Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2019-2020

Mathematical Analysis
Seminar 8

The Jacobian matrix and determinant. Let $h : \Delta \subseteq \mathbb{R}^n \to D \subseteq \mathbb{R}^n, h(x) = (h_1(x), \ldots, h_n(x))$ with all h_i admitting continuous partil derivative. Then, the Jacobian matrix of h is

$$J_h(x_1, \ldots, x_n) = \begin{pmatrix} \dfrac{\partial h_1}{\partial x_1}(x_1, \ldots, x_n) & \ldots & \dfrac{\partial h_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & & \vdots \\ \dfrac{\partial h_n}{\partial x_1}(x_1, \ldots, x_n) & \ldots & \dfrac{\partial h_n}{\partial x_n}(x_1, \ldots, x_n) \end{pmatrix}.$$

The determinant of the Jacobian matrix plays a crucial role in

Theorem (Change of variables). Let $D, \Delta \subseteq \mathbb{R}^n$ be two closed, bounded and Jordan measurable sets, while $M \subseteq \Delta$ is a set of zero Jordan measure. Also consider a map $h : \Delta \to D$ as above such that it is injective on $\Delta \setminus M$ and $\det J_h(x) \neq 0$ for all $x \in \Delta \setminus M$. If $f : D \to \mathbb{R}$ is continuous then

$$\int_D f(x)\mathrm{d}x = \int_\Delta f(h(y)) \det J_h(y)\mathrm{d}y.$$

1. Compute $I = \displaystyle\iint_D \sqrt{x^2 + y^2}\mathrm{d}x\mathrm{d}y$, where $D = \big\{(x, y) \in \mathbb{R}^2 : 2x \leq x^2 + y^2 \leq 4x,\ y \geq 0\big\}$.

2. Compute $I = \displaystyle\iiint_D \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\sqrt{x^2 + y^2 + z^2}}$, where $D = \big\{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 4,\ z \geq 0\big\}$.

3. Find the volume of

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : z \geq x^2 + y^2,\ (z - 2)^2 \geq x^2 + y^2,\ z \leq 2\right\}.$$

4. Compute $I = \displaystyle\iint_D \sqrt{x^2 + y^2}\mathrm{d}x\mathrm{d}y$, where D is delimited by

$$xy = 1,\ xy = 2,\ x^2 - y^2 = 1,\ x^2 - y^2 = 4.$$

[Hint: use the change of variables defined by $u = xy$, $v = x^2 - y^2$]