U \in \mathcal{U}(s)$ is contained in some $V \in \mathcal{V}$ and neither of different $V, V' \in \mathcal{V}$ is contained in the other.

Lemma 11.2.5. *For every $s > 0$, there is a maximal family $\mathcal{V} \subset \mathcal{U}(s)$. Every maximal family $\mathcal{V} \subset \mathcal{U}(s)$ is an $(n+1)$-colored covering of X.*

Proof. Given $s > 0$, we construct a family $\mathcal{V} \subset \mathcal{U}(s)$ by deleting every $U \in \mathcal{U}(s)$ which is contained in some other $U' \in \mathcal{U}(s)$. Now $\mathcal{V}$ is what remains. We need only to check that for every $U \in \mathcal{U}(s)$ there is a maximal $U' \in \mathcal{U}(s)$ with $U \subset U'$. Since the covering $\mathcal{U}_j$ is $(n+1)$-colored, for every $j \in \mathbb{N}$ there are only finitely many $U' \in \mathcal{U}_j$ containing U (because all of them must have different colors). Since $\mathrm{mesh}(\mathcal{U}_j) \to 0$ as $j \to \infty$, there are only finitely many $U' \in \mathcal{U}(s)$ containing U and hence there is a maximal $U' \in \mathcal{U}(s)$ among them.

Let $\mathcal{V} \subset \mathcal{U}(s)$ be a maximal family. By Lemma 11.2.4, the family $\mathcal{U}(s)$ is a covering of X, and it follows from the definition of a maximal family that $\mathcal{V}$ is also a covering of X. It follows from Theorem 11.1.1 (ii) that different $V, V' \in \mathcal{V}$ having one and the same color are disjoint. Thus $\mathcal{V}$ is $(n+1)$-colored. $\square$

Lemma 11.2.6. *There is a constant $\nu > 0$ depending only on δ, μ, r and η such that for every $s > 0$, every maximal covering $\mathcal{V} \subset \mathcal{U}(s)$ has the local capacity $\mathrm{cap}_{\mathrm{loc}}(\mathcal{V}) \ge \nu$.*

Proof. Let $\mathcal{V} \subset \mathcal{U}(s)$ be a maximal family. Given $x \in X$ we put $j = j(x) = \min\{i \in \mathbb{N} : x \in V \in \mathcal{V} \cap \mathcal{U}_i\}$. Then $\operatorname{mesh}(\mathcal{V}, x) < r^j$. We fix $V \in \mathcal{V} \cap \mathcal{U}_j$ containing x, $v \in V$ with $4|xv| \geq \operatorname{diam} V$ and note that $\operatorname{diam} V \geq \mu L(\mathcal{U}_j) \geq \delta\mu r^j$ by our assumptions.

Next we fix $\sigma > 0$ with $4\eta(4\sigma/\delta\mu) \leq 1$. Now we check that for $i \in \mathbb{N}$ with $r^{i-j} \leq \sigma$ every $U \in \mathcal{U}_i$ containing x is a member of $\mathcal{U}(s)$. There is $u \in U$ with $4|f(x)f(u)| \geq \operatorname{diam} f(U)$. We have $|xu| \leq t|xv|$ for some $t \leq 4 \operatorname{diam} U / \operatorname{diam} V \leq 4r^{i-j}/\delta\mu \leq 4\sigma/\delta\mu$. Then $\operatorname{diam} f(U) \leq 4|f(x)f(u)| \leq 4\eta(4\sigma/\delta\mu)|f(x)f(v)| \leq \operatorname{diam} f(V) \leq s$, thus $U \in \mathcal{U}(s)$.

Therefore, $L(\mathcal{V}, x) \geq L(\mathcal{U}_i) \geq \delta r^i$. Assuming that i is taken minimal with $r^{i-j} \leq \sigma$, we obtain $L(\mathcal{V}, x) \geq \delta\sigma r^{j+1}$. Thus $\frac{L(\mathcal{V},x)}{\operatorname{mesh}(\mathcal{V},x)} \geq \nu = \delta\sigma r$ for every $x \in X$ and $\operatorname{cap}_{\mathrm{loc}}(\mathcal{V}) \geq \nu$. $\square$

Lemma 11.2.7. *Given a maximal family $\mathcal{V} \subset \mathcal{U}(s)$, the $(n+1)$-colored covering $\mathcal{W} = f(\mathcal{V})$ of Y satisfies $\operatorname{diam} W \geq s/4\eta(t)$ for every $W \in \mathcal{W}$, where $t = 4/\delta\mu r$. In particular, $\mathcal{W}$ is c-balanced with $c \geq 1/4\eta(t)$.*

Proof. Note that $\operatorname{mesh}(\mathcal{W}) \leq s$ by the definition of $\mathcal{U}(s)$. Take any $W \in \mathcal{W}$ and consider $V = f^{-1}(W)$. We can assume that $V \in \mathcal{U}_j$ for some $j \in \mathbb{N}$. Then $\operatorname{diam} V \geq \mu L(\mathcal{U}_j) \geq \delta\mu r^j$ by our assumption on the sequence $\{\mathcal{U}_j\}$.

For any $U \in \mathcal{U}$ with $V \subset U$, we have $\operatorname{diam} f(U) > s$, since the family $\mathcal{V}$ is maximal. The covering $\mathcal{U}_j$ is inscribed in $\mathcal{U}_{j-1}$, and thus there is $U \in \mathcal{U}_{j-1}$ containing V, in particular, $\operatorname{diam} f(U) > s$.

Take $y \in W \subset f(U)$. There is $y' \in f(U)$ with $|yy'| \geq \operatorname{diam} f(U)/4 > s/4$. For $x = f^{-1}(y)$, $x' = f^{-1}(y')$ we have $|xx'| \leq \operatorname{diam} U \leq \operatorname{mesh}(\mathcal{U}_{j-1}) \leq r^{j-1}$. There is $v \in V$ with $|xv| \geq \operatorname{diam} V/4 \geq \delta\mu r^j/4$. Thus $|xx'| \leq r^{j-1} \leq t|xv|$ for $t = 4/\delta\mu r$. For $w = f(v) \in W$ we obtain $|yy'| \leq \eta(t)|yw| \leq \eta(t) \operatorname{diam} W$. Hence, $\operatorname{diam} W \geq s/4\eta(t)$. $\square$

Proof of Theorem 11.2.1. By Lemmas 11.2.5 and 11.2.7 for every $s > 0$, we have an open $(n+1)$-colored covering $\mathcal{W} = f(\mathcal{V})$ of Y with $cs \leq \operatorname{mesh}(\mathcal{W}) \leq s$ which is c-balanced, $c \geq 1/4\eta(t)$, where $t = 4/\delta\mu r$. Moreover, by Lemmas 11.2.6 and 11.2.2, its local capacity $\operatorname{cap}_{\mathrm{loc}}(\mathcal{W}) \geq d$, where the constant $d > 0$ depends only on η, δ, μ, r. Then, by Lemma 11.2.3, we have $\operatorname{cap}(\mathcal{W}) \geq c \cdot d$ independently of s and thus $L(\mathcal{W}) \geq c^2 \cdot d \cdot s$. This shows that $\ell\text{-}\dim Y \leq n$. $\square$

Remark 11.2.8. Separation property (ii) from Theorem 11.1.1 has only been used in the proof of Theorem 11.2.1 in a weak qualitative form; see Lemma 11.2.5.

Combining with the monotonicity theorem, we obtain

Corollary 11.2.9. *Assume that there is a quasi-symmetric $f : X \to Y$ between uniformly perfect metric spaces. Then $\ell\text{-}\dim X \leq \ell\text{-}\dim Y$.* $\square$

11.3 Supplementary results and remarks

11.3.1 The ℓ-dimension versus the topological dimension

The following proposition allows to construct examples of metric spaces with different topological and linearly controlled metric dimensions, and the distinction can be arbitrarily large.

Proposition 11.3.1. *Let X, Y be bounded metric spaces such that for every $\varepsilon > 0$ there is $A \subset X$ and a homothety $h_\varepsilon \colon A \to Y$ with ε-dense image,* $\operatorname{dist}(y, h_\varepsilon(A)) < \varepsilon$ *for every $y \in Y$. Then ℓ-$\dim X \geq \dim Y$.*

Proof. We can assume that $\dim Y > 0$, in particular, $\operatorname{diam} Y > 0$. Then we have $\lambda(\varepsilon) \geq \lambda_0 > 0$ as $\varepsilon \to 0$ for the coefficient $\lambda(\varepsilon)$ of the homothety h_ε, because X is bounded.

Assume that $n = \ell\text{-}\dim X < \dim Y$. There is $\delta > 0$ such that for every sufficiently small $\tau > 0$ there is an open covering $\mathcal{U}_\tau$ of X of multiplicity $\leq n + 1$ with $\operatorname{mesh}(\mathcal{U}_\tau) \leq \tau$ and $L(\mathcal{U}_\tau) \geq \delta\tau$.

Using the estimate $\lambda(\varepsilon) \geq \lambda_0$, we can find $\tau = \tau(\varepsilon)$ such that $\lambda(\varepsilon)\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\delta\lambda(\varepsilon)\tau(\varepsilon) \geq 4\varepsilon$. Then for the covering $\mathcal{V}_\varepsilon = h_\varepsilon(\mathcal{U}_\tau)$ of $h_\varepsilon(A)$, we have $\operatorname{mesh}(\mathcal{V}_\varepsilon) \leq \lambda(\varepsilon)\tau(\varepsilon)$ and $L(\mathcal{V}_\varepsilon) \geq \delta\lambda(\varepsilon)\tau(\varepsilon)$. Furthermore, $m(\mathcal{V}_\varepsilon) \leq n + 1$. Therefore, the family $\mathcal{V}'_\varepsilon = B_{-2\varepsilon}(\mathcal{V}_\varepsilon)$ still covers $f(A)$. Taking the ε-neighborhood in Y of every $V \in \mathcal{V}'_\varepsilon$, we obtain an open covering $\mathcal{V}$ of Y with $\operatorname{mesh}(\mathcal{V}) \leq \operatorname{mesh}(\mathcal{V}_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let us estimate the multiplicity of $\mathcal{V}$. Assume that $y \in Y$ is a common point of members $V_j \in \mathcal{V}$, $j \in J$. By the definition of $\mathcal{V}$, for every $j \in J$, there is $a_i \in A$ such that $f(a_j) \in V'_j \in \mathcal{V}'_\varepsilon$ and $|f(a_j)y| < \varepsilon$. Then the mutual distances of the points $f(a_j)$, $j \in J$, are $< 2\varepsilon$. Since $V'_j = B_{-2\varepsilon}(U_j)$ for $U_j \in \mathcal{V}_\varepsilon$, we see that every point $f(a_j)$, $j \in J$, is contained in every U_i, $i \in J$, and therefore $|J| \leq n + 1$ because the multiplicity of $\mathcal{V}_\varepsilon$ is at most $n + 1$. Hence, $m(\mathcal{V}) \leq n + 1$ and $\dim Y \leq n$, a contradiction. $\square$

As an application, we obtain the following examples. Let $Z = \{0\} \cup \{1/m : m \in \mathbb{N}\}$ be the space from the example (1) on p. 137. Then $\ell\text{-}\dim Z^n = n$ for any $n \geq 1$, while $\dim Z^n = 0$ (e.g. by the product theorem). Indeed, the spaces $X = Z^n$ and $Y = [0,1]^n$, obviously, satisfy the condition of Proposition 11.3.1, thus $\ell\text{-}\dim Z^n \geq \dim Y = n$ (we have equality here because $Z^n \subset Y$).

Further examples: Take any monotone sequence of positive $\varepsilon_k \to 0$, $\varepsilon_1 = 1/3$, and repeat the construction of the standard ternary Cantor set $K \subset [0,1]$, only removing at every k-th step, $k \geq 1$, instead of the $(1/3)^k$-length segments, the middle segments of length $s_k = \varepsilon_k l_k$, $l_1 = 1$, where the length l_{k+1} of the segments obtained after processing the k-th step is defined recurrently by $2l_{k+1} + s_k = l_k$. The resulting compact space $K_a \subset [0,1]$ is homeomorphic to K. We easily see that $\ell\text{-}\dim K = 0$. However, $X = K_a$ and $Y = [0,1]$ satisfy the condition of Proposition 11.3.1, thus

$\ell\text{-dim}\, K_a = 1$, while $\dim K_a = 0$. Similarly, one can construct 'exotic' Sierpinski carpets, Menger curves etc with the capacity dimension strictly bigger than the topological dimension. Any of those compact metric spaces is not quasi-symmetric to any locally self-similar space, in particular, it is not quasi-symmetric to the boundary (viewed with a visual metric) of a hyperbolic group. To compare, it is well known that the boundary at infinity of a typical hyperbolic group is homeomorphic to the Menger curve.

11.3.2 Quasi-symmetry invariance of the Assouad–Nagata dimension

It is proved in [LS] that the Assouad–Nagata dimension is a quasi-symmetry invariant of arbitrary metric spaces. Since for bounded metric spaces the Assouad–Nagata dimension coincides with the linearly controlled dimension, this strengthens Theorem 11.2.1 in the case of bounded spaces because then we can omit the uniform perfection condition. As the following example due to Nina Lebedeva shows, in the general case the linearly controlled dimension as well as the linearly controlled asymptotic dimension is not a quasi-symmetry invariant.

Example 11.3.2 (N. Lebedeva). Let $X = \{2^k + i : i, k \in \mathbb{N},\ i \le k\}$ and $Y = \{2^k + i/k : i, k \in \mathbb{N}, i \le k\}$ be metric spaces with metrics induced from $\mathbb{R}$. Then one easily checks that $\ell\text{-dim}\, X = 0$, $\ell\text{-asdim}\, X = 1$ and $\ell\text{-dim}\, Y = 1$, $\ell\text{-asdim}\, Y = 0$, in particular, $\text{ANdim}\, X = \text{ANdim}\, Y = 1$. On the other hand, the natural map $f : X \to Y$, $f(2^k + i) = 2^k + i/k$ is a quasi-symmetry (the map is well defined because the representation $x = 2^k + i$ is unique for every $x \in X$).

Bibliographical note. The construction of Theorem 11.1.1 fits a large array of similar constructions for various dimensions, e.g., see [Os2] for the topological dimension, [Dr2] for the asymptotic dimension, [LS] for the Assouad–Nagata dimension to name few.

Proposition 11.3.1 and its application to examples of metric spaces with different topological and linearly controlled dimensions is due to N. Lebedeva; see [BL].

Chapter 12

Linearly controlled metric dimension: Applications

12.1 Embedding into the product of trees

Here we use the whole power of Theorem 11.1.1 to prove the following embedding result.

Theorem 12.1.1. *Let X be a visual Gromov hyperbolic space whose boundary at infinity has finite ℓ-dimension, $\ell\text{-dim}(\partial_\infty X) < \infty$. Then there exists a quasi-isometric embedding $f : X \to T_1 \times \cdots \times T_n$ of X into an n-fold product of metric trees $T_1, \dots, T_n$ with $n = \ell\text{-dim}(\partial_\infty X) + 1$.*

Remark 12.1.2. On a finite product of metric spaces we consider any of standard *product metrics*, l_1, l_2 or l_∞, which are the sum, the square root of the sum of squares, or the maximal of the coordinate distances respectively. These product metrics are bilipschitz equivalent. It is only important that the distance between any two points in the product is at least the distance between their projection to any factor.

All metric trees T_k, $k = 1, \dots, n$, constructed during the proof are *simplicial* with length 1 edges, that is, metric trees which admit a triangulation. We can speak about vertices and edges of a simplicial tree. The *valence* of a vertex is the number of edges adjacent to it.

Vertices of T_k, $k = 1, \dots, n$, typically have infinite valence. The estimate of the number of tree-factors, needed for a quasi-isometric embedding given by Theorem 12.1.1, is sharp: we show in Chapter 13 that this embedding theorem is optimal in a strong sense.

The class of Gromov hyperbolic spaces to which Theorem 12.1.1 can be applied contains all visual Gromov hyperbolic spaces with doubling boundary at infinity; see Lemma 12.2.2 and Sections 2.3, 8.3.1. In particular, all hyperbolic geodesic spaces with bounded geometry as well as cobounded ones are in this class, see Corollary 2.3.7 and Theorem 8.3.9, that includes all Gromov hyperbolic groups and Hadamard manifolds with pinched negative curvature.

The idea of the embedding can be explained as follows. First of all, replacing the space X by a hyperbolic approximation of its boundary at infinity, we assume that X is a hyperbolic approximation of a metric space Z with finite ℓ-dimension. As usual, we use notation $V = \{V_j\}$ for the vertex set of X and assume that X is truncated for a bounded Z, see Chapter 6.

Theorem 11.1.1 now takes the following form.

Theorem 12.1.3. *Let Z be a bounded metric space with finite ℓ-dimension, $n = \ell\text{-dim}\, Z < \infty$.*

Then for every sufficiently small r, $r \in (0, r_0)$, there exists a sequence $\mathcal{U}_j$, $j \geq 0$, of $(n+1)$-colored (by a set A) open coverings of Z with $\mathcal{U}_0^a = \{Z\}$ for all $a \in A$ and mesh *$\mathcal{U}_j < r^j$ for every $j \in \mathbb{N}$ such that for any hyperbolic approximation X of Z with parameter r, we have the following.*

(1) *For every $v \in V_{j+1}$, $j \geq 0$, there is $U \in \mathcal{U}_j$ such that $B(v) \subset U$.*

(2) *For every $a \in A$ and for different members $U \in \mathcal{U}_j^a$, $U' \in \mathcal{U}_{j'}^a$, with $j' \leq j$ the following holds: let $B(U) = \bigcup\{B(v),\ v \in V_{j+1},\ B(v) \cap U \neq \emptyset\}$; then either $B(U) \subset U'$ or $B(U) \cap U' = \emptyset$.*

Condition (2) means that the family $\mathcal{U}^a = \bigcup_j \mathcal{U}_j^a$ has combinatorially the structure of a tree T_a for every color $a \in A$: the vertices are the members of the family and the edges are defined by the inclusion relation. We formalize this in the notion of a levelled tree, see Section 12.1.1.

It is now possible to define a map $X \to T_a$. The map is defined on the set V of vertices. A vertex v, which is just the ball $B(v) \subset X$, is mapped to the smallest $U \in \mathcal{U}^a$ such that $B(v) \subset U$. It turns out that the product map $V \to \prod_a T_a$ is quasi-isometric.

Proof of Theorem 12.1.3. Choosing $r > 0$ sufficiently small, we can assume that $k_0 \in \{-1, 0\}$ where $k_0 = k_0(\operatorname{diam} Z, r)$ is the largest integer k with $\operatorname{diam} Z < r^k$, see Section 6.4.1. Since X is truncated, the vertex sets V_j are nonempty for $j \geq k_0$ only.

By Theorem 11.1.1, there are constants $\delta, \gamma \in (0,1)$ such that for every sufficiently small $r > 0$ there exists a sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, of $(n+1)$-colored (by a set A) open coverings of Z with the following properties:

(i) mesh $\mathcal{U}_j < r^j$ and $L(\mathcal{U}_j) \geq \delta r^j$ for every $j \in \mathbb{N}$;

(ii) for every $a \in A$ and for different members $U \in \mathcal{U}_j^a$, $U' \in \mathcal{U}_{j'}^a$, with $j' \leq j$, we have either $B_s(U) \cap U' = \emptyset$, or $B_s(U) \subset U'$ for $s = \gamma r^j$.

Furthermore, we add to the sequence $\mathcal{U}_j$, $j \in \mathbb{N}$, the member $\mathcal{U}_0$ which consists of (copies of) Z for every color $a \in A$, $\mathcal{U}_0^a = \{Z\}$. Because $L(\mathcal{U}_j) \geq \delta r^j$, every ball $B_\rho(z) \subset Z$ of radius $\rho \leq \delta r^j$ is contained in some member $U \in \mathcal{U}_j$. Assuming that $r < \delta/2$, we obtain property (1) because recall that $B(v) = B_{2r^{j+1}}(v)$ for $v \in V_{j+1}$.

Finally, assume additionally that $r < \gamma/4$. Now if a color $a \in A$ and different members $U \in \mathcal{U}_j^a$, $U' \in \mathcal{U}_{j'}^a$, with $j' \leq j$ are given, we have $B(U) \subset B_s(U)$ with $s = \gamma r^j$ by the choice of r and the definition of $B(U)$. Hence, property (2). $\square$

12.1.1 Levelled trees

A poset (partially ordered set) V is called *directed* if for any $u, v \in V$ there is $w \in V$ with $u \leq w$, $v \leq w$.

A *levelled tree* T is a directed poset V, called the *vertex set* of T, together with a *level* function $\ell \colon V \to \mathbb{Z}$ which is strictly monotone in the following sense: If $v, v' \in V$ are different elements and $v \leq v'$, then $\ell(v) > \ell(v')$. In this case, v' is called an *ancestor* of v, and v is a *descendant* of v'.

We require that the following condition is satisfied:

(+) if distinct elements $v, v' \in V$ have a common descendant then one of them is an ancestor of the other.

A collection E of two point subsets of V called the edge set of T is defined by the condition: A pair of vertices (v, v') forms an edge, $(v, v') \in E$, if and only if one of its members, say v', is an ancestor of the other and the level $\ell(v')$ is maximal with this property.

If there is a vertex which has no ancestor, then such a vertex is unique by directedness, and it is called the *root* of T. Note that the root is an ancestor of every other vertex.

It follows from (+) that for every vertex $v \in V$ (except the root) there is exactly one edge (v, v') in which v is the descendant. Hence, by the uniqueness part, T has no circuit. By the existence part (together with (+)), every vertex is connected with any of its ancestors by a sequence of edges in T.

Now because V is directed, any two vertices in T are connected by a sequence of edges, i.e., T is connected. Therefore, T is a simplicial tree. Assuming that the length of every edge equals 1, we see that each levelled tree is a simplicial metric tree with length 1 edges.

12.1.2 Colored trees

Let Z be a bounded metric space with finite ℓ-dimension, $n = \ell\text{-}\dim Z < \infty$; let X be a hyperbolic approximation of Z with sufficiently small parameter $r < \min\{\operatorname{diam} Z, 1/\operatorname{diam} Z\}$ satisfying the condition of Theorem 12.1.3; and let $\{\mathcal{U}_j\}$, $j \geq 0$, be a sequence of $(n+1)$-colored (by a set A) open coverings of Z with parameter r as in Theorem 12.1.3. Then recall that $k_0 = k_0(\operatorname{diam} Z, r) = 0$ or -1. We use notation $d(v, v')$ for the distance in Z of vertices $v, v' \in V$ considered as points of Z, and $|vv'|$ for the distance in X.

For every $a \in A$, we define a rooted levelled tree T_a as follows. Its vertex set $\mathcal{U}^a$ is the disjoint union $\mathcal{U}^a = \bigcup_{j \geq 0} \mathcal{U}_j^a$ with the root $v_a = Z$ which is the unique member of $\mathcal{U}_0^a$. The partial order of $\mathcal{U}^a$ is defined by the inclusion relation, $U \leq U'$ if and only if $U \subset U'$, and this order is directed due to the existence of the root.

We say that a vertex $U \in \mathcal{U}_j^a$ has level j; this defines the level function. It follows easily from separation property (2) that the level function is strictly monotone. Then a vertex $U \in \mathcal{U}^a_j$ is a descendant of $U' \in \mathcal{U}^a_{j'}$ if $j' < j$ and $U \subset U'$.

It follows from separation property (2) that (+) is satisfied. Hence, T_a is a rooted levelled tree: a pair of vertices $U \in \mathcal{U}^a_j$, $U' \in \mathcal{U}^a_{j'}$ forms an edge of T_a if and only if it is a pair (descendant, ancestor), and the level j' of the ancestor U' is maximal with this property.

Typically, vertices of the tree T_a have infinite valence.

We use notation $|UU'|$ for the distance in T_a between its vertices U, U', and $|U|$ for the distance $|Uv_a|$. Note that $|v_a| = 0$ and $|U| \le j - k_0$ if $U \in \mathcal{U}^a_j$. Furthermore, if $U' \in \mathcal{U}^a_{j'}$ is an ancestor of $U \in \mathcal{U}^a_j$, then the level difference $j - j'$ might be arbitrarily large compared to the distance $|UU'|$ even if (U, U') is an edge of T_a.

12.1.3 A map into the product of colored trees

We now define a map $f_a : V \to T_a$, V is the vertex set of the hyperbolic approximation X of Z, as follows. The root of X is mapped into the root of T_a, $f_a(v) = v_a$ for the unique member $v \in V_{k_0}$. Given $v \in V_j$, $j > k_0$, we let $f_a(v) = U \in \mathcal{U}^a_{j'}$ be the covering element containing the ball $B(v)$, $B(v) \subset U$, and $j' \le j - 1$ is maximal with this property. By Theorem 12.1.3 (2), $f_a(v)$ is well defined.

Lemma 12.1.4. *For every $a \in A$, the map $f_a : V \to T_a$ is Lipschitz,*

$$|f_a(v) f_a(v')| \le 2|vv'| \quad \textit{for every } v, v' \in V.$$

Proof. Since the hyperbolic approximation X is geodesic, it suffices to estimate the distance $|f_a(v) f_a(v')|$ for neighbors, $|vv'| = 1$.

Assume that the edge $(v, v') \subset X$ is horizontal, i.e., $v, v' \in V_j$ for some $j \ge k_0$ and the balls $B(v)$, $B(v')$ intersect. Thus the covering elements $U = f_a(v)$, $U' = f_a(v')$ also intersect. By separation property (2), either $U = U'$ and so $f_a(v) = f_a(v')$ or these elements have different levels and one of them is contained in the other, say $U \in \mathcal{U}^a_i$, $U' \in \mathcal{U}^a_{i'}$ with $i > i'$ and $U \subset U'$. It follows from the definition of f_a that $i < j$ and from separation property (2) that any $U'' \in \mathcal{U}^a_{i''}$, $i'' < i$, intersecting U, also contains $B(v')$. Thus $(U, U') \subset T_a$ is an edge by the definition of f_a, and $|f_a(v) f_a(v')| = 1$ in this case.

Assume now that the edge $(v, v') \subset X$ is radial, say $v \in V_{j+1}$, $v' \in V_j$. Then $B(v) \subset B(v')$ and as in the previous case, $U \cap U' \ne \emptyset$. Thus we can assume that these elements have different levels, $U \in \mathcal{U}^a_i$, $U' \in \mathcal{U}^a_{i'}$ with $i \ne i'$, and one of them is contained in the other. Moreover, $i > i'$ because $B(v) \subset U'$, thus $U \subset U'$. We have $i \le j$ by definition of f_a, and as above, any ancestor $U'' \in \mathcal{U}^a_{i''}$, $i'' < i - 1$, of U separated from U by at least one generation, $|UU''| \ge 2$, also contains $B(v')$. Therefore, it follows from the definition of f_a that at most one generation can separate U from its ancestor U', and $|f_a(v) f_a(v')| \le 2$. $\square$

To prove Theorem 12.1.1, it suffices to show that the map

$$f = \prod_a f_a : V \to \prod_a T_a$$

is quasi-isometric. Indeed, by Corollaries 7.1.5 and 7.1.6, every visual hyperbolic space is roughly similar to a subspace of a hyperbolic approximation of its boundary at infinity.

The map f defined by its coordinate maps $f_a : X \to T_a$ is Lipschitz by Lemma 12.1.4. To prove that f is roughly bilipschitz, we begin with the following lemma, which is the main ingredient of the proof.

For $i \geq 0$, we denote by $T_{a,i} = \mathcal{U}_i^a$ the vertex set of T_a of level i.

Lemma 12.1.5. *Given $v \in V_{j+1}$, $j \geq 0$, for every integer i, $0 \leq i \leq j$, there is a color $a \in A$ such that* $\operatorname{dist}(f_a(v), T_{a,i}) \geq M$ *with $M + 1 \geq (j - i + 1)/|A|$. Furthermore, if for $k \leq i$ a vertex $w \in T_{a,k}$ is the lowest level vertex of the segment $f_a(v)w \subset T_a$, then $|f_a(v)w| \geq M$.*

Proof. Consider a radial geodesic $v_{i+1} \dots v_{j+1} \subset X$ with vertices $v_m \in V_m$, where $v_{j+1} = v$. This means in particular that $B(v_{m+1}) \subset B(v_m)$ for every $m = i + 1, \dots, j$. By Theorem 12.1.3 (1) for every vertex v_{m+1}, there is a covering element $U_m \in \mathcal{U}_m$ with $B(v_{m+1}) \subset U_m$, $m = i, \dots, j$.

There is a color $a \in A$ such that the set $\{U_i, \dots, U_j\}$ contains $M + 1 \geq (j - i + 1)/|A|$ members having the color a, i.e. each of those $U_m \in \mathcal{U}_m^a$. Since $B(v) \subset U_m$ for every $m \leq j$, we have $U = f_a(v) \subset U_m$ for every U_m having the color a by the definition of f_a and separation property (2).

Using again the separation property, we obtain that any path in T_a between $f_a(v)$ and the set $T_{a,i}$ must contain at least $M+1$ vertices and hence $\operatorname{dist}(f_a(v), T_{a,i}) \geq M$.

Finally, let $W \in \mathcal{U}_k^a$ be the set corresponding to the vertex w of T_a. By the assumption on w, the set W contains U and every set from the list $\{U_i, \dots, U_j\}$ having the color a. Hence, $|f_a(v)w| \geq M$. □

We say that distinct points $v \in V_j$, $v' \in V_{j'}$, $j \geq j' \geq 0$ are *horizontally close* to each other if $d(v, v') < r^{j'}$. This terminology is motivated by the fact that there is a geodesic segment $vv' \subset X$ which is almost radial. More precisely, we have

Lemma 12.1.6. *Assume that the distinct points $v, v' \in V$ are horizontally close to each other. Then their levels are different and the upper level ball is contained in the lower level ball, say $\ell(v) > \ell(v')$, $B(v) \subset B(v')$. In particular, $|vv'| \leq |\ell(v) - \ell(v')| + 1$.*

Proof. We can assume that $v \in V_j$, $v' \in V_{j'}$, $j \geq j' \geq 0$. Then $j > j'$ because v, v' are distinct and because $V_j \subset Z$ is r^j-separated for every $j \geq 0$. Furthermore, $B(v) \subset B(v')$ because $2r^j + r^{j'} < 2r^{j'}$. By Corollary 6.2.7, we have $|vv'| \leq (j - j') + 1$. □

Lemma 12.1.7. *Given $v, v' \in V$ horizontally close to each other, we have that $f_a(v)f_a(v') \subset T_a$ is a radial segment for every color $a \in A$, i.e. its lowest level vertex is one of its ends, and there is a color $a \in A$ such that*

$$|vv'| \le |A||f_a(v)f_a(v')| + \sigma,$$

where $\sigma = |A| + 1$.

Proof. We can assume that v, v' are distinct. Then, by Lemma 12.1.6, their levels are different, say $\ell(v) > \ell(v')$, and $B(v) \subset B(v')$. Thus $f_a(v) = f_a(v')$ or $f_a(v)$ is a descendant of $f_a(v')$ for every color $a \in A$. In any case, $f_a(v')$ is the lowest level vertex of the segment $f_a(v)f_a(v') \subset T_a$.

On the other hand, $f_a(v') \in T_{a,m_a}$ with $m_a \le \ell(v')$ for all $a \in A$ by the definition of f_a. By Lemma 12.1.5, we have

$$|f_a(v)f_a(v')| = \operatorname{dist}(f_a(v), T_{a,m_a}) \ge M$$

with $M + 1 \ge (\ell(v) - \ell(v'))/|A|$ for some color $a \in A$. Therefore, using again Lemma 12.1.6, we obtain

$$|vv'| \le |\ell(v) - \ell(v')| + 1 \le |A|(M + 1) + 1 \le |A||f_a(v)f_a(v')| + \sigma. \qquad \square$$

Lemma 12.1.7 shows a rough bilipschitz property of the map f in the case v, v' are horizontally close to each other. The opposite case is more complicated.

We say that vertices $v, v' \in V$ are *horizontally distinct* if $\ell(v), \ell(v') \ge 0$ and they are not horizontally close to each other, $d(v, v') \ge r^{\min\{\ell(v),\ell(v')\}}$. In this case, there is an integer l with

$$r^l \le d(v, v') < r^{l-1}.$$

We call l the *critical level* of v, v'. Note that $r^l \le \operatorname{diam} Z < r^{k_0}$ and thus $l > k_0$, in particular, $l \ge 0$. Furthermore, $l \le \min\{\ell(v), \ell(v')\}$.

Recall that a similar notion of a critical level has been used in the proof of the Assouad embedding theorem; see Chapter 8.

Lemma 12.1.8. *Let l be the critical level of horizontally distinct vertices v, $v' \in V$. Then $|vv'| \le \ell(v) + \ell(v') - 2l + 3$.*

Proof. Consider a central ancestor radial geodesic γ_v in X between the root of X and v, see Lemma 6.2.1. For the vertex $u \in \gamma_v$ of the level $l - 1$, $u \in V_{l-1}$, we have $|vu| = \ell(v) - l + 1$ and $d(v, u) \le r^{l-1}$. Thus $d(u, v') \le d(u, v) + d(v, v') < 2r^{l-1}$. Therefore, $v' \in B(u)$, the balls $B(u)$, $B(v')$ intersect and hence, $|v'u| \le \ell(v') - l + 2$ by Corollary 6.2.7. We see that $|vv'| \le |vu| + |uv'| \le \ell(v) + \ell(v') - 2l(v, v') + 3$. $\square$

Lemma 12.1.9. *Let l be the critical level of horizontally distinct $v, v' \in V$. Assume that $v \in U$, $v' \in U'$ for some elements U, $U' \in T_a$ and some color $a \in A$. Then $k < l$ for the lowest level vertex $W \in T_{a,k}$ of the segment $UU' \subset T_a$.*

Proof. The set $W \in \mathcal{U}_k^a$ contains both U and U', and we have by Theorem 12.1.3, $d(v, v') \le \operatorname{diam} W < r^k$. It follows $r^l < r^k$ and thus $k < l$. □

Lemma 12.1.10. *Given horizontally distinct $v, v' \in V$, $\ell(v) \ge \ell(v')$, there is a color $a \in A$ such that $|vv'| \le 2|A||f_a(v)w| + \sigma$, where $w \in f_a(v)f_a(v') \subset T_a$ is the lowest level vertex and $\sigma = 2|A| + 1$.*

Proof. By Lemma 12.1.9 for every color $a \in A$, any path in T_a between $f_a(v)$ and $f_a(v')$ passes through a vertex of a level $k < l$, where $l = l(v, v')$ is the critical level of v, v'.

By Lemma 12.1.5, there is a color $a \in A$ such that $\operatorname{dist}(f_a(v), T_{a,l-1}) \ge M$ with $M + 1 \ge (\ell(v) - l + 1)/|A|$. Let $w \in T_{a,k}$ be the lowest level vertex of the segment $f_a(v)f_a(v')$. By Lemma 12.1.5, we have $|f_a(v)w| \ge M$ because $k \le l - 1$. Since $\ell(v) \ge \ell(v')$, we have by Lemma 12.1.8

$$|vv'| \le 2(\ell(v) - l + 1) + 1 \le 2|A||f_a(v)w| + \sigma,$$

hence the claim. □

Proof of Theorem 12.1.1. We use also the notation $|ww'|$ for the distance between w, $w' \in \prod_c T_c$. Since the map

$$f = \prod_a f_a : V \to \prod_a T_a$$

is Lipschitz, it suffices to show that there are constants $\Lambda > 0$, $\sigma \ge 0$ depending only on $|A|$ such that

$$|vv'| \le \Lambda|f(v)f(v')| + \sigma$$

for all $v, v' \in V$.

If vertices v, v' are horizontally close to each other, then the required estimate follows from Lemma 12.1.7. If they are horizontally distinct, then the estimate follows from Lemma 12.1.10 since $|f_a(v)w| \le |f_a(v)f_a(v')|$. □

Corollary 12.1.11. *Let X be a visual Gromov hyperbolic space. Then*

$$\operatorname{asdim} X \le \ell\text{-}\operatorname{asdim} X \le \ell\text{-}\dim \partial_\infty X + 1.$$

Proof. We can assume that $\ell\text{-}\dim \partial_\infty X < \infty$. Then, by Theorem 12.1.1, there is a quasi-isometric embedding of X into the n-fold product of metric trees with $n = \ell\text{-}\dim \partial_\infty X + 1$. Using the product and monotonicity theorems for the asymptotic ℓ-dimension and Proposition 10.2.1, we obtain $\ell\text{-}\operatorname{asdim} X \le n$. □

12.2 ℓ-dimension of locally self-similar spaces

The ℓ-dimension of a metric space can be larger than the topological dimension, see Section 11.3.1. Thus in view of Theorem 12.1.1 it is important to know for which spaces these dimensions coincide. Here, we show that for locally self-similar spaces, see Section 2.3, the ℓ-dimension coincides with the topological one.

Recall that a metric space Z is locally similar to (subsets of) a metric space Y if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \leq 1/R$ there is a λ-quasi-homothetic map $f: A \to Y$ with coefficient R; see Section 2.3.

Theorem 12.2.1. *Assume that a metric space Z is locally similar to (subsets of) a compact metric space Y. Then ℓ-$\dim Z < \infty$ and ℓ-$\dim Z \leq \dim Y$.*

Recall that a metric space Z which is locally similar to (subsets of) a compact metric space Y is doubling at small scales; see Lemma 2.3.4.

Lemma 12.2.2. *Assume that a metric space Z is doubling at small scales. Then ℓ-$\dim Z < \infty$.*

Proof. By the assumption, there is $n \in \mathbb{N}$ such that every ball $B_{4r} \subset Z$ of radius $4r$ is covered by at most $n+1$ balls $B_{r/2}$ for all sufficiently small $r > 0$. We fix a maximal r-separated set $Z' \subset Z$, i.e., $|zz'| > r$ for each distinct $z, z' \in Z'$. Then the family $\mathcal{U}' = \{B_r(z) : z \in Z'\}$ is an open covering of Z.

Since every ball $B_{r/2}$ contains at most one point from Z' and $B_{4r}(z)$ is covered by at most $n+1$ balls $B_{r/2}$, the ball $B_{4r}(z)$ contains at most $n+1$ points from Z' for every $z \in Z'$. Thus, there is a coloring $\chi: Z' \to A$, $|A| = n+1$, such that $\chi(z) \neq \chi(z')$ for each $z, z' \in Z$ with $|zz'| < 4r$, cf. the proof of Theorem 8.1.1.

For $a \in A$, we let $Z'_a = \chi^{-1}(a)$ be the set of the color a. Then $|zz'| \geq 4r$ for distinct $z, z' \in Z'_a$. Putting $\mathcal{U}_a = \{B_{2r}(z) : z \in Z'_a\}$, we obtain an open $(n+1)$-colored covering $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}_a$ of Z with $\operatorname{mesh}(\mathcal{U}) \leq 4r$ and $L(\mathcal{U}) \geq r$. This shows that ℓ-$\dim Z \leq n$. $\square$

We shall use the following facts implied by the definition of a quasi-homothetic map.

Lemma 12.2.3. *Let $h: Z \to Z'$ be a λ-quasi-homothetic map with coefficient R. Let $V \subset Z$, $\widetilde{\mathcal{U}}$ be an open covering of $h(V)$ and $\mathcal{U} = h^{-1}(\widetilde{\mathcal{U}})$. Then we have:*

1. $R \operatorname{mesh}(\mathcal{U})/\lambda \leq \operatorname{mesh}(\widetilde{\mathcal{U}}) \leq \lambda R \operatorname{mesh}(\mathcal{U})$;
2. $\lambda R \cdot L(\mathcal{U}) \geq L(\widetilde{\mathcal{U}})$, *where $L(\mathcal{U})$ is the Lebesgue number of $\mathcal{U}$ as a covering of V.* $\square$

Proof of Theorem 12.2.1. It follows from Lemmas 2.3.4 and 12.2.2, that ℓ-$\dim Z = N$ is finite. We can also assume that $\dim Y = n$ is finite. There is a constant $\delta \in (0,1)$ such that for every sufficiently small $\tau > 0$ there exists an $(N+1)$-colored open covering $\mathcal{V} = \bigcup_{a \in A} \mathcal{V}^a$ of Z with $\operatorname{mesh}(\mathcal{V}) \le \tau$ and $L(\mathcal{V}) \ge \delta\tau$. It is convenient to take $A = \{0, \dots, N\}$ as the color set.

There is a constant $\lambda \ge 1$ such that for every sufficiently large $R > 1$ and every $V \subset Z$ with $\operatorname{diam} V \le 1/R$ there is a λ-quasi-homothetic map $h_V : V \to Y$ with coefficient R.

Using that Y is compact and $\dim Y = n$, we find for every $a \in A$ a finite $(n+1)$-colored open covering $\widetilde{\mathcal{U}}_a$ of Y, $\widetilde{\mathcal{U}}_a = \bigcup_{c \in C} \widetilde{\mathcal{U}}_a^c$, $|C| = n+1$, such that the following holds:

(i) $\operatorname{mesh}(\widetilde{\mathcal{U}}_0) \le \frac{\delta}{2\lambda}$;

(ii) $\operatorname{mesh}(\widetilde{\mathcal{U}}_{a+1}) \le \frac{1}{2\lambda^2} \min\{L(\widetilde{\mathcal{U}}_a), \operatorname{mesh}(\widetilde{\mathcal{U}}_a)\}$ for every $a \in A$, $a \le N-1$.

Then $l := \min\{2^{a-N} L(\widetilde{\mathcal{U}}_a) : a \in A\} > 0$ and $\operatorname{mesh}(\widetilde{\mathcal{U}}_a) \le \frac{\delta}{2\lambda}$ for every $a \in A$.

For every $V \in \mathcal{V}$, consider the slightly smaller subset $V' = B_{-\delta\tau/2}(V)$. Then the sets $Z_a = \bigcup_{V \in \mathcal{V}^a} V' \subset Z$, $a \in A$, cover Z, $Z = \bigcup_{a \in A} Z_a$ because $L(\mathcal{V}) \ge \delta\tau$.

Given $V \in \mathcal{V}$, we fix a λ-quasi-homothetic map $h_V : V \to Z$ with coefficient $R = 1/\tau$ and put $\widetilde{V} = h_V(V')$. Now for every $a \in A$, $V \in \mathcal{V}^a$ consider the family $\widetilde{\mathcal{U}}_{a,V} = \{\widetilde{U} \in \widetilde{\mathcal{U}}_a : \widetilde{V} \cap \widetilde{U} \ne \emptyset\}$, which is obviously an $(n+1)$-colored covering of $\widetilde{V}$. Then

$$\mathcal{U}_{a,V} = \{h_V^{-1}(\widetilde{U}) : \widetilde{U} \in \widetilde{\mathcal{U}}_{a,V}\}$$

is an open $(n+1)$-colored covering of V'.

Note that $U = h_V^{-1}(\widetilde{U})$ is contained in V for every $\widetilde{U} \in \widetilde{\mathcal{U}}_{a,V}$ because $\operatorname{dist}(v', Z \setminus V) > \delta\tau/2$ for every $v' \in V'$ and $\operatorname{diam} U \le \lambda\tau \operatorname{diam} \widetilde{U} \le \delta\tau/2$. Thus the family $\mathcal{U}_{a,V}$ is contained in V. Now the family $\mathcal{U}_a = \bigcup_{V \in \mathcal{V}^a} \mathcal{U}_{a,V}$ covers the set Z_a of the color a, and has the following properties:

(1) for every $a \in A$, the family $\mathcal{U}_a$ is at most $(n+1)$-colored (by C);

(2) $\operatorname{mesh}(\mathcal{U}_{a+1}) \le \frac{1}{2} \min\{L(\mathcal{U}_a), \operatorname{mesh}(\mathcal{U}_a)\}$ for every $a \in A$, $a \le N-1$ ($L(\mathcal{U}_a)$ means the Lebesgue number of $\mathcal{U}_a$ as a covering of Z_a);

(3) $\operatorname{mesh}(\mathcal{U}_a) \le \lambda\tau \operatorname{mesh}(\widetilde{\mathcal{U}}_a)$ and $L(\mathcal{U}_a) \ge (\tau/\lambda) L(\widetilde{\mathcal{U}}_a)$ for every $a \in A$.

Indeed, distinct $V_1, V_2 \in \mathcal{V}^a$ are disjoint and thus any $U_1 \in \mathcal{U}_{a,V_1}$, $U_2 \in \mathcal{U}_{a,V_2}$ are disjoint because $U_1 \subset V_1$, $U_2 \subset V_2$. This proves (1). Furthermore, for every $a \in A$, $a \le N-1$, and every $U \in \mathcal{U}_{a+1}$, we have

$$\begin{aligned} \operatorname{diam} U &\le \lambda\tau \operatorname{mesh}(\widetilde{\mathcal{U}}_{a+1}) \\ &\le \frac{1}{2\lambda R} \min\{L(\widetilde{\mathcal{U}}_a), \operatorname{mesh}(\widetilde{\mathcal{U}}_a)\} \\ &\le \frac{1}{2} \min\{L(\mathcal{U}_a), \operatorname{mesh}(\mathcal{U}_a)\} \end{aligned}$$

by Lemma 12.2.3, hence (2). Finally, (3) also follows from Lemma 12.2.3.

Now we put $\widehat{\mathcal{U}}_{-1} = \{Z\}$, $\widehat{\mathcal{U}}_0 = \mathcal{U}_0$ and assume that for some $a \in A$, we have already constructed families $\widehat{\mathcal{U}}_0, \dots, \widehat{\mathcal{U}}_a$ so that $\widehat{\mathcal{U}}_a$ is an $(n+1)$-colored covering of $Z_0 \cup \dots \cup Z_a$ and $\operatorname{mesh}(\mathcal{U}_a) \le \frac{1}{2} L(\widehat{\mathcal{U}}_{a-1})$, $\operatorname{mesh}(\widehat{\mathcal{U}}_a) \le \operatorname{mesh}(\mathcal{U}_0)$, $L(\widehat{\mathcal{U}}_a) \ge \min\{L(\mathcal{U}_a), \frac{1}{2} L(\widehat{\mathcal{U}}_{a-1})\}$. Then we have

$$\operatorname{mesh}(\mathcal{U}_{a+1}) \le \tfrac{1}{2} \min\{L(\mathcal{U}_a), \tfrac{1}{2} L(\widehat{\mathcal{U}}_{a-1})\} \le \tfrac{1}{2} L(\widehat{\mathcal{U}}_a).$$

Applying Proposition 9.6.1 to the pair of families $\widehat{\mathcal{U}}_a$, $\mathcal{U}_{a+1}$, we obtain an open $(n+1)$-colored covering $\widehat{\mathcal{U}}_{a+1}$ of $Z_0 \cup \dots \cup Z_{a+1}$ with

$$\operatorname{mesh}(\widehat{\mathcal{U}}_{a+1}) \le \max\{\operatorname{mesh}(\widehat{\mathcal{U}}_a), \operatorname{mesh}(\mathcal{U}_{a+1})\} \le \operatorname{mesh}(\mathcal{U}_0)$$

and $L(\widehat{\mathcal{U}}_{a+1}) \ge \min\{L(\mathcal{U}_{a+1}), \frac{1}{2} L(\widehat{\mathcal{U}}_a)\}$.

Proceeding by induction, we obtain an open $(n+1)$-colored covering $\mathcal{U} = \widehat{\mathcal{U}}_N$ of Z with $\operatorname{mesh}(\mathcal{U}) \le \operatorname{mesh}(\mathcal{U}_0) \le \delta\tau/2$ and $L(\mathcal{U}) \ge \min\{2^{a-N} L(\mathcal{U}_a) : a \in A\} \ge (l/\lambda)\tau$. Since we can choose $\tau > 0$ arbitrarily small and the constants δ, λ, l are independent of τ, this shows that $\ell\text{-}\dim Z \le n$. □

Corollary 12.2.4. *The ℓ-dimension of every compact, locally self-similar metric space Z is finite and coincides with its topological dimension, $\ell\text{-}\dim Z = \dim Z$.*

Proof. We have $\dim Z \le \ell\text{-}\dim Z$ for every metric space Z. By Theorem 12.2.1, $\ell\text{-}\dim Z < \infty$ and $\ell\text{-}\dim Z \le \dim Z$, hence $\ell\text{-}\dim Z = \dim Z$ is finite. □

12.3 Applications to hyperbolic spaces

Recall that the boundary at infinity of any proper hyperbolic space is compact, see e.g. Exercise 6.4.4. Now Theorem 2.3.2 and Corollary 12.2.4 give the following

Theorem 12.3.1. *The ℓ-dimension of the boundary at infinity of every cobounded, hyperbolic, proper, geodesic space X is finite and coincides with the topological dimension, $\ell\text{-}\dim \partial_\infty X = \dim \partial_\infty X$.* □

The class of spaces satisfying the condition of Theorem 12.3.1 is very large. It includes in particular all symmetric rank one spaces of noncompact type (i.e. the real, complex, quaternionic hyperbolic spaces and the Cayley hyperbolic plane), all cocompact Hadamard manifolds of negative sectional curvature, various hyperbolic buildings, etc. The most important among them is the class of Gromov hyperbolic groups; see Section 1.4.2.

Every cobounded, hyperbolic, proper, geodesic space is certainly visual. Thus, combining Theorems 12.3.1 and 12.1.1, we obtain the following.

Theorem 12.3.2. *Let X be a cobounded, hyperbolic, proper, geodesic space. Then there exists a quasi-isometric embedding $f : X \to T_1 \times \dots \times T_n$ of X into the n-fold product of metric trees $T_1, \dots, T_n$ with $n = \dim(\partial_\infty X) + 1$.* □

Now we are able to compute the asymptotic dimension and the asymptotic ℓ-dimension of any hyperbolic space from that class.

Theorem 12.3.3. *Let X be a cobounded, hyperbolic, proper, geodesic space. Then*

$$\operatorname{asdim} X = \ell\text{-}\operatorname{asdim} X = \dim \partial_\infty X + 1.$$

Proof. The estimate from below, $\operatorname{asdim} X \geq \dim \partial_\infty X + 1$, holds for a larger class of spaces; see Theorem 10.1.2.

The estimate from above, $\ell\text{-}\operatorname{asdim} X \leq \dim \partial_\infty X + 1$, follows from Corollary 12.1.11 and Theorem 12.3.1. □

12.4 Supplementary results and remarks

12.4.1 ℓ-dimension as an obstacle to quasi-symmetry

Let $Z = \{0\} \cup \{1/m : m \in \mathbb{N}\} \subset \mathbb{R}$ be the (bounded) space from the first example on p. 137. Since $\operatorname{ANdim} Z^n = \ell\text{-}\dim Z^n = n$ and $\dim Z^n = 0$ for every $n \in \mathbb{N}$ (see Section 11.3.1), it follows from quasi-symmetry invariance of the Assouad–Nagata dimension and Corollary 12.2.4 that Z^n is not quasi-symmetric to any locally self-similar space.

Standard fractal spaces like the ternary Cantor set, the Sierpinski carpet or the Menger curve are self-similar and in particular locally self-similar (see Example 2.3.1). Therefore, their ℓ-dimension coincides with the topological dimension. On the other hand, these spaces admit metrics with the ℓ-dimension strictly larger than the topological one; see Section 11.3.1. As above, any such metric is not quasi-symmetric to any locally self-similar metric.

Bibliographical note. A quasi-isometric embedding of H^n into the product of $n \geq 2$ metric trees was constructed in [BS2]. Theorem 12.1.1 is obtained in [Bu] combining ideas from [BS2] with the notion of the linearly controlled dimension. The target trees of Theorem 12.1.1 typically have infinite valence at every vertex. Quasi-isometric embeddings of hyperbolic spaces into the product of binary trees with optimal number of factors are constructed in [BDS] combining Theorem 12.1.1 with a sophisticated combinatorial argument.

The results of Sections 12.2 and 12.3 are taken from [BL].

Chapter 13

Hyperbolic dimension

The hyperbolic dimension of a metric space is a close relative of the asymptotic dimension. The main feature is that one allows coverings by unbounded sets. Of course, it would be useless to consider coverings by arbitrary unbounded sets. We do require certain conditions restricting the size of covering members. These are the large scale doubling condition and some uniformity condition.

The hyperbolic dimension possesses usual properties of dimensions like the monotonicity and product theorems. However, unlike the asymptotic dimension, the hyperbolic dimension of Euclidean spaces as well as of all doubling spaces is zero, $\operatorname{hypdim} \mathbb{R}^n = 0$ for every $n \geq 0$. What makes the hyperbolic dimension useful is that it behaves for hyperbolic spaces just like the asymptotic dimension: The main result of this chapter is the estimate $\operatorname{hypdim} \mathrm{H}^n \geq n$ for every $n \geq 2$. This result has a number of applications to nonembedding results.

13.1 Large scale doubling sets

A subset U of a metric space X is *large scale doubling* if there is a constant $N \in \mathbb{N}$ such that, for every sufficiently large $r > 1$ and for every ball $B_{2r} \subset X$ of radius $2r$, the intersection $B_{2r} \cap U$ can be covered by at most N balls of radius r. More precisely, we say that U is (N, R)-large-scale doubling or (N, R)-ls-doubling if the condition above holds for all $r \geq R$. We also say that U is N-ls-doubling if only N is of importance.

An equivalent definition is that every intersection $B_{2r} \cap U$ contains at most N points which are r-separated for all sufficiently large r.

Exercise 13.1.1. Show that the covering and separation definitions of large scale doubling sets are equivalent. What is the relation between the implied constants?

Examples 13.1.2. (1) Any Euclidean space $\mathbb{R}^n$ is (N, R)-ls-doubling for some $N = N(n)$ and $R = 0$.

(2) Let B be a bounded metric space. Then the metric product $X = B \times \mathbb{R}^n$ is (N, R)-ls-doubling for some $N = N(n)$ and $R \geq 2 \operatorname{diam} B$. We emphasize that in this example the parameter N counting the number of separated points or of covering

balls is actually independent of B, while the parameter R describing the corresponding scales tends to infinity as diam $B \to \infty$, e.g., if one takes as B an $\mathbb{R}$-tree.

Lemma 13.1.3. *Assume that $U \subset X$ is (N, R)-ls-doubling. Then for any $r \geq \rho \geq R$, we have: every intersection $B_r \cap U$ is covered by $\leq N^n$ balls B_ρ with $n = \log_2(2r/\rho)$. In particular, $B_r \cap U$ contains at most N^n points which are 2ρ-separated.*

Proof. Every intersection $B_r \cap U$ is covered by $\leq N^n$ balls of radius $r/2^n$. There is $n \in \mathbb{N}$ with
$$\frac{r}{2^n} < \rho \leq \frac{r}{2^{n-1}}.$$
Then $n \leq \log_2 \frac{2r}{\rho}$, which proves the first assertion. For the second one note that any ball B_ρ contains at most one point of any 2ρ-separated set. □

The property to be large scale doubling is a quasi-isometry invariant.

Proposition 13.1.4. *Let $f : X \to Y$ be a quasi-isometric map. Then for any large scale doubling $U \subset X$, $V \subset Y$, we have $f(U) \subset Y$, $f^{-1}(V) \subset X$ are large scale doubling quantitatively.*

Proof. We can assume that f is (a, b)-quasi-isometric for some $a \geq 1$, $b \geq 0$, and that $r \geq 2b$. The inverse image of any ball $B_{2r} \subset Y$ is contained in a ball $B_R \subset X$ of radius $R = a(4r + b)$. The image of any ball $B_\rho \subset X$ of radius $\rho = \frac{1}{a}(r - b)$ is contained in a ball $B_r \subset Y$.

By Lemma 13.1.3, the intersection $B_R \cap U$ is covered by $\leq N^k$ balls B_ρ with
$$k = \log_2 \frac{2R}{\rho} = \log_2 \frac{2a^2(4r + b)}{r - b} \leq \log_2 18a^2.$$
Thus the intersection $B_{2r} \cap f(U)$ is covered by $\leq N^k$ balls B_r, where the upper bound N^k is independent of r. □

Lemma 13.1.5. *If $U \subset X$, $V \subset Y$ are large scale doubling then $U \times V \subset X \times Y$ is large scale doubling quantitatively.*

Proof. By Proposition 13.1.4, we can consider the l_∞-product metric on $X \times Y$. Then $B_r((x, y)) = B_r(x) \times B_r(y)$ for any $(x, y) \in X \times Y$ and $r > 0$. The large scale doubling property of $U \times V$ is now obvious. □

13.2 Definition of the hyperbolic dimension

Definition 13.2.1. A covering $\mathcal{U}$ of a metric space X is called *uniformly large scale doubling* or *uniformly ls-doubling* if there exists an $N \in \mathbb{N}$ with:

(1) there exists $R \geq 0$ such that every element of the covering is (N, R)-ls-doubling;

(2) any finite union of elements of the covering is N-ls-doubling.

We also call such a covering uniformly N-ls-doubling or uniformly (N, R)-ls-doubling.

Definition 13.2.2. The *hyperbolic dimension* of X is the minimal integer $\operatorname{hypdim} X = n$ such that for every $d > 0$ there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \le n + 1$ and $L(\mathcal{U}) \ge d$ which is uniformly large scale doubling.

This is the covering definition. One can also introduce the colored and polyhedral definitions of the hyperbolic dimension, see Chapter 9.

For the hyperbolic dimension, the property set P (see Chapter 9) is identified with the interval $(0, \infty)$ and the filter $\mathcal{F}$ is generated by all subintervals $(d, \infty) \subset (0, \infty)$, $d > 0$. We say that a covering $\mathcal{U}$ of a metric space X has the property $d \in (0, \infty)$, $\mathcal{U} \in d$, if and only if $\mathcal{U}$ is open, $L(\mathcal{U}) \ge d$ and $\mathcal{U}$ being uniformly large scale doubling.

Lemma 13.2.3. *The property space P for the hyperbolic dimension satisfies the Axioms* 9.3.1.

Proof. The proof is similar to the one for the asymptotic dimension. We use the notations from Section 9.3.

1. For every natural number m, there is a lower bound $l_m \in (0, 1)$ for the Lebesgue number of the covering of any uniform polyhedron K, $\dim K + 1 \le m$, by the open stars of $\operatorname{ba} K$. We put $\lambda_m = l_m/(m+1)^2$ and define $\mathbf{ba}_m \colon P \to P$ as $\mathbf{ba}_m(t) = \lambda_m t$ for every $t > 0$. Axiom 9.3.1 (1) is satisfied because for any covering $\mathcal{U} \in t \in P$ with multiplicity $\le m$, the covering $\mathcal{U}' = \operatorname{ba}(p_{\mathcal{U}})$ is open, $L(\mathcal{U}') \ge l_m/\operatorname{Lip}(p_{\mathcal{U}}) \ge \lambda_m t$, and $\mathcal{U}'$ being inscribed in $\mathcal{U}$ is large scale doubling.

2. Axiom 9.3.1 (2) is obvious because the property to be large scale doubling is hereditary and $L(\mathcal{U}|X') \ge L(\mathcal{U})$ for every open covering $\mathcal{U}$ of X and every subspace $X' \subset X$.

3. By Remark 9.2.7, for every natural number m, there is a constant $c_m \in (0, 1)$ with the following property: given uniform polyhedra K_1, K_2, $\dim K_i + 1 \le m$, the Lebesgue number of the covering of $K_1 \times K_2$ by $\varphi^{-1}(\operatorname{st}_v)$, $v \in K_1 \times_s K_2$, is bounded from below by c_m, where $\varphi \colon K_1 \times K_2 \to K_1 \times_s K_2$ is the barycentric triangulation map. We put $\mu_m = c_m/(m+1)^2$ and define $\operatorname{prod}_m \colon P \times P \to P$ as $\operatorname{prod}_m(t_1, t_2) = \mu_m \min\{t_1, t_2\}$ for every $t_1, t_2 > 0$.

Then clearly $\operatorname{prod}_m(\mathcal{F} \times \mathcal{F}) \subset \mathcal{F}$. Furthermore, for each covering $\mathcal{U}_i \in t_i \in P$ with multiplicity $\le m$, there is by Lemma 9.2.2 a barycentric map $f_i = p_{\mathcal{U}_i} \colon X_i \to \mathcal{N}_i$ with $\operatorname{Lip}(f_i) \le (m+1)^2/t_i$, $i = 1, 2$. Then the covering $\mathcal{U}_{f_1,f_2}$ of $X_1 \times X_2$ is open and

$$L(\mathcal{U}_{f_1,f_2}) \ge c_m/\operatorname{Lip}(f_1 \times f_2) \ge \mu_m \min\{t_1, t_2\}$$

because $\operatorname{Lip}(f_1 \times f_2) \le (m+1)^2 \max\{1/t_1, 1/t_2\}$. Furthermore, the covering $\mathcal{U}_{f_1,f_2}$ is inscribed in the product covering $\mathcal{U}_1 \times \mathcal{U}_2$. Thus using Lemma 13.1.5, we easily check that $\mathcal{U}_{f_1,f_2}$ is uniformly large scale doubling.

That is, $\mathcal{U}_{f_1,f_2} \in \mathrm{prod}_m(t_1, t_2)$, and Axiom 9.3.1 (3) is satisfied. $\square$

Now using results of Section 9.3, we conclude that the three definitions of the hyperbolic dimension, the colored, covering and polyhedral ones, are equivalent, and that for the hyperbolic dimension the monotonicity and product theorems, Theorem 9.4.1 and Theorem 9.5.1, hold true. Furthermore, by the definition, we have

$$\mathrm{hypdim}\, X = 0$$

for every large scale doubling space X, and $\mathrm{hypdim}\, X \le \mathrm{asdim}\, X$ for every metric space X because any uniformly bounded covering is certainly uniformly large scale doubling. It follows from Proposition 13.1.4 that the hyperbolic dimension is a quasi-isometry invariant.

13.3 Hyperbolic dimension of hyperbolic spaces

We study here the hyperbolic dimension of hyperbolic spaces. The main result of the section, Theorem 13.3.2, is based on the fact that large scale doubling sets in CAT(-1)-spaces, when observed from distance c, look exponentially small in c if measured by the angle measure.

13.3.1 Large scale doubling sets in CAT(-1)-spaces

Let X be a CAT(-1)-space. We fix a base point $x_0 \in X$ and define the angle metric $\measuredangle_\infty$ in the geodesic boundary at infinity $\partial^g X$ as follows. Given $\xi, \xi' \in \partial^g X$, we consider the unit speed geodesic rays c_ξ, $c_{\xi'}$ from x_0 to ξ, ξ' respectively, and put

$$\measuredangle_\infty(\xi, \xi') = \lim_{s\to\infty} \measuredangle(\bar{c}_\xi(s)\bar{o}\, \bar{c}_{\xi'}(s)),$$

where $\measuredangle(\bar{c}_\xi(s)\bar{o}\, \bar{c}_{\xi'}(s))$ is the angle at $\bar{o}$ of the comparison triangle in H^2 for the triangle $c_\xi(s)x_0c_{\xi'}(s)$. By the parallelism angle formula, we have

$$\tan\left(\tfrac{1}{4}\measuredangle_\infty(\xi, \xi')\right) = e^{-\,\mathrm{dist}(\bar{o},\bar{\xi}\bar{\xi}')},$$

where $\bar{\xi}$, $\bar{\xi}' \in \partial_\infty \mathrm{H}^2$ satisfy $\measuredangle_{\bar{o}}(\bar{\xi}, \bar{\xi}') = \measuredangle_\infty(\xi, \xi')$, and $\bar{\xi}\bar{\xi}'$ is the geodesic in H^2 with the end points at infinity $\bar{\xi}$, $\bar{\xi}'$. Thus $\measuredangle_\infty(\xi, \xi') \le 4e^{-\,\mathrm{dist}(\bar{o},\bar{\xi}\bar{\xi}')}$.

The *shadow* of a set $A \subset X$ is a subset $\mathrm{sh}(A) \subset \partial^g X$ which consists of the ends ξ of all rays $x_0\xi$ intersecting A (so $\mathrm{sh}(x_0) = \partial^g X$). Given $\delta > 0$ we define *the angle δ-measure* of A, $\measuredangle_\delta A$, by

$$\measuredangle_\delta A = \inf_{\mathcal{C}} \sum_{B\in\mathcal{C}} \mathrm{diam}(\mathrm{sh}(B)),$$

where the infimum is taken over all coverings $\mathcal{C}$ of A by balls of radius $\ge \delta$ in X.

Lemma 13.3.1. *Given $N \in \mathbb{N}$, $R > 1$, there is for every sufficiently large δ a positive constant C depending only on N and δ such that if a subset $A \subset X$ is (N, R)-ls-doubling and* $\operatorname{dist}(x_0, A) \geq c > \delta$, *then*

$$\measuredangle_\delta A \leq C \cdot e^{-c/2}.$$

Proof. We take $\delta/2 \geq R$. Then by Lemma 13.1.3, every ball $B_r \subset X$ with $r > \delta/2$ contains at most $N^n = d \cdot r^k$ points of A which are δ-separated, where $k = \log_2 N$ and $d = (4/\delta)^k$. Furthermore, we can assume that $e^{c/2} \geq (c+1)^k$ for every $c > \delta$.

Take a maximal δ-separated subset $A' \subset A$. Then $A \subset \bigcup_{a \in A'} B_\delta(a)$. For any ball $B_\delta(a)$, $a \in A'$, consider $\xi, \xi' \in \operatorname{sh}(B_\delta(a))$ with $\measuredangle_\infty(\xi, \xi')$ arbitrarily close to $\operatorname{diam}(\operatorname{sh}(B_\delta(a)))$. For simplicity, we assume that $\measuredangle_\infty(\xi, \xi') = \operatorname{diam}(\operatorname{sh}(B_\delta(a)))$ because possible errors disappear in the final conclusion.

Then $\operatorname{diam}(\operatorname{sh}(B_\delta(a))) \leq 4e^{-\operatorname{dist}(\bar{o}, \bar{\xi}\bar{\xi}')}$ in the notations introduced above. We take $x \in x_0\xi \cap B_\delta(a)$, $x' \in x_0\xi' \cap B_\delta(a)$ and consider the piecewise geodesic curve γ in X which consists of the geodesic rays $x\xi$, $x'\xi'$ and the segment xx'. The curve γ connects in X the points ξ, ξ', and $\operatorname{dist}(x_0, \gamma) \geq \operatorname{dist}(x_0, B_\delta(a)) = |x_0 a| - \delta$. Furthermore, $\operatorname{dist}(x_0, \gamma) \leq \operatorname{dist}(\bar{o}, \bar{\xi}\bar{\xi}')$ by comparison with H^2. Thus

$$\operatorname{diam}(\operatorname{sh}(B_\delta(a))) \leq 4e^{\delta - |x_0 a|}$$

and

$$\measuredangle_\delta A \leq \sum_{a \in A'} \operatorname{diam}(\operatorname{sh}(B_\delta(a))).$$

Since $c > \delta$, for every $\tau \geq c + 1$, the number of points from A' whose distances to x_0 lie in the interval $[\tau - 1, \tau)$ is $\leq d \cdot \tau^k$. Thus we have

$$\begin{aligned}
\measuredangle_\delta A &\leq 4e^\delta \sum_{a \in A'} e^{-|x_0 a|} \leq 4de^\delta \Big(\sum_{q=0}^\infty (c + q + 1)^k e^{-q} \Big) e^{-c} \\
&\leq 4de^\delta \Big(\sum_{q=0}^\infty (q+1)^k e^{-q} \Big) (c+1)^k e^{-c} \leq C \cdot e^{-c/2}
\end{aligned}$$

by the choice of c. □

13.3.2 Estimate from below

Here we give a proof based on Sperner's lemma that the hyperbolic dimension of any Hadamard manifold X with sectional curvatures $K \leq -1$ is at least $n = \dim X$. Moreover, we prove the following.

Theorem 13.3.2. *Let $\mathcal{U}$ be a uniformly large scale doubling open covering of X, where X is a Hadamard manifold. Then the multiplicity of $\mathcal{U}$ is at least $n + 1$.*

Proof. Recall that X is a CAT(-1)-space. One can assume that $\mathcal{U}$ is locally finite, that $\mathcal{U}$ is uniformly (N, R_0)-ls-doubling for some $N \in \mathbb{N}$, $R_0 > 1$, and that the elements of $\mathcal{U}$ are connected. By Corollary 9.8.3, it suffices to find a continuous simplex $f : \Delta^n \to X$ coherent with $\mathcal{U}$. We fix a small positive constant α_0 so that the maximal number of α_0-separated points in the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with respect to the angle metric is $> N$. Let $W \subset S^{n-1}$ be a ball of radius $\alpha_0/4$ equipped with a structure of the $(n-1)$-dimensional standard simplex. Furthermore, let $d > 0$ be the minimal diameter of a subset $B \subset S^{n-1}$ which meets all $(n-2)$-dimensional faces of W. This d depends only on α_0.

We fix a sufficiently large δ as in Lemma 13.3.1 (one can take $\delta \geq 2R_0$). Let $C = C(N, \delta)$ be the constant from Lemma 13.3.1. Now we choose $r > \delta$ such that $Ce^{-r/2} < d$. Since $\mathcal{U}$ is locally finite, there are only finitely many elements of $\mathcal{U}$ which intersect the ball $B_r(o)$ centered at a base point $o \in X$. The union A of all those elements is N-ls-doubling by properties of the covering $\mathcal{U}$. Thus for every sufficiently large $R > r$, the intersection $A \cap S_{2R}$ contains at most N points which are R-separated, where $S_{2R} = \partial B_{2R}(o)$.

Next, we fix a maximal α_0-separated subset in S^{n-1}. Then the balls of radius $\alpha_0/4$ in $S^{n-1} \subset T_oX$ centered at its points are pairwise separated by an angle distance $\geq \alpha_0/2$. Radially projected to S_{2R} by $\exp_o : T_oX \to X$ these balls are pairwise separated by the distance $> R$ in X if R is sufficiently large (at this point, we use that X is a CAT(-1)-space). Since the number of the balls is $> N$, there is at least one such ball $W_{2R} \subset S_{2R}$ which misses the closure of A.

Using a standard $(n-1)$-simplex structure on the appropriate ball $W \subset S^{n-1}$, we introduce the induced simplex structure on W_{2R} and consider the geodesic cone $\Delta \subset X$ over W_{2R} with the vertex o as a continuous n-simplex in X. Let us check that Δ is coherent with $\mathcal{U}$. Any element of $\mathcal{U}$ which intersects the ball $B_r(o)$ misses the $(n-1)$-face $W_{2R} \subset \Delta$ by the construction. Any other element of $U \in \mathcal{U}$ is at the distance $\geq r$ from o, and thus has the angle measure $\measuredangle_\delta(U) \leq Ce^{-r/2} < d$ by Lemma 13.3.1. Since X is CAT(-1) and U is connected, the angle diameter of U observed from o is at most $\measuredangle_\delta(U)$. Hence also U cannot intersect all $(n-1)$-faces of Δ by the choice of d. Therefore, the simplex Δ is coherent with $\mathcal{U}$. □

13.4 Applications to nonembedding results

Theorem 13.4.1. *Let X be a metric space with* hypdim $X \geq p$ *and let $T_1, \dots, T_k$ be any metric trees. Then there is no quasi-isometric embedding*

$$X \to T_1 \times \cdots \times T_k \times \mathbb{R}^m$$

for $p > k$ and any $m \geq 0$.

Proof. Assume there is a quasi-isometric embedding

$$f : X \to T_1 \times \cdots \times T_k \times \mathbb{R}^m.$$

Since hypdim $\leq$ asdim, we have hypdim $T \leq 1$ for any metric tree. Thus using the quasi-isometry invariance of the hyperbolic dimension and the monotonicity and product theorems, we obtain

$$p \leq \operatorname{hypdim} X \leq \operatorname{hypdim}(T_1 \times \cdots \times T_k \times \mathbb{R}^m) \leq k. \qquad \square$$

Using Theorem 13.3.2, we obtain:

Corollary 13.4.2. *Let X be a Hadamard manifold with sectional curvatures $K \leq -1$. Then there is no quasi-isometric embedding*

$$X \to T_1 \times \cdots \times T_k \times \mathbb{R}^m$$

into the product of any $k < \dim X$ metric trees stabilized by any Euclidean factor $\mathbb{R}^m$. $\square$

This corollary shows that the embedding result obtained in Theorem 12.3.2 is optimal in a strong sense with respect to the number of tree factors: each cobounded hyperbolic Hadamard manifold admits a quasi-isometric embedding into the n-fold product of metric trees, $n = \dim X$.

One can generalize Corollary 13.4.2 as follows. Let X^n be a universal covering of a compact Riemannian n-dimensional manifold, $n \geq 2$, with nonempty geodesic boundary and constant sectional curvature -1. Then X^n satisfies the condition of Theorem 12.3.1, and hence $\dim \partial_\infty X^n = \ell\text{-}\dim \partial_\infty X^n$. Note that X^n can be obtained from the real hyperbolic space H^n by removing a countable collection of disjoint open half-spaces, and $\partial_\infty X^n \subset S^{n-1}$ is a compact, nowhere dense subset obtained from S^{n-1} by removing a countable collection of disjoint open balls. In particular, for $n = 2$, $\partial_\infty X^n \subset S^1$ is a Cantor set, for $n = 3$, $\partial_\infty X^n \subset S^2$ is a Sierpinski carpet, and for $n \geq 4$, $\partial_\infty X^n \subset S^{n-1}$ is a higher dimensional version of a Sierpinski carpet. Thus $\dim \partial_\infty X^n = n - 2$.

The space X^n contains isometrically embedded copies of H^{n-1} as boundary components. In the next chapter, we show that the k-fold product $\mathrm{H}^{n-1} \times \cdots \times \mathrm{H}^{n-1}$ contains a quasi-isometrically embedded H^p with $p = k(n-2)+1$, see Section 14.1.

Theorem 13.4.3. *Let X^n be a space as above, and let $Y_k^n = X^n \times \cdots \times X^n$ be the k-fold product, $k \geq 1$. Then there is no quasi-isometric embedding*

$$\mathrm{H}^p \to Y_k^n \times \mathbb{R}^m$$

for $p > k(n-1)$ and any $m \geq 0$.

Proof. Indeed, we have that hypdim $X^n \leq$ asdim X^n. By Theorem 12.3.3, asdim $X^n = \ell$-asdim $X^n = \dim \partial_\infty X^n + 1 = n - 1$. Furthermore, hypdim $\mathrm{H}^p \geq p$ by Theorem 13.3.2. Now applying the argument of Theorem 13.4.1, we obtain the result. $\square$

Theorem 13.4.3 implies in particular that there is no way to embed H^5 quasi-isometrically into $X^3 \times X^3 \times \mathbb{R}^m$ for any $m \geq 0$ though there is a quasi-isometric embedding $\mathrm{H}^3 \to X^3 \times X^3$ (and $\mathrm{H}^5 \to \mathrm{H}^3 \times \mathrm{H}^3$). In general, for an arbitrary $n \geq 2$, there is a quasi-isometric $\mathrm{H}^p \to X^n \times X^n$ for $p = 2n - 3$ and there is no quasi-isometric $\mathrm{H}^p \to X^n \times X^n \times \mathbb{R}^m$ for $p = 2n - 1$ and an arbitrary $m \geq 0$.

In the case $n = 2$, the space X^2 is quasi-isometric to the binary tree T whose edges all have length 1 because X^2 covers a compact hyperbolic surface with nonempty geodesic boundary. By [BDS], there is a quasi-isometric embedding $\mathrm{H}^2 \to T \times T$, and hence there is a quasi-isometric embedding $\mathrm{H}^p \to X^n \times X^n$ in the remaining case $p = 2n-2$ if $n = 2$. For $n \geq 3$, the question whether there is a quasi-isometric embedding $\mathrm{H}^{2n-2} \to X^n \times X^n$ remains open. Moreover, the same question is open for quasi-isometric

$$\mathrm{H}^{k(n-1)} \to Y_k^n, \quad n, k \geq 3.$$

13.5 Supplementary results and remarks

13.5.1 Hyperbolic dimension of general hyperbolic spaces

The proof of Theorem 13.3.2 generalizes the one of Theorem 10.1.1 for the asymptotic dimension. Recall that there is another approach to estimates from below of the asymptotic dimension of hyperbolic spaces, using the hyperbolic cone over the boundary at infinity, see Theorem 10.1.2. A more sophisticated version of that allows to prove the following, see [BS3].

Theorem 13.5.1. *Let X be a geodesic hyperbolic space, whose boundary at infinity $\partial_\infty X$ is infinite and doubling. Then*

$$\operatorname{hypdim} X \geq \dim \partial_\infty X + 1. \qquad \square$$

In that general case, one needs to use the Lebesgue number condition from the definition of the hyperbolic dimension. Similarly to Section 13.4, the embedding result in Theorem 12.3.2 is optimal in a strong sense with respect to the number of tree factors.

Corollary 13.5.2. *Let X be a geodesic hyperbolic space, whose boundary at infinity $\partial_\infty X$ is infinite and doubling. Then there is no quasi-isometric embedding*

$$X \to T_1 \times \cdots \times T_k \times \mathbb{R}^m$$

into the product of any $k < \dim \partial_\infty X + 1$ metric trees stabilized by any Euclidean factor $\mathbb{R}^m$. $\square$

Chapter 14

Hyperbolic rank and subexponential corank

In this chapter we consider other quasi-isometry invariants which are useful to prove a number of important nonembedding results in asymptotic geometry.

14.1 Hyperbolic rank

Let X be a metric space. Consider all proper geodesic hyperbolic spaces Y quasi-isometrically embedded in X and define the *hyperbolic rank* of X as

$$\operatorname{rank}_h X = \sup_Y \dim \partial_\infty Y,$$

where the supremum is taken over all such Y.

It follows from the definition that the hyperbolic rank is monotone, i.e., if there is a quasi-isometric embedding $X \to X'$, then $\operatorname{rank}_h X \le \operatorname{rank}_h X'$. Therefore, the hyperbolic rank is a quasi-isometry invariant. This invariant measures in a sense how much hyperbolicity there is in a space, i.e., how big a space is intrinsically.

If X is a proper geodesic hyperbolic space, then $\operatorname{rank}_h X = \dim \partial_\infty X$: the identity map $X \to X$ shows that $\operatorname{rank}_h X \ge \dim \partial_\infty X$; the opposite inequality follows from the stability of geodesics: any quasi-isometric embedding $Y \to X$ of a geodesic hyperbolic space Y induces an embedding $\partial_\infty Y \to \partial_\infty X$ and thus $\dim \partial_\infty X \ge \dim \partial_\infty Y$; see Theorem 5.2.17.

As an illustration for nonhyperbolic spaces, we show that

$$\operatorname{rank}_h(\mathrm{H}^2 \times \mathrm{H}^2) \ge 2.$$

The Riemannian metric of H^{n+1}, $n \ge 1$, in the solvable group model takes the form

$$ds^2 = dt^2 + e^{-2t} dx^2,$$

see Section A.4, A.5, where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ are *horospherical coordinates* in H^{n+1}, and $dx^2 = dx_1^2 + \cdots + dx_n^2$ is the Euclidean metric. The space $\mathrm{H}^{n+1}_\lambda = \lambda\, \mathrm{H}^{n+1}$ obtained from H^{n+1} by multiplying all distances by the factor $\lambda > 0$ has the constant curvature $-1/\lambda^2$ and its Riemannian metric in horospherical coordinates takes the form

$$ds^2 = dt^2 + e^{-2t/\lambda} dx^2.$$

Now take $\lambda = \sqrt{2}$ and consider the map

$$f\colon \mathrm{H}^3_\lambda \to \mathrm{H}^2 \times \mathrm{H}^2, \quad f(t, x_1, x_2) = (t/\sqrt{2}, x_1, t/\sqrt{2}, x_2)$$

in horospherical coordinates $x = (t, x_1, x_2) \in \mathrm{H}^3_\lambda$ and $x_1 \times x_2 = (t_1, x_1) \times (t_2, x_2) \in \mathrm{H}^2 \times \mathrm{H}^2$. The embedding f is isometric with respect to the Riemannian metric $ds^2 = dt^2 + e^{-t\sqrt{2}}(dx_1^2 + dx_2^2)$ on H^3_λ and the metric product $ds_1^2 + ds_2^2$, $ds_i^2 = dt_i^2 + e^{-2t_i}dx_i^2$, $i = 1, 2$, on $\mathrm{H}^2 \times \mathrm{H}^2$, that is, $|df(v)| = |v|$ for every tangent to H^3_λ.

Proposition 14.1.1. *The embedding* $f\colon \mathrm{H}^3_\lambda \to \mathrm{H}^2 \times \mathrm{H}^2$ *is quasi-isometric, in particular,* $\mathrm{rank}_h(\mathrm{H}^2 \times \mathrm{H}^2) \geq 2$.

Proof. Since f is Riemannian isometric, its leaves invariant the length of the curves. Thus $|f(x)f(x')| \leq |xx'|$ for every $x, x' \in \mathrm{H}^3_\lambda$.

To obtain the estimate from below, let us recall that for any $x, x' \in \mathrm{H}^{n+1}$ lying in a horosphere $S \subset \mathrm{H}^{n+1}$ we have

$$|xx'|_S = 2\sinh(|xx'|/2),$$

where $|xx'|_S$ is the distance between x, x' along the horosphere S; see Exercise A.3.3. Since $\mathrm{H}^{n+1}_\lambda = \lambda\,\mathrm{H}^{n+1}$, this gives

$$|xx'| \sim 2\lambda \ln |xx'|_S - 2\lambda \ln \lambda$$

for $x, x' \in S \subset \mathrm{H}^{n+1}_\lambda$. Thus

$$\begin{aligned}|xx'| &\leq 2\sqrt{2}\ln |xx'|_S + c = 2\sqrt{2}\ln |f(x)f(x')|_{f(S)} + c\\ &\leq 2\sqrt{2}\ln(2\max_i |f_i(x)f_i(x')|_{S_i}) + c\\ &\leq \sqrt{2}\cdot|f(x)f(x)| + c\end{aligned}$$

for some universal constant $c > 0$ (which may differ from part to part of the computation above) and for every $x, x' \in \mathrm{H}^3_\lambda$ lying in the same horosphere given by the equation $t = t_0$. In the general case, where x, x' are from different horospheres, the geodesic between them goes first from the upper one to the lower one almost radially, and along this part f is isometric. □

The image $Y = f(\mathrm{H}^3_\lambda) \subset \mathrm{H}^2 \times \mathrm{H}^2$ can be described as

$$Y = \{(x_1, x_2) \in \mathrm{H}^2 \times \mathrm{H}^2 : b_1(x_1) = b_2(x_2)\}$$

for appropriately chosen Busemann functions $b_1, b_2\colon \mathrm{H}^2 \to \mathbb{R}$. The straightforward generalization shows that there is a quasi-isometric embedding $\mathrm{H}^n \to \mathrm{H}^{n_1} \times \cdots \times \mathrm{H}^{n_k}$ with $n - 1 = n_1 + \cdots + n_k - k$. In particular,

$$\mathrm{rank}_h(\mathrm{H}^{n_1} \times \cdots \times \mathrm{H}^{n_k}) \geq n - 1 = n_1 + \cdots + n_k - k.$$

This raises the question whether the obtained estimates of the hyperbolic rank are optimal. To obtain estimates from above, we introduce another quasi-isometry invariant in a sense complementary to the hyperbolic rank.

14.2 Subexponential corank

A continuous map $g\colon X \to T$ of topological spaces can be regarded as a continuous foliation
$$X = \bigcup_{t \in T} g^{-1}(t)$$
of X over T. We define its *rank* as $\operatorname{rank}(g) = \sup_K \dim g(K)$, where the supremum is taken over all compact $K \subset X$. Roughly speaking, the subexponential corank of X is the minimal rank of continuous foliations $g\colon X \to T$ all fibers of which have a subexponential growth rate.

For example, for $X = \mathrm{H}^n \times \mathbb{R}^m$, the foliation $g\colon X \to \mathrm{H}^n$ given by the first factor projection is subexponential, and its rank is $n = \dim \mathrm{H}^n$. However, this is not the least rank of subexponential foliations of X. The factor H^n possesses a subexponential foliation $g_1\colon \mathrm{H}^n \to \mathbb{R}^{n-1}$ which is the projection onto a fixed horosphere $S \subset \mathrm{H}^n$ from its center (at infinity). The rank of this foliation is $n-1$, thus X has a continuous subexponential foliation of rank $n-1$.

The precise definition of the subexponential corank is somewhat complicated since we have to make the continuity condition compatible with the quasi-isometry invariance.

Assume that $\sigma \geq \delta$ and that a maximal δ-separated set $X_\delta \subset X$ are fixed. We define the *size* of $A \subset X$ (with respect to X_δ and σ) as the number
$$\operatorname{size}_{X_\delta,\sigma}(A) \in \mathbb{N} \cup \{\infty\}$$
of points $x \in X_\delta$ for which the balls $B_\sigma(x)$ intersect A. Now we can formulate the basic definition.

A continuous foliation $g\colon X \to T$ is said to be *subexponential* if, for any maximal δ-separated net $X_\delta \subset X$ (with sufficiently large δ), for any $\sigma \geq \delta$ and any $\varepsilon > 0$, there is a constant $R_0 = R_0(X_\delta, \sigma, \varepsilon) \geq 1$ such that for all $R \geq R_0$ and all $t \in T$ we have
$$\tfrac{1}{R} \ln \operatorname{size}_{X_\delta,\sigma}\big(g^{-1}(t) \cap B_R(x_0)\big) < \varepsilon$$
for some fixed point $x_0 \in X$; clearly, this property is independent of the choice of x_0.

The *subexponential corank* of a metric space X is defined as
$$\operatorname{corank} X = \sup_{Z \sim X} \inf \operatorname{rank}(g\colon Z \to T),$$
where the supremum is taken over all spaces Z quasi-isometric to X, and the infimum over all subexponential foliations of Z.

By definition, $\operatorname{corank} X$ is a quasi-isometry invariant. Taking the supremum over all Z quasi-isometric to X in the definition of $\operatorname{corank} X$ is necessary because for any discrete space Z the trivial foliation $\mathrm{id}\colon Z \to Z$ is subexponential and has rank 0, i.e. minimal possible rank.

Remark 14.2.1. The property of a foliation $g\colon X \to T$ to be subexponential means roughly speaking that every fiber $g^{-1}(t)$ has a subexponential growth rate and a bounded distortion in X. The last condition is essential. For example, all fibers of the foliation given by a Busemann function $g\colon \mathrm{H}^n \to \mathbb{R}$ are isometric to $\mathbb{R}^{n-1}$ in the induced Riemannian metric and hence they have a subexponential (in fact, polynomial) growth rate. However, this foliation is by no means subexponential because every horosphere $g^{-1}(t)$, $t \in \mathbb{R}$ is exponentially distorted in H^n and for a fixed $x_0 \in \mathrm{H}^n$ the balls $B_R(x_0)$ contain exponentially large pieces of it.

Lemma 14.2.2. *If $f : X \to Z$ is a continuous quasi-isometric map and $g\colon Z \to T$ is a continuous subexponential foliation, then $g \circ f : X \to T$ is a continuous subexponential foliation.*

Proof. Assume that f is (a,b)-quasi-isometric. We fix $x_0 \in X$ and put $z_0 = f(x_0)$. Let $\delta_0, \sigma_0 \geq \delta_0$ be the separation and the radius constants for the foliation g. We put $\delta = a(\delta_0 + b)$ and take a maximal δ-separated set $X_\delta \subset X$. Then

$$|f(x)f(x')| \geq \tfrac{1}{a}|xx'| - b \geq \tfrac{1}{a}\delta - b \geq \delta_0$$

for different $x, x' \in X_\delta$. Thus $f(X_\delta) \subset Z$ is δ_0-separated. We extend it to a maximal separated net $Z_{\delta_0} \supset f(X_\delta)$.

Furthermore, we fix $\sigma \geq \max\{\delta, \frac{1}{a}(\sigma_0 - b)\}$, and consider $t \in T$. If the ball $B_\sigma(x)$ intersects the set $(g \circ f)^{-1}(t) \cap B_R(x_0)$ for some $x \in X_\delta$, then its image $f(B_\sigma(x))$ intersects the set $g^{-1}(t) \cap B_{aR+b}(z_0)$ since $f(B_R(x_0)) \subset B_{aR+b}(z_0)$. Furthermore, $f(B_\sigma(x)) \subset B_{a\sigma+b}(f(x))$. Thus the ball $B_{a\sigma+b}(f(x))$ intersects the set $g^{-1}(t) \cap B_{aR+b}(z_0)$, and we have

$$\operatorname{size}_{X_\delta,\sigma}\left((g \circ f)^{-1}(t) \cap B_R(x_0)\right) \leq \operatorname{size}_{Z_{\delta_0},a\sigma+b}\left(g^{-1}(t) \cap B_{aR+b}(z_0)\right)$$

for every $R > 0$, $t \in T$. Fix $\varepsilon > 0$. Then, for $aR + b \geq R_0(Z_{\delta_0}, a\sigma + b, \frac{\varepsilon}{a+b})$, we get

$$\frac{1}{aR+b} \ln \operatorname{size}_{X_\delta,\sigma}\left((g \circ f)^{-1}(t) \cap B_R(x_0)\right) < \frac{\varepsilon}{a+b}.$$

Put $R_0(X_\delta, \sigma, \varepsilon) = \max\{1, \frac{1}{a}[R_0(Z_{\delta_0}, a\sigma + b, \frac{\varepsilon}{a+b}) - b]\}$. For $R \geq R_0(X_\delta, \sigma, \varepsilon)$ we have

$$\tfrac{1}{R} \ln \operatorname{size}_{X_\delta,\sigma}\left((g \circ f)^{-1}(t) \cap B_R(x_0)\right) < \varepsilon.$$

Thus the foliation $g \circ f$ is subexponential. □

14.2.1 QPC-spaces

To apply the subexponential corank to specific situations, we need to study continuous subexponential foliations of spaces. An appropriate class to study is formed by QPC-spaces. A metric space Z is called *QPC-space* if every quasi-isometric map

$f: X \to Z$ is parallel to a continuous one, i.e., if there exists a continuous map $f': X \to Z$ such that $|f(x)f'(x)| \le C < \infty$ for all $x \in X$. In that case f' is also quasi-isometric.

Lemma 14.2.3. *Suppose that X is quasi-isometric to a QPC-space Z. Then*

$$\operatorname{corank} X = \inf \operatorname{rank}(g: Z \to T),$$

where the infimum is taken over all subexponential foliations of Z.

Proof. If X' is quasi-isometric to X then X' is quasi-isometric to Z. Thus there is a continuous quasi-isometry $X' \to Z$. By Lemma 14.2.2, $\inf \operatorname{rank}(g': X' \to T') \le \inf \operatorname{rank}(g: Z \to T)$. □

Lemma 14.2.4. *Every proper Hadamard space X is QPC.*

Proof. Assume there is an (a,b)-quasi-isometric map $f: Y \to X$. We take a maximal δ-separated net $Y_\delta \subset Y$ with $a\delta + b \ge \delta_0 > 0$ and note that every ball $B_{2\delta}(\alpha)$, $\alpha \in Y_\delta$, contains only finitely many elements of the net Y_δ. This is so because the set $f(Y_\delta) \subset X$ is δ_0-separated, the space X is proper and hence the ball $B_{2a\delta+b}(f(\alpha)) \supset f(B_{2\delta}(\alpha))$ intersects $f(Y_\delta)$ over a finite set. Thus the nerve $\mathcal{N}$ of the covering $\mathcal{A} = \{B_\delta(\alpha) : \alpha \in Y_\delta\}$ of Y is a locally finite simplicial complex.

Choosing a barycentric map associated with $\mathcal{A}$, we obtain a continuous map $g: Y \to \mathcal{N}$, see Section 9.2.2. Note that $g(y)$ lies in the simplex Δ_y spanned by $\{\alpha \in Y_\delta : |\alpha y| < \delta\}$. Next, we identify Y_δ with the 0-skeleton of $\mathcal{N}$ and extend $f|Y_\delta: \mathcal{N}_0 \to X$ to a continuous $\bar{f}: \mathcal{N} \to X$ using the convexity of X and acting by induction on the dimension of the skeletons. Then $\bar{f}(\Delta_y) \subset B_{a\delta+b}(f(y))$ and thus $\bar{f} \circ g: Y \to X$ is a continuous map parallel to f. □

14.2.2 Properties of the subexponential corank

We list some properties of the corank which easily follow from the definition. They are in parts similar to properties of the hyperbolic dimension.

(1) Monotonicity: If X is quasi-isometric to QPC and X' is quasi-isometrically embedded in X, then $\operatorname{corank} X' \le \operatorname{corank} X$.
Indeed, one can assume that X is QPC. If Z is quasi-isometric to X' then there is a continuous quasi-isometric map $f: Z \to X$. By Lemma 14.2.3, $\operatorname{corank} X = \inf \operatorname{rank}(g: X \to T)$, where the infimum is taken over all subexponential foliations. By Lemma 14.2.2, every subexponential foliation $g: X \to T$ induces a subexponential foliation $g \circ f: Z \to T$. Furthermore $\operatorname{rank}(g \circ f) \le \operatorname{rank}(g)$. Hence $\operatorname{corank} X' \le \operatorname{corank} X$.

(2) The product theorem: If the metric product $X_1 \times X_2$ is a QPC-space, then $\operatorname{corank}(X_1 \times X_2) \le \operatorname{corank} X_1 + \operatorname{corank} X_2$.
Indeed, in this case both X_1, X_2 are QPC, and the product of subexponential foliations $g_i: X_i \to T_i$, $i = 1, 2$, is a subexponential foliation $g_1 \times g_2: X_1 \times X_2 \to T_1 \times T_2$

with $\operatorname{rank}(g_1 \times g_2) \le \operatorname{rank}(g_1) + \operatorname{rank}(g_2)$ by the product theorem for the topological dimension.

(3) $\operatorname{corank} \mathbb{R}^n = 0$ for every $n \ge 0$.

(4) For an n-dimensional Hadamard manifold X, we have $\operatorname{corank} X \le n - 1$. Indeed, the projection to a fixed horosphere from its center is a subexponential foliation of rank $n - 1$.

Theorem 14.2.5. *Assume that a metric space X is quasi-isometric to a QPC-space. Then*

$$\operatorname{rank}_h X \le \operatorname{corank} X.$$

For the proof we need the following notion. Let X be a (truncated) hyperbolic approximation (with parameter $r \le 1/6$) of a compact metric space Z (see Chapter 6, Section 6.4.1). The *extended hyperbolic approximation* $\widehat{X}$ of Z is a simplicial polyhedron which contains X as the 1-skeleton and which is defined in the same way as X with the only difference that for each $k \ge k_0$ any collection of vertices $v \in V_k$ with nonempty intersection of the balls $B(v)$ spans a simplex of $\widehat{X}$. We consider a path metric on $\widehat{X}$ for which every simplex is isometric to the standard *spherical simplex* with edge length 1 (in the sphere of radius $2/\pi$). For every $k \ge k_0$, the combinatorial sphere S_k is a subpolyhedron of $\widehat{X}$ spanned by V_k.

The inclusion $X \subset \widehat{X}$ is a quasi-isometry since the distances between any vertices $v, v' \in V$ in $\widehat{X}$ and in its 1-skeleton X differ only by a universal constant. This is because any geodesic segment in $\widehat{X}$ of length > 2 which misses the 0-skeleton contains conjugate points and cannot be a minimizer. Thus the extended hyperbolic approximation $\widehat{X}$ of Z is hyperbolic since X is hyperbolic, and $\partial_\infty \widehat{X} = \partial_\infty X = Z$. The advantage of $\widehat{X}$ over X is that for every $k \ge k_0$ we have a continuous barycentric map $p_k \colon Z \to \widehat{X}$ associated with the covering $\{B(v) : v \in V_k\}$ of Z (the covering is finite because Z is compact). The image $p_k(Z) \subset \widehat{X}$ lies in the combinatorial sphere S_k, and S_k is a subset of the metric annulus $\{x \in \widehat{X} : k \le |ox| \le k + 1/2\}$, where o is the root of X.

Proof of Theorem 14.2.5. We can assume that X is QPC. Let $f : Y \to X$ be a quasi-isometric embedding of a proper geodesic hyperbolic space Y into X. By Theorem 7.1.2, we can replace Y first by a hyperbolic approximation of $Z = \partial_\infty Y$, taken with a visual metric d, and then by the extended hyperbolic approximation according to the discussion above. Furthermore, we can also assume that f is continuous.

By Lemma 14.2.2, any continuous subexponential foliation $g \colon X \to T$ induces a continuous subexponential foliation $g \circ f : Y \to T$. Thus it suffices to show that

$$\dim Z \le \operatorname{rank}(g \colon Y \to T)$$

for any subexponential foliation g.

The idea of the proof is simple. For $k \ge k_0$, we consider a barycentric map $p_k \colon Z \to S_k$ into the combinatorial sphere S_k (of the level k). Let $K_k = g(S_k) \subset T$

be a compact set. Its dimension is at most $\operatorname{rank}(g)$ by the definition. On the other hand, any open covering of K_k is lifted by $h_k = g \circ p_k : Z \to T$ to an open covering of Z with the same multiplicity. The main step of the proof is to show that if the mesh of a covering of K_k is sufficiently small and the level k is large, then the mesh of the induced covering Z is arbitrarily small, which implies the required inequality. The fact that the mesh of the induced covering is small easily follows from the fact that the foliation g is subexponential.

Now we look at this argument in details. Given an open covering $\mathcal{O}$ of K_k, there is an inscribed finite closed covering $\mathcal{C}$ with multiplicity $\le n+1$, $n = \operatorname{rank}(g)$.

We take a maximal δ-separated net $Y_\delta \subset Y$ (we can assume that Y_δ is the subset of the vertex set V of Y) and $\sigma \ge \delta$ for which the subexponential growth rate condition of the foliation is fulfilled, and fix $\varepsilon \in (0,1)$.

In what follows, we shorten the notation $\operatorname{size}_{Y_\delta,\sigma} =: \operatorname{size}$. For every $R \ge R_0(Y_\delta, \sigma, \varepsilon)$ we have

$$\tfrac{1}{R} \ln \operatorname{size}(g^{-1}(t) \cap B_R(o)) < \varepsilon,$$

where $o \in Y$ is the root.

Lemma 14.2.6. *Let $k \ge R_0(Y_\delta, \sigma, \varepsilon)$ be given. Then there is an open covering $\mathcal{O}_k = \{U_t : t \in K_k\}$ of the compact space K_k such that*

$$\tfrac{1}{k} \ln \operatorname{size}(g^{-1}(U_t) \cap S_k) < 2\varepsilon$$

for every $t \in K_k$.

Proof. Take $R = k + 1/2$. The set $g^{-1}(t) \cap S_k$ is covered by

$$N(k,t) = \operatorname{size}(g^{-1}(t) \cap S_k)$$

open balls $B_\sigma(y)$ with $y \in Y_\delta$. Since $S_k \subset B_R(o)$, we have

$$\tfrac{1}{k} \ln N(k,t) \le \tfrac{2}{R} \ln \operatorname{size}(g^{-1}(t) \cap B_R(o)) < 2\varepsilon.$$

Let $W \subset Y$ be the union of the mentioned balls. We claim that there is a neighborhood U_t of $t \in K_k$ in T such that $g^{-1}(U_t) \cap S_k \subset W$. If this is not the case, then there is a sequence $y_i \in S_k \setminus W$ for which $g(y_i) \to t$. One can assume that $y_i \to y_\infty \in S_k \setminus W$. By continuity, $g(y_\infty) = \lim g(y_i) = t$. This contradicts the fact that $y_\infty \notin g^{-1}(t)$. Thus the covering $\mathcal{O}_k = \{U_t : t \in K_k\}$ fulfills the requirements. $\square$

We take a finite closed covering $\mathcal{C}_k$ with multiplicity $\le n+1$ inscribed in $\mathcal{O}_k$, and consider the finite closed covering $\mathcal{A}_k = g^{-1}(\mathcal{C}_k)$ of the combinatorial sphere S_k. Its multiplicity is $\le n+1$. Then, by Lemma 14.2.6, $\operatorname{size} A < e^{2k\varepsilon}$ for any $A \in \mathcal{A}_k$. Unfortunately, this does not mean that the diameter (along the sphere S_k) of A is subexponential – the property which we would like to have. Thus we modify the covering $\mathcal{A}_k$ as follows. For every $A \in \mathcal{A}_k$, we consider the covering by those open balls $B_\sigma(y)$, $y \in Y_\delta$ which intersect A. Let $D \subset Y_\delta$ be the set of all $y \in Y_\delta$ with $B_\sigma(y) \cap A \neq \emptyset$. We say that points $y, y' \in D$ are in one component if they are connected by a sequence in D for which the appropriate balls B_σ related to consecutive members of the sequence intersect.

Then D consists of finitely many components α since there are only finitely many points of the net Y_δ in the ball $B_{R+\sigma}(o)$, $R = k + 1/2$. Thus $A = \bigcup_\alpha A_\alpha$ is the finite disjoint union of closed sets $A_\alpha = A \cap \left(\bigcup_{y\in\alpha} B_\sigma(y)\right)$, which we now consider as the covering members.

For every $A_\alpha \subset A$, any $x, x' \in A_\alpha$ can be connected in Y by a piecewise geodesic path whose vertices lie in the σ-neighborhood of S_k. Such a path has at most $e^{2k\varepsilon}$ edges all of length $\le 2\sigma$, and all edges, except maybe the initial and the final ones, lie in the 1-skeleton of Y (these edges are geodesics in Y between the centers of balls $B_\sigma(y)$).

Finally, we consider the finite closed covering $\mathcal{B}_k = p_k^{-1}(\mathcal{A}_k)$ of Z. Its multiplicity is $\le n + 1$. To complete the proof of Theorem 14.2.5, it remains to show that the mesh of this covering is arbitrarily small as $k \to \infty$.

For $B \in \mathcal{B}_k$, we take $\xi, \xi' \in B$ with $d(\xi, \xi') = \operatorname{diam} B$. The points $p_k(\xi)$, $p_k(\xi') \in A_\alpha$ lie in open stars st_v, $\mathrm{st}_{v'}$ of some vertices $v, v' \in V_k$ respectively.

Let $r \le 1/6$ be the parameter of Y.

Lemma 14.2.7. *Assume that a piecewise geodesic path $\gamma \subset Y$ between $v, v' \in V_k$ lies in the annulus $\{k - \sigma \le |oy| \le k + \sigma\}$ and consists of p geodesic edges having the length $\le 2\sigma$. Then*

$$d(v, v') \le c p^{-\log_2 r} \cdot r^k,$$

where the constant $c = c(r, \sigma)$ is independent of k and of the path γ.

Proof. We can assume that γ lies in the 1-skeleton of Y. The idea is to project γ down at lower levels until it collapses to a point, and estimate how many steps one needs for that.

First, using Lemma 6.2.3, we can project γ to a horizontal path γ' in the highest level $S_{k'}$ below γ, $k' \ge k - 4\sigma$, without increasing its length $L(\gamma') \le L(\gamma) \le 2p\sigma$. Next, let $v_{i-1}v_i, v_iv_{i+1} \subset S_{k'}$ be adjacent horizontal edges. As in Lemma 6.2.1, we find $w \in V_{k'-1}$ so that v_{i-1}, v_i, v_{i+1} are connected with w by radial edges. Applying this construction together with Lemma 6.2.3 to γ', we find a projection $\gamma'' \subset S_{k'-1}$ of γ', whose length is at most half of the length $L(\gamma')$. After at most $m = \log_2 L(\gamma')$ steps γ' is projected to a point in $S_{k'-m}$. Since during all projections, the end points v, v' are moving along radial geodesics, we have $k' - m \le (v|v')_o + c$. Thus we obtain

$$k \le k' + 4\sigma \le (v|v')_o + \log_2 L(\gamma') + 4\sigma + c.$$

Since $d(v, v') \le c \cdot r^{(v|v')_o}$, we obtain

$$d(v, v') \le c \cdot p^{-\log_2 r} \cdot r^k. \qquad \square$$

Applying Lemma 14.2.7, we obtain

$$\begin{aligned} d(\xi,\xi') &\le d(\xi,v) + d(v,v') + d(v',\xi') \\ &\le 2r^k + c \cdot p^{-\log_2 r} r^k \end{aligned}$$

with $p \le e^{2k\varepsilon}$ by the estimates above. Thus for sufficiently small ε we have mesh $\mathcal{B}_k \to 0$ as $k \to \infty$. This completes the proof of Theorem 14.2.5. □

14.3 Applications to nonembedding results

14.3.1 Subexponential corank of a product

Let $X = \mathrm{H}^{n_1} \times \cdots \times \mathrm{H}^{n_k}$, where $n_1, \ldots, n_k \ge 2$. By Theorem 14.2.5 and property (4) of the subexponential corank, we have corank $\mathrm{H}^{n_i} = n_i - 1$. Thus

$$\operatorname{corank} X \times \mathbb{R}^m \le \sum_i (n_i - 1) = \dim X - k$$

for every $m \ge 0$. In particular, for $p - 1 > \dim X - k$ there is no quasi-isometric embedding $\mathrm{H}^p \to X \times \mathbb{R}^m$. This estimate is the best possible because for $p = \dim X - (k-1)$ there is a quasi-isometric embedding $\mathrm{H}^p \to X$; see Section 14.1.

14.3.2 Subexponential corank of symmetric spaces

Let X be a Riemannian symmetric space of noncompact type. Then

$$\operatorname{corank} X \le \dim X - \operatorname{rank} X.$$

Therefore, by Theorem 14.2.5, there is no quasi-isometric embedding $Y \to X$ for any proper geodesic hyperbolic space Y with $\dim \partial_\infty Y > \dim X - \operatorname{rank} X$, in particular, there is no quasi-isometric embedding $\mathrm{H}^p \to X \times \mathbb{R}^m$ with $p - 1 > \dim X - \operatorname{rank} X$ and any $m \ge 0$.

Indeed, an Iwasawa decomposition $G = NAK$ of the connected component of the identity in its isometry group allows to identify X with the solvable group NA. Fix $x_0 \in X$ and consider the orbit $T = Nx_0 \subset X$. Then the map $g\colon X \to T$ given by $g(x) = nx_0$, where $x = nax_0$ for $n \in N$, $a \in A$, is a subexponential foliation of rank $\dim T = \dim N = \dim X - \operatorname{rank} X$. Its fibers $g^{-1}(t)$ are geodesic rank X-flats.

14.4 Subexponential corank versus hyperbolic dimension

The subexponential corank as well as the hyperbolic dimension may serve as obstacles to quasi-isometric embeddings. We have the following estimates of the hyperbolic rank from above

$$\operatorname{rank}_h X \le \operatorname{corank} X,$$
$$\operatorname{rank}_h X \le \operatorname{hypdim} X - 1,$$

where the first estimate holds for all QPC-spaces, see Theorem 14.2.5, and the second one is obtained as follows: If $Y \to X$ is a quasi-isometric embedding of a hyperbolic space Y, then $\operatorname{hypdim} Y \le \operatorname{hypdim} X$ and, since $\operatorname{hypdim} Y \ge \dim \partial_\infty Y + 1$ by Theorem 13.5.1, we get the estimate.

However, the ranges of these invariants are different. For example, $\operatorname{corank}(\mathrm{H}^2 \times \mathrm{H}^2) \le 2$ which implies by Theorem 14.2.5 that there is no quasi-isometric embedding $\mathrm{H}^4 \to \mathrm{H}^2 \times \mathrm{H}^2 \times \mathbb{R}^m$ for any $m \ge 0$. The estimate $\operatorname{hypdim}(\mathrm{H}^2 \times \mathrm{H}^2) \le 4$ prohibits quasi-isometric embeddings $\mathrm{H}^5 \to \mathrm{H}^2 \times \mathrm{H}^2 \times \mathbb{R}^m$ for any $m \ge 0$, but it gives no information about the existence of embeddings $\mathrm{H}^4 \to \mathrm{H}^2 \times \mathrm{H}^2 \times \mathbb{R}^m$ (the precise value of $\operatorname{hypdim}(\mathrm{H}^2 \times \mathrm{H}^2)$ is not known, it must be 3 or 4, and very likely it is 4). Furthermore, the hyperbolic dimension unlike the subexponential corank is hard to compute for nonhyperbolic spaces.

On the other hand, in many cases the hyperbolic dimension gives better nonembedding results than the subexponential corank. For example, by Theorem 13.4.1, there is no quasi-isometric embedding

$$\mathrm{H}^3 \to T_1 \times T_2 \times \mathbb{R}^m$$

for any metric trees T_1, T_2 and any $m \ge 0$ because $\operatorname{hypdim} \mathrm{H}^3 \ge 3$ and $\operatorname{hypdim} T_i \le 1$. If we take trees with exponential growth rate, e.g. binary trees, then $\operatorname{corank} T_i \ge 1$ because any continuous map of T_i into a zero dimensional space must be constant and thus there is no continuous subexponential foliation $T_i \to T$ of rank 0. Hence, $\operatorname{corank}(T_1 \times T_2 \times \mathbb{R}^m) \le 2$, which prohibits quasi-isometric embeddings $\mathrm{H}^4 \to T_1 \times T_2 \times \mathbb{R}^m$ but gives no information about the existence of embeddings $\mathrm{H}^3 \to T_1 \times T_2 \times \mathbb{R}^m$.

In general, the situation can be described as follows. There are four quasi-isometry invariants or better two pairs of invariants relevant to the nonembedding problem:

$$(\operatorname{rank}_h, \operatorname{corank}) \quad \text{and} \quad (\text{t-rank}, \operatorname{hypdim}).$$

The hyperbolic rank measures how large a space is intrinsically and the subexponential corank is complementary to it, that is, the subexponential corank serves as a tool to estimate the hyperbolic rank from above.

The t-rank (t stands for trees) of a metric space X is the minimal k so that X can be quasi-isometrically embedded into the product $T_1 \times \cdots \times T_k \times \mathbb{R}^m$ for some $m \ge 0$. The t-rank measures how large a space is extrinsically. The hyperbolic dimension is complementary to the t-rank in the sense that it serves as a tool to estimate the t-rank from below,

$$\operatorname{hypdim} X \le \text{t-}\operatorname{rank} X$$

for every metric space X.

14.5 Supplementary results and remarks

14.5.1 Hyperbolic rank of some nonhyperbolic spaces

We discuss the hyperbolic rank of Riemannian symmetric spaces and of products.

Let X be a Riemannian symmetric space of noncompact type. It is proven in [Le] that $\operatorname{rank}_h X \geq \dim X - \operatorname{rank} X$. Thus by Section 14.3.2,

$$\operatorname{rank}_h X = \dim X - \operatorname{rank} X = \operatorname{corank} X.$$

The product $X = X_1 \times \cdots \times X_m$ of Hadamard manifolds with pinched negative curvature, $-k^2 \leq K \leq -1$, always has

$$\operatorname{rank}_h X \geq \sum_{i=1}^{m} \operatorname{rank}_h X_i.$$

This is a generalization of Proposition 14.1.1 and is proven in [Gr2], [BrFa] for X_i being real hyperbolic manifolds and in [FS1] for the general case. Using properties of the subexponential corank and Theorem 14.2.5, we obtain the product formula

$$\operatorname{rank}_h X = \sum_{i=1}^{m} \operatorname{rank}_h X_i = \sum_{i=1}^{m} \operatorname{corank} X_i = \operatorname{corank} X.$$

However a product formula of the type $\operatorname{rank}_h \Pi_i X_i = \sum_i \operatorname{rank}_h X_i$, does not hold in general. In Chapter 10, we have described a quasi-isometric embedding of the hyperbolic plane H^2 into the product $T_a \times T_b$ of two metric simplicial trees. Thus we obtain $\operatorname{rank}_h(T_a \times T_b) \geq 1$ (actually equality holds). On the other hand, $\dim \partial_\infty T_a = \dim \partial_\infty T_b = 0$ and therefore $\operatorname{rank}_h T_a = \operatorname{rank}_h T_b = 0$. Hence we have

$$\operatorname{rank}_h(T_a \times T_b) > \operatorname{rank}_h T_a + \operatorname{rank}_h T_b.$$

In this example, the trees T_a, T_b have infinite valence at every vertex. According to [BDS], there is a quasi-isometric embedding of H^n into the n-fold product X_n of binary trees. Thus $\operatorname{rank}_h X_n \geq n - 1$, while $\operatorname{rank}_h T = 0$ for every binary tree T.

For a further discussion of the hyperbolic rank of products it is useful to review the hyperbolic product construction.

14.5.2 Hyperbolic product of hyperbolic spaces

The embedding described in Proposition 14.1.1 is a particular case of a general construction which associates to any hyperbolic spaces X_1, X_2 a hyperbolic subspace $Y \subset X_1 \times X_2$ as follows. On $X_1 \times X_2$ consider the l_∞ product metric, i.e.

$$|(x_1, x_2), (y_1, y_2)| := \max\{|x_1 y_1|, |x_2 y_2|\} \quad \text{for all } x_\nu, y_\nu \in X_\nu,\ \nu = 1, 2.$$

Recall that for $a, b, c \in \mathbb{R}$ and $c \geq 0$ we have the notation

$$a \doteq_c b \quad \text{if and only if} \quad |a-b| \leq c.$$

Given two pointed hyperbolic metric spaces (X_1, o_1) and (X_2, o_2) as well as a number $\Delta \geq 0$, we write $o := (o_1, o_2) \in X_1 \times X_2$ and define

$$Y_{\Delta,o} := \{(x_1, x_2) \in X_1 \times X_2 : |o_1 x_1| \doteq_\Delta |o_2 x_2|\}.$$

The space $Y_{\Delta,o} \subset X_1 \times X_2$ is endowed with the restriction of the maximum metric on $X_1 \times X_2$.

Theorem 14.5.1 (FS1). *If X_1, X_2 are hyperbolic, then $Y_{\Delta,o}$ is hyperbolic.* □

In order to investigate the boundary of $Y_{\Delta,o}$ one needs more structure: Let $k \geq 0$. A *k-rough geodesic* is a map $\gamma\colon I \to X$ from an interval $I \subset \mathbb{R}$ to a metric space X with

$$|\gamma(s)\gamma(t)| \doteq_k |s-t| \quad \text{for all } s, t \in I.$$

The space X is called *k-roughly geodesic* if for every pair $x, y \in X$ there exists a k-rough geodesic $\gamma\colon [0, |xy|] \to X$ with $\gamma(0) = x$ and $\gamma(|xy|) = y$. X is called *roughly geodesic* if X is k-roughly geodesic for some $k \geq 0$.

Theorem 14.5.2 (FS1). *If X_1, X_2 are δ-hyperbolic and k-roughly geodesic, then there exists $\Delta_0 \geq 0$ such that for all $\Delta \geq \Delta_0$ the space $Y_{\Delta,o}$ is roughly geodesic and the boundary $\partial_\infty Y_{\Delta,o}$ is naturally homeomorphic to $\partial_\infty X_1 \times \partial_\infty X_2$.* □

There is also a version of these theorems where the basepoints $o_\nu \in X_\nu$ are replaced by Busemann functions based at points $z_\nu \in \partial_\infty X_\nu$.

The space $Y_{\Delta,0}$ is called the *hyperbolic product* of the pointed spaces (X_ν, o_ν). For simplicity, we denote the hyperbolic product by $X_1 \times_h X_2$. The operation $\times_h$ can be considered as a natural product construction in the class of hyperbolic spaces.

Since $X_1 \times_h X_2 \subset X_1 \times X_2$ we clearly have

$$\operatorname{rank}_h(X_1 \times_h X_2) \leq \operatorname{rank}_h(X_1 \times X_2).$$

By Theorem 14.5.2 we have for roughly geodesic hyperbolic spaces X_ν that

$$\operatorname{rank}_h(X_1 \times_h X_2) = \dim(\partial_\infty X_1 \times \partial_\infty X_2).$$

By the product theorem in dimension theory we obtain

$$\operatorname{rank}_h(X_1 \times_h X_2) \leq \operatorname{rank}_h X_1 + \operatorname{rank}_h X_2.$$

We now show that there are cases of strict inequality for both estimates of $\operatorname{rank}_h(X_1 \times_h X_2)$. It is proven in [Dr4] that for every prime p, there exists a hyperbolic Coxeter group Γ_p with a Pontryagin surface Π_p as boundary at infinity. It is well known that for primes $p \neq q$ we have $\dim(\Pi_p \times \Pi_q) = \dim \Pi_p + \dim \Pi_q - 1$. Thus we obtain

$$\operatorname{rank}_h(\Gamma_p \times_h \Gamma_q) < \operatorname{rank}_h(\Gamma_p) + \operatorname{rank}_h(\Gamma_q).$$

We already remarked in Section 14.5.1 that $\operatorname{rank}_h(T \times T) \geq 1$ for the binary tree T. Since $\operatorname{rank}_h(T \times_h T) \leq \operatorname{rank}_h T + \operatorname{rank}_h T = 0$, we have

$$\operatorname{rank}_h(T \times_h T) < \operatorname{rank}_h(T \times T).$$

We now come back to the discussion of the hyperbolic rank of a product of hyperbolic spaces. We have already seen in the last section that for Hadamard manifolds X_i with pinched negative curvature we have

$$\operatorname{rank}_h(X_1 \times X_2) = \operatorname{rank}_h X_1 + \operatorname{rank}_h X_2,$$

but, for example, for the binary tree we have

$$\operatorname{rank}_h(T \times T) > \operatorname{rank}_h T + \operatorname{rank}_h T.$$

We do not know whether there are spaces X_1 and X_2 with $\operatorname{rank}_h(X_1 \times X_2) < \operatorname{rank}_h X_1 + \operatorname{rank}_h X_2$. In analogy with the hyperbolic product a possible candidate for this situation is $\Gamma_p \times \Gamma_q$. We do not know whether the strict inequality holds in that case. However using the equality obtained in [Leb]

$$\operatorname{asdim}(\Gamma \times \Gamma') = \dim(\partial_\infty \Gamma \times \partial_\infty \Gamma') + 2,$$

which holds for all hyperbolic groups Γ, Γ', we can estimate

$$\begin{aligned}
\operatorname{rank}_h(\Gamma_p \times \Gamma_q) &\leq \operatorname{hypdim}(\Gamma_p \times \Gamma_q) - 1 \\
&\leq \operatorname{asdim}(\Gamma_p \times \Gamma_q) - 1 = \dim(\Pi_p \times \Pi_q) + 1 \\
&= \dim \Pi_p + \dim \Pi_q = \operatorname{rank}_h \Gamma_p + \operatorname{rank}_h \Gamma_q.
\end{aligned}$$

Here, the first inequality follows from Theorem 13.5.1.

Bibliographical note. The notion of the hyperbolic rank of a metric spaces is introduced in [Gr1] in a slightly stronger form: one takes into account all hyperbolic spaces. The results of this chapter are based on [BS1] where a slightly weaker version of Theorem 14.2.5 is obtained for the hyperbolic rank, which takes into account only CAT(-1)-spaces.

Appendix

Models of the hyperbolic space H^n

Here we consider various models of the real hyperbolic space H^n and explain the classical result that there is one-to-one correspondence between the isometries of H^n and the Möbius transformations of the unit sphere S^{n-1}. Furthermore, we give a proof of another classical result which characterizes the Möbius transformations as ones preserving the cross-ratio.

A.1 The pseudo-spherical model

For simplicity, we assume first that $n = 2$. Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form given by

$$g(v) = x^2 + y^2 - z^2, \quad v = (x, y, z) \in \mathbb{R}^3.$$

Consider the unit pseudo-sphere (more precisely, its upper component)

$$A = \{(x, y, z) : x^2 + y^2 - z^2 = -1,\ z > 0\}.$$

We introduce the pseudo-spherical coordinates on A

$$x = \sinh\chi\cos\varphi, \quad y = \sinh\chi\sin\varphi, \quad z = \cosh\chi,$$

where $0 \le \varphi < 2\pi$, $\chi \ge 0$. We have $x^2 + y^2 - z^2 = \sinh^2\chi - \cosh^2\chi \equiv -1$ and computing the induced quadratic form on A, we obtain

$$ds_A^2 = dx^2 + dy^2 - dz^2 = d\chi^2 + \sinh^2\chi d\varphi^2.$$

We see that the form ds_A^2 is positive definite and thus is a Riemannian metric on A. Its Gaussian curvature is

$$K = -\frac{(\sinh\chi)''}{\sinh\chi} \equiv -1.$$

The space A with the Riemannian metric ds_A^2 is called the *pseudo-spherical model* of the hyperbolic plane H^2. The geodesics in A are the intersections of A with 2-dimensional linear subspaces $E \subset \mathbb{R}^3$.

A.2 The unit disc model

We introduce the polar coordinates (r, φ), $0 \le r < 1$, $0 \le \varphi < 2\pi$ in the unit disc $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$:

$$x = r\cos\varphi, \quad y = r\sin\varphi.$$

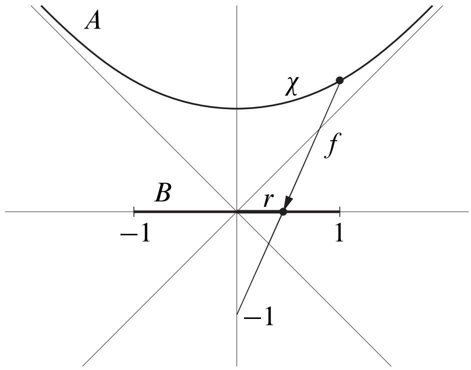


Figure A.1. The map $f : A \to B$.

The map $f : A \to B$, $f(\chi, \varphi) = (r, \varphi)$, where

$$\frac{\sinh\chi}{r} = \frac{\cosh\chi + 1}{1},$$

introduces the structure of the hyperbolic plane H^2 on B. Let us find the corresponding metric ds_B^2. We have

$$\left(\frac{\sinh\chi}{r} - 1\right)^2 = \cosh^2\chi = \sinh^2\chi + 1,$$

whence

$$\left(\frac{1}{r^2} - 1\right)\sinh\chi = \frac{2}{r}$$

and consequently

$$\sinh\chi = \frac{2r}{1 - r^2}.$$

On the other hand

$$r = \frac{\sinh\chi}{\cosh\chi + 1},$$

thus

$$dr = \frac{\cosh\chi(\cosh\chi + 1) - \sinh^2\chi}{(\cosh\chi + 1)^2}d\chi = \frac{d\chi}{\cosh\chi + 1}$$

and $d\chi = (\cosh\chi + 1)dr = \frac{\sinh\chi}{r}dr = \frac{2}{1-r^2}dr$. It follows that

$$\begin{aligned} ds_B^2 &= d\chi^2 + \sinh^2\chi d\varphi^2 \\ &= \frac{4}{(1-r^2)^2}dr^2 + \frac{4r^2}{(1-r^2)^2}d\varphi^2 \\ &= \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\varphi^2), \end{aligned}$$

or

$$ds_B^2 = \frac{4}{(1-(x^2+y^2))^2}(dx^2 + dy^2).$$

The space B with the Riemannian metric ds_B^2 is called the *unit disc model* of H^2. An isomorphism between the models A and B is given by the map f.

Exercise A.2.1. Using the isomorphism f show that any geodesic in B is either a diameter of B or the arc of a circle in $\mathbb{R}^2$ orthogonal to the boundary circle S^1 of B.

A.3 The upper half-plane model

We denote by $C = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ the upper half-plane and consider the map $g\colon B \to C$ given in complex numbers by the fractional linear transformation

$$g(z) = w = \imath\frac{1-z}{1+z},$$

where $z = x + \imath y$. The map g introduces the structure of H^2 on C. Let us find the corresponding metric ds_C^2. Let $\bar{z} = x - \imath y$ be the conjugate to z. Then $x^2 + y^2 = z\bar{z}$, $dx^2 + dy^2 = dz \cdot d\bar{z}$ and thus

$$ds_C^2 = \frac{4dz \cdot d\bar{z}}{(1-z\bar{z})^2}.$$

Since

$$z = \frac{\imath - w}{\imath + w} \quad \text{and} \quad \bar{z} = \frac{\imath + \overline{w}}{\imath - \overline{w}},$$

we have

$$dz = -\frac{2\imath dw}{(\imath + w)^2} \quad \text{and} \quad d\bar{z} = \frac{2\imath d\overline{w}}{(\imath - \overline{w})^2}.$$

Using that we obtain

$$1 - z\bar{z} = \frac{2\imath(w - \overline{w})}{(\imath + w)(\imath - \overline{w})}$$

and find

$$\begin{aligned} ds_C^2 &= \frac{4}{(1-z\bar{z})^2} dz \cdot d\bar{z} \\ &= -\frac{(\imath + w)^2(\imath - \bar{w})^2}{(w-\bar{w})^2} \frac{4dw \cdot d\bar{w}}{(\imath + w)^2(\imath - \bar{w})^2} \\ &= -\frac{4dw \cdot d\bar{w}}{(w-\bar{w})^2} = \frac{1}{v^2} dw \cdot d\bar{w}. \end{aligned}$$

The space C with the Riemannian metric

$$ds_C^2 = \frac{1}{v^2}(du^2 + dv^2)$$

is called the *upper half-plane model* of H^2. An isomorphism between the models B and C is given by the map g.

Exercise A.3.1. It is well known that any fractional linear map of $\mathbb{C}$ transforms a generalized circle (i.e. a circle or a line) into a generalized circle. Using this and Exercise A.2.1 show that any geodesic in C is either a vertical half-line or the arc of a circle in $\mathbb{C}$ orthogonal to the boundary real line $\mathbb{R}$ of C.

A.3.1 The angle of parallelism

Consider a right-angled infinite triangle $ab\xi \subset \mathrm{H}^2$ with $\xi \in \partial_\infty \mathrm{H}^2$ and $\measuredangle_a(b,\xi) = \pi/2$. Then the angle $\alpha = \measuredangle_b(a,\xi)$ is called the *angle of parallelism*. There is an important formula relating the angle of parallelism with the distance $d = |ab|$.

Lemma A.3.2. *Under the condition above, we have*

$$\tan(\alpha/2) = \exp(-d).$$

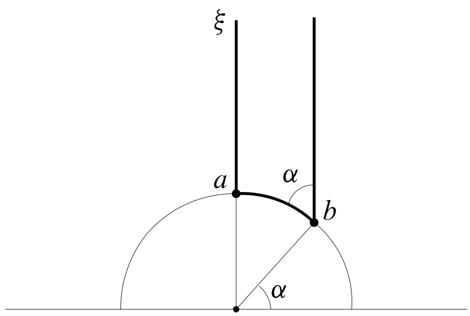


Figure A.2. Parallelism angle.

Proof. It is convenient to use the upper half-plane model C. Assuming that the side ab lies on the half-circle $\{(u,v) \in C : u^2 + v^2 = 1\}$ so that a lies in $\{u = 0\}$, and $\xi = \infty$, we obtain that the sides $a\xi$, $b\xi$ are vertical rays in the model C. The ray $b\xi$ forms the angle α with the half-circle. Thus

$$|ab| = \int_{\alpha}^{\pi/2} \frac{dt}{\sin t} = -\ln\tan\frac{\alpha}{2}. \qquad \square$$

Exercise A.3.3. Show using the angle parallelism formula that

$$|xx'|_S = 2\sinh(|xx'|/2)$$

for a horosphere $S \subset \mathrm{H}^2$ and for all $x, x' \in S$, where $|xx'|_S$ is the distance between x, x' along S.

A.4 The solvable group model

We fix $\lambda \neq 0$ and consider on $\mathbb{R}^2 = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}\}$ a group structure of the semi-direct product $S_2 = \mathbb{R} \ltimes \mathbb{R}$ given by multiplication

$$(t,x) * (t',x') = (t+t', x + e^{\lambda t}x').$$

It is easy to see that S_2 is a solvable group. It acts on itself by the left translations $L_{(t,x)} \colon S_2 \to S_2$, $L_{(t,x)}(t',x') = (t,x) * (t',x')$. A metric on S_2 is *left invariant* if the left translation $L_{(t,x)}$ is an isometry for every $(t,x) \in S_2$.

To find a left invariant metric on S_2, we note that the curve $(0,x')$, $x' \in \mathbb{R}$ is shifted to the curve $L_{(t,x)}(0,x') = (t, x + e^{\lambda t}x')$. The tangent vector dx' at $(0,0)$ is shifted to the vector $dL_{(t,x)}(dx') = (0, e^{\lambda t}dx')$. Furthermore, the curve $(t',0)$, $t' \in \mathbb{R}$ is shifted to the curve $L_{(t,x)}(t',0) = (t+t', x)$, and the tangent vector dt' at $(0,0)$ is shifted to the vector

$$dL_{(t,x)}(dt') = dt'$$

at (t',x'). Assuming that the vectors dx' and dt' are orthogonal at zero with respect to a Riemannian metric $\langle\cdot,\cdot\rangle$ on S_2, we find that the condition to be left invariant is reduced to the equality

$$\langle e^{\lambda t}dx', e^{\lambda t}dx'\rangle_{(t,x)} = \langle dx', dx'\rangle_{(0,0)},$$

or $\langle dx', dx'\rangle_{(t,x)} = e^{-2\lambda t}\langle dx', dx'\rangle_{(0,0)}$. In other words, the metric

$$ds^2 = dt^2 + e^{-2\lambda t}dx^2$$

is left invariant on S_2. Its Gaussian curvature is constant $K = -\frac{(e^{-\lambda t})''}{e^{-\lambda t}} = -\lambda^2$ and for $\lambda = 1$ it equals -1. This gives one more model of the hyperbolic plane H^2, which we call the *solvable group model* and denote by D,

$$ds_D^2 = dt^2 + e^{-2t}dx^2.$$

The horosphere $\{(0, x) : x \in \mathbb{R}\}$ is an abelian subgroup in S_2.

The map $h\colon C \to D$, $h(u, v) = (\ln v, u)$ is an isomorphism between C and D: for $t = \ln v$, $x = u$ we have $dt = \frac{dv}{v}$, $dx = du$ and $v = e^t$. Thus

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + e^{2t} dt^2}{e^{2t}} = dt^2 + e^{-2t} dx^2.$$

A.5 Generalizations to an arbitrary dimension

The quadratic form $g(v) = x_0^2 - (x_1^2 + \cdots + x_n^2)$, $v = \{x_0, x_1, \ldots, x_n\}$ induces on $A^n \subset \mathbb{R}^{n+1}$, given by

$$g(v) = 1$$

and the condition $x_0 > 0$, the metric with the constant sectional curvature -1 which has the form

$$ds_A^2 = d\chi^2 + \sinh^2 \chi d\omega^2$$

in the pseudo-spherical coordinates $x_0 = \cosh \chi$, $x_i = \sinh \chi f_i(\omega)$, $i \geq 1$, where $d\omega^2$ is the metric of the unit sphere in $\mathbb{R}^n$. This is the pseudo-spherical model of the hyperbolic space H^n.

The model of H^n in the unit disc $B^n = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 < 1\}$ is given by the metric

$$ds_B^2 = \frac{4}{(1 - r^2)^2}(dr^2 + r^2 d\omega^2),$$

where (r, ω) are spherical coordinates in $\mathbb{R}^n$. An isomorphism $f : A^n \to B^n$ between these models is given by the same formula as for $n = 2$. The boundary at infinity $\partial_\infty \mathrm{H}^n$ is the boundary sphere for the ball B^n.

The upper half-space model of H^n in the upper half-space

$$C^n = \{(x_1, \ldots, x_n) : x_n > 0\}$$

is given by the metric

$$ds_C^2 = \frac{1}{x_n^2}(dx_1 + \cdots + dx_n^2).$$

Here the boundary at infinity $\partial_\infty \mathrm{H}^n$ is the subspace $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_n) : x_n = 0\}$ complemented by a point ∞. An isomorphism $g\colon B^n \to C^n$ can be obtained by composing the inversion $\varphi_0\colon \mathbb{R}^n \to \mathbb{R}^n$ with respect to the sphere of radius 2 centered at $(-1, 0, \ldots, 0)$, and an appropriate similitude $\varphi_1\colon \mathbb{R}^n \to \mathbb{R}^n$. This is slightly different from the case $n = 2$, where g leaves invariant the orientation.

Finally, the metric

$$ds_D^2 = dt^2 + e^{-2t} dx^2$$

on the solvable group $S_n = \mathbb{R} \ltimes \mathbb{R}^{n-1}$ with multiplication

$$(t, x) * (t', x') = (t + t', x + e^t x')$$

is left invariant and has the constant sectional curvature -1, i.e., it gives another model of the hyperbolic space H^n. As an isomorphism $h\colon C^n \to D^n$ between the models C^n and D^n one can take the map given by

$$h(x_1, \dots, x_n) = (t = \ln x_n, x = (x_1, \dots, x_{n-1}))\,.$$

A.6 Möbius transformations

The boundary at infinity $\partial_\infty \mathrm{H}^{n+1}$ of the hyperbolic space H^{n+1} possesses a canonical conformal structure of the unit sphere $S^n \subset \mathbb{R}^{n+1}$. There is an important relation between the isometries of H^{n+1} and the Möbius transformations of the sphere S^n.

A.6.1 Inversions and isometries

Let $S_a(z) \subset \mathbb{R}^n$ be the sphere of radius a centered at z. The inversion $\varphi\colon \mathbb{R}^n \setminus z \to \mathbb{R}^n \setminus z$ with respect to $S_a(z)$ is defined by the condition that the point $\varphi(x)$ lies in the ray $(zx\rangle$ with vertex z which contains x and

$$|z\varphi(x)| \cdot |zx| = a^2.$$

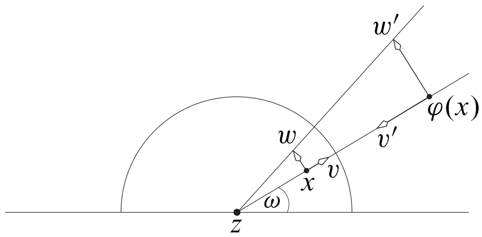
If one compactifies $\mathbb{R}^n$ by a point ∞, $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$, then φ is extended to the involution $\varphi\colon \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$, $\varphi(z) = \infty$, $\varphi(\infty) = z$. Note that $\widehat{\mathbb{R}}^n$ can be identified with the sphere $S^n \subset \mathbb{R}^{n+1}$ by the stereographic projection. We also define the inversion with respect to a hyperplane $E \subset \mathbb{R}^n$ as the reflection with respect to E (the hyperplane E can be viewed as the sphere $S_\infty(\xi)$ of infinite radius centered at $\xi \in \partial_\infty \mathbb{R}^n$).

Theorem A.6.1. *Let* $\mathrm{H}^{n+1} = \{(x_1, \dots, x_{n+1}) : x_{n+1} > 0\} \subset \mathbb{R}^{n+1}$ *be the upper half-space model of the hyperbolic space. Then the restriction of the inversion* φ *with respect to the sphere* $S_a(z) \subset \mathbb{R}^{n+1}$ *to* H^{n+1} *is an isometry of* H^{n+1} *for every* $z \in \partial_\infty \mathrm{H}^{n+1}$ *and every radius* a.

Proof. For $z = \infty$ this is obvious since such an inversion is the reflection with respect to the vertical hyperplane. Thus we assume that $z = (0, \dots, 0)$. Fix $x \in \mathrm{H}^{n+1}$ and show that the differential $d\varphi\colon T_x \mathrm{H}^{n+1} \to T_{\varphi(x)} \mathrm{H}^{n+1}$ is an isometry of corresponding tangent spaces.

Let $v \in T_x \mathrm{H}^{n+1}$ be a radial vector, i.e. tangent to the ray $(zx\rangle$, and let $w \in T_x \mathrm{H}^{n+1}$ be a tangential vector, i.e. tangent to the sphere centered at z and containing x. Denote by $v' = d\varphi(v)$, $w' = d\varphi(w) \in T_{\varphi(x)} \mathrm{H}^{n+1}$ the images of that vectors. Then v' is radial, and w' tangential. It follows from similarity of corresponding triangles that

$$\frac{|w'|_e}{|w|_e} = \frac{|z\varphi(x)|}{|zx|} = \frac{a^2}{|zx|^2} =: \lambda\,.$$



We find $\frac{|v'|_e}{|v|_e}$, assuming that $|v|_e = 1$. The expression

$$\frac{1}{t}(|z\varphi(x+t)| - |z\varphi(x)|) = \frac{a^2}{t}\left(\frac{1}{|z(x+t)|} - \frac{1}{|zx|}\right) = -\frac{a^2}{|zx| \cdot |z(x+t)|}$$

tends to $-\frac{a^2}{|zx|^2} = -\lambda$ as $t \to 0$. Thus $\frac{|v'|_e}{|v|_e} = \lambda$. This implies that $\varphi\colon \widehat{\mathbb{R}}^{n+1} \to \widehat{\mathbb{R}}^{n+1}$ is a conformal map, $|d\varphi(u)|_e = \lambda |u|_e$ for all tangent vectors at x.

Let $\omega \in (0, \pi/2]$ be the angle between the ray $(zx\rangle$ and the plane $x_{n+1} = 0$. Then the hyperbolic length $|u|_h = \frac{1}{|zx|\sin\omega}|u|_e$. Thus for the image u' of $u \in T_x\,\mathrm{H}^{n+1}$ under the differential $d\varphi$ we have

$$\begin{aligned}|u'|_h &= \frac{1}{|z\varphi(x)|\sin\omega}|u'|_e = \frac{|zx|}{a^2\sin\omega}|u'|_e \\ &= \frac{|zx|}{a^2\sin\omega}\lambda|u|_e = \frac{1}{|zx|\sin\omega}|u|_e = |u|_h.\end{aligned}$$

This proves that the inversion φ is an isometry of H^{n+1}. □

Since the sphere $\widehat{\mathbb{R}}^n$ coincides with the boundary at infinity $\partial_\infty \mathrm{H}^{n+1}$ of H^{n+1} in the upper half-space model, any inversion of $\widehat{\mathbb{R}}^n$ is extended to an inversion of the upper half-space H^{n+1}, which by Theorem A.6.1 is the reflection with respect to the corresponding hyperplane in the hyperbolic geometry. Recall that every isometry of H^{n+1} can be represented as the composition of finitely many reflections.

A *Möbius transformation* of $\widehat{\mathbb{R}}^n$ is by definition the composition of finitely many inversions. Thus Theorem A.6.1 implies

Corollary A.6.2. *Any Möbius transformation $\widehat{\varphi}\colon \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ is extended to an isometry $\varphi\colon \mathrm{H}^{n+1} \to \mathrm{H}^{n+1}$. Every isometry of H^{n+1} can be obtained in such a way.* □

A.7 Cross-ratio

Let $a, b, c, d \in \widehat{\mathbb{R}}^n$ be pairwise distinct points. Their (classical) cross-ratio is defined by

$$[a,b,c,d] = \frac{|ac| \cdot |bd|}{|ab| \cdot |cd|}$$

(distances are Euclidean). If one of the points coincides with ∞, then the factors containing it cancel out. For example,

$$[\infty, b, c, d] = \frac{|bd|}{|cd|}.$$

Theorem A.7.1. *The cross-ratio is invariant under any Möbius transformation.*

Proof. Assume that $a, b, c, d \in \widehat{\mathbb{R}}^n$ are pairwise distinct, and let $\varphi\colon \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ be Möbius. We have to check that

$$[\varphi(a), \varphi(b), \varphi(c), \varphi(d)] = [a, b, c, d].$$

One can assume that φ is the inversion with respect to the sphere $S_r(o)$. The triangles oab and $o\varphi(b)\varphi(a)$ are similar, since $|o\varphi(a)| = \frac{r^2}{|oa|}$, $|o\varphi(b)| = \frac{r^2}{|ob|}$ and thus

$$\frac{|o\varphi(a)|}{|o\varphi(b)|} = \frac{|ob|}{|oa|}.$$

Therefore,

$$\frac{|\varphi(a)\varphi(b)|}{|ab|} = \frac{|\varphi(a)o|}{|ob|} = \frac{|\varphi(b)o|}{|oa|}.$$

Then

$$|\varphi(a)\varphi(c)| = |ac|\frac{|\varphi(a)o|}{|oc|}; \quad |\varphi(b)\varphi(d)| = |bd|\frac{|\varphi(d)o|}{|ob|}$$

and

$$|\varphi(a)\varphi(b)| = |ab|\frac{|\varphi(a)o|}{|ob|}; \quad |\varphi(c)\varphi(d)| = |cd|\frac{|\varphi(d)o|}{|oc|}.$$

Thus

$$\begin{aligned}
[\varphi(a), \varphi(b), \varphi(c), \varphi(d)] &= \frac{|\varphi(a)\varphi(c)| \cdot |\varphi(b)\varphi(d)|}{|\varphi(a)\varphi(b)| \cdot |\varphi(c)\varphi(d)|} \\
&= \frac{|ac| \cdot |bd|}{|ab| \cdot |cd|} \frac{|\varphi(a)o|}{|oc|} \frac{|ob|}{|\varphi(a)o|} \frac{|\varphi(d)o|}{|ob|} \frac{|oc|}{|\varphi(d)o|} \\
&= [a, b, c, d].
\end{aligned}$$

The case when one of the points coincides with ∞ we leave as an exercise to the reader. $\square$

Theorem A.7.2. *Assume that a map $\varphi\colon \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ leaves invariant the cross-ratio. Then φ is Möbius.*

Proof. Composing with an appropriate Möbius transformation and using Theorem A.7.1, one can assume that $\varphi(\infty) = \infty$. Then $\varphi(\mathbb{R}^n) = \mathbb{R}^n$. Fix $c, d \in \mathbb{R}^n$ and consider the points $a, b \in \mathbb{R}^n$ as variables. Then

$$\begin{aligned}\frac{|ab|}{|cd|} &= \frac{[\infty, b, c, d]}{[a, b, \infty, d]} = \frac{[\varphi(\infty), \varphi(b), \varphi(c), \varphi(d)]}{[\varphi(a), \varphi(b), \varphi(\infty), \varphi(d)]} \\ &= \frac{[\infty, \varphi(b), \varphi(c), \varphi(d)]}{[\varphi(a), \varphi(b), \infty, \varphi(d)]} = \frac{|\varphi(a)\varphi(b)|}{|\varphi(c)\varphi(d)|}.\end{aligned}$$

Hence the ratio

$$\frac{|\varphi(a)\varphi(b)|}{|ab|} = \lambda = \frac{|\varphi(c)\varphi(d)|}{|cd|}$$

is independent of a, b, and thus φ is a similitude. $\square$

A.7.1 Cross-ratio in hyperbolic geometry

Consider the sphere $\widehat{\mathbb{R}}^n$ as the boundary at infinity $\partial_\infty \mathrm{H}^{n+1}$. Any distinct points $a, b \in \widehat{\mathbb{R}}^n$ are connected by the unique geodesic $ab \subset \mathrm{H}^{n+1}$.

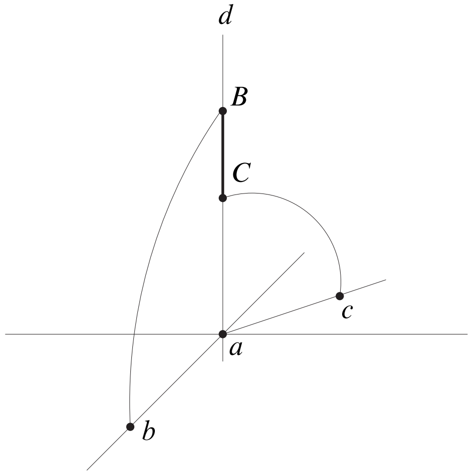


Figure A.3. Cross-ratio.

Theorem A.7.3. *Given pairwise distinct points $a, b, c, d \in \widehat{\mathbb{R}}^n$ let B, $C \in ad \subset \mathrm{H}^{n+1}$ be the projections of b, c respectively to the geodesic ad. Then for the hyperbolic distance between B and C we have*

$$|BC|_h = |\ln [a, b, c, d]|.$$

Proof. One can assume that $d = \infty$. Then in the upper half-space model the points B and C lie in the vertical ray $[a, d)$. Since the geodesics in this model are either vertical rays or vertical half-circles centered at infinity, we see that the Euclidean distances $|aB| = |ab|$ and $|aC| = |ac|$. Thus

$$|BC|_h = \left|\ln \frac{|ac|}{|ab|}\right| = |\ln[a, b, c, d]|.$$

$\square$

Historical note. The models B^n, C^n, D^n of the hyperbolic space H^n for $n \geq 2$ appeared among others in 1868 in a memoir by M. Beltrami, [Belt]. That is 14 years before the famous papers by H. Poincaré on Fuchsian functions, where the unit disk and the upper half-space models in dimensions 2 and 3 appeared, fractional linear transformations as product of inversions were represented and interpreted as motions of two and three dimensional hyperbolic geometry. For more details see [Po] and [Mi].

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