3.10.2019

Algebra Course 1

Linear Algebra

Structure : Chapter 1 - Preliminaries
Chapter 2 - Vector spaces
Chapter 3 - Matrices and Linear Sistems
Chapter 4 - Coding Theory - introduction

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Seminar : - min. attendance : 75%

- bonus points - up to 0.5p (5x0.1p)
- bonus projects - course : up to 1p (5x0.2p)

Exam

Partial exam 1: Week 8

Partial exam 2: Week 14

Final grade:

$$G = 1 + \underset{4p}{P_1} + \underset{5p}{P_2} + \underset{\text{bonus}}{B}$$

Chapter 1 Preliminaries

1 Relations

Def By a (binary) relation we mean a triple:

$r = (A, B, R)$, where A, B are sets and $R \subseteq A \times B$

A — domain, B — codomain, R — graph; $A \times B = \{(a,b) \mid a \in A, b \in B\}$

if $A = B$ then r is called homogeneous

Def Let $r = (A, B, R)$ be a relation and $X \subseteq A$

then $r(X) \overset{not}{=} \{b \in B \mid \exists x \in X : (x, b) \in R\}$

called the relation class of X with respect to R

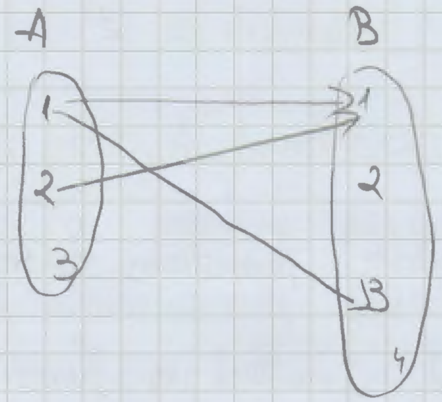
if $X = \{x\}$ then we denote $r\langle x\rangle \overset{not}{=} r(\{x\}) =$
$= \{b \in B \mid (x, b) \in R\}$

Notation $(a, b) \in R \equiv a \; r \; b$

Remark: In case of relations defined on finite sets we may use diagrames to picture that

$r = (A, B, R)$ $\quad A = \{1, 2, 3\}$ $\quad B = \{1, 2, 3, 4\}$

$R = \{(1,1), (1, 3), (2, 1)\}$



$r\langle 1\rangle = \{1, 3\}$

$r(\{1,2\}) = \{1,3\}$

Ex a) $r = (C, P, R)$

C – the set of children

P – the set of parents

$R = \{(c,p) \in C \times P \mid c \text{ is a child of } p\}$

b) $r = (\mathbb{R}, \mathbb{R}, R)$

$R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$

c) divisibility of – on $\mathbb{N}, \mathbb{Z}$

paralelism – $\parallel$, $\perp$ on lines, $\equiv$ $\sim$ on Δ

d) $\varnothing(A, B, \varnothing)$ – the void relation

$u = (A, B, A \times B)$ – the universal relation

e) $\delta_A = (A, A, \Delta_A)$

$\Delta_A = \{(a,a) \mid a \in A\}$ – the equality relation

f) Every function is a relation

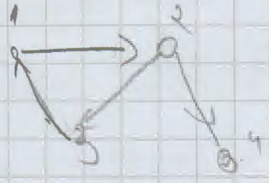
$f: A \to B \rightsquigarrow (A, B, G_f)$

where $G_f = \{(a, f(a)) \mid a \in A\}$

g) Every directed graph is a relation

V – set of vertices

T - set of vertices; E - set of edges -> (V, V, E)



$V = \{1,2,3,4\}$

$E = \{(1,2), (2,3), (2,4), (3,1)\}$

$\boxed{2}$ Functions – how do we relate functions to relations?

Def: A relation $r = (A, B, R)$ is called a function if ~~for elem~~ $\forall a \in A$, $|r\langle a\rangle| = 1$

number of elem $r\langle a\rangle$

$f : A \to B$

$\forall a \in A \quad |f\langle a\rangle| = 1$

this unique elem. will be $f(a)$

Ex – relation that is not a function, check Remark

Homework: recall: injective, surjective, bijective

$\boxed{3}$ Equivalence relations and partitions

Def Equivalence relations

$r = (A, A, R)$ is called an equivalence relation if it take the following 3 properties

(r) reflexivity: $\forall a \in A$, $a \, r \, a$

(t) transitivity: $a, b, c \in A$ with $a \, r \, b$ and $b \, r \, c$, we have $a \, r \, c$

(s) symetry $\forall a, b \in A$ with $a \, r \, b$, we have $b \, r \, a$

Notation $E(A)$ – the set of equivalence relations on a set A

a. $\delta_A = (A, A, \Delta_A) \in E(A)$ equality

b. $\equiv$ of triangles is ~~an~~ equivalence relation on a set of Δ

<u>Def</u> Let $A \neq \emptyset$ be a set

By a ~~partion~~ partition on A we mean a family $(A_i)_{i\in I}$ of non-empty subsets of A such that:

- $\bigcup_{i\in I} A_i = A$
- $\forall\, i,j \in I,\ i \neq j$, we have $A_i \cap A_j = \emptyset$

Notation: $P(A)$ the set of all partitions on a set A

<u>Ex</u> (a) $A = \{1,2,3,4\}$

$\{A_1, A_2, A_3\}$ is a partition of A if:

$A_1 = \{1,2\}$

$A_2 = \{3\}$

$A_3 = \{4\}$

(b) O - the set of odd integers

E - the set of even integers

$\{O, E\}$ is a partition of $\mathbb{Z}$

(c) $\{\{x\} \mid x \in \mathbb{Z}\}$ - is a partition of $\mathbb{Z}$

Theorem:

(1) Let r be $r \in E(A)$

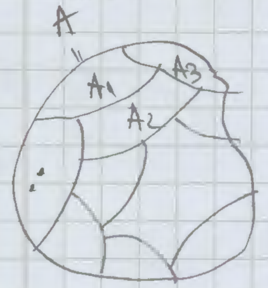
Denote $A/r \overset{not}{=} \{r\langle a\rangle \mid a \in A\}$ - the quotion set of a by r

then $A/r \in P(A)$

(2) Let $\pi = (A_i)_{i\in I} \in P(A)$. Define a relation r_π on A by

def

$x, y \in A \ , \ x \, r_{\bar{\pi}} \, y \overset{\text{def}}{\iff} \exists \, i \in I : x, y \in A_i$



Then $r_{\bar{\pi}}$ is an equivalent relation on set A

$r_{\bar{\pi}} \in E(A)$

(3) There exists a bijection

$F : E(A) \to P(A) \ , \quad F(r) = A/r$

with inverse

$G : P(A) \to E(A) \ , \ G(\bar{\pi}) = r_{\bar{\pi}}$

10.10.2019

Course 2

$\boxed{4}$ Operations (composition law)

Definition : by an operation or composition law on a set A we mean a function

$f : A \times A \to A$

Example : "+" is an operation on all numerical sets

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

"$-$" is an op. on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

"$\cdot$" is an op on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

"$/$" is an op on $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

Definition: Let $(A, \cdot)$ be a set together with an operation

(Associative law): $\forall a, b, c \in A \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$

(Commutative law) $\forall a, b \in A \quad a \cdot b = b \cdot a$

(Identity law) $\exists \, e \in A$ s.t $a \cdot e = e \cdot a = a$

(Symmetric law) For any a from A exists a' from A such that a*a'=a'*a=e, where exists e from A identity element

Lemma: Let $(A, \cdot)$

(i) The identity element is unique

(ii) Assume that " $\cdot$ " is associative and $a \in A$ has a symmetric. Then a has a unique symmetric

Definition let $(A, \cdot)$ and $B \subseteq A$

Then B is called a stable subset (B is closed under " $\cdot$ " in A) is $\forall b_1, b_2 \in B$, $b_1 \cdot b_2 \in B$

Another point of view

$\varphi : A \times A \to A$, $B \subseteq A$ stable subset

$\forall b_1, b_2 \in B$, $\varphi(b_1, b_2) \in B$

$\Rightarrow \varphi' = \varphi|_{B \times B} : B \times B \to B \Rightarrow \varphi|_{B \times B}$ is an op. on B

Remark: Assoc. law, commutative law transfer to stable subsets (they are defined by using only the quantifier $\forall$)

ex $(\mathbb{Z}, +)$, $\mathbb{N} \subseteq \mathbb{Z}$ is a stable subset

$\boxed{5}$ Groups and rings

Definition let $(A, \cdot)$

Then it is called:

(i) semigroup if " $\cdot$ " associative

(ii) monoid if " $\cdot$ " associative and $\exists$ identity elem.

(iii) group if " $\cdot$ " associative

$\exists$ identity elem.

all elem. are symmetrical

have a symmetric (inverse

If "$\cdot$" is also commutative, we have a commutative semigroup, monoid, group

A commutative group is also called <u>abelian</u>.

Remark: The identity elem. will usually be denoted by 1 and the inverse of an elem. $a \in A$ will usually be denoted by a^{-1}.

Examples: (a) "$-$" is ~~not~~ associative on $\mathbb{Z}$

(b) $(\mathbb{N}^*, +)$ is a semigroup but not a monoid

(c) $(\mathbb{N}, +)$ is a monoid but not a group

(d) $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ are groups.

$(\mathbb{Q}^*, \cdot)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$ are groups

(e) Let $A = \{e\}$ be a single-elem. set

exists unique $\exists!$ op on A defined by $e \cdot e = e$

$(\{e\}, \cdot)$ is a group called the trivial group

(f) Let $n \in \mathbb{N}$, $n \geq 2$. Then

$(\mathbb{Z}_n, +)$ is an abelian group

where $\forall \hat{x}, \hat{y} \in \mathbb{Z}_n$,

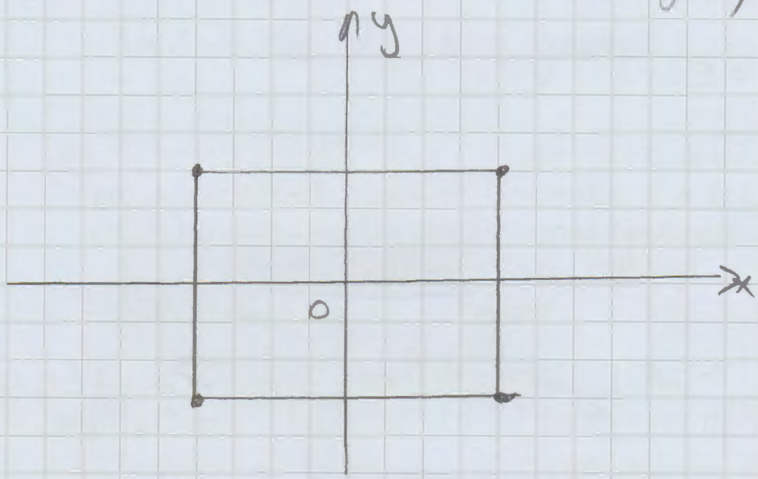
$$\hat{x} + \hat{y} = \widehat{x+y}$$

This is called - the group of residue classes modulo n

(g) Let $K = \{e, a, b, c\}$ and consider the operation given by table:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\Rightarrow$ $(K, \cdot)$ is an abelian group Klein's group



Definition let $(A, +, \cdot)$ (a set A together with 2 op denoted by "$+$", "$\cdot$")

Then it is called a:

(i) Ring if $\begin{cases} (A,+) \text{ is an abelian group} \\ (A,\cdot) \text{ is a semigroup} \\ \forall a,b,c \in A \quad a\cdot(b+c) = a\cdot b + a\cdot c \\ \qquad\qquad\qquad (b+c)\cdot a = b\cdot a + c\cdot a \end{cases}$

(distributive laws)

(ii) unitary ring (ring with identity)

if $(A,+,\cdot)$ ring where $(A,\cdot)$ monoid

we denote by 1 the identity elem)

exp $\to$ (iii) division ring (or skew field)

if $\begin{cases} (A,+) \text{ abelian group (w. identity 0)} \\ (A^*, \cdot) \text{ group } (A^* = A \text{ without the identity elem}) \\ \qquad\qquad\qquad A \setminus \{0\} \\ \text{Distributive laws} \end{cases}$

(iv) field (comp comutativ) if

it is a commutative division ring

Ex (a) $(\mathbb{Z}, +, \cdot)$ is a commutative unitary ring

(b) $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ fields

(c) Let $A = \{e\}$ be a single elem set

we define $e + e = e$

$e \cdot e = e$

$\Rightarrow (\{e\}, +, \cdot)$ is a commutative unitary ring called a trivial ring

(d) Let $n \in \mathbb{N}$, $n \geq 2$. we define on $\mathbb{Z}_n$ the op

$$\hat{x} + \hat{y} = \widehat{x+y} \qquad \forall \hat{x}, \hat{y} \in \mathbb{Z}_5$$
$$\hat{x} \cdot \hat{y} = \widehat{x \cdot y}$$

then $(\mathbb{Z}_n, +, \cdot)$ is a commutative unitary ring

(note that $(\mathbb{Z}_n, +, \cdot)$ field $\Leftrightarrow$ n prime)

(e) Let $(R, +, \cdot)$ be a commutative unitary ring != {0}

Then $(R[X], +, \cdot)$ is a commutative unitary ring

polynomials with coefficients in R

(f) Let $(R, +, \cdot)$ be a ring, $n \geq 2$ $n \in \mathbb{N}$

Then $(M_n(R), +, \cdot)$ is a ring

$\boxed{6}$ Subgroups and subrings

Def Let $(G, \cdot)$ be a group

Then $H \subseteq G$ is called a subgroup (denote $H \leq G$) if

$$\begin{cases} H \neq \emptyset \quad (1 \in H) \\ \forall x, y \in H, \ x \cdot y \in H \\ \forall x \in H, \ x^{-1} \in H \end{cases}$$

Theorem The following are equivalent for a group $(G, \cdot)$ and $H \leq G$

(1) $H \leq G$

(2) i) $H \neq \emptyset$ $(1 \in H)$

(2) i) H != null set, (1 belongs to H)
ii) For any x, y from H, x*y^-1 is also in H

(3) (i) H stable subset

(ii) $(H, \cdot)$ is a group

<u>ex</u>: (a) Let $(G, \cdot)$ be a group. Then $\{1\}$ and G are subgroups of $(G, \cdot)$

(b) $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$

Definition Let $(R, +, \cdot)$ be a ring

Then $A \subseteq R$ is called a subring (denoted $A \leq R$) if

$$\begin{cases} A \neq \emptyset & (0 \in A) \\ \forall x, y \in A, & x - y \in A \\ \forall x, y \in A, & x \cdot y \in A \end{cases}$$

Let $(K, +, \cdot)$ be a field. Then $A \subseteq K$ is a subfield (denoted $A \leq K$) if

$$\begin{cases} |A| \geq 2 & (0, 1 \in A) \\ \forall x, y \in A, & x - y \in A \\ \forall x, y \in A \text{ with } y \neq 0, & x \cdot y^{-1} \in A \end{cases}$$

Theorem: Let $(R, +, \cdot)$ be a ring (field). Then $A \subseteq R$ is a subring (subfield) if:

(i) A stable subset of $(R, +, \cdot)$

(ii) $(A, +, \cdot)$ ring (field)

Example (a) Let $(R, +, \cdot)$ be a ring. Then $\{0\}$, R are subrings of $(R, +, \cdot)$

(b) $\mathbb{Z}$ is a subring of $(\mathbb{Q}, +, \cdot)$

$\mathbb{Q}$ is a subfield of $(\mathbb{R}, +, \cdot)$

(c) $2\mathbb{Z} \overset{\text{not}}{=} \{2k \mid k \in \mathbb{Z}\}$ is a subring of $(\mathbb{Z}, +, \cdot)$ without identity

HW - recall group/ring homo morphism (isomorphism)

Course 3 17. 10-2019

Chapter 2: Vector spac

1 Basic definition, examples and properties

Def Let $(K, \oplus, \cdot)$ be a field

By a K-vector space (or K-linear space, or vector spac over K) we mean: an abelian group $(V, \oplus)$ together with so-called "external operation"

+ not the same

$$\varphi : K \times V \to V$$

$$\varphi(k, v) \overset{not}{=\!=} k \cdot v \quad ! \text{ not-commutative}$$

satisfying the axioms:

$(L1)$ $k \cdot (v_1 + v_2) = k \cdot v_1 + k \cdot v_2$

$(L2)$ $(k_1 + k_2) \cdot v = k_1 v + k_2 \cdot v$

(L_3) $(k_1 \cdot k_2) \cdot v = k_1 \cdot (k_2 \cdot v)$

(L_4) $1 \cdot v = v$

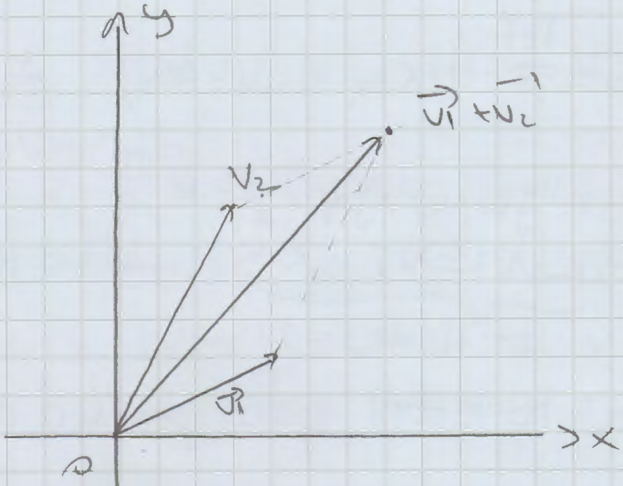
$\forall\, k, k_1, k_2 \in K,\ \forall\, v, v_1, v_2 \in V$

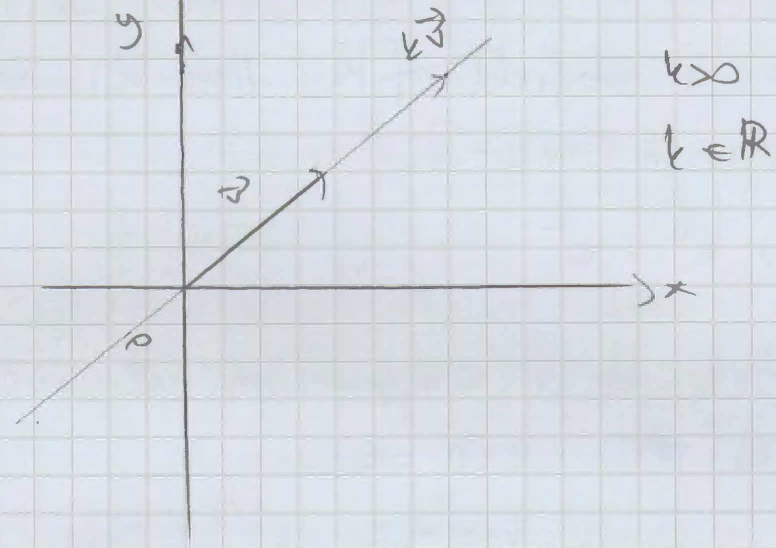
The elements of V are called vectors and the elem of K are called <u>scalars</u>.

Notation ${}_K V$ (or $(V, K, +, \cdot)$)

left K vect spaces

<u>Ex</u> (a) Let V_2 be the set of all vectors (from Physics) in plane having the origin in a fixed point O of the plane





$$\left.\begin{array}{l}\vec{v_1} \longleftrightarrow (x_1; y_1) \\ \vec{v_2} \longleftrightarrow (x_2; y_2)\end{array}\right\} \Rightarrow \vec{v_1}+\vec{v_2} \longleftrightarrow (x_1+x_2, y_1+y_2)$$

$$\vec{v} \longleftrightarrow (x,y) \Rightarrow k\cdot\vec{v} \longleftrightarrow (k\cdot x, k\cdot y)$$

$V_2 \longleftrightarrow \mathbb{R}\times\mathbb{R} \overset{\text{not}}{=} \mathbb{R}^2$; V_2 is an $\mathbb{R}$ vector space

Similarly, the set V_3 of all vectors in space having the origin in a fixed point O form an $\mathbb{R}$-vector space

(b) Let K be a field and $n\in\mathbb{N}^*$. We define

- $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \overset{\text{def}}{=}$

$= (x_1+y_1, \dots, x_n+y_n)$

• $k \cdot (x_1, \dots, x_n) \overset{def}{=} (k \cdot x_1, \dots, k \cdot x_n)$

$\forall (x_1, \dots, x_n), (y_1, \dots, y_n) \in K^n$ ← V dif'ni'tie

(where $K^n = \underbrace{K \times K \times \dots \times K}_{n \text{ times}}$), $\forall k \in K$

Then K^n is a K-vector space, called the canonical K-vector space

(c) Let A be a subfield of K. Then K is an A-vector space (the op are the same ; "+")

eg. ${}_{\mathbb{Q}}\mathbb{R}$, ${}_{\mathbb{R}}\mathbb{C}$, ${}_{\mathbb{Q}}\mathbb{C}$

(d) Let $V = \{e\}$ be a single-elem. set and K a field. We define: $e + e = e$

$k \cdot e = e$, $\forall k \in K$

Then $V = \{e\}$ is a K-vector space, called the trivial K vector space, and we denote it by $\{0\}$

(e) Let $m, n \in \mathbb{N}$ $m, n \geq 2$

Define on $M_{m,n}(K)$

(the set of all matrices $m \times n$ with entries in K) the usual addition and multiplication by a scalar

Then $M_{m,n}(K)$ is a K-vector space

(f) Let $K[x]$ be the set of all polynomials with coefficients in a field K. Define on $K[x]$ the usual addition and scalar multiplication

Then $K[x]$ is a K vector-space

<u>Theorem</u> Let V be a K-vector space. Then:

(i) $k \cdot 0_V = 0_K \cdot v = 0_V$

(ii) $k \cdot (-v) = (-k) \cdot v = -k \cdot v$

(iii) $k \cdot (v - v') = kv - k \cdot v'$ $\quad \forall k, k_1, k_2 \in K$

(iv) $(k_1 - k_2) \cdot v = k_1 v - k_2 \cdot v$ $\quad \forall v, v_1, v_2 \in V$

Proof (i) • $k \cdot 0 + k \cdot v \overset{L_4}{=} k \cdot (0 + v) =$

$= k \cdot v$

$k \cdot 0 + k \cdot v = k \cdot v \quad | -k \cdot v$

$k \cdot 0 = 0$

• $0 \cdot v + k \cdot v \overset{L_2}{=} (0 + k) \cdot v = k \cdot v$

$0 \cdot v + k \cdot v = k \cdot v \quad | -k \cdot v$

$0 \cdot v = 0$

(ii) • $k \cdot (-v) + k \cdot v \overset{L_1}{=} k(-v + v) = k \cdot 0 \overset{(i)}{=} 0$

$k \cdot (-v) + k \cdot v = 0 \quad | -k \cdot v$

$k \cdot (-v) = -k \cdot v$

• $(-k) \cdot v + k \cdot v = (-k + k) \cdot v = 0 \cdot v \overset{(i)}{=} 0.$

$(-k) \cdot v + k \cdot v = 0 \quad + kv$

$(-k) \cdot v = -kv$

(iii) $k \cdot (v - v') + k \cdot v' \overset{L_1}{=} k(v - v' + v') = k \cdot v$

$k \cdot (v - v') + k \cdot v' = k \cdot v$

$k(v - v') = k \cdot v - k \cdot v'$

(iv) Similarly

Theorem 2 Let V be a K-vector space and $v \in V, k \in K$

Then $k \cdot v = 0 \iff k = 0$ or $v = 0$

(no 0 divisors)

proof „$\Leftarrow$" By theorem 1

„$\Rightarrow$" Assume that $k \cdot v = 0$

if $k=0$, then we are done

assume $k \neq 0 \Rightarrow \exists k^{-1} \in K$

$k \cdot v = 0 \quad | \cdot k^{-1}$

$k^{-1}(k \cdot v) = k^{-1} \cdot 0$

$(k^{-1} \cdot k) \cdot v = 0$ by theorem 1 and L_3

$1 \cdot V = 0$

$\overset{L(4)}{\Longrightarrow} \quad V = 0$

[2] Subspaces

<u>Def</u> Let V be a K vector space and $S \subseteq V$

Then S is called a subspace of $_K V$ iff

$$\begin{cases} S \neq \emptyset \\ \forall v_1, v_2 \in S \quad v_1 + v_2 \in S \\ \forall k \in K, \forall v \in S, k \cdot v \in S \end{cases} \quad \text{not } S \leq_K V$$

Theorem Let V be a K-vector space and $S \subseteq V$

Then S is a subspace of $V \Leftrightarrow$

$$\begin{cases} S \neq \emptyset \quad (0 \in S) \\ \forall k_1, k_2 \in K, \forall v_1, v_2 \in S, k_1 \cdot v_1 + k_2 \cdot v_2 \in S \end{cases}$$

$\Leftrightarrow$ $\begin{cases} S \text{ is a stable subset of } V \text{ w.r.sp to "+" \& "}\cdot\text{" } \\ \text{scalars} \\ S \text{ is a } K\text{-vector space} \end{cases}$

Example: (a) Let V be a K-vectorspace. Then $\{0\}$ and V are subspaces of V

(b) Consider the real vector space V_2

Its subspaces are the following:

$\begin{cases} \bullet \ \{0\} \\ \bullet \ \text{any line passing through (the origin) } O \\ \bullet \ V_2 \end{cases}$

The subspaces of V_3:

$\begin{cases} \bullet \ \{0\} \\ \bullet \ \text{Any line passing through } O \\ \bullet \ \text{any plane passing through } O \\ \bullet \ V_3 \end{cases}$

(c) Let $n \in \mathbb{N}$ Denote

$$K_n[x] = \{ f \in K[x] \mid \text{degree}(f) \leq n \}$$

Then $K_n[x] \leq_K K[x]$

Theorem Let V be a K-vector space and let $(S_i)_{i \in I}$ be a family of subspaces of V

Then $\bigcap_{i \in I} S_i \leq_K V$

Proof • $0 \in \bigcap_{i \in I} S_i$

Let $k_1, k_2 \in K$ and $v_1, v_2 \in \bigcap_{i \in I} S_i \Rightarrow \left. \begin{array}{l} v_1, v_2 \in S_i, \ \forall i \in I \\ S_i \leq_K V \end{array} \right\} \Rightarrow$

th

$\Rightarrow k_1 \cdot v_1 + k_2 \cdot v_2 \in S_i \quad \forall i \in I \Rightarrow k_1 \cdot v_1 + k_2 \cdot v_2 \in \bigcap_{i \in I} S_i$

Hence set intersection Si, i from I <= kV (intersection = reversed U)

Rk: The union of subspaces is <u>NOT</u> a subspace in general

- for instance, take the union of 2 lines passing through O in V_2

24. 10. 2019

<u>Course 4</u>

<u>General problem</u> $V \to K$ v.s. X vectors in V

Given a set of vectors X in a K vector space V, complete it in a minimal way with some other vectors in order to get a subspace of V

<u>Def</u>: Let V be a K-vector space and $X \subseteq V$

Then, we denote

$$\langle X \rangle \overset{not}{=} \bigcap \{ S \leq_K V \mid X \subseteq S \}$$

Note that $\langle X \rangle \leq_K V$ because it is an intersection of subspaces of V. Also, note that $\langle X \rangle$ is the "smallest" (with respect to inclusion) subspace of V containing X

$\langle X \rangle$ - the subspace generated by X. In this setting, X is called a generating set for $\langle X \rangle$.

If $V = \langle X \rangle$ for some $X \subseteq V$, then V is said to be generated by X

If X is finite, then V is called finitely generated

When $X = \{ a \}$, then we denote $\langle a \rangle = \langle \{ a \} \rangle$

$X = \{ v_1, \dots, v_n \}$ $\quad \langle v_1, \dots, v_n \rangle = \langle \{ v_1, \dots, v_n \} \rangle$

Remark: $<\emptyset> = \{0\}$

Theorem: Let V be a K-vector space and $\emptyset \neq X \subseteq V$. Then:

$$<X> = \{ k_1 \cdot v_1 + \dots + k_n \cdot v_n \mid k_1, \dots, k_n \in K, \; v_1, \dots, v_n \in X, \; n \in \mathbb{N}^* \}$$

- the set of all finite linear combinations of vectors from X $(k_1 \cdot v_1 + \dots + k_n \cdot v_n)$

Proof: Denote $L = \{ k_1 \cdot v_1 + \dots + k_n \cdot v_n / k_1, \dots, k_n \in K, v_1, \dots, v_n \in X, n \in \mathbb{N}^* \}$

In order to show that $<X> = L$, it is enough to prove that L is the smallest subspace of V containing X.

Step 1: $L \leq_K V$

- $L \neq \emptyset$, because $\exists\, v \in X \neq \emptyset$ and $0 = 0 \cdot v \in L$
- Let $k, k' \in K$ and $v, v' \in L$.

We prove that $k \cdot v + k' \cdot v' \in L$

We have $v = \sum_{i=1}^{n} k_i \cdot v_i$, $v' = \sum_{j=1}^{m} k_j' \cdot v_j$

$\Rightarrow k \cdot v + k' \cdot v' = \sum_{i=1}^{n} (k \cdot k_i) \cdot v_i + \sum_{j=1}^{m} (k' \cdot k_j') \cdot v_j \in L$

Hence $L \leq_K V$

Step 2: $X \subseteq L$

We have $\forall v \in X \neq \emptyset$

$v = 1 \cdot v \in L$

Hence $X \subseteq L$

Step 3: We show that if $S \leq_K V$ with $X \subseteq S$, then $L \subseteq S$

Let $S \leq_K V$ with $X \leq S$

$\Rightarrow \forall v_1, \ldots, v_n \in X, \forall k_1, \ldots k_n \in K$,
we have $k_1 \cdot v_1 + \ldots + k_n \cdot v_n \in S$ because $S \leq_K V$

Hence any finite linear combination of vectors from X belongs to S

$\Rightarrow L \subseteq S$

Corollary: Let V be a K-vector space and $v_1, \ldots, v_n \in V$

Then $\langle v_1, \ldots, v_n \rangle = \{ k_1 v_1 + \ldots + k_n v_n \mid k_1, \ldots, k_n \in K \}$

Example: Consider the canonical (real) vector space $\mathbb{R}^3$ and $v_1 = (1,0,0)$
$v_2 = (0,1,0)$
$v_3 = (0,0,1)$

$\Rightarrow \langle v_1, v_2, v_3 \rangle = \{ k_1 \cdot v_1 + k_2 v_2 + k_3 \cdot v_3 \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$= \{ k_1 (1,0,0) + k_2 (0,1,0) + k_3 (0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$= \{ (k_1, k_2, k_3) \mid k_1, k_2, k_3 \in \mathbb{R} \} = \mathbb{R}^3$

Hence $\mathbb{R}^3$ is a finitely generated $\mathbb{R}$-vector space

General problem:

Decompose a vector space into subspaces

Def Let V be a K vector-space and $S,T \leq_K V$

Then $S+T \overset{not}{=} \{ s+t / s\in S,\ t\in T \}$

is called the sum of S and T

Theorem Let V be a K-vector space and $S,T \leq_K V$

Then $S+T = \langle S \cup T \rangle$ In particular, $S+T$ is a subspace of V $(S+T \leq_K V)$

Proof: $\boxed{\subseteq}$ Let $v \in S+T \Rightarrow v = s+t$ for some $s\in S$ and $t \in T$

$v = 1\cdot s + 1\cdot t \in \langle S \cup T \rangle$

$\boxed{\supseteq}$ Let $v \in \langle S \cup T \rangle$ Then $v = \sum_{i=1}^{n} k_i \cdot v_i$ for some

$v_i \in S \cup T$

Denote $I = \{ i \in \{1,\dots,n\} / v_i \in S \}$, $J = \{1,\dots,n\} \setminus I$

$v = \underbrace{\sum_{i\in I} k_i \cdot \underset{\underset{S}{\cap}}{v_i}}_{\underset{S}{\cap}} + \underbrace{\sum_{j\in J} k_j \cdot \underset{\underset{T}{\cap}}{v_j}}_{\underset{T}{\cap}} \in S+T$ (we've used $S,T \leq_K V$)

Def: Let V be a K-vector space and $S,T \leq_K V$

Then we denote $V = S \oplus T$ if $V = S+T$ and $S\cap T = \{0\}$

In this case, we say that V is the <u>direct sum</u> of S,T

Theorem: Let V be a K-vector-space and $S,T \leq_K V$

Then $V = S \oplus T \Leftrightarrow \forall v \in V, \exists!\ s \in S$ and $t \in T$ s.t. $v = s+t$

Proof: $\boxed{\Longrightarrow}$ Suppose that $V = S \oplus T \Rightarrow$

$\Rightarrow V = S + T$ and $S \cap T = \{0\}$

$\Rightarrow \forall v \in V$, $\exists s \in S$ and $t \in T$ such that $v = s + t$

For uniqueness, assume that $\exists s' \in S$ and $t' \in T$ such that
$v = s' + t'$

$$\Rightarrow s + t = s' + t' \Rightarrow \underbrace{s - s'}_{\in S} = \underbrace{t' - t}_{\in T} \in S \cap T = \{0\}$$

$$\Rightarrow \begin{cases} s = s' \\ t = t' \end{cases}$$

$\boxed{\Longleftarrow}$ suppose that $\forall v \in V, \exists! s \in S, t \in T$ s.t
$v = s + t$

We show that $S \cap T = \{0\}$

let $v \in S \cap T$

$$v = \underset{\in S}{v} + \underset{\in T}{0} = \underset{\in S}{0} + \underset{\in T}{v} \xrightarrow{\text{unigness}} v = 0 \Rightarrow S \cap T = \{0\}$$

Example: let $S = \{(x, 0) / x \in \mathbb{R}\}$ Then $\mathbb{R}^2 = S \oplus T$
$T = \{(0, y) / y \in \mathbb{R}\}$

$$\forall (x, y) \in \mathbb{R}^2, (x, y) = \underset{\in S}{(x, 0)} + \underset{\in T}{(0, y)} \in S + T$$

$S \cap T = \{(0, 0)\}$

Using the theorem: $\forall (x, y) \in \mathbb{R}^2$, $\exists!$ $s \in S$ and $t \in T$
s.t $(x, y) = s + t$

$(x, y) = (a, 0) + (0, b)$

$(x, y) = (a, b)$

3 Linear maps

Definition: Let V and V' be K vector spaces. Then $f: V \to V'$ is called a K-linear map if:

$$\begin{cases} \forall v_1, v_2 \in V,\ f(v_1+v_2) = f(v_1) + f(v_2) \\ \forall k \in K,\ \forall v \in V,\ f(k \cdot v) = k \cdot f(v) \end{cases}$$

A K-linear map $f: V \to V'$ is called

- isomorphism if it is bijective
- endomorfism if $V = V'$
- automorfism if $V = V'$ and f bijective

Notation

- We denote $V \simeq V'$ if $\exists$ isomorphism between V and V'
- $End_K(V)$ - the set of all endomorphisms of V
- $Aut_K(V)$ - the set of all automorphisms of V

Remark. Every K-linear map $f: V \to V'$ is a group homomorphism between the abelian groups $(V, +)$ and $(V', +)$

$\Rightarrow f(0) = 0'$ and $f(-v) = -f(v)$ $\forall v \in V$

Theorem $f: V \to V'$ is a K-linear map $\Leftrightarrow \forall k_1, k_2 \in K$, $\forall v_1, v_2 \in V$ $f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$

Example: Let $S \leq_K V$. Then $i: S \to V$, $i(u) = u$ is a K linear map called the inclusion K-linear map

Course 5

Recall:

Definition: Let $f: V \to V'$ be a function beetwen K-vector spaces V and V'. Then f is called a K-linear map if $\forall v_1, v_2 \in V$, $f(v_1+v_2) = f(v_1) + f(v_2)$

$\forall k \in K, \forall v \in V, f(k \cdot v) = k \cdot f(v)$

$\Leftrightarrow \forall k_1, k_2 \in K, \forall v_1, v_2 \in V, f(k_1 v_1 + k_2 v_2) = k_1 f(v_1)+k_2 f(v_2)$

Definition: Let $f: V \to V'$ be a K-linear map. Then

$\mathrm{Ker} f \overset{not}{=} \{ v \in V \mid f(v) = 0' \}$ is called the kernel (nucleul) of f

$\mathrm{Im} f \overset{not}{=} \{ f(v) \mid v \in V \}$ is called the image of f

Theorem: Let $f: V \to V'$ be a K-linear map. Then $\mathrm{Ker} f \leq V$ and $\mathrm{Im} f \leq V'$

Proof: • $0 \in \mathrm{Ker} f$, because $f(0) = 0'$

• Let $k_1, k_2 \in K$ and and $v_1, v_2 \in \mathrm{Ker} f$

We show that $k_1 v_1 + k_2 v_2 \in \mathrm{Ker} f$

We have $f(k_1 \cdot v_1 + k_2 v_2) = k_1 \cdot f(v_1) + k_2 f(v_2)$

$= k_1 \cdot 0' + k_2 \cdot 0' = 0'$

Hence $\mathrm{Ker} f \leq V$

• $0' = f(0) \in \mathrm{Im} f$

• Let $k_1, k_2 \in V$ and $v_1', v_2' \in \mathrm{Im} f$

We have $k_1 \cdot v_1' + k_2 \cdot v_2' = k_1 \cdot f(v_1) + k_2 \cdot f(v_2)$ for some

for some v1, v2 in V (because v1', v2' in Im f)

$= f(k_1 \cdot v_1 + k_2 \cdot v_2) \in \operatorname{Im} f$

Hence $\operatorname{Im} f \leq V'$

Theorem Let $f: V \to V'$ be a K linear-map and $X \subseteq V$

Then $f(\langle X \rangle) = \langle f(X) \rangle$

$\boxed{14}$ Linear independence and basis

$f(x)$

Definition : Let V be a K-vector space and $v_1, \dots, v_m \in V$. Then $v_1, \dots, v_m$ are called <u>linearly independent</u> (or $\{v_1, \dots v_m\}$ is linearly independent) if for every $k_1, \dots, k_m \in K$ s.t.

$$k_1 v_1 + \dots + k_m v_m = 0 \quad \text{we must have } k_1 = \dots = k_m = 0$$

The vectors $v_1, \dots, v_m \in V$ are called <u>linearly dependent</u> if they are not linearly independent, that is, $\exists k_1, \dots k_m \in K$ <u>not</u> all zero such that

$$k_1 v_1 + \dots + k_m \cdot v_m = 0$$

Theorem Let V be a K-vector space and $v_1, \dots, v_m \in V$. Then $v_1, \dots, v_m$ are linearly dependent $\Leftrightarrow$

$\exists \ j \in \{1, \dots, m\}$ s.t.

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{m} k_i v_i$$

Proof: $\boxed{\Longrightarrow}$ Assume that $v_1, \dots, v_m$ are linearly dependent $\Rightarrow \exists \ k_1, \dots, k_m \in K$ <u>not</u> all zero. such that $k_1 \cdot v_1 + \dots + k_m \cdot v_m = 0 \Rightarrow \exists \ j \in \{1, \dots, m\}$ s.t. $k_j \neq 0$ and

$k_1 v_1 + \dots + k_{j-1} v_{j-1} + k_j v_j + k_{j+1} v_{j+1} + \dots + k_n v_n = 0$

$$\Rightarrow k_j v_j = -\sum_{\substack{i=1 \\ j \neq i}}^{n} k_i v_i \quad | \cdot k^{-1} \exists$$

$$\Rightarrow v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} (-k_j^{-1} \cdot k_i) \cdot v_i$$

$\boxed{\Leftarrow}$ Assume that $\exists j \in \{1, \dots, n\}$ s.t.

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i v_i$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} k_i v_i - v_j = 0$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} k_i \cdot v_i + (-1) \cdot v_j = 0$$

This is a linear combination of ~~the~~ $v_1, \dots, v_n$ equal to 0 but not all scalars are $0 \Rightarrow v_1, \dots, v_n$ lin. dependent

Theorem: let $n \in \mathbb{N}$, $n \geq 2$

(i) 2 vectors in the canonical vector space K^n are linearly dependent $\Leftrightarrow$ if their components are respectively proportional

(ii) n vectors in ${}_K K^n$ are linearly dependent $\Leftrightarrow$ if the determinant consisting of their components is zero

Proof: (i) let $v_1, v_2 \in K^n$, say $v_1 = (x_{11}, x_{21}, \dots x_{n1})$
$v_2 = (x_{12}, x_{22}, \dots, x_{n2})$

$\overset{Th}{\Rightarrow}$ one of them is a linear combination of the Other one say $v_1 = k \cdot v_2$

$$(x_{11}, \dots x_{n1}) = k \, (x_{12}, \dots x_{n2})$$

$$\Rightarrow \begin{cases} x_{11} = kx_2 \\ \dots \quad \dots \\ x_{n1} = k \, x_{n2} \end{cases}$$

(ii) Let $\begin{cases} v_1 = x_{11}, \dots, x_{n1} \\ v_2 = x_{12}, x_{22}, \dots, x_{n2}) \\ \vdots \\ v_n = (x_{1n}, x_{2n}, \dots x_{nn}) \end{cases}$

$v_1, \dots, v_n$ are linearly dependent in K^n

$\Leftrightarrow$ $k_1, \dots, k_n \in K$ not all zero s.t.

$k_1 v_1 + \dots + k_n \cdot v_n = 0$

$k_1 (x_{11}, x_{21}, \dots, x_{n1}) + \dots + k_n (x_{1n}, x_{2n}, \dots x_n) = (0, \dots 0) \in K^n$

$\Leftrightarrow \exists \, k_1, \dots k_n \in K$ not all zero s.t

$$(S) \begin{cases} k_1 \cdot x_{11} + \dots + k_n x_{1n} = 0 \\ \dots \\ k_1 x_{1n} + \dots + k_n x_{nn} = 0 \end{cases}$$

$\Leftrightarrow$ determinant of (S) is zero

<u>Examples</u> (a) Consider the real vector space V_2

- v linearly dependent $\Leftrightarrow v = 0$
- v_1, v_2 linearly dependent $\Leftrightarrow v_1, v_2$ are collinear
- any 3 vect in K are linearly dependent

Consider ${}_{\mathbb{R}}V_3$

- v linearly dependent $\Leftrightarrow$ $v = 0$
- v_1, v_2 $\Leftrightarrow v_1, v_2$ col.
- v1, v2, v3 <=> v1,v2, v3 are in the same plane

• any 4 (or more) vectors in V_3 are linearly dependent

(b) Consider ${}_K K^n$. Let

$e_1 = (1, 0, \dots, 0)$
$e_2 = (0, 1, \dots, 0)$ $\quad \in K^n$
$\dots$
$e_n = (0, 0, \dots, 1)$

let $k_1, \dots, k_n \in K$ s.t.
$k_1 e_1 + \dots + k_n e_n = 0$
$\Rightarrow k_1(1, \dots, 0) + k_2(0, 1, \dots, 0) + \dots + k_n(0, \dots, 1) = (0, \dots, 0) \in K^n$
$\Rightarrow k_1 = \dots = k_n = 0$
Hence $e_1, \dots, e_n$ are linearly independent

c) Let ${}_K M_2(K)$ Then

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ..

are linearly independent

(d) Let ${}_K K_n[X] = \{ f \in K[X] \mid \deg(f) \leq n \}$

Then $1, X, X^2, \dots X^n$ are l. independent

Def: Let V be a K-vector space. By a list of vectors we mean an n-tuple $(v_1, \dots, v_n) \in V^n$
A list $B = (v_1, \dots, v_n)$ of vectors in V is called a basis for V if:
(i) B is linearly independent in V
(ii) $V = \langle B \rangle$, that is, B is a system of generators for

Theorem: Any vector space has (at least) a basis

Theorem: Let V be a K-vector space and $B=(v_1,\dots,v_n)$ be a list of vectors in V. Then B is a basis of $V \Leftrightarrow$
$\Leftrightarrow \forall v \in V, \exists! \, k_1,\dots,k_n \in K$ s.t. $v = k_1 \cdot v_1 + \dots + k_n v_n$

are called the coord. of v in the basis B

Proof: $\boxed{\Rightarrow}$ Assume that B is a basis of V

$\Rightarrow \begin{cases} B \text{ is l. independent in } V \\ V = \langle B \rangle \end{cases}$

$\Downarrow$

$\forall v \in V, \exists \, k_1,\dots,k_n \in K$ s.t. $v = k_1 v_1 + \dots + k_n v_n$ (1)

- for uniqueness, suppose that we also have
  $v = k_1' \cdot v_1 + \dots + k_n' v_n$ (2)

$\Rightarrow k_1 v_1 + \dots + k_n v_n = k_1' \cdot v_1 + \dots + k_n' \cdot v_n = 0$

$(k_1 - k_1') \cdot v_1 + \dots + (k_n - k_n') \cdot v_n = 0$

$\xRightarrow[\text{independent}]{B \text{ is linearly}} \begin{cases} k_1 - k_1' = 0 \\ \vdots \\ k_n - k_n' = 0 \end{cases} \Rightarrow k_i = k_i', \forall i \in \{1,\dots,n\}$

$\boxed{\Leftarrow}$ Assume that $\forall v \in V, \exists! \, k_1,\dots,k_n \in K$ s.t.
$v = k_1 \cdot v_1 + \dots + k_n \cdot v_n$

$\Rightarrow V = \langle B \rangle$

We prove that B is linearly independent

Let $k_1,\dots,k_n \in K$ be such that

$k_1 \cdot v_1 + \dots + k_n v_n = 0$

but $0 = 0 \cdot v_1 + \dots + 0 \cdot v_n$

Uniqueness of writing 0 as a linear comb

but 0=0*v1+...+0*vn

$\Rightarrow k_1 = \ldots = k_m = 0$

Hence B is linearly independent and so B is a basis of V

<u>Examples</u> (a) Consider ${}_K K^n$ (canonical)

Then $E = (e_1, \ldots, e_n)$ is a basis of ${}_K K^n$,

where $\begin{cases} e_1 = (1, 0, \ldots 0) \\ \ldots \\ e_n = (0, \ldots, 1) \end{cases}$

E is linearly independent and K^n is generated by E

($K^n = \langle E \rangle$) because $\forall (x_1, \ldots, x_n) \in K^n$, $v = x_1 e_1 + \ldots + x_n e_n$

E is called the canonical basis of ${}_K K^n$

(b) Consider ${}_{\mathbb{R}}V_2$ ($\simeq {}_{\mathbb{R}}\mathbb{R}^2$)

$\begin{cases} \vec{i} = (1, 0) \\ \vec{j} = (0, 1) \end{cases}$ form a basis of ${}_{\mathbb{R}}V_2$

Consider ${}_{\mathbb{R}}V_3$

$(\vec{i}, \vec{j}, \vec{k})$ basis of ${}_{\mathbb{R}}V_3$, where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$

(c) Consider ${}_K M_2(K)$

(E_1, E_2, E_3, E_4) is a basis of $M_2(K)$

where $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(d) Consider ${}_K K_n[X]$

$(1, X, X^2, \ldots X^n)$ is a basis of ${}_K K_n[X]$

because $\forall f \in K_n[X]$, $\exists!\, a_0, \ldots, a_n \in K$ s.t.

f = a0*1+a1*x+...+anx^n

Course 6

Partial exam
21^{st} Nov. 2019
13. 50
Courses 1-6
Seminars 1-7
4p → 1p theory
→ 3p ex

5 Dimension

Theorem (STEINITZ)
Let V be a K-vector space
$X=(x_1,\dots,x_m)$ be a linearly independent list in V,
$Y=(y_1,\dots,y_n)$ be a system of generators for V. Then

- $m \le n$
- m vectors from Y may be replaced by the vectors of X obtaining again a syst. of generators for V.

Corollary. Any 2 basis of K-vector space have the same number of vectors (we consider finitely generated K-vector spaces)

Proof. Let $B=(v_1,\dots,v_m)$, $B'=(v_1',\dots,v_n')$ be basis of a K vector space

$\left.\begin{array}{l} B \text{ is linearly independent} \\ B' \text{ is a system of generators} \end{array}\right\} \xRightarrow{\text{Steinitz}} m \le n$

$\left.\begin{array}{l} B' \text{ is linearly independent} \\ B \text{ is a system of generators} \end{array}\right\} \xRightarrow{\text{Steinitz}} n \le m$

Hence $n=m$

Definition: By the dimension of a K-vector space V, we mean the number of vectors of any of its bases

Notation $\dim_K V$

Examples:

(a) Consider the trivial K vector space $V = \{0\}$

Then $\emptyset$ is a basis of V so $\dim_K V = 0$

(b) Consider ${}_{\mathbb{R}}V_3$ ($\underset{\mathbb{R}}{\simeq} \mathbb{R}^3$)

Its subspaces are:

- V_3 : $\dim_{\mathbb{R}} V_3 = 3$

(a basis is $\vec{i}, \vec{j}, \vec{k}$)

- any plane passing through 0

$\rightarrow \dim_{\mathbb{R}} V_3 = 2$

- any line passing through 0

$\rightarrow \dim_{\mathbb{R}} V_3 = 1$

- $\{0\}$ $\rightarrow \dim_{\mathbb{R}} \{0\} = 0$

(c) Consider the canonical K-vector K^n. It has the canonical basis $E = (e_1, \dots, e_n)$, where $e_1 = (1, 0, \dots, 0)$

$\dots\dots$

$e_n = (0, \dots, 1)$

$\Rightarrow \dim_K K^n = n$

(d) $\dim_K M_{m,n}(K) = m \cdot n$

(e) $\dim_K K_n[X] = n+1$ (a basis is $(1, X, X^2, \dots, X^n)$)

Theorem Let V be a K-vect space. The following are equivalent

(i) $\dim_K V = n$

(ii) The maximum number of linearly independent vectors in V is n

(iii) The minimum number of vectors of a system of generators for V is n.

Proof: $\boxed{(i) \Rightarrow (ii)}$ Suppose that $\dim_K V = n$

So V has a basis $B = (v_1, \dots, v_n)$

$\Rightarrow \exists$ linearly independent list with n vectors, namely B

Let $X = (x_1, \dots, x_m)$ be a linearly independent list in V

View B as a system of generators for V

$\xRightarrow{\text{Steinitz}} m \leq n$

$\boxed{(ii) \Rightarrow (i)}$ Suppose that the max. number of linearly independent vectors in V is n

Consider a basis $B = (v_1, \dots, v_m)$ of $V \Rightarrow \dim_K V = m$

$$\left.\begin{array}{l} \text{View } B \text{ as linearly independent list} \xRightarrow[\text{hypothesis}]{} m \leq n \\ \text{View } B \text{ as a system of generators} \xRightarrow[\text{Steinitz}]{} n \leq m \end{array}\right\} \Rightarrow m = n$$

$\boxed{(i) \Leftrightarrow (iii)}$ Homework

Theorem. Let V be a K-vector space with $\dim_K V = n$. Let $X = (u_1, \dots, u_n)$ be a list in V. Then X is linearly independent in $V \Leftrightarrow X$ is a system of generators for V

Proof. $\boxed{\Rightarrow}$ Suppose that X is linearly independent in V

Let $B = (v_1, \dots, v_n)$ be a basis of $V \Rightarrow B$ is a system of generators for V

Steinitz $\Longrightarrow$ n vectors from B (so all of them) may be replaced by the vectors of X obtaining again a system of generators

$\Rightarrow$ X is a system of generators for V

$\boxed{\Longleftarrow}$ Suppose that X is a system of generators

Assume that X is linearly dependent

$\Rightarrow \exists\, j \in \{1, \dots, n\}$ s.t. $\boxed{u_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i \cdot u_i}$

We have $V = \langle X \rangle = \langle u_1, \dots, u_n \rangle \overset{!}{=}$

$\overset{!}{=} \langle u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n \rangle \Rightarrow$

$\Rightarrow V$ has a system of generators with $n-1$ vectors

But $\dim_K V = n \Rightarrow$ the min. number of vectors in a syst. of generators for V is n

$\Rightarrow$ contradiction. Hence X is linearly independent

<u>Corollary</u> n vectors in the canonical K-vector space K^n form a basis $\Leftrightarrow$ they are linearly independent

$\Leftrightarrow$ the det. of their components is non-zero

<u>Theorem</u>: Let V be a K vector space, and $S \leq_K V$

(i) Any linearly independent list in V can be completed to a basis of V

(ii) Any basis of S can be completed o a basis of V

(iii) $\dim_K S \leq \dim_K V$

(iv) $\dim_K S = \dim_K V \Leftrightarrow S = V$

Proof: (i) Let $X = (u_1, \dots, u_m)$ be a linearly independent list in V. Let $B = (v_1, \dots, v_n)$ be a basis of V

Steinitz $\Rightarrow m \leq n$ and m vectors from B can be replaced by those from X obtaining again a system of generators for V.

By reordering them if necessary, let us assume that the first m vectors from B are replaced by those of X. $\Rightarrow (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ is a system of generators

$\left.\begin{array}{l} \\ \text{but } \dim_K V = n \end{array}\right\} \Rightarrow (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ is

linearly independent in V $\Rightarrow$ it is a basis of V

<u>Corollary</u> Let V be a K v.s. and $S \leq_K V$
Then $\exists\, \bar{S} \leq_K V$ s.t. $V = S \oplus \underset{\substack{\uparrow \\ \text{complement of } S}}{\bar{S}}$

Proof: Let $B = (u_1, \dots, u_m)$ be a basis of S
$B' = (v_1, \dots, v_n)$ be a basis of V

Complete B as a basis of V
$(u_1, \dots, u_m, v_{m+1}, \dots, v_n)$

Then $\bar{S} = \langle v_{m+1}, \dots, v_n \rangle$ [...]

<u>Theorem</u> Let V and V' be K v.s. Then $V \simeq V' \Leftrightarrow$
$\dim_K V = \dim_K V'$

Corollary. Let V be a K.v.s. with $\dim_K V = n$

Then $V \simeq K^n$

$\boxed{6}$ Dimension formulas

Theorem (1^{st} dimension formula)

Let $f: V \to V'$ be a K linear map. Then

$\dim_K V = \dim_K \operatorname{Ker} f + \dim_K \operatorname{Im} f$.

Theorem (2^{nd} dimension formula)

Let V be a K.v.s., $S, T \leq_K V$

Then $\dim_K S + \dim_K T = \dim_K (S+T) + \dim_K (S \cap T)$