Mathematical Analysis
Seminar 3

1. Let $x = (1, 2, -1)$ and $y = (2, 0, 1) \in \mathbb{R}^3$.
a) Computer the following: $x + y, \quad x - y, \quad \|x\|, \quad \|y\|, \quad x \cdot y$.
b) Find z such that $z \perp x$, $z \perp y$ and $\|z\| = 1$.
c) Plot x, y, $x + y$ and $x - y$. (Homework)

2. Let $x = (0, 2)$ and $y = (2, 1) \in \mathbb{R}^2$. Find the intersection of the segment $[x, y]$ with the sphere $S = \{z \in \mathbb{R}^3 : \|z\| = 2\}$.

3. Prove that the following properties hold for any $x, y \in \mathbb{R}^n$:
a) $\|x + y\|^2 - \|x - y\|^2 = 4\langle x, y\rangle$.
b) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (the generalized parallelogram identity).

4. Let $x, y \in \mathbb{R}^n$. Prove that the following statements are equivalent:
1° $\langle x, y\rangle = 0$ (i.e., x and y are orthogonal).
2° $\|x + y\| = \|x - y\|$.
3° $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

5. Find the second order partial derivatives of the following functions:
a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \cos x \cos y - \sin x \sin y$.
b) $f : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$, $f(x, y) = x^y$.
c) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = (x + y + z)[(2^x)^y]^z$.
d) $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$, $f(x, y, z) = xe^y/z$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x, y) = f(x^2 + y^2),\ \forall\, (x, y) \in \mathbb{R}^2.$$

Prove that for any $(x, y) \in \mathbb{R}^2$ we have

$$y\frac{\partial g}{\partial x}(x, y) - x\frac{\partial g}{\partial y}(x, y) = 0.$$

7. Find the gradient $\nabla f(c)$ and the Hessian matrix $\nabla^2 f(c)$ in the following cases:
a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 y^3$ and $c = (1, 1)$.
b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = e^{xyz}$ and $c = 0_3$.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be partially differentiable functions. Prove that

$$\nabla(fg)(c) = f(c)\nabla g(c) + g(c)\nabla f(c),\ \forall\, c \in \mathbb{R}^n.$$

Homework

9. Study whether the function $f\colon \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x, y) = \dfrac{xy}{x^2 + y^2}$ has a limit at 0_2.

10. Study the continuity and the partial differentiability at 0_2 for $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq 0_2 \\ 0, & \text{if } (x, y) = 0_2. \end{cases}$$