Corollary. Let V be a K-v.s. with $\dim_K V = n$

Then $V \simeq K^n$

$\boxed{6}$ Dimension formulas

Theorem (1^{st} dimension formula)

Let $f: V \to V'$ be a K linear map. Then

$$\dim_K V = \dim_K \operatorname{Ker} f + \dim_K \operatorname{Im} f.$$

Theorem (2^{nd} dimension formula)

Let V be a K-v.s., $S, T \leq_K V$

Then $\dim_K S + \dim_K T = \dim_K (S+T) + \dim_K (S \cap T)$

$$\dim_K (S+T) = \dim_K S + \dim_K T - \dim_K (S \cap T)$$

14. 11. 2019

Course 7

Chapter 3 Matrices and linear systems

$\boxed{1}$ Elementary operations

Definition: Let V be a K-vector space. Then by an elementary operation we mean any of the following functions:

- $\varepsilon_{ij} : V^n \to V^n$ ($n \in \mathbb{N}^*$ fixed)

  $\varepsilon_{ij}(v_1, \dots, v_i, \dots, v_j \dots, v_n) =$

  $= (v_1, \dots, \underset{\text{pos } i}{v_j}, \dots, \underset{\text{pos } j}{v_i}, \dots, v_n)$ - vectors are interchanged

- $\varepsilon_{i\alpha} : V^n \to V^n$

  $\varepsilon_{i\alpha}(v_1, \dots, v_i, \dots, v_n) = (v_1, \dots, \alpha v_i, \dots, v_n)$

- $\mathcal{E}_{ij\alpha} : V^n \to V^n, \alpha \in K$

$$\mathcal{E}_{ij\alpha}(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = (v_1, \dots, \underset{\overset{\uparrow}{\text{pos } i}}{v_i}, \dots, \underset{\overset{\uparrow}{\text{pos } j}}{\alpha v_i + v_j}, \dots, v_n)$$

Lemma Let V be a K vect space and $n \in \mathbb{N}^*$.

Then V^n is a K vect. space with respect to the operations:

$$\begin{cases} (v_1, \dots, v_n) + (v_1', \dots, v_n') = (v_1 + v_1', \dots, v_n + v_n') \\ k \cdot (v_1, \dots, v_n) = (k v_1, \dots, k v_n) \end{cases}$$

$\forall (v_1, \dots, v_n), (v_1', \dots, v_n') \in V^n$

$\forall k \in K$

Theorem With the previous notation $\mathcal{E}_{ij}, \mathcal{E}_{i\alpha}, \mathcal{E}_{ij\alpha} \in Aut(V$

Definition Let $X = (v_1, \dots, v_n)$, $X' = (v_1', \dots, v_n')$ be lists of vectors in a K v.s V. Then X and X' are equivalent (not $X \sim X'$) if one of them can be obtained from the other one by a finite number of elementary operations

Remarks

(a) $X \sim X' \Leftrightarrow X' \sim X$

(b) '$\sim$' is an equivalence relation on lists of vectors

Theorem Let V be a K v.s. and X, X' be lists s.t. $X \sim X'$. (with n vectors). Then:

(i) X is linearly independent in $V \Leftrightarrow$ so is X'

(ii) X is a system of generators for $V \Leftrightarrow$ so is X'

(iii) X is a basis of $V \Leftrightarrow$ so is V'

- Let $A \in M_{m,n}(K)$, say $A = (a_{ij})$

We may view A as a list of column-vectors $(a^1, a^2, \dots, a^n)$, where $a^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, $j = \overline{1,n}$

Also, we may view A as a list of row-vectors $(a_1, a_2, \dots, a_m)$, where $a_i = (a_{i1} \quad a_{i2} \quad a_{in})$

$\forall i = \overline{1,m}$

Say $A = (a^1, \dots, a^n)$. Then the elementary operations on the set A become:

- interchange 2 columns
- multiply a column by a scalar
- multiply a column by $\alpha \in K$ and add it to another column.

Theorem The value of an elementary operation applied on $A = (a_{ij}) \in M_{m,n}(K)$, seen as a list of column vectors $(a^1, \dots, a^n)$, is equal to A multiplied on the right hand side by the matrix obtained from the identity matrix I_m, also seen as a list of column vectors by applying the same elem. operation.

E.g. take ε_{12}: $\varepsilon_{12}(A) = \varepsilon_{12}(a^1, \dots, a^n) = (a^2, a^1, \dots, a^n) =$

$$= \begin{pmatrix} a_{12} & a_{11} & a_{13} & \dots & a_{1n} \\ a_{22} & a_{21} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m2} & a_{m1} & a_{m3} & & a_{mn} \end{pmatrix} \overset{!}{=}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} =$$

$$= A \cdot \mathcal{E}_{12}(I_n)$$

Def: With the previous notation, $\mathcal{E}_{ij}(I_m)$, $\mathcal{E}_{i\alpha}(I_m)$, $\mathcal{E}_{ij\alpha}(I_m)$ are called elementary matrices.

Note that they are invertible!

Definition: We say that $A \in \mathcal{M}_{m,n}(K)$ is in echelon form (formă eşalon) with r non-zero rows if: (1) the first r rows of A are non-zero

(2) $0 \leq N(1) < N(2) < \ldots\ldots$

where $N(i)$ denotes the number of zero elements from the beginning of row i $(i=\overline{1,m})$

Theorem Every $A \neq O_{m,n}$ is equivalent to a matrix in echelon form

Example

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & -2 & 6 \\ -1 & 1 & 1 & 0 \end{pmatrix} \overset{R_2 \leftarrow (-3)R_1+R_2}{\sim} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \underset{R_3 \leftarrow R_1+R_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & \enclose{circle}{-1} & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \underset{R_3 \leftarrow 2R_2+R_3}{\sim} \left(\boxed{\begin{matrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{matrix}} \right)$$

$\boxed{2}$ Applications of elem. operations

Theorem Let A be $\in \mathcal{M}_{m,n}(K)$. Then:

$$\operatorname{rank}(A) = \dim_K \langle a^1, \dots, a^n \rangle = \dim_K \langle a_1, \dots, a_m \rangle$$

Theorem Let $A \in \mathcal{M}_{m,n}(K)$ having an echelon form C with r non-zero rows.
Then $\operatorname{rank}(A) = \operatorname{rank}(C) = r$ //

Example The rank of A from the previous example is 3 (the number of non zero rows of an echelon form of A)

Theorem Let $A \in \mathcal{M}_n(K)$ with $\det A \neq 0$. Then A is equivalent to I_n and A^{-1} is obtained from I_n by applying the same elem. operations as one does to obtain I_n from A.

Example $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$$\det A = 1 \neq 0 \Rightarrow \exists A^{-1}$$

$$\left(\underbrace{\begin{matrix} \textcircled{0} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{matrix}}_{A} \ \vdots\ \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \right) \underset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array} \right)$$

$$\overset{R_2 \leftarrow (-1) \cdot R_1 + R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & \textcircled{-1} & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array} \right) \overset{R_3 \leftarrow R_2 + R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right)$$

$$\underset{R_1 \leftarrow (-1)R_3+R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & -1 & \cancel{2} \\ 0 & \boxed{-1} & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \underset{R_1 \leftarrow 2R_2+R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\underset{R_2 \leftarrow (-1)\cdot R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \quad \underbrace{\qquad}_{I_3} \; \underbrace{\qquad}_{A^{-1}}$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Course 9

Definition : Let V be a K vector space and $B = (v_1, \ldots, v_n)$ a basis of V, and $X = (u_1, \ldots, u_m)$ a list of vectors in V. Then we may write (uniquely) :

$$\begin{cases} u_1 = \boxed{a_{11}} \cdot v_1 + \ldots + \boxed{a_{1n}} \cdot v_n \\ \vdots \\ u_m = \boxed{a_{m1}} \cdot v_1 + \ldots + \boxed{a_{mn}} \cdot v_n \end{cases}$$

Then we denote : $[X]_B = (a_{ij}) \in M_{m,n}(K)$

and we call it the matrix of X in the basis B.

Example : Consider the canonical vector space ${}_{\mathbb{R}}\mathbb{R}^4$ and $X = (u_1, u_2, u_3)$ where $u_1 = (1,2,3,4)$

$u_2 = (5,6,7,8)$

$u_3 = (9,10,11,12)$

$$[X]_E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

Theorem Let V be a K vector space, B a basis of V, and X a list of vectors in V. Then $\dim\langle X\rangle =$

$$= \operatorname{rank}([X]_B)$$

and a basis of $\langle X\rangle$ consists of the non-zero rows from an echelon form of $[X]_B$.

13] Matrix of a linear map

Definition. Let V be a K vector space, $B=(v_1,\dots,v_n)$ a basis of V and $v\in V$. Then we may uniquely write

$$v = k_1v_1+\dots+k_nv_n$$

Then we denote by $[v]_B = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in \mathcal{M}_{n,1}(K)$ and we call it the matrix of v in the basis B.

Definition Let $f: V\to V'$ be a K linear map, let $B=(v_1,\dots,v_n)$ be a basis of V, let $B'=(v_1',\dots,v_m')$ be a basis of V'.

Then, we may uniquely write:

$$\begin{cases} f(v_1) = a_{11}v_1' + a_{21}\cdot v_2' + \dots + a_{m1}\cdot v_m' \\ f(v_2) = a_{12}\cdot v_1' + a_{22}\cdot v_2' + \dots + a_{m2}\cdot v_m' \\ \vdots \\ f(v_n) = a_{1n}v_1' + a_{2n}v_2' + \dots + a_{mn}v_m' \end{cases}$$

Then we denote

$$[f]_{BB'} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \in \mathcal{M}_{m,n}(K)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ f(v_1) & f(v_2) & f(v_n) \end{array}$$

and we call it the matrix of f in the bases B, B'

If $V = V'$ and $B = B'$, then we denote

$$[f]_B = [f]_{BB}$$

$$f(v_j) = \sum_{i=1}^{m} a_{ij} v_i' \quad , \forall\ j = \overline{1,m}$$

<u>Example</u> let $f: \mathbb{R}^4 \to \mathbb{R}^3$, $f(x,y,z,t) = (x+y+z, y+z+t, z+t+x)$
This is an $\mathbb{R}$-linear map. Consider the canonical bases E of$_{\mathbb{R}}$ $\mathbb{R}^4$ and E' of$_{\mathbb{R}}$ $\mathbb{R}^3$. Let us compute $[f]_{EE'}$

$$(1,0,0)+(0,0,1)$$

$$f(e_1) = f(1,0,0,0) = (1,0,1) = e_1' + e_3'$$

$$f(e_2) = f(0,1,0,0) = (1,1,0) = e_1' + e_2'$$

$$f(e_3) = f(0,0,1,0) = (1,1,1) = e_1' + e_2' + e_3'$$

$$f(e_4) = f(0,0,1,0) = (0,1,1) = e_2' + e_3'$$

$$\Rightarrow [f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$f(1,0,0) = (1,0,2)$$

Conversely, given $[f]_{EE'}$, one may recover the definition of f.

$\forall\ v = (x,y,z,t) \in \mathbb{R}^4$, $f(v) =$

$$f(v) = f(x \cdot e_1 + y \cdot e_2 + z \cdot e_3 + t \cdot e_4) \overset{f\ K\ l.map}{=}$$

$$= x\,\underline{\underline{f(e_1)}} + y\,\underline{\underline{f(e_2)}} + z\,\underline{\underline{f(e_3)}} + t\,\underline{\underline{f(e_4)}} =$$

$$= x \cdot (1,0,1) + y(1,1,0) + z(1,1,1) + t(0,1,1)$$

$$= (x+y+z,\ y+z+t,\ x+z+t)$$

<u>Theorem</u> Let $f: V \to V'$ be a K linear map.
$B = (v_1, \dots, v_n)$ be a basis of V, $B' = (v_1', \dots, v_m')$ a basis of V', and $v \in V$. Then

$$[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$$

Proof: Let $[f]_{BB'} = (a_{ij}) \in M_{m,n}(K)$

$\forall j \in \overline{1,m}$, $f(v_j) = \sum_{i=1}^{m} a_{ij} \cdot v_i'$ (1)

Let $v = \sum_{j=1}^{n} k_j v_j$, $k_j \in K$ (2)

$\cdot\ f(v) = \sum_{i=1}^{m} \underset{coef}{\underset{\sim}{k_i'}} \cdot v_i'$, $k_i' \in K$ (3)

On the other hand, we have:

$$f(v) \overset{(2)}{=} f\Big(\sum_{j=1}^{n} k_j \cdot v_j\Big) \underset{K\text{-lin.}}{=} \sum_{j=1}^{n} k_j \cdot f(v_j) \overset{(1)}{=}$$

$$= \sum_{j=1}^{n} k_j \cdot \Big(\sum_{i=1}^{m} a_{ij} \cdot v_i'\Big) = \sum_{i=1}^{m} \underbrace{\Big(\sum_{j=1}^{n} a_{ij} \cdot k_j\Big)}_{coef} \cdot v_i' \quad (4)$$

(3), (4) are writings of the same $f(v)$ as linear combinations of the vectors in the basis B'. But such a writing is unique

$$\Rightarrow k_i' = \sum_{j=1}^{n} a_{ij} \cdot k_j \quad , i = \overline{1,n}$$

$$\begin{pmatrix} k_1' \\ \vdots \\ k_m' \end{pmatrix} = (a_{ij}) \cdot \begin{pmatrix} k_1 \\ \vdots \\ k_n' \end{pmatrix}$$

$$\Rightarrow [f(v)]_{B'} = [f]_{BB'} \cdot [u]_B$$

Theorem Let $f: V \to V'$ be a K linear map. Then $\dim(\mathrm{Im} f) = \mathrm{rank}([f]_{BB'})$ in any pair of bases B, B' of V, V' respectively

- $\dim(\mathrm{Im} f)$ is also denoted by $\mathrm{rank}(f)$ and called the rank of f.

Theorum Let V, V', V'' be K v.s., $B = (v_1, \dots, v_n)$ be a basis of V, $B' = (v_1', \dots, v_m')$ a basis of V', $B'' = (v_1'', \dots, v_p'')$ a basis of V''. Then $\forall f, g \in \operatorname{Hom}_K(V, V') \overset{not}{=} \{ f : V \to V' \mid f \text{ is a } K \text{ lin. map} \}$, $\forall h \in \operatorname{Hom}_K(V', V'')$, we have

$$\begin{cases} [f+g]_{BB'} = [f]_{BB'} + [g]_{BB'} \\ [k \cdot f]_{BB'} = k \cdot [f]_{BB'} \\ [h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'} \end{cases}$$

and $\operatorname{Hom}_K(V, V')$ is a K vect. space.

without proof

Proof: Denote

- $[f]_{BB'} = (a_{ij}) \in M_{m,n}(K)$
  $\forall j = \overline{1,n}$, $f(v_j) = \sum_{i=1}^{m} a_{ij} \cdot v_i'$
- $[g]_{BB'} = (b_{ij}) \in M_{m,n}(K)$
  $\forall j \in \overline{1,n}$, $g(v_j) = \sum_{i=1}^{m} b_{ij} \cdot v_i'$
- $[h]_{B'B''} = (c_{ki}) \in M_{pm}(K)$
  $\forall i = \overline{1,m}$ $h(v_i') = \sum_{k=1}^{p} c_{ki} \cdot v_k''$

We have $(f+g)(v_j) = f(v_j) + g(v_j) =$

$= \sum_{i=1}^{m} a_{ij} \cdot v_i' + \sum_{i=1}^{m} b_{ij} \cdot v_i' =$

$= \sum_{i=1}^{m} (a_{ij} + b_{ij}) \cdot v_i'$, $\forall j = \overline{1,n}$

$$\Rightarrow [f+g]_{BB'} = (a_{ij}+b_{ij}) = [f]_{BB'} + [g]_{BB'}$$
$$a_{ij} \quad + \quad b_{ij}'$$

$$\cdot (k\cdot f)(v_j) = k\cdot f(v_j) = k\cdot \sum_{i=1}^{m} a_{ij}\cdot v_i' =$$
$$= \sum_{i=1}^{m} (k\cdot a_{ij})\cdot v_i' \Rightarrow [k\cdot f]_{BB'} = k\cdot a_{ij} = k\cdot [f]_{BB'}$$

$$\cdot (h\circ f)(v_j) = h\left(\sum_{i=1}^{m} a_{ij}\cdot v_i'\right) =$$
$$= \sum_{i=1}^{m} a_{ij}\cdot h(v_i') =$$
$$= \sum_{i=1}^{m} a_{ij}\cdot\left(\sum_{k=1}^{p} c_{ki}\cdot v_k''\right) =$$
$$= \sum_{k=1}^{p}\left(\sum_{i=1}^{m} c_{ki}\cdot a_{ij}\right)\cdot v_k''$$
$$\Rightarrow [h\circ f]_{BB''} = \sum_{i=1}^{m} c_{ki}\cdot a_{ij} = (c_{ki})\cdot(a_{ij}) = [h]_{B'B''}[f]_{BB'}$$

Corollary Let V and V' be K vector spaces. with $\dim_K V = n$ and $\dim_K V' = m$. Then the map

$$\varphi : \mathrm{Hom}_K(V, V') \to M_{m,n}(K)$$
$$\varphi(f) = [f]_{BB'}$$

where B, B' are bases of V, V', respectively, is an isomorphism of K vector spaces.

- φ bijective
- $\forall\ f, g \in \mathrm{Hom}_K(V, V')$, $\varphi(f+g) = \varphi(f) + \varphi(g)$
- $\forall\ k \in K$, $\forall\ f \in \mathrm{Hom}_K(V, V')$, $\varphi(k\cdot f) - k\cdot\varphi(f)$

Corollary Let V be a K vector space wit $\dim_K V = n$

Then: $\varphi : \mathrm{End}_K(V) \to M_n(K)$

$$\varphi(f) = [f]_B$$

where B is a basis of V, is an isomorphism of K vector spaces and an isomorphism between the rings $(\mathrm{End}_K(V),+,\cdot)$ and $(M_n(K),+,\cdot)$

<u>Corollary</u> Let V be a K vector space and $\dim_K V = n$ and $f \in \mathrm{End}_K(V)$. Then $f \in \mathrm{Aut}_K(V) \Leftrightarrow$

$\Leftrightarrow$ f is invertible in the ring $(\mathrm{End}_K(V),+,\circ)$

$\overset{\text{cor}}{\Leftrightarrow}$ $T(f)=[f]_B$ is invertible in the ring $(M_n(K),+,\cdot)$

$\Leftrightarrow$ $\det [f]_B \neq 0$

<u>Course 10</u>

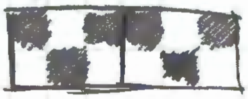
$\boxed{14}$ Change of bases

<u>Definition</u>: Let V be a K vector space, $B=(v_1,\dots,v_n)$, $B'=(v_1',\dots,v_n')$ be bases of V.
Then we may write uniquely

$$v_1' = t_{11}\cdot v_1 + t_{21}\cdot v_2 + \dots + t_{n1}\cdot v_n$$
$$v_2' = t_{12}\, v_1 + t_{22}\cdot v_2 + \dots + t_{n2}\, v_n$$
$$v_n' = t_{1n}\, v_1 + t_{2n}\, v_2 + \dots + t_{nn}\, v_n$$

Then $T_{BB'} = (t_{ij}) \in M_n(K)$ is called the change matrix from B to B'

<u>Remark</u>. We put the set of coordinates on the columns of $T_{BB'}$.



<u>Theorem</u> With the above notation, $T_{BB'}$ is invertible and $T_{BB'}^{-1} = T_{B'B}$

Proof: $v_j' = \sum_{i=1}^{n} t_{ij} v_i$, $\forall j=\overline{1,n}$ (1)

Denote $S = T_{B'B} = (s_{ki}) \in M_n(K)$

$$v_i = \sum_{k=1}^{n} s_{ki} \cdot v_k' \quad , i=\overline{1,n} \qquad (2)$$

$$(1), (2) \Rightarrow \underset{\underset{B'}{\cap}}{v_j'} = \sum_{i=1}^{n} t_{ij} \sum_{k=1}^{n} s_{ki} \cdot v_k' = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} s_{ki} \cdot t_{ij} \right) \cdot v_k'$$

$v_j' = 1 \cdot v_j'$ unique writing

$$\Rightarrow \sum_{i=1}^{n} s_{ki} \cdot t_{ij} = \begin{cases} 1, & \text{if } k=j \\ 0, & \text{if } k \neq j \end{cases} \qquad j,k=\overline{1,n}$$

$$\Rightarrow \quad S \cdot T_{BB'} = I_n$$

$\Rightarrow T_{BB'}$ has S as a left inverse

Similarly, S is also a right inverse of $T_{BB'}$

$$\Rightarrow T_{BB'}^{-1} = S = T_{BB'}$$

Theorem With the above notation we have:

$$[v]_B = T_{BB'} \cdot [v]_{B'} \quad , \forall v \in V$$

Proof: $T_{BB'} = (t_{ij}) \in M_n(K)$

(1) $v_j' = \sum_{i=1}^{n} t_{ij} \cdot v_i \quad , j=\overline{1,n}$

(2) $v = \sum_{i=1}^{n} k_i v_i$

(3) $v = \sum_{j=1}^{n} k_j' \cdot v_j'$

Replace (1) into (3) to get:

$$v = \sum_{j=1}^{n} k_j' \cdot \sum_{i=1}^{n} t_{ij} \cdot v_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} t_{ij} \cdot k_j' \right) \cdot v_i \quad (4)$$

2), (4) and the unique writing of v as a linear combination of the vectors in B

$$\Rightarrow k_i = \sum_{j=1}^{m} t_{ij} \cdot k_j \qquad i=\overline{1,n}$$

$$\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = (t_{ij}) \cdot \begin{pmatrix} k_1' \\ \vdots \\ k_m' \end{pmatrix}$$

$$\Rightarrow [v]_B = T_{BB'} \cdot [v]_{B'}$$

<u>Theorem</u> Let $f \in End_{\mathbb{R}}(V)$, B, B' be bases of V

Then: $[f]_{B'} = T_{BB'}^{-1} [f]_B \cdot T_{BB'}$

<u>Example</u> In the canonical real vector space ${}_{\mathbb{R}}\mathbb{R}^3$, consider the basis

$E = (e_1, e_2, e_3)$ - the canonical basis

$B = (v_1, v_2, v_3)$, where $v_1 = (0,1,1)$

$v_2 = (1,1,2)$

$v_3 = (1,1,1)$

~~$T_{BB'} = ?$~~

~~$T_{B'B} = ?$~~

$T_{EB} = ?$

$T_{BE} = ?$

$$\begin{cases} v_1 = e_2 + e_3 \\ v_2 = e_1 + e_2 + 2e_3 \\ v_3 = e_1 + e_2 + e_3 \end{cases} \Rightarrow T_{EB} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$T_{BE} = T_{EB}^{-1} = \ldots$ OR express the vectors of E as linear comb of vectors of B

$$\begin{cases} e_1 = -v_1 + v_3 \\ e_2 = v_1 - v_2 + v_3 \\ e_3 = v_2 - v_3 \end{cases}$$

$$T_{BE} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Now take a vector $u = (1,2,3)$

$$[u]_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad [u]_B = ?$$

$$[u]_B = T_{BE} \cdot [v]_E =$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Now let $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x,y,z) = (x+y,\ y-z,\ z+x)$$

$$f(e_1) = f(1,0,0) = (1,0,1)$$

$$f(e_2) = f(0,1,0) = (1,1,0)$$

$$f(e_3) = f(0,0,1) = (0,-1,1)$$

$$\Rightarrow [f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$[f]_B = T^{-1}_{EB} \cdot [f]_E \cdot T_{EB} =$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} =$$

$$= \dots = \begin{pmatrix} -1 & -3 & -2 \\ 1 & 4 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

$\boxed{5}$ Eigenvectors and eigenvalues
(vectori proprii şi valori proprii)

Definition Let $f \in \mathrm{End}_K V$.
Then a non-zero $v \in V$ is called an eigenvector of f if $\exists\, \lambda \in K$ such that $f(v) = \lambda \cdot v$

Remark Assume that $\exists\, \lambda, \lambda' \in K$ s.t.
$f(v) = \lambda \cdot v$ and $f(v) = \lambda' \cdot v$

$$\Rightarrow \lambda \cdot v = \lambda' \cdot v$$

$$\Rightarrow (\lambda - \lambda')\underset{0}{\underset{\neq}{v}} = 0 \Rightarrow \lambda - \lambda' = 0$$

$$\lambda = \lambda'$$

λ is called the eigenvalue of f corresponding to the eigenvector v

For an eigenvalue $\lambda \in K$, denote

$V(\lambda) = \{v \in V \mid f(v) = \lambda \cdot v\}$ - the set consisting of 0 and all eigenvectors of f having eigenvalue λ

Theorem Let $f \in \mathrm{End}_K(V)$ and λ be an eigenvalue of f. Then $V(\lambda) \leq_K V$, called the eigenspace of λ or the characteristic subspace of λ

Proof: $\cdot\ 0 \in V(\lambda) \neq \emptyset$

$\cdot$ Let $k_1, k_2 \in K$ and $v_1, v_2 \in V(\lambda)$

We show that $k_1 v_1 + k_2 v_2 \in V(\lambda)$

$$f(k_1v_1 + k_2v_2) = k_1 f(v_1) + k_2 f(v_2) =$$

$$= k_1(\lambda v_1) + k_2(\lambda \cdot v_2) =$$

$$= \lambda(k_1v_1 + k_2v_2) \Rightarrow k_1v_1 + k_2v_2 \in V(\lambda)$$

Theorem Let $f \in End_K(V)$, $B=(v_1,\dots,v_n)$ be a basis of V, $A=[f]_B=(a_{ij})\in \mathcal{M}(K)$

Then $\lambda\in K$ is an eigenvalue of f $\Longleftrightarrow$

$$\underbrace{\begin{vmatrix} \boxed{a_{11}-\lambda} & a_{12} & \cdots & a_{1n} \\ a_{21} & \boxed{a_{22}-\lambda} & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & \boxed{a_{nn}-\lambda} \end{vmatrix}}_{\text{characteristic det}} = 0 \quad \text{- characteristic equation}$$

$K\in\{\mathbb{R},\mathbb{C}\}$

Remark: Note that the eigenvalues of f do <u>not</u> depend on the choice of the basis B!

Proof of theorem:

$\lambda\in K$ is an eigenvalue of $f \Rightarrow \exists\ 0\neq v\in V$ such that $f(v)=\lambda\cdot v$

We have $f(v)=\lambda\cdot v \Leftrightarrow f(v)-\lambda\cdot v=0$

$$\Rightarrow [f(v)-\lambda\cdot v]_B=[0]_B \Rightarrow [f(v)]-\lambda\cdot[v]_B=[0]_B$$

$$\Leftrightarrow [f]_B\cdot[v]_B-\lambda[v]_B=[0]_B$$

$$\Leftrightarrow ([f]_B-\lambda I_n)[v]_B=[0]_B$$

$$\Leftrightarrow \underset{(*)}{(A-\lambda I_n)[v]_B}=[0]_B$$

we may uniquely write

$v=x_1v_1+\dots+x_nv_n$

for some $x_1,\dots,x_n\in K$

$$\Rightarrow [v]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then $(*) \Leftrightarrow (A - \lambda I_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in M_{n,1}(K)$

$$\Leftrightarrow \quad (S) \begin{cases} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}\cdot x_n = 0 \\ a_{21}\cdot x_1 + (a_{22}-\lambda)x_2 \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \dots + (a_{nn}-\lambda)x_n = 0 \end{cases}$$

characteristic system

Hence $\lambda \in K$ is an eigenvalue of $f \Leftrightarrow$

$\Leftrightarrow$ S) has a non-zero solution

$\Leftrightarrow \det S = 0 \Rightarrow |A - \lambda I_n| = 0$

Theorem Let $f \in \mathrm{End}_K V$ ~~and~~ with $\dim_K V = n$ and assume f has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then

$$[f]_B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where B is a basis consisting of the eigenvectors of f.

Example Let $f \in \mathrm{End}_{\mathbb{R}} \mathbb{R}^3$ $f(x,y,z) = (2x, y+2z, -y+4z)$

$f(e_1) = (2,0,0)$

$f(e_2) = (0,1,-1)$

$f(e_3) = (0,2,4)$

$$\Rightarrow A = [f]_E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix}$$

$\lambda \in \mathbb{R}$ is an eigenvalue of f

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{cases} \lambda_1 = \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$$

I $\lambda_1 = \lambda_2 = 2$. Then the eigenvectors of f having the eigenvalues $\lambda_1 = \lambda_2 = 2$ are the solutions of the system

$$\begin{pmatrix} 2-\lambda_1 & 0 & 0 \\ 0 & 1-\lambda_1 & 2 \\ 0 & -1 & 4-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -x_2 + 2x_3 = 0 \\ -x_2 + 2x_3 = 0 \end{cases} \Rightarrow \quad x_2 = 2x_3$$

The solutions are $(x_1, 2x_3, x_3)$ $x_1, x_3 \in \mathbb{R}$

$V(2) = \{(x_1, 2x_3, x_3) \mid x_1, x_2 \in \mathbb{R}\} =$

$= <(1,0,0), (0,+2,1)>$

II $\lambda_3 = 3$ Homework

password: coding 19

Lecture 11

$$\begin{tikzcd}
& 100 \text{ not a code word} & \\
000 \arrow[ur] & 110 \arrow[u] & 101 \arrow[ul, dash]
\end{tikzcd}$$

$111 \rightarrow 101$ - discovers 2, corrects 1

(n, k, d) codes

min hamming distance between code words

1) $$\begin{array}{l|l}
x^6 + x^4 & \underline{x^4 + x^3 + x^2 + 1} \\
\underline{-x^6 - x^5 - x^4 - x^2} & x^2 - x + 1 \\
-x^5 - x^2 & \\
\underline{x^5 + x^4 + x^3 + x} & \\
/\ x^4 + x^3 - x^2 + x & \\
\underline{-x^4 - x^3 - x^2 - 1} & \\
/\ \ /\ \underset{\overset{\shortparallel}{0}}{-2}x^2 + x - 1 & = x + 1
\end{array}$$

- code words - polynomials divisible by p

2. $100110 \rightsquigarrow m = 1 + x^3 + x^4$

$$\begin{array}{l|l}
x^4 + x^3 + 1 & \underline{x^3 + x + 1} \\
\underline{x^4 + x^2 + x} & x + 1 \\
/\ x^3 + x^2 + x + 1 & \\
\underline{x^3 + x + 1} & \\
x^2 &
\end{array}$$

$\Rightarrow p \nmid u \Rightarrow$

$\Rightarrow u$ is not a code word

Course 13

Linear systems of equations

Throughout K will be a field

$$(S)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in K, \forall i = \overline{1,m}, \forall j = \overline{1,n}$

$x_1, \dots, x_n \in K$ unknowns

$A = (a_{ij}) \in \mathcal{M}_{m,n}(K)$ is called the matrix of S

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$
is called the augmented (extended) matrix

Denote $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1}(K)$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathcal{M}_{m,1}(K)$

$(S)\quad A \cdot X = b$

We know there is a K-linear map associated to A

$A \in \mathcal{M}_{m,n}(K) \rightsquigarrow f_A \in \mathrm{Hom}_K(K^n, K^m)$

such that $[f_A]_{E,E'} = A$, where E and E' are the canonical bases of K^n and K^m respectively

Denote $x = (x_1, \dots, x_n) \in K^n$

$b = (b_1, \dots, b_m) \in K^m$

We have:

- $[f_A(x)]_{E'} = [f_A]_{EE'} \cdot [x]_E = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = [b]_{E'}$

Hence :

(S) $f_A(x)=b$

Denote by (S_0) the corresponding homogeneous system

(S_0) $A\cdot x=0$

(S_0) $f_A(x)=0$

Denote $S=\{x\in M_{n,1}(K) / A\cdot x=b\}$

or $S=\{x\in K^n / f_A(x)=b\}$

the set of solutions of S

Denote $S_0=\{x\in M_{n,1}(K) / A\cdot x=0\}$

$S_0=\{x\in K^n / f_A(x)=0\}$ - kernel

Theorum With the above notation, $S_0 \le K^n$ and $\dim_K S_0 = n - \text{rank}(A)$

Proof : $S_0 = \text{Ker}(f_A) \le K^n$ and

$\dim_K S_0 = \dim_K \text{Ker} f_A = \dim_K K^n - \dim_K \text{Im} f_A =$

$= n - \text{rank}(A)$

Theorum let $x'\in S$ (that is, a solution of (S)). Then:

$$S = x' + S_0 \overset{\text{not}}{=} \{x'+x^0 / x^0\in S_0\}$$

Proof: $\boxed{\subseteq}$ Let $x^2\in S \Rightarrow A\cdot x^2=b$ $\}\Rightarrow$

but $x_1\in S \Rightarrow A\cdot x^1=b$

$\Rightarrow A\cdot x^2 = A\cdot x^1 \Rightarrow A(x^2-x^1)=0 \Rightarrow x^2-x_1\in S_0$

$\Rightarrow x^2-x^1 = x^0\in S_0 \Rightarrow x^2 = x^1+x^0$ ~~with~~ $\in S_1+S_0$

$\boxed{2}$ Let $x^2 \in x_1 + S_0 \Rightarrow x^2 = x^1 + x^0$ with $x^0 \in S_0$

We have $A \cdot x^2 = \underbrace{A \cdot x^1} + \underbrace{A \cdot x^0} - b + 0 = b \Rightarrow x^2 \in S$

Definition : (S) is called
compatible if $S \neq \emptyset$

A compatible system (S) is called

- **determinate** if $|S| = 1$
- **non-determinate** if $|S| > 1$

Remark (1) (S) is compatible $\Leftrightarrow b \in \operatorname{Im} f_A$

(2) (S_0) is compatible $\Leftrightarrow 0 \in \operatorname{Im} f_A$ always TRUE

Theorem (Kronecker - Capelli)
(S) is compatible $\Leftrightarrow \operatorname{rank}(\bar{A}) = \operatorname{rank} A$

Proof (S) is compatible $\Leftrightarrow b \in \operatorname{Im} f_A = f_A(K^n)$
$\Leftrightarrow b \in f_A(\langle e_1, \dots, e_n \rangle) \Leftrightarrow$

* $(e_1, \dots, e_n)$ canonical basis of ${}_K K^n)$ *

$\Leftrightarrow b \in \langle f_A(e_1), \dots, f_A(e_n) \rangle$

$\Leftrightarrow b \in \langle \underbrace{a^1, \dots, a^n}_{\text{the colums of } A} \rangle$

$\overset{!}{\Leftrightarrow} \langle a^1, \dots, a^n, \underset{?}{b} \rangle = \langle a^1, \dots, a^n \rangle \Leftrightarrow$
is a linear comb of others

$\Leftrightarrow \dim_K(\langle a^1, \dots, a^n, b \rangle) = \dim_K \langle a^1, \dots, a^n \rangle$

$\Leftrightarrow \operatorname{rank} \bar{A} = \operatorname{rank} A$