Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2019-2020

Mathematical Analysis
Seminar 12

1. Prove that for any $x \in [-1, +\infty)$ and $n \in \mathbb{N}$ we have

$$(1+x)^n \geq 1 + nx \quad \text{(Bernoulli's Inequality)}.$$

Deduce that, whenever $m \in \mathbb{N}$ is even, the following inequality holds for all $y \in \mathbb{R}$:

$$(1+y)^m \geq 1 + my.$$

2. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n := \left(1 + \frac{1}{n}\right)^n.$$

a) Using Bernoulli's Inequality prove that $\dfrac{x_{n+1}}{x_n} > 1$ for all $n \in \mathbb{N}$.
b) Using Newton's Binomial Formula prove that $x_n < 3$ for all $n \in \mathbb{N}$.
Hint: notice that $\dbinom{n}{k} \leq \dfrac{n^k}{2^{k-1}}$ for all $k \in \mathbb{N}$, $k \leq n$.
c) Deduce that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by e (Euler's number), show that $2.71 < e \leq 3$.
d) Similarly to a) prove that the sequence $(y_n)_{n\in\mathbb{N}}$, defined for all $n \in \mathbb{N}$ by

$$y_n := \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right) x_n,$$

is strictly decreasing. Then, observing that $x_n < y_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$, deduce that $e < 2.72$.

3. Consider the sequence $(\gamma_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$\gamma_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$$

a) Using the fact that $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$, prove that $(\gamma_n)_{n\in\mathbb{N}}$ is strictly decreasing and bounded below by 0.
b) Deduce that $(\gamma_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by γ (Euler's constant, also known as the Euler-Mascheroni constant), show that $\gamma < 0.58$.
c) Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n := \gamma_n + \ln n - \ln(n+1)$$

is strictly increasing. Then, observing that $x_n < \gamma_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \gamma_n$, deduce that $\gamma > 0.57$.