Mathematical Analysis
Seminar 10

The **Euler integrals** of first and second kind (also called **beta** and **gamma** functions) are special functions[1] given by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x, \quad a,b>0$$

and

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\mathrm{d}x, \quad a>0.$$

We will use the fact that

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a,b>0$$

holds true (for a proof see [D. Popa, Calcul integral]).

1. a) Check the convergence of $B(a,b)$ (for fixed a,b);

b) Check the convergence of $\Gamma(a)$ (for fixed a).

2. Prove that

a) $B(a,b) = B(b,a)$ for $a,b>0$;

b) $B(a,b) = \displaystyle\int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}\mathrm{d}y$ for $a,y>0$;

c) $\Gamma(a+1) = a\Gamma(a)$ for $a>0$;

d) $\displaystyle\int_0^\infty x^{a-1}e^{-xy}\mathrm{d}x = \frac{\Gamma(a)}{y^a}$;

e) $\Gamma(\frac{1}{2}) = 2\displaystyle\int_0^\infty e^{-x^2}\mathrm{d}x$;

e) $\Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin \pi a}$ for $0<a<1$.

3. Compute $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)}}$ in two ways (using the the properties of B, Γ and directly).

[1] In this case, functions defined by means of parametric improper integrals.