$\overline{T}$ (ABEL II) The sum of the power series $\sum_{n=1}^{\infty} a_n x^n$

that is the function $S(x) = \sum_{n=1}^{\infty} a_n x^n$ is cont. at $x = R$ if $\sum_{n=1}^{\infty} a_n R^n$ conv

The most famous example $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots$

10.01.2020

Course 13

Recap

I Diff Calculus

II Integral Calculus

III Sequences and series

I Diff. calculus

1\. Diff calc for f of one variable

Continuity, Diff-ability ← NO ε, δ definitions

$\overline{T}$ Weierstrass $f: [a,b] \to \mathbb{R}$ cont.

then f reaches its extremal values (and all values between those)

$\overline{T}$ Fermat x^* local min/max for f, f diff $\Rightarrow f'(x^*) = 0$

$\overline{T}$ Rolle

$f: [a,b] \to \mathbb{R}$

$\left.\begin{array}{l} \text{cont on } [a,b], \text{ diff on } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists\, c \in (a,b)$ $f'(c) = 0$

$\overline{T}$ Lagrange

cont on $[a,b]$, diff on (a,b) $\Rightarrow \exists\, c \in (a,b)$

s.t. $f(b) - f(a) = f'(c)(a-b)$

Q 1. Give an example of f and x^*

s.t. $f'(x^*) = 0$ but x^* is not a local min/max.

Ex 2 Let $f:[a,b]\to\mathbb{R}$, f' is cont on $[a,b]$

prove that $\exists\ L>0$ s.t.

$$|f(x)-f(y)| \le L\,|x-y| \quad \forall\, x,y\in[a,b]$$

T Taylor if all $(m+1)$ derivatives are cont.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$

$$+ \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + \frac{f^{(m+1)}(\xi)}{(m+1)!}(x-x_0)^{m+1}$$

for some $\xi \in (x_0, x)$

Yes: IDEA OF PROOF

→ Papa p 58 →

283 def Calculus for f of several variables

$f(x) \longleftrightarrow f(x_1,\dots,x_n)$

AIM : replicate the $f(x)$ calculus

Ex 3 Use the Taylor formula to prove the fact that

$f'(x^*)=0$, $f''(x^*) \gtrless 0$ min max

you need : · The Geometry of the Euclidian space $\mathbb{R}^n$

· operate with $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ $\quad x+y$, λx

! → inner product $x\cdot y = \sum_{i=1}^{n} x_i y_i$ $\quad$ norm $\|x\| = \sqrt{x\cdot x}$

orthogonality $x\cdot y = 0 \Leftrightarrow x\perp y$ $\quad n=1\ \|x\|=|x|$ abs value

- the Ball $B_r(x^*) = \{x\in\mathbb{R}^n \,/\, \|x-x^*\| < r\}$ convex

- distance $\operatorname{dist}(x,y) = \|x-y\|$

- The Segment $[x,y] = \{(1-\alpha)x + \alpha y \quad \alpha\in[0,1]\}$

$x,y\in\mathbb{R}^m$

Convexity

convex

· Partial Derivatives

$$\frac{\partial f}{\partial x_i}, \quad \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

Th Schwartz $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ (cont)

Th Fermat

$$\left.\begin{array}{l} x^* \text{ local min/max} \\ f: \mathbb{R}^n \to \mathbb{R} \end{array}\right\} \Rightarrow \nabla f(x^*) = 0_n$$

Th Lagrange

$f: C \subseteq \mathbb{R}^n \to \mathbb{R}$ f diff

C convex, closed, bounded

$\exists\, c \in [x,y] : f(x) - f(y) = \nabla f(c)(x-y)$

Ex: Prove Th Lagrange for f of several var using the Th Lagrange for f of one var

$x_1, \dots, x_n : [a,b] \to \mathbb{R}$

$x: [a,b] \to \mathbb{R}^n$

$x(t) = (x_1(t), \dots, x_n(t)) \quad t \in [a,b]$

$f: \mathbb{R}^n \to \mathbb{R}$ (look at $F = f \circ x$)

$$\frac{d}{dt} f(x(t)) = \nabla f(x(t)) \cdot \frac{dx}{dt}(t)$$

Linear functions $T: \mathbb{R}^n \to \mathbb{R}$ ← Freshet diff

$T_a(x) = a \cdot x \quad a \in \mathbb{R}^n$ NO Def & Theorems

$d_f(x^*)(x) = \nabla f(x^*) \cdot x$ ⟵ YES: concept exists & notation

Higher order F-differentials

Quadratic functions $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

$(a_{ij})_{i,j=\overline{1,n}}$ $n \times n$ matrix

$a_{ij} \in \mathbb{R}$

Sylvester pos/neg def

YES $k=2 \to$ Hessian Matrix

- Optimality Conditions . YES
- Constraint Optimization $f(x,y) \to \min, \max$

$g(x,y) = 0$

Lagrange functions $\to$ Lagr. mult. method

I Diff calculus

1. $f: \mathbb{R} \to \mathbb{R}$ $\quad f: [a,b] \to \mathbb{R}$

2.
3. $\left| \; f: \mathbb{R}^n \to \mathbb{R} \right.$

NO

4. Applications : Least Squares
Gradient Desent & Nesterov
Constrained Optimization
only $L = \ldots$

II Integral calculus

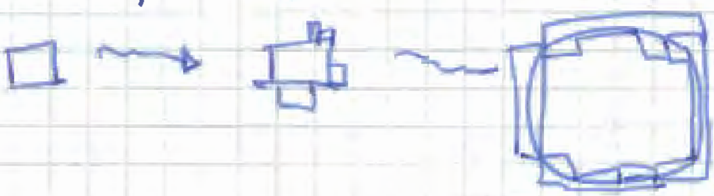
5. Antiderivatives & Riemann Integral

$F' = f$ $\quad F$ antiderivatives

Riemann Sum & geometric meaning

No: ε def of Riemann Integrability

6. Multiple integrals & Jordan measure Yes



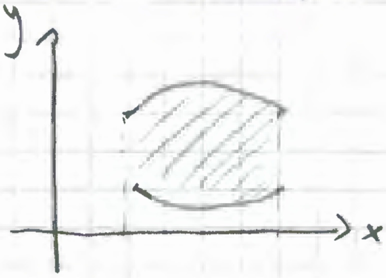
Riemann Sum for multiple integrals YES

$\uparrow$ formal Def : NO

How to comput $\iiint$ =?

$\Uparrow$ Fubini $\int_{A\times B} f(x,y)\, dx\, dy = \int_A \left(\int_B f(x,y)\, dy \right) dx$

Simple domains
in $\mathbb{R}^2$



Changes of coord

polar coord , spherical coord.

YES $\uparrow$

Formula NO

NO 7. Curvilinear Integrals

8. Improper integrals (w. parameter)

$\int_0^{\infty} e^{-x}\, dx$, $\int_0^1 \frac{1}{x}\, dx$

eg 5 $\int_0^1 x^p\, dx$ and $\int_1^{\infty} x^p\, dx$

$p \in \mathbb{Z}$? conv / div

Conv criteria :

Comparison tests

$I(y) = \int_a^b f(x,y)\, dx$

$I'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx$

9. Application of improper integrals

Full detail $\int_0^{\infty} e^{-x^2}\, dx$, $\int_0^{\infty} \frac{\sin x}{x}\, dx$, $\int_0^{\infty} e^{-x} \frac{\sin x}{x}\, dx$

NO

Euler's Integrals : Γ, B Functions

Def of Γ, B
Properties of Γ } YES

III Sequences & Series

10. Seq & Series of numbers

Def Cauchy Sequence YES

$\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ the "e" - sequence

Series $(a_n)_{n\in\mathbb{N}^*}$ sequence

$\sum_{n=1}^{\infty} a_n \rightsquigarrow S_n = a_1+\dots+a_n$

conv/div $(S_n)_{n\in\mathbb{N}^*}$ conv/div

$\sum_{n=1}^{\infty} q^n$, with $|q|<1$ - geometric series conv
PROOF
↑

$\sum_{n\geq 1} \frac{1}{n}$ harmonic series, DIV (Proof YES)

$\sum_{n\geq 1} \frac{1}{n^2}$ CONV (proof YES)

Th I Conv test for series

$\sum_{n\geq 1} f(n) \sim \int_1^{\infty} f(x)\,dx$ (Proof YES)

Convergence Test : Comparison Tests (yes)
Cauchy's Condensation Test (yes)
Others: NO

11. Seq & Series of Functions

$f_n(x) \xrightarrow{n\to\infty} f(x)$

Def: pointwise & unif. conv. (YES)

Ex 6. Example of $(f_n)_{n \in \mathbb{N}^*}$ which converges pointwise but not uniformly on $\mathcal{F}$.

Power Serie : NO

12: NO Fourier Integrals

Only $\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$

$\cos nx \sin mx$

$\sin \quad \sin$

$\int_{-\pi}^{\pi} (\sin nx)^2 dx$, $\int_{-\pi}^{\pi} (\cos nx)^2 dx$