Course 7

7. Extensions of the Riemann integral I

§ 7.1. The Riemann Stieltjes integral

$f:[a,b]\to\mathbb{R}$ cont Δ = division of $[a,b]$

$\Delta = \{a = x_0 < x_1 \dots < x_n = b\}$

ξ a system of intermediate points

$\xi_i \in [x_{i-1}, x_i]$

The Riemann sum $\sigma(f, \Delta, \xi) = \left(\sum_{i=1}^{n} f(\xi_i)(g(x_i)\right)$

$$= \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

$g:[a,b]\to\mathbb{R}$

The R-Stieltjes sum $\sigma^{RS}(f, g, \Delta, \xi) = \sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1}))$

$$\downarrow$$

$$\int_a^b f(x)\, dg(x)$$

T f cont and $g = g_1 - g_2$ where g_1, g_2 non-decreasing

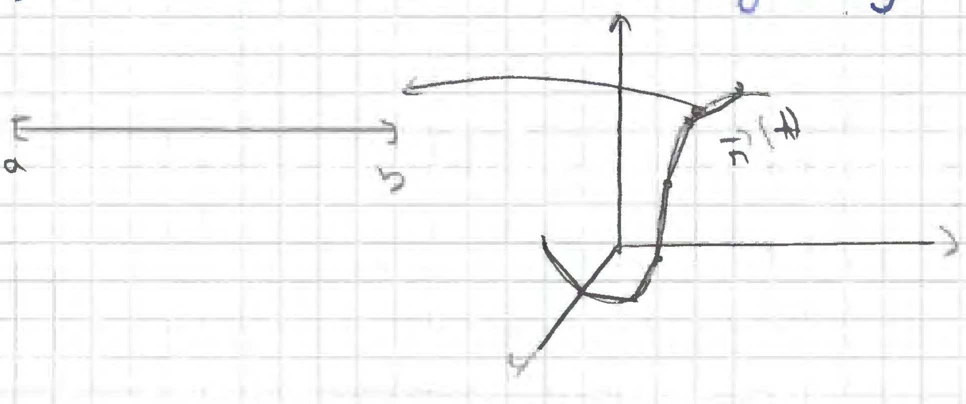
(g is of BV) bounded var.

then $\int_a^b f(x)\, dx$ exists

Rk a) $g(x) = x$ you recover Riemann $\int$

b) g diff-able then $\int_a^b f(x)\, dg(x) = \int_a^b f(x)\cdot g'(x)\, dx$

§ 7.2 Curves in $\mathbb{R}^3$ (or more generally in $\mathbb{R}^n$)



curvilinear integrals = line int = path integrals

<u>Def</u>: a Parametrized path is a cont function

$\gamma : [a,b] \to \mathbb{R}^3$

$$\gamma \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad \text{or} \quad \gamma : \vec{r} = \vec{r}(t) \quad t \in [a,b]$$

$\vec{r} = (x, y, z)$ parameters

- γ is smooth if x, y, z diff-able

$$\frac{d\vec{r}}{dt}(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t) \right)$$

and $\frac{d\vec{r}}{dt}(t) \neq 0_{\mathbb{R}^3}$, $\forall t \in [a,b]$

- γ is rectifiable if $\exists M > 0$ s.t. $\forall \Delta$ division of $[a,b]$

$$\sum_{i=1}^{m} \| \vec{r}(t_i) - \vec{r}(t_{i-1}) \| < M \quad \Delta = \{ a = t_0 < t_1 < \dots < t_m = b \}$$

(γ can be approximated polygonal line)

[T] If γ is smooth $\Rightarrow \gamma$ rectif and it has length

$$\ell(\gamma) = \int_a^b \left\| \frac{d\vec{r}}{dt}(t) \right\| dt$$

- $\gamma_1 : [a,b] \to \mathbb{R}^3$ and $\gamma_2 : [c,d] \to \mathbb{R}^3$ are <u>equivalent</u> if there exists a function $\varphi : [a,b] \to [c,d]$ bijective $\varphi(a) = c$, $\varphi(b) = d$, φ and φ_1 diff-able and

$$\gamma_1 = \gamma_2 \circ \varphi$$

A curve is the class of all equivalent ~~of~~ paths

does not depend on the way we parametrize it

§ 7.3 Curvilinear integrals

- The curvilinear integral

$$\underset{\uparrow}{\text{arc length}} \quad s(t) = \int_a^{t} \left\| \frac{d\vec{r}}{dt}(t) \right\| dt$$

a natural parametrization for a curve

A. The curvilinear integral of a <u>scalar field</u>

$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ cont scalar field

γ curve (smooth) $\gamma : [a,b] \to \mathbb{R}$ and $\gamma([a,b]) \subset D$

$$\int_\gamma f(\vec{r})\, ds = \int_a^b f(\vec{r}(t)) \left\| \frac{d\vec{r}}{dt}(t) \right\| dt$$

B. The curvilinear integral of a vector field

$\vec{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ vector field

$\vec{F} = (f_1, f_2, f_3)$

γ - curve

$$\int_\gamma \vec{F}(\vec{r})\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}(t)\, dt$$

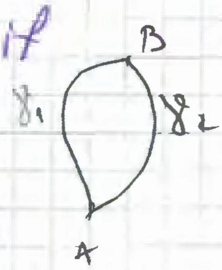
Physical meaning : WORK

$$\boxed{d\vec{r} = \frac{d\vec{r}}{dt}\, dt}$$

§ 7.4. Integrals that don't depend on the path (curve)

<u>Def</u> A vector field $\vec{F}$ is conservative if

$$\int_{\gamma_1} \vec{F}\, d\vec{r} = \int_{\gamma_2} \vec{F}\, d\vec{r} \quad \text{whenever} \quad \gamma_{AB}$$



<u>Def</u> A vector field $\vec{F}$ is of potential type if there exists an energy (a potential)

$\mathcal{E} : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ diff-able and $\nabla \mathcal{E}(\vec{r}) = \vec{F}(\vec{r})$

$\boxed{U}$ Given that $\vec{F}$, $\mathcal{E}$, γ smooth enough

- $\vec{F}$ conservative $\Leftrightarrow$ $\vec{F}$ potential type
- $\int_\gamma \vec{F}(\vec{r})\, d\vec{r} = \mathcal{E}(\vec{r}_B) - \mathcal{E}(\vec{r}_A)$

γ is smooth and connects A and B

$\vec{F}(\vec{r}) = \nabla \mathcal{E}(\vec{r})$

$$\mathcal{E}(\vec{r}_B) - \mathcal{E}(\vec{r}_A) = \int_\gamma \nabla \mathcal{E}(\vec{r})\, d\vec{r}$$

Course 8

8. Extensions of the Riemann Integral II

Improper integral

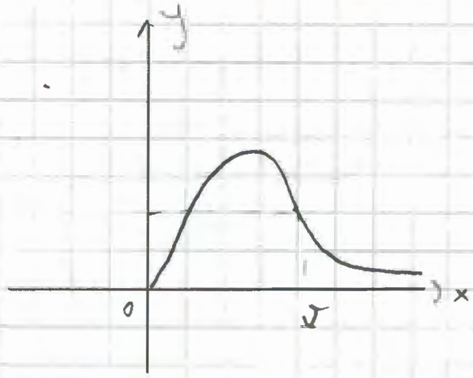
Why probability?

Speed distribution $f(v) = Cv^2 e^{-v^2}$

$v = \|\vec{v}\|$

Probability deduction

$$\int_0^\infty Cv^2 e^{-v^2}\, dv$$



§ 8.1. Improper integrals

"improper integrals are limits"

<u>Ex 1</u> $\displaystyle\int_0^\infty e^{-x}\, dx = \lim_{t\to\infty} \int_0^t e^{-x}\, dx = \lim_{t\to\infty} (1 - e^{-t}) = 1$

<u>Ex 2</u> $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\, dx = \lim_{\substack{t\to 0\\ t<0}} \int_{-1}^{t} \frac{1}{x^2}\, dx = \lim_{t\to 0} -\frac{1}{x}\Big|_{-1}^{t} =$

$\displaystyle= \lim_{\substack{t\to 0\\ t<0}} -\frac{1}{t} + 1 = +\infty$

Ex 3 $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$

$$\int_0^\infty \frac{1}{x}\,dx = \int_0^1 + \int_1^\infty$$

Def $f:[a,b)\to\mathbb{R}$ ($b=\infty$ is admitted) $a\in\mathbb{R}$, $a\neq\pm\infty$

Riemann integrable on any $[a;\tilde{b}]$ $\tilde{b}<b$

if $\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx$ exists and is finite, then we call $\int_a^b f(x)\,dx$ CONV (convergent)

the improper integral

Otherwise $\int_a^b f(x)\,dx$ is DIV (divergent)

§ 8.2. ~~The~~ Testing the convergence of improper integrals

T (Cauchy) $f:[a;b)\to\mathbb{R}$ int on any $[a,t]$, $t<b$

The improper $\int_a^b f(x)\,dx$ is CNV iff $\forall \varepsilon>0$

$\exists\, b_\varepsilon < b$ s.t. for any $b_\varepsilon < t < b$

$\left|\int_{b_\varepsilon}^t f(x)\,dx\right| < \varepsilon$ (The improper integral is convergent if $\left|\int_{b_\varepsilon}^t\right|$ can be made arbitraly small)

T (Comparison I)

$f,g:(a,b)\to\mathbb{R}$ both Riemann int on $(a,t]$ $t<b$

and $0\le f(x)\le g(x)$. Then

(i) $\int_a^b g$ conv $\Rightarrow \int_a^b f$ conv

(ii) $\int_a^b f$ div $\Rightarrow \int_a g$ div

$\boxed{U}$ (Comparison II)

$\rightarrow$ same ass on f, g

but $g(x) \neq 0 \quad \forall x \in [a,b)$

and $\lim_{x \to b} \frac{f(x)}{g(x)} = l < \infty$

(i) $l \neq 0$, then $\int_a^b f$ and $\int_a^b g$ have the same nature

(div div or conv conv)

(ii) $l = 0$ then $\int_a^b g$ conv $\Rightarrow \int_a^b |f|$ conv

(this is called abs conv) absolutely

ex 1 Is $\int_0^\infty e^{-x} \cdot (\sin x)^3 \, dx$ conv?

we know that $\int_0^\infty e^{-x} \, dx = 1$ conv

and $|e^{-x} (\sin x)^3| = e^{-x} |\sin x|^3 \leq e^{-x}$

so comp I applies

Rk $\int_1^\infty x^\alpha \, dx$ is conv $\alpha < -1$

div $\alpha \geq -1$

$\int_0^1 x^\beta$ conv $\beta > -1$

div $\beta \leq -1$

Rk everything (def, II, ...) works the same way

on $[a,b)$ or $(a,b]$

Homework $\int_{-\infty}^{\infty} e^{-|x|}\, dx = ?$

Rk Int. by parts and change of variables both work if the improper int are conv

§ 8.3. Improper integrals with parameter

$f : [a,b) \times [c,d] \qquad F(y) = \int_a^{b} f(x, y)\, dx$

parameter

$$\boxed{\begin{array}{l} \int_a^t f(x)\, dx \xrightarrow{t \to b} \int_a^b f(x)\, dx \\ \int_a^t f(x,y)\, dx \xrightarrow{t \to b} F(y) \end{array}}$$

Def The improper int with params converges uniformly (w.r.t. y) to F iff $\forall \varepsilon > 0$, $\exists\, b_\varepsilon < b$ s.t. $\forall\, b_\varepsilon < t < b$ and $\forall y \in [c,d]$

$$\left| \int_{b_\varepsilon}^{t} f(x,y)\, dx - F(y) \right| < \varepsilon$$

T (Continuity) $f : [a,b) \times [c,d] \to \mathbb{R}$
if f cont (as a function of 2 var) and if $\int_a^b f(x,y)\, dx$ is UNIF (uniformly) CONV
then $F(y) = \int_a^b f(x,y)\, dx$ is cont (in y)

T (Diff w.r.t y) $f : [a,b) \times [c,d] \to \mathbb{R}$ cont and $\frac{\partial f}{\partial y}$ is cont and also $\int_a^b f(x,y)\, dx$,
$\int_a^b \frac{\partial f}{\partial y}(x,y)\, dx$ "u conv", then
$F(y) = \int_a^b f(x,y)\, dx$ is diff-able (w. resp to y)

and $F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx$

[I] (Integrating w.r.t y) f cont and $\int_a^b f(x,y)\,dx$ u-conv

then $F(y)$ integrable and

$$\int_c^d F(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy =$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Ex 1 $\quad f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

f diff able? $\quad f^{(n)}(0) = ?$

! $\quad f(x) = \int_0^1 e^{xy}\,dy$

Ca de obicei, Viorel șterge .....

$$f(y) = \begin{cases} \dfrac{e^y - 1}{y}, & y \neq 0 \\ 1, & y = 0 \end{cases}$$

! $\quad f(y) = \int_0^1 e^{xy}\,dx$

$$f'(y) = \int_0^1 \frac{\partial}{\partial y}(e^{xy}) = \int_0^1 x e^{xy}\,dx$$

$$f''(y) = \int_0^1 \frac{\partial}{\partial y}(x e^{xy}) = \int_0^1 x \cdot x e^{xy} = \int_0^1 x^2 e^{xy}$$

$$f^{(n)}(y) = \int_0^1 x^n e^{xy}\,dx$$

$$y = 0 \Rightarrow f^{(n)}(0) = \int_0^1 x^n\,dx = \frac{1}{n+1}$$

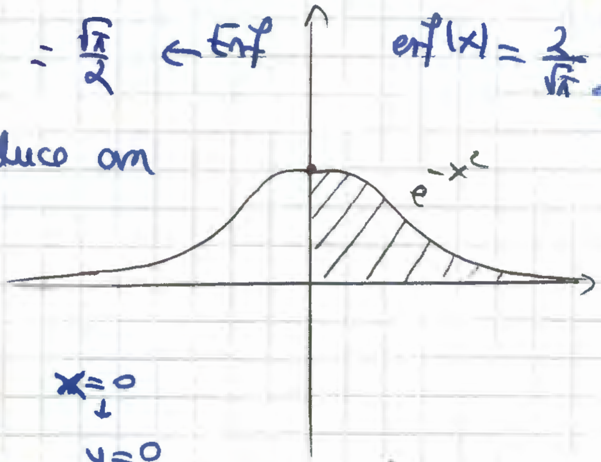
$$\int_0^\infty e^{-x^2}\,dx \quad , \quad \int_0^\infty \frac{\sin x}{x}\,dx$$

29.11.2019

Course 9

9. Improper integrals: Applications. The Beta and Gamma functions (of Euler)

§ 9.1 $I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ ← Erf $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$



Step 1: Key idea: introduce an (artificial) additional parameter $t \geq 0$

$$x = t\cdot y \qquad dx = t\,dy \qquad x=0 \to y=0, \quad x=\infty \to y=\infty$$

$$I = \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-t^2y^2}\,t\,dy$$

Step 2: Multiply by e^{-t^2} and integrate w.r.t. t

$$I = t\int_0^\infty e^{-t^2y^2}\,dy \quad \Big|\cdot e^{-t^2}$$

$$I\cdot e^{-t^2} = e^{-t^2}\cdot t\int_0^\infty e^{-t^2y^2}\,dy \quad \Big|\int_0^\infty dt$$

$$I\cdot\underbrace{\int_0^\infty e^{-t^2}\,dt}_{I} = \int_0^\infty t e^{-t^2}\left(\int_0^\infty e^{-t^2y^2}\,dy\right)dt$$

$$I^2 = \int_0^\infty\left(\int_0^\infty t\cdot e^{-t^2}\cdot e^{-t^2y^2}\,dy\right)dt$$

Step 3: Change the order of integr. and compute the inner integral

$$(*)\quad I^2 = \int_0^\infty \underbrace{\left(\int_0^\infty t\cdot e^{-t^2(1+y^2)}\,dt\right)}_{J(y)}dy$$

$$J(y) = \int_0^\infty t\cdot e^{-t^2(1+y^2)}\,dt$$

$$\frac{\partial}{\partial t}\left(e^{-t^2(1+y^2)}\right) \underset{\substack{\text{chain}\\\text{rule}}}{=} e^{-t^2(1+y^2)} \cdot \left(-2t(1+y^2)\right)$$

$$J(y) = \int_0^\infty t \cdot e^{-t^2(1+y^2)}\,dt = \frac{1}{-2(1+y^2)} \int_0^\infty \frac{\partial}{\partial t}\left(e^{-t^2(1+y^2)}\right) dt =$$

$$= -\frac{1}{2(1+y^2)} \cdot \underbrace{e^{-t^2(1+y^2)}\Big|_{t=0}^{t=\infty}}_{0-1}$$

$$J(y) = \frac{1}{2} \cdot \frac{1}{1+y^2} \quad (**)$$

Step 4 : Combining preview steps given (by inserting $**$ in $*$)

$$I^2 = \int_0^\infty J(y)\,dy = \int_0^\infty \frac{1}{2} \cdot \frac{1}{1+y^2}\,dy = \frac{1}{2} \operatorname{arctg} y \Big|_{y=0}^{y=\infty} =$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

$$\boxed{I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$$

§ 92. $I = \int_0^\infty \frac{\sin x}{x}\,dx$

Step 1: Key idea : analyze the parametric family of integrals $y \geq 0$

$$I(y) = \int_0^\infty e^{-x \cdot y} \cdot \frac{\sin x}{x}\,dx$$ by diff w.r.t y

$$I'(y) = \int_0^\infty \frac{\partial}{\partial y}\left(e^{-xy} \cdot \frac{\sin x}{x}\right) dx$$

$$= \int_0^\infty -\cancel{x} \cdot e^{-xy} \cdot \frac{\sin x}{\cancel{x}}\, dx =$$

$$= -\int_0^\infty e^{-xy} \cdot \sin x \, dx$$

Compute $I'(y)$ (integrate by parts)

$$\frac{\partial}{\partial x}(e^{-xy}) = e^{-xy} \cdot (-y)$$

$$I'(y) = \frac{1}{y}\int_0^\infty \frac{\partial}{\partial x}(e^{-xy}) \cdot \sin x \, dx =$$

$$= \frac{1}{y} \cdot \Big(\underbrace{e^{-xy} \cdot \sin x \Big|_0^\infty}_{0-0} - \int_0^\infty e^{-xy} \cdot \cos x \, dx \Big)$$

$$I'(y) = -\frac{1}{y} \int_0^\infty e^{-xy} \cdot \cos x \, dx =$$

$$= \frac{1}{y^2} \int_0^\infty \frac{\partial}{\partial x}(e^{-xy}) \cdot \cos x \, dx =$$

$$= \frac{1}{y^2} \Big(\underbrace{e^{-xy} \cdot \cos x \Big|_{x=0}^{x=\infty}}_{0-1} + \underbrace{\int_0^\infty e^{-xy} \sin x \, dx}_{-I'(y)} \Big)$$

$$I'(y) = \frac{1}{y^2} \cdot (-1 - I'(y))$$

$$y^2 I'(y) + 1 + I'(y) = 0$$

$$I'(y) = -\frac{1}{1+y^2}$$

$$I(y) = \int -\frac{1}{1+y^2}\, dy = -\operatorname{arctg} y + C$$

Step 4: We find C by explicitly computing $I(1)$

$$I(1) = \int_0^\infty e^{-x} \cdot \frac{\sin x}{x}\, dx = \frac{\pi}{4} \left(\cancel{\frac{*}{**}}\right)$$

key obs: $\boxed{\frac{\sin x}{x} = \int_0^1 \cos xy \, dy}$

$$I(1) = \int_0^\infty e^{-x}\left(\int_0^1 \cos xy \, dy\right) dx =$$

$$= \int_0^1 \underbrace{\left(\int_0^\infty e^{-x} \cos xy \, dx\right)}_{J(y)} dy$$

~~Remark~~, Homework: use partial ~~deriv~~ integr to show that $J(y) = \int_0^\infty e^{-x} \cos xy \, dx = \frac{1}{1+y^2}$

$$I(1) = \int_0^1 \frac{1}{1+y^2}\, dy = \arctan y \Big|_{y=0}^{y=1} = \frac{\pi}{4}$$

$$I(y) = \int -\frac{1}{1+y^2}\, dy = -\arctan y + C$$

$$\frac{\pi}{4} \ \cancel{*} = I_1 = -\arctan 1 + C$$

$$C = \frac{\pi}{2}$$

$$\Rightarrow I(y) = -\arctan y + \frac{\pi}{2}$$

$$\Rightarrow I(0) = \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Homework: Try to find C by using the limit

$$\lim_{y\to\infty} I(y)$$

$$\int_0^\infty e^{-xy} \frac{\sin x}{x}\, dx$$

§ 9.3. $I = \int_0^\infty e^{-x} \cdot \frac{\sin x}{x}\, dx = \frac{\pi}{4}$

Homework $I = \int_0^\infty \underline{x^2 \cdot e^{-x^2}}\, dx = ?$

Maxwell - Boltzmann

§ 9.4. The Beta and Gamma functions (Euler)

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \qquad a,b>0$$

("two param. family of integrals")

$$\Gamma(a) = \int_0^\infty x^{a-1}\cdot e^{-x}\,dx \qquad a>0$$

- Volume of n-dim ball of radius R

$$\boxed{V_n(R) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\cdot R^n}$$

T (Properties of $B(a,b)$)

(i) $B(a,b) = B(b,a)$, $\forall a,b>0$

(ii) $B(a,b) = \dfrac{b-1}{a+b-1}B(a,b-1)$ $\quad a,b>1$

(iii) $B(a,n) = \dfrac{(n-1)!}{a(a+1)\dots(a+n-1)}$ $\quad a>0,\ n\in\mathbb{N}^*$

(iv) $B(m,n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$, $m,n\in\mathbb{N}^*$

T (Properties of $\Gamma(a)$)

(i) $\Gamma(a+1) = a\Gamma(a)$, $a>0$

(ii) $\Gamma(n+1) = n!$ $\quad n\in\mathbb{N}$

(iii) $B(a,b) = \dfrac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$ $\quad a,b>0$

(iv) $\Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin(a\pi)}$ $\quad a\in(0,1)$

(v) $\Gamma\left(\frac{1}{2}\right) = 2\displaystyle\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}$

(vi) Stirling

Proof: (ii) $\underbrace{\int_0^\infty e^{-x}\,dx}_{\Gamma(1)} = \lim_{t\to\infty}(1-e^{-t}) = 1$

$$\Gamma(m+1) \overset{(i)}{=} m\Gamma(m) = m\cdot(m-1)\cdot\Gamma(m-1) = \dots$$
$$= m(m-1)\cdot\ldots\cdot 2\cdot 1\,\Gamma(1) = m!\cdot 1 = m!$$

Course 10

Part III Sequences & Series

10. Series of numbers

Infinite sums (and sums of infinitesimals)

$1+(-1)+1+(-1)+\dots$

$(1-1)+(1-1)+\dots = 0$ ↳ rigorous def

$1+(-1+1)+(-1+1)+\dots = 1$ 19th century

§ 10.1. Sequences and series of (real) numbers

What is a sequence (of numbers)?

A sequence is a map $\mathbb{N}^* \to \mathbb{R}$

$(a_m)_{m\in\mathbb{N}^*}$ $\mathbb{N}^* \ni m \longmapsto a_m \in \mathbb{R}$

Recall

Def a sequence $(a_m)_{m\in\mathbb{N}^*}$ converges to the limit $l\in\mathbb{R}$ if $\forall\, \varepsilon>0\ \exists\, N(\varepsilon)\in\mathbb{N}^*$ such that for any $m \geq N(\varepsilon)$ we have $|a_m - l| < \varepsilon$

Def $(a_m)_{m\in\mathbb{N}^*}$ is called *fundamental* (or Cauchy) sequence if $\forall\, \varepsilon>0\ \exists\, N(\varepsilon)\in\mathbb{N}^*$ s.t. $\forall\, m,n \geq N(\varepsilon)$ $|a_m - a_n| < \varepsilon$ (obviously any conv. sequence is Cauchy)

$\boxed{T}_1$ In $\mathbb{R}$ (that is for $(a_n)_{n\in\mathbb{N}^*}$ with $a_n \in \mathbb{R}$)
any Cauchy sequence converges ($(a_n)_{n\in\mathbb{N}^*}$ conv $\Leftrightarrow$
$(a_n)_{n\in\mathbb{N}^*}$ Cauchy sequence)

Let $(a_n)_{n\in\mathbb{N}^*}$ be a seq of numbers and $s_n = a_1 + \dots + a_n$
$(s_n)_{n\in\mathbb{N}^*}$ (the sequence of partial sums)

$\boxed{\text{Def}}$ A <u>series</u> is a pair $((a_n)_{n\in\mathbb{N}^*}, (s_n)_{n\in\mathbb{N}^*})$ and is
denoted by $\sum_{n=1}^{\infty} a_n$

$\boxed{\text{Def}}$ A series $\sum_{n=1}^{\infty} a_n$ converges (CONV) if $(s_n)_{n\in\mathbb{N}^*}$ converges

$s_n \xrightarrow{n\to\infty} s$ the sum of the series

if $\lim_{n\to\infty} s_n$ is ∞ or does not exist then
$\sum_{n=1}^{\infty} a_n$ diverges (DV)

Two questions: • Does $\sum_{n\in\mathbb{N}^*} a_n$ conv?

• What is the sum of $\sum_{n\in\mathbb{N}^*} a_n$?

$\boxed{T}_1$

Based on def of Cauchy seq. and $\boxed{T}_1$
$\boxed{T}$ (fundamental CONV criterion of Cauchy)
$(s_n)_{n\in\mathbb{N}^*}$ Cauchy sequence $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ conv
$(\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}^* : |a_{N(\varepsilon)+1} + a_{N(\varepsilon)+2} + \dots + a_{N(\varepsilon)+p}| < \varepsilon$
$\forall p \in \mathbb{N}$

§ 10.2. Series of positive numbers

Standing assumption: $a_n \geq 0 \quad \forall n \in \mathbb{N}^*$

T (Integral crit of Cauchy)

$f : [1; +\infty) \to \mathbb{R}_+$ decreasing

we define $(f_n)_{n \in \mathbb{N}^*}$ by $f_n = f(n)$

$$\sum_{n=1}^{\infty} f_n \text{ CONV} \Leftrightarrow \int_1^{\infty} f(x)\,dx \text{ CONV}$$

$$\text{DIV} \Leftrightarrow \text{DIV}$$

Proof: $S_n = f_1 + \dots + f_n \qquad (S_n)_{n \in \mathbb{N}^*}$

f decreasing $f_{n+1} = f(n+1) \leq f(x) \leq f(n) \leq f_n$ for $x \in [n; n+1]$

we integrate w.r.t. x $\quad f_{n+1} \leq f(x) \leq f_n \quad \cdot \int_n^{n+1} dx$

we obtain $\quad f_{n+1} \leq \int_n^{n+1} f(x)\,dx \leq f_n$

add all these up from 1 to n

$$f_{n+1} + f_n + \dots + f_2 \leq \int_1^{n+1} f(x)\,dx \leq f_n + f_{n-1} + \dots + f_1$$

$$S_{n+1} - f_1 \leq \int_1^{n+1} f(x)\,dx \leq S_n$$

Recall that

$(a_n)_{n \in \mathbb{N}^*}$ bounded and increasing (decreasing)

$\Rightarrow (a_n)$ convergent

Examples:

a) the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ DIV

($\overset{\text{T Cauchy}}{\Longleftrightarrow}$ $\int_1^{\infty} \frac{1}{x}\,dx$ DIV)

b) generalized harm series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONV

with $p = 2$

$\Leftrightarrow \left(\int_1^{\infty} \frac{1}{x^2}\,dx \text{ CONV} \right)$

c) the geometric series ($|q|<1$)

$$\sum_{n=1}^{\infty} q^n \Leftrightarrow \left(\int_1^{\infty} q^x \, dx \text{ conv} \right)$$

<u>HW</u> : compute $\int_1^{\infty} q^x \, dx$ (and discuss $|q| \gtrless 1$ cases)

T (Comparison I) $a_n \le b_n$, $\forall n \ge N_0 \in \mathbb{N}^*$

$$\sum_{n=1}^{\infty} a_n \text{ DIV} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ DIV}$$

$$\sum_{n=1}^{\infty} b_n \text{ CONV} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ CONV}$$

Proof: HW

T (Comparison II) $\frac{a_n}{b_n} \xrightarrow{n \to +\infty} L < \infty$

if $L \neq 0$ both $\sum a_n$, $\sum b_n$ are CONV or both DIV

if $L = 0$ $\sum_{n=1}^{\infty} b_n$ CONV $\Rightarrow \sum_{n=1}^{\infty} a_n$ CONV

$\sum_{n=1}^{\infty} a_n$ DIV $\Rightarrow \sum_{n=1}^{\infty} b_n$ DIV

T (Cauchy's root test for conv)

- if $\sqrt[n]{a_n} \le q \in [0,1)$ $\forall n \ge N_0 \in \mathbb{N}^*$
  then $\sum_{n=1}^{\infty} a_n$ CONV
- if $\sqrt[n_k]{a_{n_k}} \ge 1$ $(a_{n_k}) \subset (a_n)_{n \in \mathbb{N}}$ (subsequence)
  then $\sum_{n=1}^{\infty} a_n$ DIV

(a_{n_k} and a_n have the "same number of elem")

Proof (Sketch) $\sqrt[n]{a_n} \le q \Rightarrow a_n \le q^n \mid \sum_{n=1}^{\infty}$

geom series CONV

I (D'Alembert's ratio test)

(i) $\frac{a_{n+1}}{a_n} \le q \in (0,1) \ \forall n \geq N_0 \in \mathbb{N}^*$

then $\sum_{n=1}^{\infty} a_n$ CONV

(ii) DIV (see [D. Popa, Calculus])

Proof: for simplicity take $N_0 = 1$

$a_{n+1} \le q \cdot a_n$

$a_n \le q \cdot a_{n-1}$

$\vdots$

$a_2 \le q \cdot a_1$

$a_{n+1} \le q^n a_1$

compare with geom series

Limit versions:

Cauchy's root test

$\lim_{n\to\infty} \sqrt[n]{a_n} = L < \infty$

D'alembert

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < \infty$

(1) $L<1 \Rightarrow \sum a_n$ CONV

(2) $L>1 \Rightarrow \sum a_n$ DIV

(3) $L=1 \Rightarrow$ test not efficient

II (Raabe – Duhamel)
1801–1859 1797–1872

(i) If $\exists q>1$ and $N_0 \in \mathbb{N}^*$ such that

$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q \qquad \forall n \geq N_0$

then $\sum_{n=1}^{\infty} a_n$ CONV

(ii) ———— | $<q$ and DIV

§ 10.3. Alternate series

T (ABEL - DIRICHLET)

Niels Henrik

if $(a_n)_{n\in\mathbb{N}^*}$ decreasing and with $a_n \xrightarrow{n\to\infty} 0$

and $(b_n)_{n\in\mathbb{N}^*}$ $(T_n)_{n\in\mathbb{N}^*}$ defined by $T_n = b_1 + \dots + b_n$ is bounded $(T_n \leq M, \forall n\in\mathbb{N}^*)$

Then $\sum_{n=1}^{\infty} a_n b_n$ CONV

As a consequence

T If $(a_n)_{n\in\mathbb{N}^*}$ decreasing and $a_n \xrightarrow{\infty} 0$

then $\sum_{n=1}^{\infty} (-1)^n a_n$ CONV

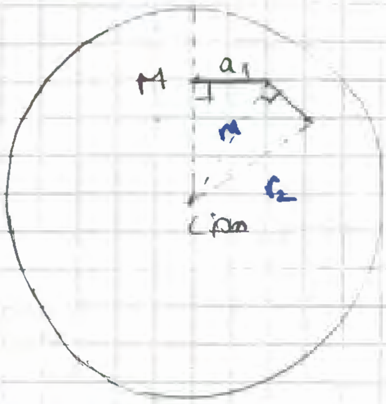
idea for proof

Apply Abel - Dirichlet to a_n and $b_n = (-1)^n$

13.12.2019

Lecture 11

The Lion and Man Pursuit Probl.



Lion & Man have same speed

Lion Strategy: Try to stay on the radius that connects the center to the current pos of Man

Man strategy

(1) $\sum_{n=1}^{\infty} a_n = \infty$ otherwise, you've been caught

$s = \frac{a_n}{t_n}$ $t_n = \frac{1}{s} a_n$ $\sum_{n=1}^{\infty} t_n = \frac{1}{s} \sum_{n=1}^{\infty} a_n$

time

total life time in the arena

(2) $r_1^2 = r_0^2 + a_1^2$

$r_2^2 = r_1^2 + a_2^2 = r_0^2 + a_1^2 + a_2^2$

. . . .

$r_n^2 = r_0^2 + a_1^2 + a_2^2 + \dots + a_n^2$

$r < R$

$\Rightarrow \sum_{n=1}^{\infty} a_n^2 \leq \underbrace{R^2 - r_0^2}_{\text{const}} < \infty$

(1) $\sum_{n=1}^{\infty} a_n = \infty$

(2) $\sum_{n=1}^{\infty} a_n^2 = \text{finite} < \infty$

you can take

$\sum_{n=1}^{\infty} \frac{1}{n}$ DIV

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv

Important Remark

Let $(a_n)_{n \in \mathbb{N}^*}$ Then $a_n \xrightarrow{n \to \infty} 0$ is a necessary cond for $\sum_{n=1}^{\infty} a_n$ to conv

(This means : $\sum_{n=1}^{\infty} a_n$ conv $\Rightarrow$ $a_n \xrightarrow{n \to \infty} 0$

$a_n \not\to 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ DIV)

11. Sequences and series of Functions

$f, f_n : [a,b] \to \mathbb{R}$ $\quad n = 1, 2, \dots$

CONVERGENCE w.r.t. n?

<u>Def</u> [pointwise conv] For any fixed x

$\lim_{n \to \infty} f_n(x) = f(x)$

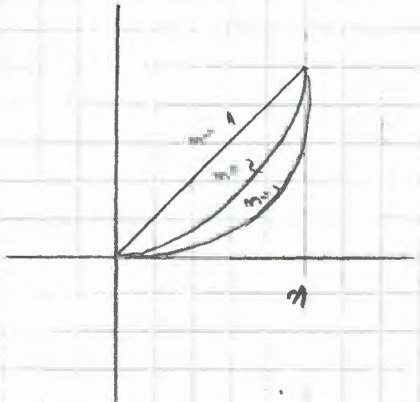
$\forall \varepsilon > 0 \ \exists N(\varepsilon, x) \in \mathbb{N}^*$ s.t. for any $n \geq N(\varepsilon, x)$

$|f_n(x) - f(x)| < \varepsilon$

Ex 1 $f_n(x) = x^n$, $f_n : [0;1] \to \mathbb{R}$ $n = 1, 2, \ldots$
all are cont

$$\lim_{n\to\infty} f_n(x) = \left.\begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}\right\} = f(x)$$

1! discont?



p.w. conv does not preserve continuity!

Def [uniform conv] $f_n \xrightarrow{u} f$ if
$\forall \varepsilon > 0$ $\exists N(\varepsilon) \in \mathbb{N}^*$ s.t. for any $n \geq N(\varepsilon)$
$|f_n(x) - f(x)| < \varepsilon$, $\forall x \in [a,b]$

Hence $f_n \xrightarrow{P} f$ (pointwise)
but $f_n \not\xrightarrow{u} f$ does not converge uniformly

T (Continuity) $f_n : [a,b] \to \mathbb{R}$ all cont. and $f_n \xrightarrow{u} f$
$f : [a,b] \to \mathbb{R}$, then f is continuous

T (Riemann integral) $(f_n)_{n\in\mathbb{N}^*}$ all cont and $f_n \xrightarrow{u} f$ then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

T (Differentiability) $(f_n)_{n\in\mathbb{N}^*}$ diff-able
and (i) $f_n \xrightarrow{P} f$
(ii) $f_n' \xrightarrow{u} g$
then f is diff-able
and $f' = g$

§ 11.2. Series of functions

$f_n : [a,b] \to \mathbb{R}$

$\sum_{n=1}^{\infty} f_n$ series of functions

sequence of partial sums $s_n(x) = f_1(x) + \dots + f_n(x)$
(functions) $s_n : [a,b] \to \mathbb{R}$

- We say the series converges pointwise or uniformly if $(s_n)_{n\in\mathbb{N}^*}$ have p. or u. conv. property
p-CONV or m-CONV

Motivation: The Taylor Formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n + R_n$$

if we simplify $x_0 = 0$

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)\, x^k + R_n$$

Can we let $n \to \infty$ ($\& R_n \to 0$)

Answer is Yes and we have $f(x) = \underbrace{\sum_{n=1}^{\infty} a_n x^n}_{\text{power series}}$ (but not for any x!)

(§ 11.3. Power series)

$\boxed{T}$ (WEIERSTRASS) if $\sum_{n=1}^{\infty} f_n$ and $(a_n)_{n\in\mathbb{N}}$ $a_n \geq 0$
$f_n : [a,b] \to \mathbb{R}$

and (i) $\sum_{n=1}^{\infty} a_n$ CONV

(ii) $|f_n(x)| \leq a_n$ $\forall n \geq n_0 \in \mathbb{N}^*, \forall x \in [a,b]$

then $\sum_{n=1}^{\infty} f_n$ m-CONV

$\overline{\text{T}}$ (ABEL II) The sum of the power series $\sum_{n=1}^{\infty} a_n x^n$

that is the function $S(x) = \sum_{n=1}^{\infty} a_n x^n$ is cont. at $x = R$ if $\sum_{n=1}^{\infty} a_n R^n$ conv

The most famous example $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots$

10.01.2020

Course 13

Recap

I Diff Calculus

II Integral Calculus

III Sequences and series

I Diff. calculus

1. Diff calc for f of one variable

Continuity, Diff-ability ← NO ε, δ definitions

$\overline{\text{T}}$ Weierstrass $f : [a,b] \to \mathbb{R}$ cont.

then f reaches its extremal values (and all values between those

$\overline{\text{T}}$ Fermat x^* local min/max for f $\Rightarrow f'(x^*) = 0$

$\overline{\text{T}}$ Rolle f diff

$f : [a,b] \to \mathbb{R}$

$\left.\begin{array}{l} \text{cont on } [a,b], \text{ diff on } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists\, c \in (a,b)$ $f'(c) = 0$

$\overline{\text{T}}$ Lagrange

cont on $[a,b]$, diff on (a,b) $\Rightarrow \exists\, c \in (a,b)$

s.t $f(b) - f(a) = f'(c)(a-b)$

Q 1. Give an example of f and x^*

s.t. $f'(x^*) = 0$ but x^* is not a local min/max.

x^3

Fermat's theorem

§ 11.3. Power series

T (ABEL I) $\sum_{n=1}^{\infty} a_n x^n$ power series $\exists$ $R \in [0; +\infty]$ s.t.

the power series n-CONV $\forall x \in [0, R]$

Proof (Sketch) : $R = 0$ nothing to prove ✓

if $R > 0$ s.t. $\sum_{n=1}^{\infty} a_n R^n < \infty$ then this implies

$|a_n R^n| \leq M$, $\forall n \in \mathbb{N}^*$

for $|x| < R$ rewrite $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n R^n \cdot \frac{x^n}{R^n} \leq$

$\sum_{n=1}^{\infty} \underbrace{|a_n R^n|}_{\leq M} \cdot \left|\frac{x}{R}\right|^n \leq M \underbrace{\sum_{n=1}^{\infty} q^n}_{\text{CONV}}$

q^n with $0 < q < 1$

<u>Def</u> [radius of convergence] The radius of conv. for a given power series $\sum_{n=1}^{\infty} a_n x^n$ is the largest $R > 0$ predicted by T Abel I

T (CAUCHY - HADAMARD)

if $\lim_{n \to \infty} \sqrt[n]{a_n}$ exists then the radius of conv

for $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{a_n}}$

with $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof: based on Cauchy's root crit for numerical series

Rk! if $\lim_{n \to \infty} \frac{a_n}{a_{n+1}}$ exists then $\underbrace{R}_{\text{radius of conv}} = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$

§ 11.3. Power series

T (ABEL I) $\sum_{n=1}^{\infty} a_n x^n$ power series $\exists$ $R \in [0;+\infty]$ s.t.

the power series n-CONV $\forall x \in [0, R]$

Proof (Sketch): $R=0$ nothing to prove ✓

if $R>0$ s.t. $\sum_{n=1}^{\infty} a_n R^n < \infty$ then this implies
$|a_n R^n| \leq M$, $\forall n \in \mathbb{N}^*$

for $|x|<R$ rewrite $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n R^n \cdot \frac{x^n}{R^n} \leq$

$\sum_{n=1}^{\infty} \underbrace{|a_n R^n|}_{\leq M} \cdot \left|\frac{x}{R}\right|^n \leq M \underbrace{\sum_{n=1}^{\infty} q^n}_{\text{CONV}}$

q^n with $0<q<1$

Def [radius of convergence] The radius of conv. for a given power series $\sum_{n=1}^{\infty} a_n x^n$ is the largest $R>0$ predicted by T Abel I

T (CAUCHY - HADAMARD)
if $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists then the radius of conv for $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{a_n}}$

with $\frac{1}{0}=\infty$ and $\frac{1}{\infty}=0$

Proof: based on Cauchy's root crit for numerical series

Rk! if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ exists then $\underbrace{R}_{\text{radius of conv}} = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$