Babeş-Bolyai University, Faculty of Mathematics and Computer Science

Bachelor, Computer Science, Groups 911-917, Academic Year 2019-2020

Mathematical Analysis
Seminar 9

1. Study the convergence of and (if convergent) compute the improper integrals

a) $\displaystyle\int_0^\infty \frac{\operatorname{arctg} x}{1+x^2}\mathrm{d}x$;

b) $\displaystyle\int_2^\infty \frac{x-1}{x^2+x+1}\mathrm{d}x$;

c) $\displaystyle\int_0^1 (\ln x)^2\,\mathrm{d}x$.

2. a) Study the convergence of and compute $\displaystyle\int_0^\infty x^2 e^{-x}\mathrm{d}x$.

b) Prove that $\displaystyle\int_0^\infty P(x)e^{-x}\mathrm{d}x = P(0)+P'(0)+\cdots+P^{(n)}(0)$ where P is a polynomial function of degree $n \in \mathbb{N}$.

3. Compute the following integrals by embedding them in parametrized families and differentiating with respect to the parameter[1]

a) $\displaystyle\int_0^1 \frac{x^5-1}{\ln x}\mathrm{d}x$; [Hint: consider $I(a)=\int_0^1 \frac{x^a-1}{\ln x}\mathrm{d}x$]

b) $\displaystyle\int_0^\infty \frac{\operatorname{arctg} 2x}{x(1+x^2)}\mathrm{d}x$. [Here $I(y)=\int_0^\infty \frac{\operatorname{arctg} xy}{x(1+x^2)}\mathrm{d}x$]

4. The **Euler integrals** of first and second kind (also called **beta** and **gamma** functions) are special functions[2] given by

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x, \quad a,b>0 \quad \text{and} \quad \Gamma(a)=\int_0^\infty x^{a-1}e^{-x}\mathrm{d}x, \quad a>0.$$

a) Check the convergence of $B(a,b)$ (for fixed a,b);

b) Check the convergence of $\Gamma(a)$ (for fixed a);

c) Prove that $\Gamma(a+1)=a\Gamma(a)$ for $a>0$ and $\Gamma(n+1)=n!$ for $n\in\mathbb{N}$;

d) Prove that $\Gamma(\frac{1}{2})=\displaystyle\int_0^\infty e^{-x^2}\mathrm{d}x$.

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[1] Assume that all improper (parametric) integrals are convergent.

[2] in this case functions defined by means of parametric improper integrals