Math Analysis

I Differencial Calculus

1. Differencial Calculus for real function of one variable

INFINITESIMAL

1.1. Limits and continuity

Def $f : I \subseteq \mathbb{R} \to \mathbb{R}$ has a limit l in a point $x_0 \in I$
if ~~given~~ $\forall\, \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ such that $|x - x_0| < \delta(\varepsilon)$
for any $x \in I$
we have $|f(x) - f(x_0)| < \varepsilon$

$\varepsilon, \delta \to$ CAUCHY

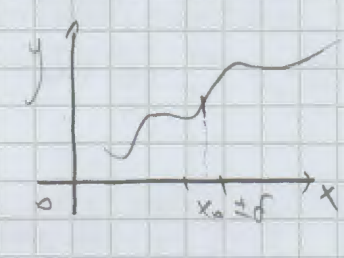
not $\lim\limits_{x \to x_0} f(x) := l$

Def f is cont in x_0
if $\lim\limits_{x \to x_0} f(x) = f(x_0)$

counter ex

$$f(x) = \begin{cases} 0, & x \le 1 \\ 1, & x > 1 \end{cases}$$

Geom : cont = draw it at once



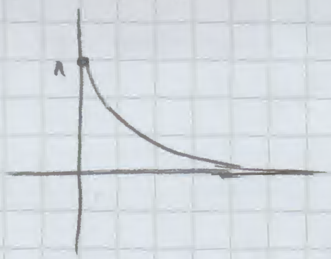
Th (Weierstrass) $f : [a;b] \to \mathbb{R}$ continuous
then $f([a,b]) := \{ n \in \mathbb{R} : \exists\, x \in [a;b], f(x) = n \}$
$= [m ; M]$ (f reaches its m and M)

$M = \max(f(x))$

$m = \min(f(x))$

Counter example

$f : [0, +\infty) \to \mathbb{R}$, $f(x) = e^{-x}$



Rk: f, g cont

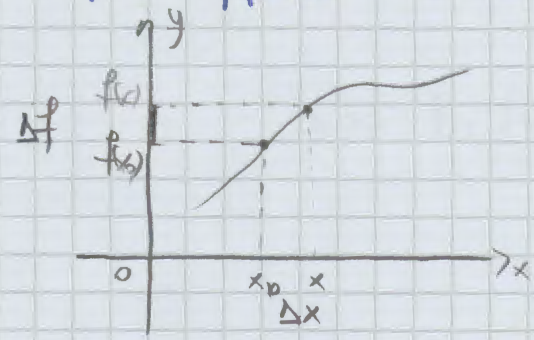
$c_1 f + c_2 f$ cont
etc

1.2. The derivate of a function. Differentiability

Def $f : [a;b] \to \mathbb{R}$ has derivative at $x_0 \in (a,b)$
if the limit $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists

Notation $f'(x_0)$

Def if the limit is finite, then the function
f is differentiable in x_0



$\frac{\Delta f}{\Delta x} \xrightarrow{\Delta x \to 0}$?

Leibniz Notation

$f'(x) \overset{Not}{=} \frac{df}{dx}(x)$

Def f is differentiable on $[a;b]$ if diff in any $x \in [a,b]$

Chain Rule $f(g(x))' = f'(g(x)) \cdot g'(x)$

$$\frac{d f(g(x))}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{df}{dx}$$

Def (higher derivatives are introduced inductively)

$$f^{(k+1)}(x) = (f^{(k)})'(x)$$

$$\frac{d^{k+1} f}{d x^{k+1}} = \underbrace{\frac{d}{dx}}_{\text{diff opperation}} \cdot \left(\frac{d^k f}{d \cdot x^k}\right)$$

Su

AIM of Part I : OPTIMIZATION

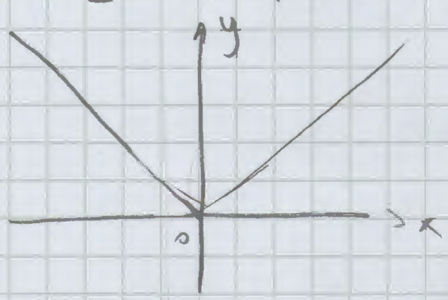
Ex Box with fixed volume and minimal surface

1.3 Derivatives and extremal points

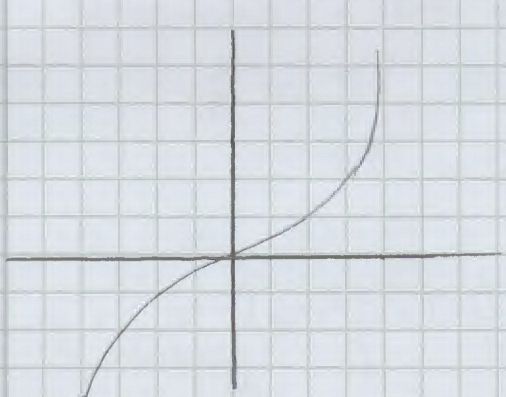
Def $f: [a;b] \to \mathbb{R}$, $x_0 \in [a;b]$ is a local min/max of f if $\exists\, \varepsilon > 0$ such that for all $x \in (x_0 - \varepsilon; x_0 + \varepsilon)$ $f(x) \overset{min}{\geq} f(x_0)$, $\underset{max}{\leq}$

Def $x_0 \in (a;b)$ stationary point for a diff function $f: [a;b] \to \mathbb{R}$, if $f'(x_0) = 0$

Ex 1) $f(x) = |x|$ has a min in $x_0 = 0$ but is not diff



2) $f(x) = x^3$

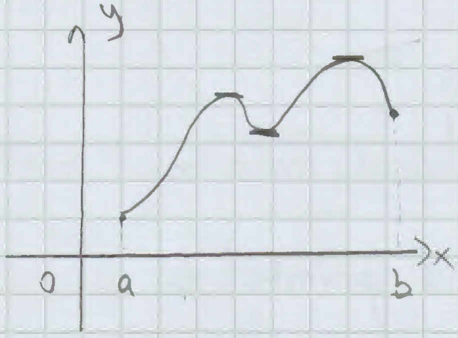


f is differentiable in $x_0 = 0$

$f'(x_0) = 0$

x_0 is not a max/min

T (Fermat) $f:[a;b]\to\mathbb{R}$ is diff-able, x_0 is local min/max, then $f'(x_0)=0$



Mid-point theorems

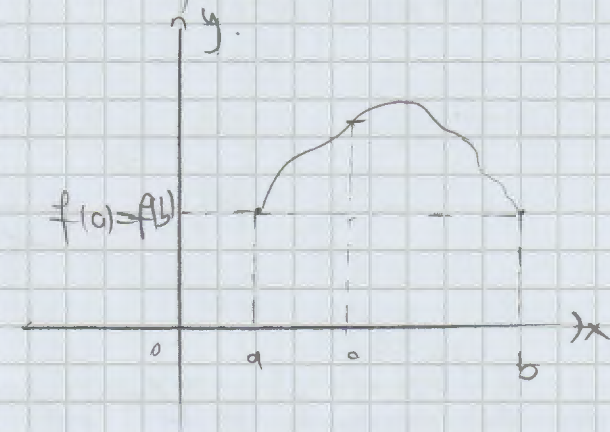
T (Rolle) $f:[a;b]\to\mathbb{R}$ with

- $f(a)=f(b)$
- f cont on $[a;b]$
- f diff on $(a;b)$

then $\exists\ c\in(a;b)$ s.t. $f'(c)=0$

Proof used T Fermat & T Weierstrass

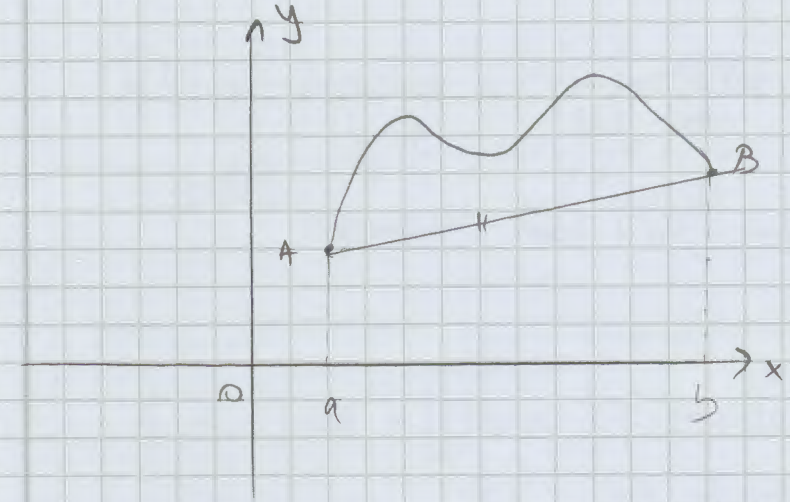
D. Popa vol 1



T (Lagrange) $f:[a;b]\to\mathbb{R}$

- f cont on $[a;b]$
- f diff on $(a;b)$

then $\exists\ c\in(a;b)$ s.t. $f(b)-f(a)=f'(c)(b-a)$



Proof: use $\boxed{T}$ Rolle

$$F(x) = (b-a)f(x) - x(f(b)-f(a))$$

$$F(a) = bf(a) - \cancel{af(a)} - af(b) + \cancel{af(a)} =$$
$$= bf(a) - af(b)$$

$$F(b) = \cancel{bf(b)} - af(b) - \cancel{bf(b)} + bf(a) =$$
$$= bf(a) - af(b)$$

We can apply Rolle

$$F'(x) = (b-a)f'(x) - (f(b)-f(a))$$

$F'(c) = 0$ for some $c \in (a;b)$

so this means

$$(b-a)f'(c) - (f(b)-f(a)) = 0$$
$$f(b)-f(a) = (b-a)f'(c)$$

Consequence: f cont. diff (f' cont)

$f'(x) \geq 0 \rightarrow f$ non decreasing (charact. monotonity

$\leq 0 \rightarrow f$ non increasing in the f' cont case)

Proof (Home) Idea: Lagrange

$\boxed{T}$ Cauchy $f, g : [a,b] \to \mathbb{R}$

· $g'(x) \neq 0$

f,g cont on [a, b]
f,g diff on (a,b)

· f cont on $[a,b]$

then $\exists\, c \in (a;b)$ $\quad \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

- there are also other mid-term theorems
  - → D. Pompeiu
  - → D.M. Ican

1.4. The Taylor Formula

polynomial

IDEA/GOAL: Approximate a given function by a polynomia

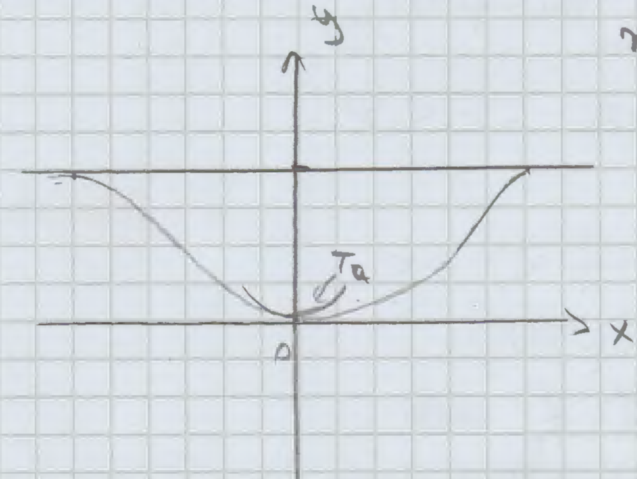
$\boxed{T}$ (Taylor) $f:[a;b]\to\mathbb{R}$ $n+1$ times diff.

There exists a point $\xi \in (x_0, x)$ such that

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Taylor polynomial } T_n} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange Reminder } R_n} \to \text{small}$$

$n \to \infty \Rightarrow n! \gg n < \infty$

Ex:



$f(x) = \dfrac{x^2}{1+x^2}$

approx f by polynoms of degree 2

Rk: $\boxed{T}$ Lagrange $\equiv$ Taylor with $n=1$

! Rk: $x_0 = 0$ $\quad e^x = 1 + x + \dfrac{x^2}{2} + \dots + \dfrac{x^n}{n!} + R_n$

Proof (Sketch) – for simplicity $x = x_0$

Leibniz-Newton $f(x) - f(0) = \int_0^x s' f'(s)\, ds =$

$= s f(s) \Big|_0^x - \int_0^x s f''(s)\, ds =$ we want to do it more complicated

---

$f(x) - f(0) = \int_0^x (s-x)' f'(s)\, ds =$

$= (s-x) f'(s) \Big|_0^x - \int_0^x (s-x) f''(s)\, ds =$

$= 0 + x f'(0) - \int_0^x (s-x) f''(s)\, ds$

$f(x) = f(0) + f'(0) \cdot x - \underbrace{\int_0^x (s-x) f''(s)}_{\text{integral rem.}}$

2 steps:

1. general formula

$f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \dots + \frac{f^{(n)}(0)}{n!} \cdot x^n +$

$\underbrace{+ (-1)^n \int_0^x (s-x)^n \cdot \frac{f^{(n+1)}(s)}{(n+1)!}\, ds}_{\bar{R}_n}$

2. show that $\overline{R_n} = R_n$

Homework: f cont, diff (f' cont)

- $f'(x) \geq 0 \rightarrow f$ non decreasing
- $f'(x) \leq 0 \rightarrow f$ non increasing

$f : I \rightarrow \mathbb{R}$

Let $x_1, x_2 \in I$ s.t. $x_1 < x_2$

- we apply "L" Lagrange on $[x_1; x_2]$

$$f(x_2)-f(x_1)=(x_2-x_1)\cdot f'(c)\ ,\ c\in[x_1;x_2]$$

$$\left.\begin{array}{l} c\in[x_1;x_2]\subset I \Rightarrow f'(c)\geq 0 \\ x_1<x_2 \Rightarrow x_2-x_1>0 \end{array}\right\} \Rightarrow (x_2-x_1)\cdot f'(c)\geq 0$$

$$\Rightarrow f(x_2)-f(x_1)\geq 0$$

$$f(x_1)\leq f(x_2)$$

$\Rightarrow f$ non decreasing on I

Let $x_1, x_2 \in I$ s.t. $x_1 \quad x_2$

- we apply $\boxed{T}$ Lagrange on $[x_1;x_2]$

$$f(x_2)-f(x_1)=(x_2-x_1)\cdot f'(c)$$

$$\left.\begin{array}{l} c\in(x_1;x_2)\subset I \Rightarrow f'(c)\leq 0 \\ x_1<x_2 \Rightarrow x_2-x_1\geq 0 \end{array}\right\} \Rightarrow (x_2-x_1)\cdot f'(c)\leq 0$$

$$\Rightarrow f(x_2)-f(x_1)\leq 0 \Rightarrow f(x_1)\geq f(x_2)$$

$\Rightarrow f$ non increasing on I

Course 2

2. Calculus for functions of several variables I

(Partial derivatives)

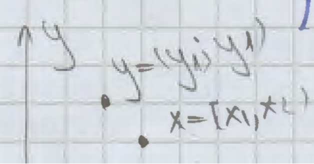
Why fs of several var? $\qquad f(x)\ , x\in\mathbb{R} \Leftarrow$ NOT ENOUGH

NOT ENOUGH

Vol. of cone $\quad V(r,h)=\dfrac{\pi r^2 h}{3}$

2.1. The geometry of the Euclidian space $\mathbb{R}^n$ $(n\geq 1)$

- Operations on $\mathbb{R}^n$



axe de coordonate

<=

elem of R -> $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$

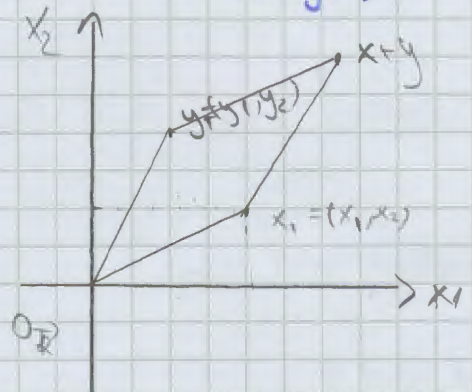
$x_k \in \mathbb{R}$ $\quad k = \overline{1,n}$
component

$y = (y_1, y_2, \dots, y_n)$

- Addition "+"

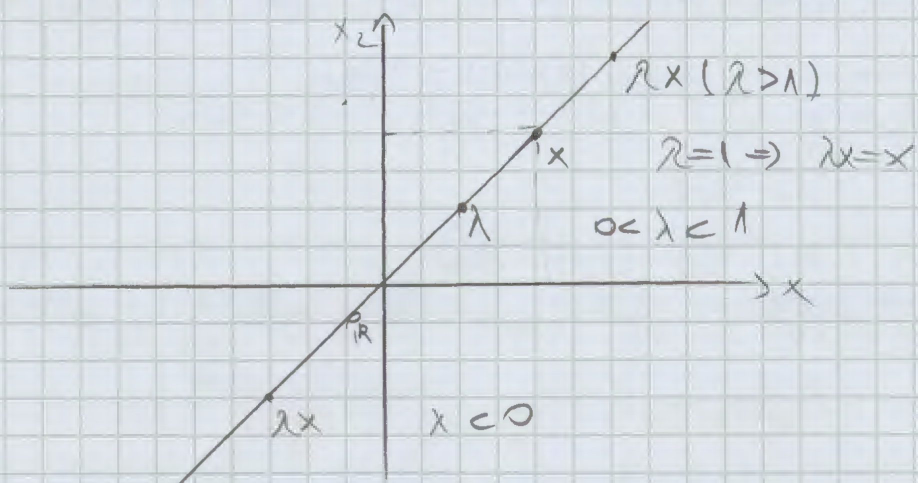
$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n) =$

$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$



- Multiplication by a scalar $\lambda \in \mathbb{R}$

$\lambda x = \lambda(x_1, x_2, \dots x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$



! Consider these objects to be matrices

- Inner product (Ro: produs scalar) "dot product"

$x \cdot y = (x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) =$
$= x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

RK: a) if n = 1 (inner product =standard mult on R
b) x*(y+z)=x*y+x*z
x*y=y*x

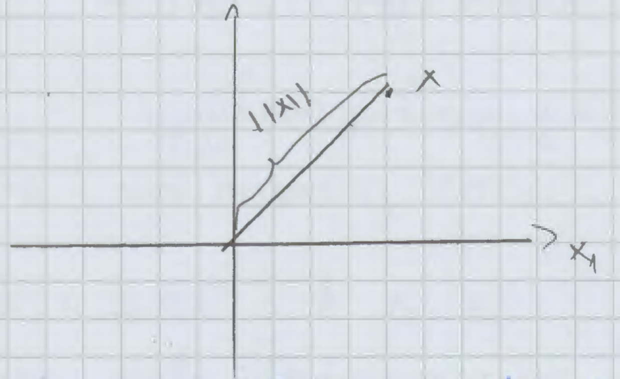
inner product = standard mult. on $\mathbb{R}$

$\lambda x \cdot y = (\lambda x) \cdot y = x \cdot \lambda y$

- Distance and length

<u>the Euclidian norm</u>

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$



"length of segment enclosing x and $0_{\mathbb{R}^n}$"

<u>The distance between x and y</u>

$\|x-y\|$ $(=: d(x,y))$ $\{ \|x-y\| = \|y-x\|$

Rk: if $n=1$ $\|x\| = |x|$ abs. value of x

$n=1$, then $\|x-y\| = |x-y|$

Hw: Prove that:

(N1) $\|x\| = 0 \Leftrightarrow x = 0_{\mathbb{R}^n}$

(N2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}$

(N3) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$

triangle inequality

(d1) $d(x,y) = 0 \Leftrightarrow x = y$

(d2) $d(x,y) = d(y,x)$, $\forall x, y \in \mathbb{R}^n$

(d3) $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x, y \in \mathbb{R}^n$

- Orthogonality

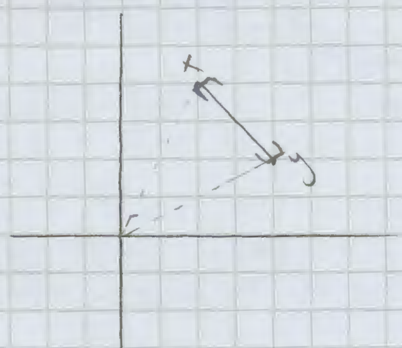
we say that $x \perp y$ (orthogonal) $\Leftrightarrow x \cdot y = 0_{\mathbb{R}}$

$\theta_{x,y} = \arccos\left(\frac{x \cdot y}{\|x\| \cdot \|y\|}\right)$ angle $\measuredangle x, y$

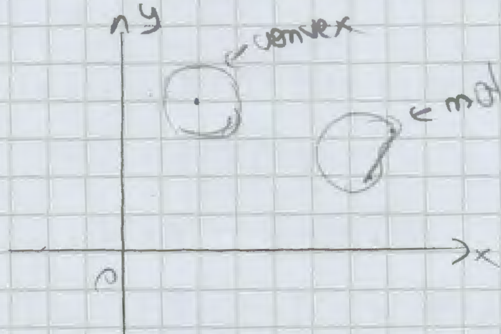
• Segments and convex sets

(x,y) $\qquad\qquad\qquad\qquad \alpha\in(0;1)$

$$[x,y] = \{ z \in \mathbb{R}^n : \exists\, \alpha\in[0;1] \text{ s.t. } \cancel{z=\alpha x+(1-\alpha)y} \;\; \underset{\substack{\uparrow \\ \text{avg}}}{z} = (1-\alpha)x + \alpha y \}$$



$C \subseteq \mathbb{R}^n$ is convex $\Leftrightarrow$ $\forall x,y\in C$ we have $[x;y] \subset C$

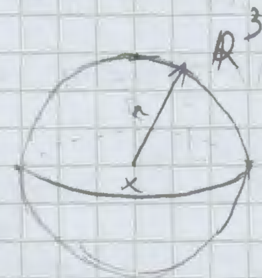


• The ball

$$B_r(x) = \{ z\in\mathbb{R}^n : \|z-x\| < r \} \quad \text{open ball}$$

r: radius

$$\overline{B_r}(x) = \{ z\in\mathbb{R}^n : \|z-x\| \le r \} \quad \text{closed ball}$$



Hw: prove that $B_1(0)$ is a convex subset of $\mathbb{R}^n$

Rk: if $n=1$ $\quad B_r(x) = (x-r, x+r)$

The canonical basis of $\mathbb{R}^n$ (and decompositions)

What are the simplest elem of $\mathbb{R}^n$

$0_{\mathbb{R}^n} = (0, 0, \dots, 0)$ $\quad 0_{\mathbb{R}^n} + x = x + 0_{\mathbb{R}^n} = x$

$0_{\mathbb{R}^n} \cdot x = 0_{\mathbb{R}^n}$

$$\left.\begin{array}{l} e_1 = (1, 0, \dots, 0) \\ e_2 = (0, 1, \dots, 0) \\ \dots \\ e_n = (0, 0, 0, \dots, 0, 1) \end{array}\right\} \quad Rk: \quad \|e_i\| = 1, \quad e_i \, e_j = 0 \quad \forall i \neq j$$

for any $x = (x_1, x_2, \dots, x_n) = (x_1, 0, \dots, 0) + (0, x_2, \dots, 0) + \dots$

$+ (0, 0, \dots x_n) =$

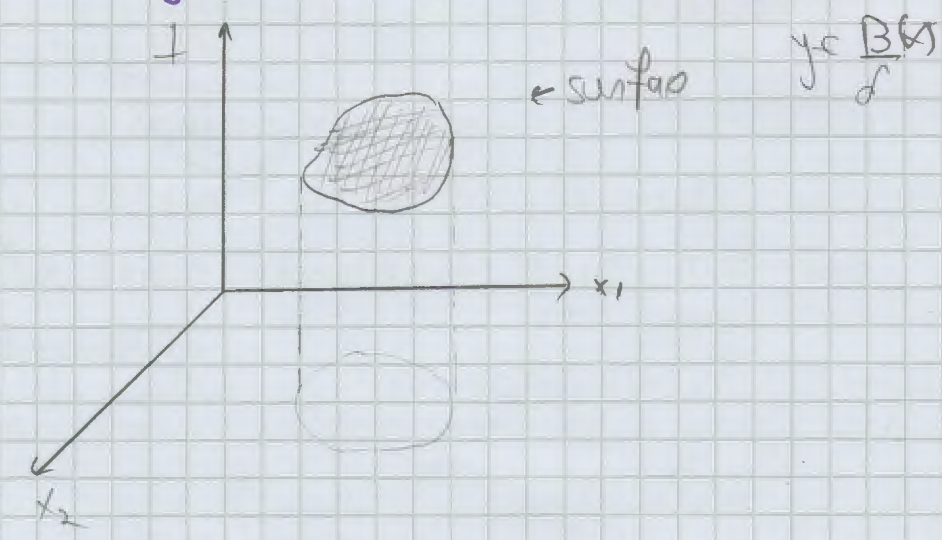
$= x_1 (1, 0, \dots, 0) + x_2 (0, 1, \dots, 0) + \dots + x_n (0, \dots, 0, 1)$

$= x_1 e_1 + x_2 e_2 + \dots + x_n e_n =$

$\overset{\text{claim}}{=} \underbrace{(x \cdot e_1)}_{\text{projection}} e_1 + (x \cdot e_2) e_2 + \dots (x \cdot e_n) e_n$

HW: prove $x \cdot e_i = x_i$

Def $f : \mathbb{R}^n \to \mathbb{R}$ is continuous in a given point $x = (x_1, \dots x_n)$ is $\forall \varepsilon > 0 \ \exists \ \delta = \delta(x, \varepsilon) > 0$ s.t. $\forall y \in \mathbb{R}^n$ with the proper[ty] that $\|x - y\| < \delta$, we have $|f(x) - f(y)| < \varepsilon$



2.2. Partial derivatives and the Gradient of a function

Def $f: \mathbb{R}^n \to \mathbb{R}$ $f(x_1, \dots, x_n)$

we say that f has a partial derivative with respect to $\underset{k\text{th variable}}{x_k}$ in a point $a = (a_1, \dots, a_n)$ if

$$\lim_{x_k \to a_k} f(x_k)$$

$$\frac{\partial f}{\partial x_k}(a) \overset{\text{not}}{=} \lim_{x_k \to a_k} \frac{f(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n) - f(a_1, \dots, a_{k-1}, a_k, \dots, a_n)}{x_k - a_k}$$

exists

f is diff-able w r t x_k in a if the limit exists and is finite

The IDEA : fix all but the k-th the diff-ate function of single variable

$f|a$.

Higher order partial derivatives

$$\frac{\partial^2 f}{\partial x \, \partial x_j} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x_j}\right)$$

T (Schwarz H. A.)

if $f: \mathbb{R}^n \to \mathbb{R}$ admits partial derivatives of order 2 wrt x_i, x_j in a ball around $a = (a_1, \dots, a_n)$ and if these derivatives are continuous, then

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \, \partial x_i}(a)$$ (order doesn't matter)

Partial derivatives give only partial info

The gradient of f

mable $\rightarrow$ $\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right) \in \mathbb{R}^n$

The directional derivative $\frac{\partial f}{\partial y}(a) = \nabla f(a) \cdot y$

in the direction $y \in \mathbb{R}^n$

$\boxed{T}$ (special chain rule)

$f: \mathbb{R}^n \to \mathbb{R}$ has cont $\frac{\partial f}{\partial x_i}$ and

define a curve on $\mathbb{R}^n$ $\overrightarrow{x_1, \dots, x_n} : [a;b] \subset \mathbb{R} \to \mathbb{R}$ diff

then

$F: [a;b] \to \mathbb{R}$ $F(t) = f(x_1(t), \dots, x_n(t))$

$F = f \circ (x_1, \dots, x_n)$

$$\frac{dF}{dt} = \nabla f(x_1(t), \dots x_n(t)) \cdot \left(\frac{dx_1}{dt}, \dots \frac{dx_n}{dt} \right)$$

$$= \nabla f \cdot \frac{dx}{dt}$$

$\boxed{T}$ (Lagrange for $f(x_1, \dots, x_n)$)

$C \subseteq \mathbb{R}^n$ convex $a, b \in C$, $a \neq b$

$f: C \to \mathbb{R}$ has cont partial derivatives

Then $\underbrace{f(b) - f(a) = \nabla f(c) \cdot (b-a)}_{\text{there exists } c \in [a,b] \text{ s.t.}}$

f'(c)(b-a)

Lagrange $f: [a,b] \subset \mathbb{R} \to \mathbb{R}$, $\exists$ $c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$

Proof: idea: Apply $\boxed{T}$ (Lagrange) and chain rule to

$F(t) = f(a + t(b-a))$ to $[0;1]$

Course 3

Calculus for functions of several variables II (The Fréchet differential / Optimality)

3.1. Recap

- $\mathbb{R}^n$ $\quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$ elem of $\mathbb{R}^n$

$x \cdot y = x_1 y_1 + \dots + x_n y_n$ inner product

$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ - how far x is from the origin

the norms of x

$d(x,y) = \|x - y\|$ distance (or metric)

The open ball $B_r(x) = \{ y \in \mathbb{R}^n : \|x - y\| < r \}$

- $\frac{\partial f}{\partial x_i}$ $\qquad f : \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial f}{\partial x_i}(a) = \frac{d}{dx} f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$

$x = a_i$ usual deriv for $f(a, \dots, \cdot, \dots a)$

The gradient of f

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$

The $\mathbb{R}^n$ counterpart of f'

T (Lagrange $\mathbb{R}^n$)

$f : C \subseteq \mathbb{R}^n \to \mathbb{R}$, all $\frac{\partial f}{\partial x_i}$ cont

convex

$a, b \in C \quad a \neq b$

Then $\exists\, c \in [a; b]$ s.t. $f(b) - f(a) = \nabla f(c)(b - a)$

Consequence :

$f : B_1(x^*) \to \mathbb{R}$, all $\frac{\partial f}{\partial x}$ cont

and $\nabla f(x) = 0_{\mathbb{R}^n}$, $\forall\, x \in B_1(x^*)$

Then f is constant on $B_1(x^*)$

PROOF - TL

3.2. Linear functions

Def Let $T : \mathbb{R}^n \to \mathbb{R}$ linear if

(i) $T(x+y) = T(x) + T(y)$, $\forall x, y \in \mathbb{R}^n$

(ii) $T(\alpha x) = \alpha T(x)$, $\forall \alpha \in \mathbb{R}$, $x \in \mathbb{R}^3$

Rk: · (i) & (ii) $\Leftrightarrow$ $T(\underbrace{\alpha x + \beta y}_{\text{linear combination}}) = \alpha T(x) + \beta T(y)$

· iii) & induction

$$\text{iii})_k \quad T\left(\sum_{j=1}^{k} \alpha_j v_j\right) = \sum_{j=1}^{k} \alpha_j T(v_j)$$

· $T(0_{\mathbb{R}^n}) = 0$ (take $\alpha = 0$ in ii)

T (Representation of linear functions)

$T : \mathbb{R}^n \to \mathbb{R}$ linear Then

$\exists!\, a_T \in \mathbb{R}^n$ s.t. $T(x) = a_T \cdot x$

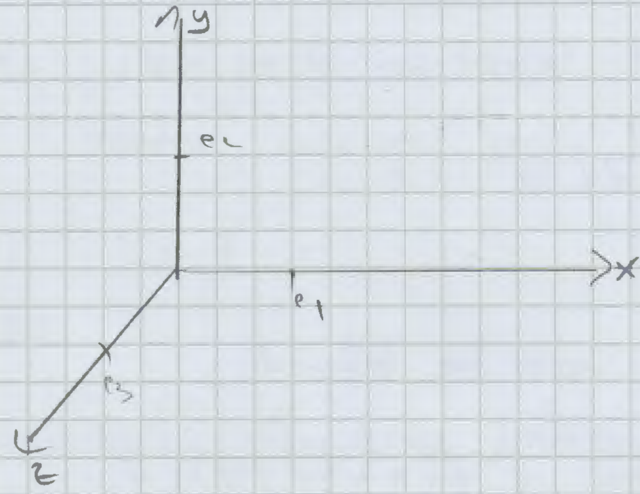
IDEA of PROOF

$e_1 = (1, 0, \dots, 0)$

$e_2 = (0, 1, \dots, 0)$

$\vdots$

$e_n = (0, 0, \dots, 1)$



$\forall\, x=(x_1,\dots,x_n)$ we have

$$x = \sum_{i=1}^{n} x_i e_i \quad \text{to this apply } T$$

$$T(x) = T\Big(\sum_{i=1}^{n} x_i e_i\Big) = \underbrace{\sum_{i=1}^{n} x_i T(e_i)}_{\text{innerproduct}} = x \cdot a_T = a_T \cdot x$$

uniqueness

$$a_T = (T(e_1), T(e_2), \dots, T(e_n))$$

3.3. The Fréchet differential - 1878-1973

Def $f : \mathbb{R}^n \to \mathbb{R}$ is called F-diff at a point $x^* \in \mathbb{R}^n$ if $\exists$ a linear function $T : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\cancel{\lim_{h\to 0}}$$

$$\lim_{x \to x^*} \frac{|f(x) - f(x^*) - T(x - x^*)|}{\|x - x^*\|} = 0$$

if such a T exists not: $df(x^*) = T$

$\boxed{T}$ $df(x^*)$ is unique (if it exists)

$\boxed{T}$ f F-diff $\Rightarrow$ f cont at x^*

$\boxed{T}$ $f : \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial f}{\partial x_i}$ cont

then $\underbrace{df(x^*)}_{T}(x) = \nabla f(x^*) \cdot x$

- higher order F-diff can be defined

$$d^{(k+1)} f(x^*)(x) = \frac{d}{dt}\left(d^k f(x^* + tx)(x) \right) \Big|_{t=0}$$

$\boxed{T}$ f is $n+1$ times F-diff then $\xi \in [x; x^*]$

$$f(x) = f(x^*) + \frac{df(x^*)(x-x^*)}{1!} + \frac{d^2 f(x^*)(x-x^*)}{2!} + \dots + \frac{d^n f(x^*)(x-x^*)}{n!} + \frac{d^{n+1} f(\xi)(x-x^*)}{(n+1)!}$$

3.4. Optimality conditions

Def $f : \mathbb{R}^n \to \mathbb{R}$

x^* local min / max for f if there exists $r > 0$ s.t.

$f(x) \overset{\min}{\underset{\max}{\geq / \leq}} f(x^*)$, $\forall x \in B_r(x^*)$

Def: f has all $\frac{\partial f}{\partial x_i}$ cont

x^* stationary (crit' point) for f if

$\nabla f(x^*) = 0_{\mathbb{R}^n}$ $(\Leftrightarrow df(x^*) = 0_T)$

$\boxed{T}$ Fermat ($\mathbb{R}^n$)

$f : B_r(x^*) \to \mathbb{R}$ f F-diff at x^*

x^* local min/max $\Rightarrow \nabla f(x^*) = 0_{\mathbb{R}^n}$

necessary optimal condition not reff

Rk: There exist nontrivial critical points that are neither min nor max

1D $\cup$ $f''>0$ $\cap$ $f''<0$
$(\mathbb{R})$

• Quadratic functions (forms) - forme pătratice

<u>Def</u> $Q:\mathbb{R}^n \to \mathbb{R}$ quadratic if

$\exists A = (a_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ with $a_{ij} = a_{ji}$

s.t. $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ $a_{ij} \cdot x_i \cdot x_j$

$x = (x_1, \dots, x_n)$

- generalises $f(x) = ax^2 \in \mathbb{R}$

<u>Def</u>: Q quadratic then Q is:
- positive definite if $Q(x) > 0$, $\forall x \neq 0_{\mathbb{R}^n}$
- negative $Q(x) < 0$

$\boxed{\text{Th}}$ (Sylvester)

• Q is pos def

$$a_{11} > 0 \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots & \dots \\ \dots \\ a_{n1} & \dots & \dots & a_{nn} \end{vmatrix} > 0$$

• Q is neg def

$$a_{11} < 0 \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, (-1)^n \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots \\ \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} > 0$$

Rk: Let f be twice F-diff. Then

$d^2 f(x^*)$ is a quadratic form with

$$A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x^*)\right)_{i,j=\overline{1,n}} \quad \text{Hessian matrix}$$

$\boxed{\text{II}}$ - sufficient Opt cond

f with al $\frac{\partial^2 f}{\partial x_i \partial x_j}$ cont

and x^* s.t. $\nabla f(x^*) = 0_{\mathbb{R}^n}$

then
- $d^2 f(x^*)$ is pos def $\Rightarrow$ x^* min
- $d^2 f(x^*)$ neg def $\Rightarrow$ x^* max
- $d^2 f(x^*)$ change sign $\Rightarrow$ x^* neither min nor max

$$H_f(x^*) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x^*)\right) \quad \text{Sylvester prop.}$$

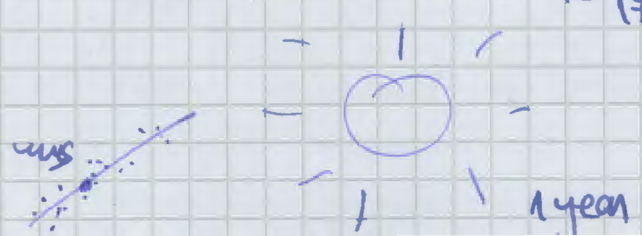
IDEA: Taylor with 3 terms

<u>Course 4</u>

4. Applications. Method of Least Squares. Gradient Decent. Lagrange Multiplier Method

IDEA K.F. Gauß: Predicted the position of Ceres

Probl. Ceres (asteroid) ~ 1795



The Model (Linear Regression)

- measurements

x	$x_1 \;\ldots\; x_i \ldots\; x_n$
y	$y_1 \;\cdots\; y_i \cdots\; y_n$

- a function (a family of functions)

$$f_{a,b}(x) = ax+b$$

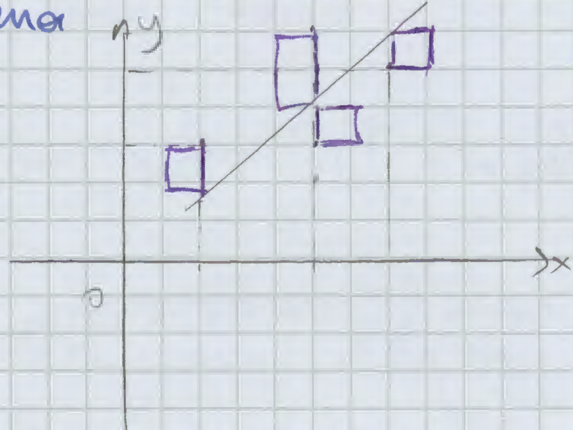
AIM: find the param. values a^*, b^* such that
f_{a^*,b^*} "fits the data" (in the best possible way)

This is an Optimiz. Probl!!

$$E(a,b) = \sum_{i=1}^{n} (y_i - (ax_i+b))^2 \to \min!$$

error

quadratic, diff-able, one min



To find a^*, b^* that minimize E use lecture #3

$$\nabla E(a^*, b^*) = 0$$

$$\begin{cases} \dfrac{\partial E}{\partial a}(a^*, b^*) = 0 \\ \dfrac{\partial E}{\partial b}(a^*, b^*) = 0 \end{cases} \Rightarrow \begin{cases} \sum_{i=1}^{n} (y_i - (a^*x_i + b^*)) \cdot x_i = 0 \\ \sum_{i=1}^{n} (y_i - (a^*x_i + b^*)) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sum x_i^2 \, a^* + \sum x_i \, b^* = \sum x_i y_i \\ \sum x_i \, a^* + n b^* = \sum y_i \end{cases}$$

Notations

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$a^* = \frac{\sum_{i=1}^{n}(x_i-\bar{x})\, y_i}{\sum(x_i-\bar{x})} \qquad b^* = \frac{\sum_{i=1}^{n} x_i\,(\bar{y}x_i - \bar{x}y_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

2. The gradient descent algorithm

const L
E

$E : \mathbb{R}^n \to \mathbb{R}$ convex, ∇E Lipschitz cont with co

(GD) $x^{k+1} = x^k - h \cdot \nabla E(x^k)$ $\quad 0 < h < \frac{1}{L_E}$

elem $k+1$

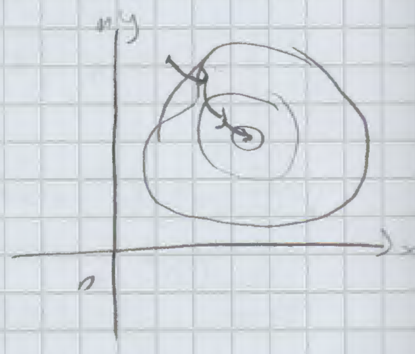
step size

"HOPE" $x^k \xrightarrow{k\to\infty} x^*$

minimizer of E (i.e. $\nabla E(x^*) = 0$)

IDEA: Cauchy

- you take a step of "length h" in the direction of the gradient



- rate of convergence

$$|E(x^k) - E(x^*)| \leq \frac{C(L_E)}{k}$$

From GD to Gradient Flows

in (GD) $\frac{x^{k+1}-x^k}{h} = -\nabla E(x^k)$

let $h \to 0$ $\qquad \lim_{h\to 0} \frac{x(t+h)-x(t)}{h} = -\nabla E(x(t))$

$x(kh) = x^k$

set $kh = t$ $\quad \leftarrow$ "continuous flow"

Gradient Flow $\qquad \frac{dx}{dt}(x) = -\nabla E(x(t))$ GF

cont. version of G.D.

$x(t)$ a trajectory

Nesterov : Accelerated Gradient Decent

$$x^{k+1} = y^k - h\nabla E(y^k)$$
$$y^k = x^k + \underset{\left(\frac{k-1}{k+2}\right)}{\alpha_k}\,(x^k - x^{k-1})$$

rate of conv $\quad |E(x^k) - E(x^*)| \leq \frac{C}{k^2}$

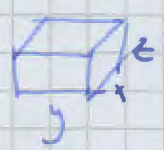
2016 Claim that Nesterov's Accel GD is a discrete version of :

$$\frac{d^2x}{dt^2}(t) + \frac{r}{t}\frac{dx}{dt}(t) = -\nabla E(x(t))$$

Diff Eq.

3. The Lagrange Multiplier Method

- the box of ~~value~~ volume 1 and least surface



$Vol = xyz = 1 \qquad g(x,y,z) = xyz - 1$

$Sur = 2(xy + yz + xz) \rightarrow$ minimize

more gen
subject to $\left\{\begin{array}{l} f \to \min! \\ g = 0 \end{array}\right.$ conditional optimization

we will look at $\left\{\begin{array}{l} f(x,y) \to \min \\ g(x,y) = 0 \end{array}\right.$

Curves in the plane

- parametric description

$$C \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [a, b]$$

Example: the circle

$$x = \cos t \quad t \in [0; 2$$
$$y = \sin t$$

[0; 2pi]

- implicit description (using an eq)

$F(x,y) = 0$ (*) $\quad x^2 + y^2 - 1 = 0$ $\quad x^2 = \cos^2 t$

$y^2 = \sin^2 t$

You may not be able to solve

The Folium of Descartes

$x^3 + y^3 - 3xy = 0$ ~ 1638

$\boxed{\text{U}}$ The implicit function Thm — gives a local param. for an implicit curve)

Ass that $F : A \times B \to \mathbb{R}$ satisfies:

(i) $F(x^*, y^*) = 0$

(ii) $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ cont on $A \times B$

(iii) $\frac{\partial F}{\partial y}(x^*, y^*) \neq 0$

Then $\exists I^*, J^*$ and $f : I^* \to J^*$ s.t

(a) $f(x^*) = y^*$

(b) $F(x, f(x)) = 0, \forall x \in I^*$

(c) f is diff-able on I^* and

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}$$

- Level sets (level curves)

$C_\lambda^f = \{ (x,y) \in \mathbb{R}^2 : f(x,y) = \lambda \}$ (may be empty)

Rk : The gradient ∇f is orthogonal to level curves

Ass. that C_λ^f is param C_λ^f $\begin{cases} x = f(t) \\ y = f(t) \end{cases}$

$f(x(t), y(t)) = \lambda$ (holds)

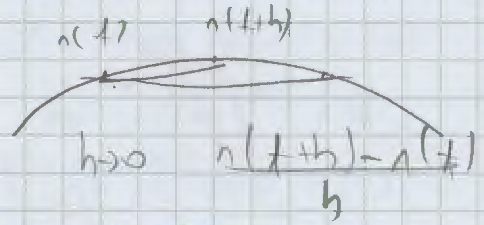
x, y, f diff-able

$$\underbrace{\frac{d}{dt} f(x(t), y(t))}_{} = 0 \quad [f = \lambda \text{ const}]$$

$$= \nabla f(x(t), y(t)) \cdot \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right) \quad \text{i.e.}$$

$$\nabla f(x(t), y(t)) \perp \underbrace{\left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)}_{\text{tangent to } C_\lambda^f}$$

"velocity"



Informal 'U' (Lagrange multipl. meth)

Ass x^*, y^* min and for $f(x,y) \to \min!$

$g(x,y) = 0$

then $\exists\, \lambda^* \in \mathbb{R}$ s.t.

Lagrange multiplier

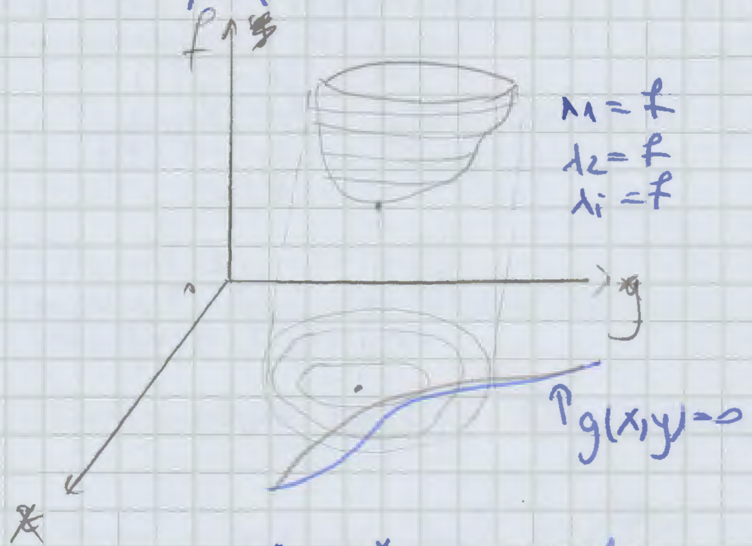
(x^*, y^*, λ^*) is a critical point for the associated Lagrangian

$$L(x, y, \lambda) = f(x,y) - \lambda\, g(x,y)$$

- addition variable λ and new function L

but a standard (no constraint) Optimiz. problem

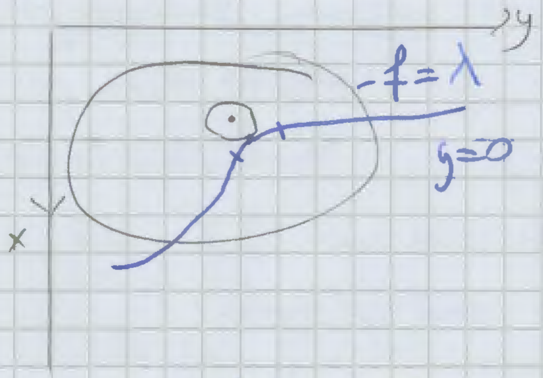
iDEA of proof



ass x^*, y^* is cond min (is part $g(x^*,y^*)=0$)

apply Implicit function Thm to get a local param of $g=0$ around (x^*,y^*) $\to$ $h : I \subset \mathbb{R}$, $h(t) = f(x(t),$ y(A))

on I, $x(t), y(t)$ exist



has a min at t^* for whi

$x(t^*) = x^*$ and $y(t^*) = y^*$

$h'(t^*) = 0 = \frac{dh}{dt}\Big|_{t=t^*}$

$0 = \frac{dh}{dt}\Big|_{t=t^*} \overset{\text{chain rule}}{=} Df(x(t),y(t))\Big|_{t=t^*} \cdot \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)\Big|$ t = A*

$Df(x^*,y^*) \perp C_0^g$: the curve $g=0$

but we know

$Df(x^*,y^*) \perp C_{\lambda^*}^f$

$\lambda^* = f(x^*,y^*)$

$\Rightarrow$ C_0^g, $C_{\lambda^*}^f$ have the same tangent

$\nabla f = \lambda \nabla g$

Course 5

Part I: Differential Calulus (Recap)

1. $f: \mathbb{R} \to \mathbb{R}$

local extrema vs critical points

$f(x^*) \overset{max}{\geq} f(x)$ $\quad f'(x^*) = 0$

$\underset{m}{\leq} f(x)$

$\forall x \in [x^* - \varepsilon; x^* + \varepsilon]$

'u' Fermat

x^* local min/max
f diff at x^* | $f'(x^*) = 0$

Nec. opt cond

'u' Lagrange

$f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a;b)$

$f(b) = f(a) + \frac{f'(c)}{1!}(b-a)$ hence $n \to \infty$

IDEA TAYLOR POLYN

$f(x) \approx T_n(x)$

locally around x_0

'u' Taylor

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Taylor polynomial of degree } n}$$

$+ f^{(n+1)}(c)$ $\quad \exists c \in (x, x_0)$

$\boxed{T}$ (Apply Taylor with a quadratic polyn.)

Ass $f'(x^*) = 0$

(i.e. x^* ~~cont~~ crit point)

$f''(x^*) > 0 \Rightarrow x^*$ min $\cup$

$f''(x^*) < 0 \Rightarrow x^*$ max $\cap$

Nec & Suf cond of opt

2 & 3 $f: \mathbb{R}^n \to \mathbb{R}$

· The geom. of $\mathbb{R}^n$

balls $B_1(a) = \{x \in \mathbb{R}^n : \|x-a\| < 1\}$

$x = (x_1, \dots, x_n) \in \mathbb{R}$, $x_i \in \mathbb{R}$

$d(x,y) = \|x-y\| = \sqrt{(x-y)(x-y)}$

The inner product $\xrightarrow{\text{defines}}$ distances (length)

$x \cdot y = x_1 y_1 + \dots + x_n y_n$ angles

· Partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_j x_i}$ $\left(= \frac{\partial^2 f}{\partial x_i x_j} \right)$ ← Schwartz

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \to$ gradient $\longrightarrow f'$

The Hessian $H_f(x) = \left(\frac{\partial^2 f}{\partial x_j \partial x_i} \right)_{i,j = \overline{1,n}} \longleftrightarrow f''$

$\nabla^2 f(x)$

$\boxed{T}$ Fermat $\mathbb{R}^n$

$\nabla f(x^*) = 0_{\mathbb{R}^n}$ nec opt cond

$\boxed{T}$ nec & suf opt cond $\mathbb{R}^n$

$\nabla f(x^*) = 0_{\mathbb{R}^n}$

$H_f(x^*)$ pos def $\Rightarrow x^*$ local min

neg def $\Rightarrow x^*$ local max

$\boxed{\text{U}}$ Lagrange $\mathbb{R}^n$

$f(b) - f(a) = \nabla f(c)\,(b-a) \qquad c \in [a;b]$ segm.

1. Appl of Optimization $\leftarrow$ Machine Language-training
   - Least Squares (fitting functions to data) neural network
   - Opt Algorithm (Gradient Decent, Nesterov)

- Constrain Optimization

  $f(x, y, \ldots) \to \min!$

  subject to $g(x, y, \ldots) = 0$

Ex Box of fixed vol = 1

minimal surface

Surface $= f(x,y,z) = 2xy + 2xz + 2yz \to \min!$

Vol $= xyz - 1$

IDEA: use Lagrange multipliers

" $\boxed{\text{LI}}$ " (without regularity assumptions)

if (x^*, y^*) ~~...~~ cond min / max for (P)

Then:

Where L is the Lagrangian

$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$\lambda \in \mathbb{R}$ Lagrange multiplier

$$(*)\quad \begin{cases} \dfrac{\partial L}{\partial x}(x^*, y^*, \lambda^*) = 0 \\ \dfrac{\partial L}{\partial y}(x^*, y^*, \lambda^*) = 0 \\ \dfrac{\partial L}{\partial \lambda^*}(x^*, y^*, \lambda^*) = 0 \end{cases}$$

$\nabla_{x,y,\lambda} L = 0$

$(*)$ nec cond for constr. opt

for suf cond you need to look at
$d^2L(x^*, y^*)$ pos de $f \to$ min
neg def $\to$ max

<u>Rk</u> $\mathbb{R}^n$ cannot be ordered for $n \geq 2$
$x \overset{?}{\leq} y$ in $\mathbb{R}$
$x \leq y \Leftrightarrow x_i \leq y_i$, $\forall i = \overline{1,n}$

Part II : Integral Calculus

5. Antiderivatives and the Rieman integral

5.1. Antiderivatives (or primitive functions or indefinite integral

~~Diff~~: Easy   Int = NOT

<u>Def</u> F is an antid of f
if F diff and $F' = f$ this is an equation
$F = ?$ given $F' = f$

Elem. functions: x^p, a^x, $\sin x$
Simple functions: combine Elem f's using $+, -, \times, \div$
a finite n. of times (e^x)

Derivatives of simple fs and simple
Anti derivative may not be simple

$\int e^{x^2} dx$, $\int \frac{\sin x}{x} dx$, $\int \frac{1}{\ln x} dx$

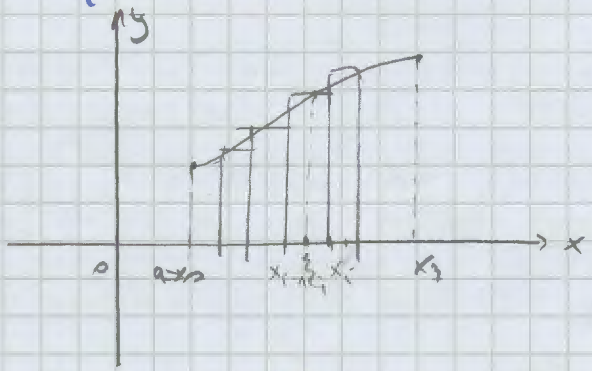
5.2. The Riemann integral
$[a;b] \subseteq \mathbb{R}$ cl compact

Δ - a division on $[a,b]$

$\Delta = \{ a = x_0, x_1, \dots, x_i, \dots, x_n = b \}$ with $x_i < x_{i+1}$

$\xi = \{ \xi_1, \dots, \xi_n \}$ - set of intermediate points

$\xi_i \in [x_{i-1}, x_i]$



~~Area ≈ σ(f, Δ, ξ~~

Area $\approx \sigma(f, \Delta, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$ Riemann Sum

Δ_2 is a refinement of Δ_1 if

Δ_2 contains more points and $\Delta_1 \subset \Delta_2$

<u>Def</u> $f : [a,b] \to \mathbb{R}$ is Riemann integrable

if $\exists\, I \in \mathbb{R}$ s.t.

$\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ and $\exists\, \Delta_\delta$ with

$\max_i |x_i - x_{i-1}| < \delta$ and ξ corresp to Δ_δ :

$$|I - \sigma(f, \Delta, \xi)| < \varepsilon$$

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \varepsilon$$

if f Riemann integrable, we denote I (called Riemann integral) by = $I = \int_a^b f(x)\,dx$

$\boxed{T}$ f cont $\Rightarrow f$ integrable

$\boxed{T}$ Leibniz - Newton

f R integrable and F antideriv of f

then $\int_a^b f(x)\,dx = F(b) - F(a)$

T Ass that f cont

Then $F(x) = \int_a^x f(s)\,ds$ is an antiderivative of f

Properties:

- f, g R. int then so is $\alpha f + \beta g$ $(\alpha, \beta \in \mathbb{R})$

$$\int_a^b \alpha f(x) + \beta (g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

- $f(x) \le g(x) \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
- if $|f(x)|$ is R-int

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$$

- f cont, g R integrable. Then $\exists$ c

$$\int_a^b f(x) \cdot g(x)\,dx = f(c) \int_a^b g(x)\,dx$$

- $\int_a^b f \cdot g' = fg \Big|_a^b - \int_a^b f' \cdot g$
- f cont u diff $: [a,b] \to J$

$f : J \to \mathbb{R}$

$$\int_a^b f(u(x)) \cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(t)\,dt$$

Course 6

6. Multiple integrals

$f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ $\qquad \int_a^b f(x)\,dx$ RIEMANN

today: $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ $\qquad \int_D f(x)\,dx$???

6.1. Jordan measurability measure

Aim: generalize length (1D), area (2D), volume (3D) to $\mathbb{R}^n$ (arbitrary dim) context

· Let $R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$

n-dimensional (closed) rectangle (or simply - "box")

$a_i < b_i$

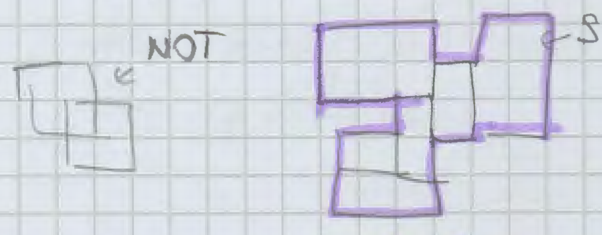
$i = \overline{1, n}$

· The measure of R is $m(R) = (b_1 - a_1)(b_2 - a_2) \dots (b_n - a_n)$

Def $S \subset \mathbb{R}^n$ is called simple if $\exists$ ~~k≥0~~, $k \in \mathbb{N}$ and R_i rectangles $i = \overline{1, k}$ s.t.

(*) $S = \bigcup_{i=1}^{k} R_i$ and $\text{int } R_i \cap \overset{\text{int}}{R_j} = \emptyset$ $\forall i \neq j$

where $\text{int } R = (a_1, b_1) \times \dots \times (a_n, b_n)$



· if S is simple (*) then the measure of S is $m(S) = m(R_1) + \dots + m(R_n)$

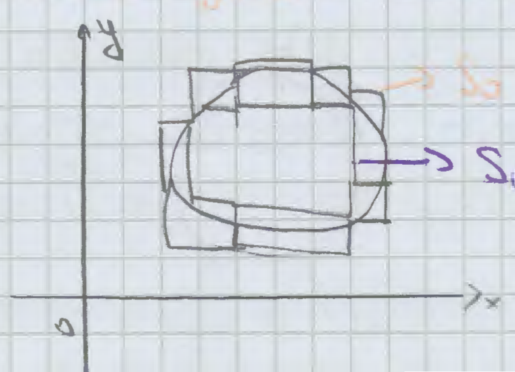
· Let $D \subset \mathbb{R}^n$ bounded ($\exists r > 0$ s.t. $D \subset B_r(0_{\mathbb{R}^n})$)
contained in a large enough ball

$m_i(D) = \sup\{ m(S_i) : S_i \text{ simple and } S_i \subseteq D \}$

$m_e(D) = \inf\{ m(S_o) : S_o \text{ simple and } S_o \supseteq D \}$

- → Inner Jordan measure of D

- → Outer Jordan measure of D



Def A bounded set $D \subset \mathbb{R}^n$ is Jordan measurable if $m_i(D) = m_o(D)$. This common value is denoted by $m(D)$. $m(\emptyset) = 0$

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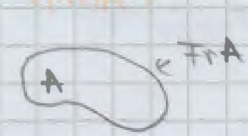
T (Properties of the Jordan measure)
let $A, B \subset \mathbb{R}^n$ be Jordan measurable
(i) $\operatorname{int} A \cap \operatorname{int} B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$
(ii) $A \subseteq B \Rightarrow m(A) \leq m(B)$
(iii) $m(B) = 0 \Rightarrow m(A \cup B) = m(A)$
(iv) $m(A) = m(\operatorname{int} A)$

T (Charact. of Jordan measurability)
A is Jordan meas. if $\operatorname{Fr} A = A \setminus \operatorname{int} A$ is Jord. meas. and $m(\operatorname{Fr} A) = 0$

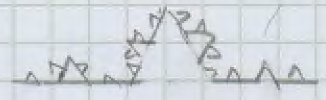
T (in $\mathbb{R}^2$)



A is J.m iff Fr A is a piecewise smooth curve

Rk: From Jordan to Lesbegue
· singleton (single point)
$D = \{x\}$ $\quad m(D) = 0$
has zero measures
· $[0;1] \cap \mathbb{Q}$ (reunion of countable singletons)
is <u>not</u> J.m
$\Rightarrow$ Lesbegue

- Koch Snowflake



$m \to \infty$

not J.m

Fractals

6.2. The (multiple) Riemann integral

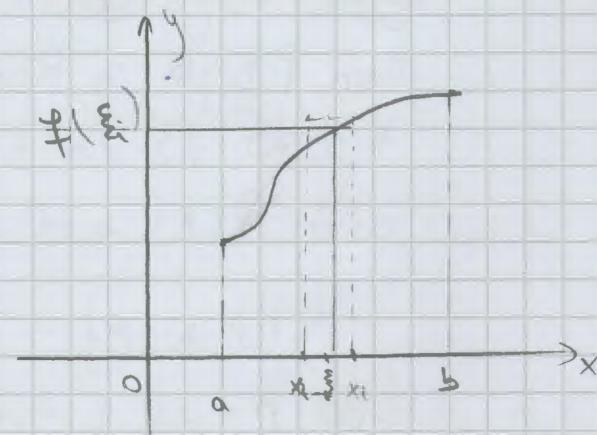
Riemann Sum $\quad f:[a,b] \to \mathbb{R} \qquad \int_a^b f(x)\,dx$

$$\sigma(f, \Delta, \xi) = \sum_{i=1}^{m} f(\xi_i)(x_i - x_{i-1}) \xrightarrow{\|\Delta\| \to 0} I \in \mathbb{R}$$

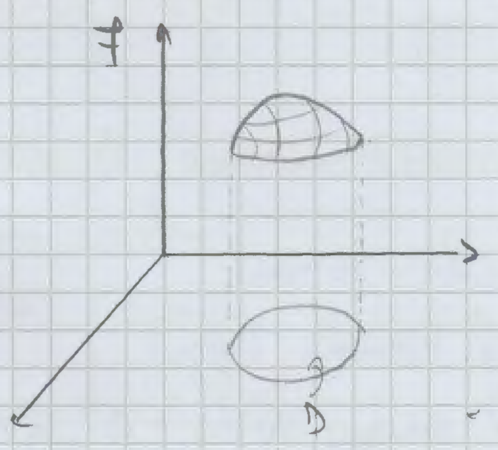
$\Delta = \{x_0 = a, x_1, \ldots x_i, \ldots, x_m = b\}$

$\xi = \{\xi_1, \ldots, \xi_m\}$

$\xi_i \in [x_{i-1}; x_i]$



$f: D \subset \mathbb{R}^2 \to \mathbb{R}$



graph of f is a surface

D = proj of this surface

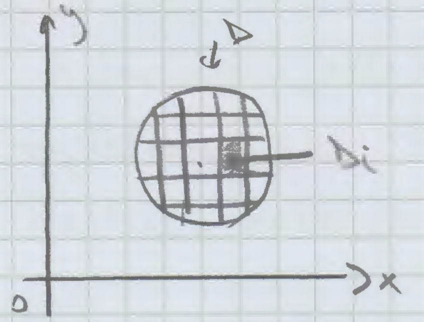
Δ a division of D → in the sense that $\Delta = \{D_1, \ldots, D_m\}$

$D_i \subset D$

$D = \bigcup_{i=1}^{m} D_i \qquad \text{int } D_i \cap \text{int } D_j = \emptyset \quad i \neq j$

$\nu(\Delta) = \max_i \delta(D_i)$

ro(D) = sup {||x-y||} x,y from D



$\xi = \{\xi_1, \dots, \xi_n\} \in \Delta_i$

The Riemann Sum of $f : \Delta \to \mathbb{R}$ is

$$\sigma(f, \Delta, \xi) = \sum_{i=1}^{n} f(\xi_i)\,(m\Delta_i)$$

Def Let Δ be a bounded Jordan meas. subset of $\mathbb{R}^n$

Then $f : \Delta \to \mathbb{R}$ is Riemann integrable if $\exists\, I \in \mathbb{R}$ s.t.

$\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ and for any Δ (fine enough)

with $\nu(\Delta) < \delta$ and any assoc ξ

$$|\sigma(f, \Delta, \xi) - I| < \varepsilon$$

$$I \overset{\text{Not}}{=} \int_\Delta f(x)\,dx \quad \left(\overset{\text{OR}}{=} \int_\Delta f(x_1, \dots, x_n)\, dx_1, dx_2, \dots, dx_n \right)$$

$\iint \quad \iiint$

$\mathbb{R}^2$ $\square$ dx_2, dx_1

$dx = dx_1 \cdot dx_2$

Properties of the Riemann Integral

Ass $f, g : \Delta \to \mathbb{R}$ are Riemann Integral

- $\int_\Delta (\alpha f(x) + \beta g(x)) = \alpha \int_\Delta f(x)\,dx + \beta \int_\Delta g(x)\,dx$
- if $\Delta = \Delta_1 \cup \Delta_2$ $\quad \operatorname{int}\Delta_1 \cap \operatorname{int}\Delta_2 = \emptyset$

$$\int_\Delta f(x)\,dx = \int_{\Delta_1} f(x)\,dx + \int_{\Delta_2} f(x)\,dx$$

- $f(x) \le g(x) \Rightarrow \int_\Delta f(x)\,dx \le \int_\Delta g(x)\,dx$

$$\left|\int_D f(x)\,dx\right| \le \int_D |f(x)|\,dx$$

- $$\int_D f(x)\cdot g(x)\,dx = f(c)\int_D g(x)\,dx$$

$\exists c$ s.t.

- works only if D compact, convex f cont.

6.3. Computing multiple integrals

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A. $\boxed{T}$ (Fubini)

$f : A\times B \to \mathbb{R}$ $\quad f = f(x,y)$ $\quad x\in A, y\in B$

if $f(\cdot,y)$ and $f(x,\cdot)$ are integrable, then

$$\int_{A\times B} f(x,y)\,dx\,dy = \int_A \left(\int_B f(x,\underline{y})\,\underline{dy}\right) dx =$$

$$= \int_B \left(\int_A f(\underline{x},y)\,\underline{dx}\right) dy$$

Ex $\displaystyle\int_D xy\,dx\,dy = \int_0^1\left(\int_0^1 xy\,dy\right)dx = \underbrace{\int_0^1 x\,dx}_{\frac12}\underbrace{\int_0^1 y\,dy}_{\frac12} = \frac14$

$D = [0;1]\times[0;1]$

B. Simple domains (Domains which are simple w.r.t one direction)

$$\int_D f(x,y)\,dx\,dy = \int_a^b\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right)dx$$