1 Examples for Lectures 1 – 6

Example 1.1 *Compute the* ***finite difference table*** *for the following given data*

x	1	2	3	4	5
y	4	13	34	73	136

For theory, see Lecture 1, slides 11–13.

The "maximum order" for the finite difference table is nr. points -1, so, in our case it is 4. (we have 5 data). On the *first column*, we should put *the function's values*, in our case y.

y	$\Delta_h^1 y$	$\Delta_h^2 y$	$\Delta_h^3 y$	$\Delta_h^4 y$
4	13 - 4 = 9	21-9 = 12	18-12 =6	6-6=0
13	34 - 13 = 21	39-21 = 18	24-18=6	
34	73 - 34 = 39	63-39 = 24		
73	136 - 73 = 63			
136				

Example 1.2 *Compute the* ***divided difference table*** *for the following given data*

x	0	1	3	4
f	1	3	49	129

For theory, see Lecture 1, slides 15–17.

The "maximum order" for the divided difference table is nr. points -1, so, in our case it is 3. (we have 4 data). On the *first column*, we should put *the nodes* (x), on the *second column*, we should put *the values of the function on the nodes* (f).

x	f	$\mathcal{D}^1 f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
$x_0 = 0$	1	$\frac{f_1-f_0}{x_1-x_0} = \frac{3-1}{1-0} = 2$	$\frac{\mathcal{D}^1 f_1-\mathcal{D}^1 f_0}{x_2-x_0} = \frac{23-2}{3-0} = 7$	$\frac{\mathcal{D}^2 f_1-\mathcal{D}^2 f_0}{x_3-x_0} = \frac{19-7}{4-0} = 3$
$x_1 = 1$	3	$\frac{f_2-f_1}{x_2-x_1} = \frac{49-3}{3-1} = 23$	$\frac{\mathcal{D}^1 f_2-\mathcal{D}^1 f_1}{x_3-x_1} = \frac{80-23}{4-1} = 19$	
$x_2 = 3$	49	$\frac{f_3-f_2}{x_3-x_2} = \frac{129-49}{4-3} = 80$		
$x_3 = 4$	129			

Remark 1.3 For the 1st order ($\mathcal{D}^1 f$), you divide by the difference of consecutive nodes ($x_1 - x_0$, $x_2 - x_1$ and so on...). For the 2nd order, you "skip" a node ($x_2 - x_0$, $x_3 - x_1$,...). Then you "skip" 2 nodes ($x_3 - x_0$), and so on...

Example 1.4 *Compute the* ***Lagrange polynomial*** *for the following data using*

x	3	4	5
y	1	2	4

1. *the fundamental formula*

   The fundamental formula for the Lagrange polynomial is:

$$L_m f(x) = \sum_{i=0}^{m} l_i(x) f(x_i) \tag{1.1}$$

   considering $m+1$ interpolation nodes given x_i, $i = 0, ..., m$, and the values of a function f on these nodes, $f(x_i)$, $i = 0, ..., m$.

Remark 1.5 ***The values of the polynomial $L_m f$ on the nodes should be the same as the values of the function!!*** (this is what interpolation means). So, $L_m f(x_i) = f(x_i)$, $i = 0, ..., m$. (this is how you could check if your computations were correct.)

The fundamental interpolation polynomials l_i are defined as

$$l_i(x) = \frac{(x - x_0)(x - x_1) \cdot ... \cdot (x - x_{i-1})(x - x_{i+1}) \cdot ... \cdot (x - x_m)}{(x_i - x_0)(x_i - x_1) \cdot ... \cdot (x_i - x_{i-1})(x_i - x_{i+1}) \cdot ... \cdot (x_i - x_m)}$$

Remark 1.6 *On the numerator, the term $(x - x_i)$ is missing, on the denominator the term $(x_i - x_i)$ is missing!*

So, in our case $x_0 = 3$, $x_1 = 4$, $x_2 = 5$ and $f(x_0) = 1$, $f(x_1) = 2$, $f(x_2) = 4$.
For $l_0(x)$, the terms that contain x_0 will be missing from the numer. and denom.

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 4)(x - 5)}{(3 - 4)(3 - 5)} = \frac{x^2 - 9x + 20}{2}$$

For $l_1(x)$, the terms that contain x_1 will be missing from the numer. and denom.

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 3)(x - 5)}{(4 - 3)(4 - 5)} = \frac{x^2 - 8x + 15}{-1} = -x^2 + 8x - 15$$

For $l_2(x)$, the terms that contain x_2 will be missing from the numer. and denom.

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 3)(x - 4)}{(5 - 3)(5 - 4)} = \frac{x^2 - 7x + 12}{2}$$

Now we substitute them in the eq. (1.1), for $m = 2$ and get

$$\begin{aligned} L_2 f(x) &= l_0(x) \cdot f(x_0) + l_1(x) \cdot f(x_1) + l_2(x) \cdot f(x_2) = \\ &= \frac{x^2 - 9x + 20}{2} \cdot 1 + (-x^2 + 8x - 15) \cdot 2 + \frac{x^2 - 7x + 12}{2} \cdot 4 = \\ &= \frac{x^2 - 9x + 20 - 4x^2 + 32x - 60 + 4x^2 - 28x + 48}{2} = \\ &= \frac{x^2 - 5x + 8}{2}. \end{aligned}$$

Remark 1.7 *If you have to approximate $f(3.5)$ using this polynomial, you simply have to compute*

$$L_2 f(3.5) = \frac{(3.5)^2 - 5 \cdot (3.5) + 8}{2} = ...$$

2. *the barycentric formula*
   The barycentric formula for the Lagrange polynomial is

$$L_m f(x) = \frac{\sum\limits_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum\limits_{i=0}^{m} \frac{A_i}{x - x_i}}, \tag{1.2}$$

with

$$A_i = \frac{1}{u_i(x_i)} = \frac{1}{\prod\limits_{j=0, j \neq i}^{m} (x_i - x_j)}.$$

First, let us compute A_0, A_1, A_2:

$$A_0 = \frac{1}{(x_0 - x_1)(x_0 - x_2)} \text{ (the term } (x_0 - x_0) \text{ is missing), so } A_0 = \frac{1}{(3 - 4)(3 - 5)} = \frac{1}{2}.$$

$$A_1 = \frac{1}{(x_1 - x_0)(x_1 - x_2)} \text{ (the term } (x_1 - x_1) \text{ is missing), so } A_1 = \frac{1}{(4-3)(4-5)} = -1.$$

$$A_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} \text{ (the term } (x_2 - x_2) \text{ is missing), so } A_2 = \frac{1}{(5-3)(5-4)} = \frac{1}{2}.$$

We also need $\frac{A_i}{x - x_i}$, so

$$\frac{A_0}{x - x_0} = \frac{\frac{1}{2}}{x-3} = \frac{1}{2(x-3)}, \quad \frac{A_1}{x - x_1} = -\frac{1}{x-4}, \quad \frac{A_2}{x - x_2} = \frac{1}{2(x-5)}$$

The numerator is

$$\begin{aligned}
N &= \frac{A_0}{x - x_0} \cdot f(x_0) + \frac{A_1}{x - x_1} \cdot f(x_1) + \frac{A_2}{x - x_2} \cdot f(x_2) = \\
&= \frac{1}{2(x-3)} - \frac{2}{x-4} + \frac{4}{2(x-5)} = \frac{(x-4)(x-5) - 4(x-3)(x-5) + 4(x-3)(x-4)}{2(x-3)(x-4)(x-5)} = \\
&= \frac{x^2 - 5x + 8}{2(x-3)(x-4)(x-5)}.
\end{aligned}$$

The denominator is

$$\begin{aligned}
M &= \frac{A_0}{x - x_0} + \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} = \frac{1}{2(x-3)} - \frac{1}{x-4} + \frac{1}{2(x-5)} = \\
&= \frac{(x-4)(x-5) - 2(x-3)(x-5) + (x-3)(x-4)}{2(x-3)(x-4)(x-5)} = \\
&= \frac{2}{2(x-3)(x-4)(x-5)}.
\end{aligned}$$

So, with the barycentric formula $L_2 f(x)$ is

$$L_2 f(x) = \frac{N}{M} = \frac{\frac{x^2-5x+8}{2(x-3)(x-4)(x-5)}}{\frac{2}{2(x-3)(x-4)(x-5)}} = \frac{x^2 - 5x + 8}{2},$$

as we previously obtained using the fundamental formula. ☺

For the limit of error of the Lagrange polynomial, see Lecture 2, slide 12.

Example 1.8 *Use the* ***Aitken's method*** *to approximate $f(3)$ using the data*

x_k	0	1	2	4
$f(x_k)$	1	1	2	5

We have to construct the table

x_0	f_{00}			
x_1	f_{10}	f_{11}		
x_2	f_{20}	f_{21}	f_{22}	
x_3	f_{30}	f_{31}	f_{32}	$\mathbf{f_{33}}$

where $f_{i0} = f(x_i)$, $i = 0, \ldots m$ and $f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}$ $\quad i = 1, \ldots, m, \ j = 0, \ldots, i-1.$
We have $x_0 = 0, \ x_1 = 1, \ x_2 = 2, \ x_3 = 4$. So, we get

$$f_{00} = f(x_0) = 1, \quad f_{10} = f(x_1) = 1, \quad f_{20} = f(x_2) = 2, \quad f_{30} = f(x_3) = 5.$$

In our case, since we want to approx. f(3), x=3.

$$f_{11} = f_{1,0+1} = \frac{1}{x_1 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = 1$$

$$f_{21} = f_{2,0+1} = \frac{1}{x_2 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{20} & x_2 - x \end{vmatrix} = \frac{1}{2} \cdot \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} = \frac{5}{2}$$

$$f_{31} = f_{3,0+1} = \frac{1}{x_3 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{30} & x_3 - x \end{vmatrix} = \frac{1}{4} \cdot \begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} = 4$$

$$f_{22} = f_{2,1+1} = \frac{1}{x_2 - x_1}\begin{vmatrix} f_{11} & x_1 - x \\ f_{21} & x_2 - x \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -2 \\ \frac{5}{2} & -1 \end{vmatrix} = 4$$

$$f_{32} = f_{3,1+1} = \frac{1}{x_3 - x_1}\begin{vmatrix} f_{11} & x_1 - x \\ f_{31} & x_3 - x \end{vmatrix} = \frac{1}{3} \cdot \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} = 3$$

$$f_{33} = f_{3,2+1} = \frac{1}{x_3 - x_2}\begin{vmatrix} f_{22} & x_2 - x \\ f_{32} & x_3 - x \end{vmatrix} = \frac{1}{2} \cdot \begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} = \frac{7}{2}.$$

The approximation for $f(3)$ will be f_{33}, so, $\frac{7}{2}$.

Example 1.9 *Construct the Lagrange polynomial in the* ***Newton form*** *for the data* $\begin{array}{c|ccc} x & 3 & 4 & 5 \\ \hline f & 1 & 2 & 4 \end{array}$.

The Newton form of the Lagrange pol. is

$$N_m f(x) = f(x_0) + \sum_{i=1}^{m}(x - x_0) \cdot \ldots \cdot (x - x_{i-1})(\mathcal{D}^i f)(x_0) \tag{1.3}$$

for some given interpolation nodes x_i, $i = 0, ..., m$ and $(\mathcal{D}^i f)(x_0)$ being the *divided difference of order i for* x_0. (**you will use only the first row of the divided diff. table to get the value at** x_0).

First, we have to construct the div. diff. table.

x	f	$\mathcal{D}^1 f$	$\mathcal{D}^2 f$
$x_0 = 3$	1	$\frac{f_1 - f_0}{x_1 - x_0} = \frac{2-1}{4-3} = 1$	$\frac{\mathcal{D}^1 f_1 - \mathcal{D}^1 f_0}{x_2 - x_0} = \frac{2-1}{5-3} = \frac{1}{2}$
$x_1 = 4$	2	$\frac{f_2 - f_1}{x_2 - x_1} = \frac{4-2}{5-4} = 2$	
$x_2 = 5$	4		

So, we have

$$\begin{aligned} N_2(f) &= f(x_0) + (x - x_0) \cdot \mathcal{D}^1 f(x_0) + (x - x_0)(x - x_1) \cdot \mathcal{D}^2 f(x_0) \\ &= 1 + (x-3) \cdot 1 + (x-3)(x-4) \cdot \frac{1}{2} = 1 + x - 3 + \frac{1}{2}x^2 - \frac{7}{2}x + 6 = \\ &= \frac{1}{2}x^2 - \frac{5}{2}x + 4. \end{aligned}$$

Remark 1.10 This is the same polynomial that we obtained by computing with the fundamental and barycentric Lagrange formulae.

Example 1.11 *Construct the interpolation polynomial that approximates the data*

x	y	y'
-1	2	-4
1	2	–

We have $x_0 = -1$, $x_1 = 1$, $f(x_0) = 2$, $f(x_1) = 2$, $f'(-1) = -4$. We have some information about the derivative of f, we cannot use Lagrange interpolation, so we are left with either Hermite or Birkhoff. Since no derivative's order is skipped for the 2 nodes (we have f(-1), f'(-1) -> max. order for derivative is 1 and we have f(1) -> max. order is 0), we have to use **Hermite interpolation**. (The derivative's maximum order doesn't have to be the same for each node. It is important not to skip any order from 0 to the max.)

$$m = 1,\ \ r_0 = 1 \text{ (max. order of derivative for } x_0),\ \ r_1 = 0 \text{ (max. order of derivative for } x_1)$$

$$\Longrightarrow\ n = r_0 + r_1 + m = 2.$$

So our polynomial will have **the degree 2**. For theory, see Lecture 4, slides 1, 2, 3.

$$H_n f(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) \cdot f^{(j)}(x_k)$$

where $f^{(j)}(x_k)$ denotes the derivative of order j of function f at the node x_k.
In our case, we have

$$H_2 f(x) = \sum_{k=0}^{1} \sum_{j=0}^{r_k} h_{kj}(x) \cdot f^{(j)}(x_k) = h_{00}(x) \cdot f(x_0) + h_{01}(x) \cdot f'(x_0) + h_{10}(x) \cdot f(x_1) \tag{1.4}$$

The unknowns here are the polynomials h_{00}, h_{01} and h_{10}. Since the Hermite pol. should have the degree 2, we will write each of these 3 polynomials as a second degree pol. $(ax^2 + bx + c)$ and determine the coeffs. a, b, c in each case, by using the following properties for the Hermite fundamental polynomials h_{kj}:

$$h_{kj}^{(p)}(x_\nu) = 0,\ \text{when } k \neq \nu,\ p = 0, \dots, r_\nu$$
$$h_{kj}^{(p)}(x_k) = \delta_{jp},\ \text{when }\ p = 0, \dots, r_k,\ j = 0, \dots, r_k,\ \text{and } \nu, k = 0, \dots, m$$
$$\delta_{jp} = 0\ (j \neq p) \text{ and } 1\ (j = p).$$

So, $h_{01}(x_1) = 0$ (because $0 \neq 1$).
$h_{11}(x_1) = 0$ (we have $1 = 1$, but $h_{11}(x_1) = h_{11}^{(0)}(x_1)$ and $0 \neq 1$).
$h_{11}'(x_1) = 1$ (because $1{=}1$ and $1{=}1$).

Remark 1.12 *The same properties are used for the Birkhoff fundamental interpolation polynomials.*

Coming back to our problem, let us find h_{00}, h_{01} and h_{10}.

- $h_{00}(x) = ax^2 + bx + c$ and $h_{00}'(x) = 2ax + b$

$$\begin{cases} h_{00}(x_0) = 1 \\ h_{00}'(x_0) = 0 \\ h_{00}(x_1) = 0 \end{cases} \Longrightarrow \begin{cases} h_{00}(-1) = a - b + c = 1 \\ h_{00}'(-1) = -2a + b = 0 \\ h_{00}(1) = a + b + c = 0 \end{cases} \Longrightarrow \begin{cases} b = 2a \\ -a + c = 1 \\ 3a + c = 0 \end{cases} \Longrightarrow \begin{cases} a = -\dfrac{1}{4} \\ b = -\dfrac{1}{2} \\ c = \dfrac{3}{4} \end{cases}$$

$$\Longrightarrow\ h_{00}(x) = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4}.$$

- $h_{01}(x) = ax^2 + bx + c$ and $h_{01}'(x) = 2ax + b$

$$\begin{cases} h_{01}(x_0) = 0 \\ h_{01}'(x_0) = 1 \\ h_{01}(x_1) = 0 \end{cases} \Longrightarrow \begin{cases} h_{01}(-1) = a - b + c = 0 \\ h_{01}'(-1) = -2a + b = 1 \\ h_{01}(1) = a + b + c = 0 \end{cases} \Longrightarrow \begin{cases} b = 1 + 2a \\ -a + c = 1 \\ 3a + c = -1 \end{cases} \Longrightarrow \begin{cases} a = -\dfrac{1}{2} \\ b = 0 \\ c = \dfrac{1}{2} \end{cases}$$

$$\Longrightarrow\ h_{01}(x) = -\frac{1}{2}x^2 + \frac{1}{2}.$$

- $h_{10}(x) = ax^2 + bx + c$ and $h'_{10}(x) = 2ax + b$

$$\begin{cases} h_{10}(x_0) = 0 \\ h'_{10}(x_0) = 0 \\ h_{10}(x_1) = 1 \end{cases} \Longrightarrow \begin{cases} h_{10}(-1) = a - b + c = 0 \\ h'_{10}(-1) = -2a + b = 0 \\ h_{10}(1) = a + b + c = 1 \end{cases} \Longrightarrow \begin{cases} b = 2a \\ -a + c = 0 \\ 3a + c = 1 \end{cases} \Longrightarrow \begin{cases} a = \frac{1}{4} \\ b = \frac{1}{2} \\ c = \frac{1}{4} \end{cases}$$

$$\Longrightarrow h_{10}(x) = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}.$$

Going back at eq. (1.4), we obtain

$$\begin{aligned} H_2(f) &= 2 \cdot (-\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4}) - 4 \cdot (-\frac{1}{2}x^2 + \frac{1}{2}) + 2 \cdot (\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}) = \\ &= -\frac{1}{2}x^2 - x + \frac{3}{2} + 2x^2 - 2 + \frac{1}{2}x^2 + x + \frac{1}{2} = \\ &= 2x^2. \end{aligned}$$

Indeed, this polynomial satisfies the interpolation properties

$$H_2f(-1) = 2 = f(-1), \quad H'_2f(-1) = -4 = f'(-1), \quad H_2f(1) = 2 = f(1).$$

Remark 1.13 *If you need to approximate $f(0)$, you have to compute $H_2f(0) = 2 \cdot 0^2 = 0$.*

For limit of the error, see Lecture 4, slide 6.

Example 1.14 *Consider the double nodes $x_0 = -1$ and $x_1 = 1$. Consider also $f(-1) = -3$, $f'(-1) = 10, f(1) = 1$, $f'(1) = 2$. Find the* ***Hermite interpolation polynomial using the divided difference table for double nodes****.*

Remark 1.15 *Hermite interpolation for double nodes can be used only when you know the values of f and f' **for all the nodes**!*

First, we should compute the divided difference table with double nodes.
$z_0 = x_0$, $z_1 = x_0$, $z_2 = x_1$, $z_3 = x_1$. You should also double the values of f. The difference appears when you compute the divided difference of first order. At the odd positions you have to put the derivative of f at the corresponding node. The other entries are computed the same.

x	f	$\mathcal{D}^1 f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
z_0 = -1	-3	$f'(-1) = 10$	$\frac{\mathcal{D}^1 f_1 - \mathcal{D}^1 f_0}{z_2 - z_0} = \frac{2-10}{1-(-1)} = -4$	$\frac{\mathcal{D}^2 f_1 - \mathcal{D}^2 f_0}{z_3 - z_0} = \frac{0-(-4)}{1-(-1)} = 2$
z_1 = - 1	-3	$\frac{f_2 - f_1}{z_2 - z_1} = \frac{1-(-3)}{1-(-1)} = 2$	$\frac{\mathcal{D}^1 f_2 - \mathcal{D}^1 f_1}{z_3 - z_1} = \frac{2-2}{1-(-1)} = 0$	
z_2 = 1	1	$f'(1) = 2$		
z_3 = 1	1			

Next, we will use the Newton form:

$$H_{2m+1}f(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0) \cdot \ldots \cdot (x - z_{i-1})(\mathcal{D}^i f)(z_0) \tag{1.5}$$

that is in our case

$$\begin{aligned} H_3f(x) &= -3 + (x - z_0) \cdot (\mathcal{D}^1 f)(z_0) + (x - z_0)(x - z_1) \cdot (\mathcal{D}^2 f)(z_0) + (x - z_0)(x - z_1)(z - z_2) \cdot (\mathcal{D}^3 f)(z_0) = \\ &= -3 + (x + 1) \cdot 10 + (x + 1)^2 \cdot (-4) + (x + 1)^2 (x - 1) \cdot 2 \\ &= -3 + 10x + 10 - 4x^2 - 8x - 4 + 2x^3 - 2x^2 + 4x^2 - 4x + 2x - 2 = \\ &= 2x^3 - 2x^2 + 1. \end{aligned}$$

Example 1.16 *Approximate $f(\frac{1}{2})$ knowing the following information: $x_0 = 0,\ x_1 = 1,\ f(x_0) = 1,\ f'(x_0) = 2$ and $f'(x_1) = -1$.*

Since for x_1 the value of f is skipped, we cannot use Hermite interpolation, so we have to use **Birkhoff interpolation**.

$I_0 = \{0, 1\}$ (for x_0 we have the val. for f and f') $I_1 = \{1\}$ (for x_1 we know only f'). So, the **degree of the Birkhoff pol.** is

$$n = |I_0| + |I_1| - 1 = 2 + 1 - 1 = 2 \quad (|I| \text{ denotes the cardinal of the set I = nr. of elements in I})$$

. Using similar reasoning as in Hermite interpolation, we have

$$B_n f(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) \cdot f^{(j)}(x_k)$$

where $f^{(j)}(x_k)$ denotes the derivative of order j of function f at the node x_k.

In our case, we have

$$B_2 f(x) = \sum_{k=0}^{1} \sum_{j \in I_k} b_{kj}(x) \cdot f^{(j)}(x_k) = b_{00}(x) \cdot f(x_0) + b_{01}(x) \cdot f'(x_0) + b_{11}(x) \cdot f'(x_1) \tag{1.6}$$

The unknowns here are the polynomials b_{00}, b_{01} and b_{11}. Since the Birkhoff pol. should have the degree 2, we will write each of these 3 polynomials as a second degree pol. $(ax^2 + bx + c)$ and determine the coeffs. a, b, c in each case, by using the fundamental properties we used for the Hermite polynomials h_{kj}:

- $b_{00}(x) = ax^2 + bx + c$ and $b'_{00}(x) = 2ax + b$

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_0) = 0 \\ b'_{00}(x_1) = 0 \end{cases} \implies \begin{cases} b_{00}(0) = c = 1 \\ b'_{00}(0) = b = 0 \\ b'_{00}(1) = 2a + b = 0 \implies a = 0 \end{cases}$$

$$\implies b_{00}(x) = 1.$$

- $b_{01}(x) = ax^2 + bx + c$ and $b'_{01}(x) = 2ax + b$

$$\begin{cases} b_{01}(x_0) = 0 \\ b'_{01}(x_0) = 1 \\ b'_{01}(x_1) = 0 \end{cases} \implies \begin{cases} b_{01}(0) = c = 0 \\ b'_{01}(0) = b = 1 \\ b'_{01}(1) = 2a + b = 0 \implies a = -\dfrac{1}{2} \end{cases}$$

$$\implies b_{01}(x) = -\frac{1}{2}x^2 + x.$$

- $b_{11}(x) = ax^2 + bx + c$ and $b'_{11}(x) = 2ax + b$

$$\begin{cases} b_{11}(x_0) = c = 0 \\ b'_{11}(x_0) = b = 0 \\ b'_{11}(x_1) = 2a + b = 1 \implies a = \dfrac{1}{2} \end{cases}$$

$$\implies b_{11}(x) = \frac{1}{2}x^2.$$

Going back at eq. (1.6), we obtain

$$\begin{aligned} B_2(f) &= 1 \cdot 1 + 2 \cdot (-\frac{1}{2}x^2 + x) - 1 \cdot \frac{1}{2}x^2 == 1 - x^2 + 2x - \frac{1}{2}x^2 = \\ &= -\frac{3}{2}x^2 + 2x + 1 \end{aligned}$$

Indeed, this polynomial satisfies the interpolation properties

$$B_2 f(0) = 1 = f(0), \quad B_2' f(0) = 2 = f'(0), \quad B_2' f(1) = -1 = f'(1).$$

For the approximation of $f(\frac{1}{2})$, we have to compute $B_2 f(\frac{1}{2}) = -\frac{3}{2} \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 1 = \frac{13}{8}$.

Example 1.17 *Fit the data from the table with*

x	-1	0	1
y	1	1	2

a) the best least squares line

The best least squares line is obtained by considering the linear polynomial (degree = 1) $P(x) = ax + b$, with the unknowns a and b. They can be found from the system

$$\begin{cases} a \sum_{i=0}^{m} x_i^2 + b \sum_{i=0}^{m} x_i = \sum_{i=0}^{m} x_i y_i \\ a \sum_{i=0}^{m} x_i + b(m+1) = \sum_{i=0}^{m} y_i \end{cases}$$

In our case, $m = 2$ (we have x_0, x_1, x_2). We also have

	x_i	y_i	x_i^2	$x_i y_i$
	-1	1	1	-1
	0	1	0	0
	1	2	1	2
$\sum$	0	4	2	1

so the system becomes

$$\begin{cases} 2 \cdot a + 0 \cdot b = 1 \\ 0 \cdot a + 3 \cdot b = 4 \end{cases} \Longrightarrow \begin{cases} a = \dfrac{1}{2} \\ b = \dfrac{4}{3} \end{cases}$$

so $P(x) = \frac{1}{2}x + \frac{4}{3}$.

b) the best least squares polynomial of degree 2

$P(x) = a_0 + a_1 x + a_2 x^2$, with the unknowns a_0, a_1, a_2. They can be found from the system

$$\begin{cases} a_0 \sum_{i=0}^{m} x_i^0 + a_1 \sum_{i=0}^{m} x_i^1 + a_2 \sum_{i=0}^{m} x_i^2 = \sum_{i=0}^{m} x_i^0 y_i \\ a_0 \sum_{i=0}^{m} x_i^1 + a_1 \sum_{i=0}^{m} x_i^2 + a_2 \sum_{i=0}^{m} x_i^3 = \sum_{i=0}^{m} x_i^1 y_i \\ a_0 \sum_{i=0}^{m} x_i^2 + a_1 \sum_{i=0}^{m} x_i^3 + a_2 \sum_{i=0}^{m} x_i^4 = \sum_{i=0}^{m} x_i^2 y_i \end{cases}$$

In our case, $m = 2$ (we have x_0, x_1, x_2). We also have

	x_i	y_i	x_i^2	$x_i y_i$	x_i^3	x_i^4	$x_i^2 y_i$
	-1	1	1	-1	-1	1	1
	0	1	0	0	0	0	0
	1	2	1	2	1	1	2
$\sum$	0	4	2	1	0	2	3

so the system becomes

$$\begin{cases} 3 \cdot a_0 + 0 \cdot a_1 + 2 \cdot a_2 = 4 \\ 0 \cdot a_0 + 2 \cdot a_1 + 0 \cdot a_2 = 1 \\ 2 \cdot a_0 + 0 \cdot a_1 + 2 \cdot a_2 = 3 \end{cases} \Longrightarrow \begin{cases} a_0 = 1 \\ a_1 = \dfrac{1}{2} \\ a_2 = \dfrac{1}{2} \end{cases}$$

so $P(x) = 1 + \frac{1}{2}x + \frac{1}{2}x^2$.

For the general case (polynomial of order n), see the system that is obtained in Lecture 6, slide 10.

Example 1.18 *For the points* $(1,2)$, $(2,3)$, $(3,5)$ *construct a cubic spline that passes through them, in the following cases:*

*a) the spline is **natural**;*

*b) the spline is **clamped** and S'(1)=2, S'(3)=1.*

For a function $f : [a,b] \to \mathbb{R}$ whose values on the nodes $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ are known, a cubic spline S satisfies the following properties:

1. it is a cubic polynomial $S_j(x)$ on the interval $[x_j, x_{j+1}]$, $\forall j = \overline{0, n-1}$

$$S(x) = \begin{cases} S_0(x),\, x \in [x_0, x_1] \\ S_1(x),\, x \in [x_1, x_2] \\ \ldots \\ S_{n-1}(x),\, x \in [x_{n-1}, x_n] \end{cases}$$

with

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

The unknowns a_j, b_j, c_j, d_j are found from the following relations

2. $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$, $j = \overline{0, n-1}$;
3. $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$, $j = \overline{0, n-2}$;
4. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$, $j = \overline{0, n-2}$;
5. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$, $j = \overline{0, n-2}$;
6. one of the following conditions is satisfied
   a) $S''(x_0) = S''(x_n) = 0$ ($\iff$ $S''_0(x_0) = S''_{n-1}(x_n) = 0$) - natural spline
   b) $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ ($\iff$ $S'_0(x_0) = f'(x_0)$, $S'_{n-1}(x_n) = f'(x_n)$) - clamped spline
   c) $S_0(x) = S_1(x)$ and $S_{n-2}(x) = S_{n-1}(x)$ - de Boor spline

a) For our exercise, we first construct the **natural cubic spline**.
$x_0 = 1.$ $x_1 = 2$, $x_2 = 3$ and $f(x_0) = 1$, $f(x_1) = 2$, $f(x_2) = 5$ and $n = 2$. We will have

$$1.\ S(x) = \begin{cases} S_0(x),\, x \in [1,2] \\ S_1(x),\, x \in [2,3] \end{cases} \text{ with } \begin{cases} S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \\ S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3 \end{cases}$$

We also need the first and second derivatives for S_0 and S_1, that are

$$\begin{matrix} S'_0(x) = b_0 + 2c_0(x-1) + 3d_0(x-1)^2 \\ S'_1(x) = b_1 + 2c_1(x-2) + 3d_1(x-2)^2 \end{matrix} \text{ and } \begin{matrix} S''_0(x) = 2c_0 + 6d_0(x-1) \\ S''_1(x) = 2c_1 + 6d_1(x-2) \end{matrix}$$

Conditions 2–6 give us

$$\begin{cases} S_0(x_0) = f(x_0) \\ S_0(x_1) = f(x_1) \\ S_1(x_1) = f(x_1) \\ S_1(x_2) = f(x_2) \\ S_0(x_1) = S_1(x_1) \\ S_0'(x_1) = S_1'(x_1) \\ S_0''(x_1) = S_1''(x_1) \\ \color{red}{S_0''(x_0) = 0} \\ \color{red}{S_1''(x_2) = 0} \end{cases} \iff \begin{cases} a_0 = f(1) = 2 \\ a_0 + b_0 + c_0 + d_0 = f(2) = 3 \\ a_1 = f(2) = 3 \\ a_1 + b_1 + c_1 + d_1 = f(3) = 5 \\ a_0 + b_0 + c_0 + d_0 = a_1 = 3 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \\ 2c_0 = 0 \implies c_0 = 0 \\ 2c_1 + 6d_1 = 0 \implies c_1 + 3d_1 = 0 \end{cases}$$

For solving the system, see the pdf file "Cubic Splines - Examples (natural + clamped)" in MS Teams file.

The solution of the system is: $a_0 = 2$, $b_0 = \frac{3}{4}$, $c_0 = 0$, $d_0 = \frac{1}{4}$, $a_1 = 3$, $b_1 = \frac{3}{2}$, $c_1 = \frac{3}{4}$, $d_1 = -\frac{1}{4}$, so the spline S is

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, & x \in [2,3] \end{cases}.$$

b) We construct now the **clamped cubic spline**.

$x_0 = 1$. $x_1 = 2$, $x_2 = 3$ and $f(x_0) = 1$, $f(x_1) = 2$, $f(x_2) = 5$ and $n = 2$. We will have

$$1.\ S(x) = \begin{cases} S_0(x), & x \in [1,2] \\ S_1(x), & x \in [2,3] \end{cases} \text{ with } \begin{cases} S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \\ S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3 \end{cases}$$

We also need the first and second derivatives for S_0 and S_1, that are

$$\begin{matrix} S_0'(x) = b_0 + 2c_0(x-1) + 3d_0(x-1)^2 \\ S_1'(x) = b_1 + 2c_1(x-2) + 3d_1(x-2)^2 \end{matrix} \text{ and } \begin{matrix} S_0''(x) = 2c_0 + 6d_0(x-1) \\ S_1''(x) = 2c_1 + 6d_1(x-2) \end{matrix}$$

Conditions 2–6 give us

$$\begin{cases} S_0(x_0) = f(x_0) \\ S_0(x_1) = f(x_1) \\ S_1(x_1) = f(x_1) \\ S_1(x_2) = f(x_2) \\ S_0(x_1) = S_1(x_1) \\ S_0'(x_1) = S_1'(x_1) \\ S_0''(x_1) = S_1''(x_1) \\ \color{blue}{S_0'(x_0) = f'(x_0)} \\ \color{blue}{S_1'(x_2) = f'(x_2)} \end{cases} \iff \begin{cases} a_0 = f(1) = 2 \\ a_0 + b_0 + c_0 + d_0 = f(2) = 3 \\ a_1 = f(2) = 3 \\ a_1 + b_1 + c_1 + d_1 = f(3) = 5 \\ a_0 + b_0 + c_0 + d_0 = a_1 = 3 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \\ b_0 = 2 \\ b_1 + 2c_1 + 3d_1 = 1 \end{cases}$$

For solving the system, see the pdf file "Cubic Splines - Examples (natural + clamped)" in MS Teams Files.

The solution of the system is: $a_0 = 2$, $b_0 = 2$, $c_0 = -\frac{5}{2}$, $d_0 = \frac{3}{2}$, $a_1 = 3$, $b_1 = \frac{3}{2}$, $c_1 = 2$, $d_1 = -\frac{3}{2}$, so the spline S is

$$S(x) = \begin{cases} 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3, & x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3, & x \in [2,3] \end{cases}.$$

Remark 1.19 *The changes appear only when we impose the conditions for the natural spline (the last two relations written with red) and for the clamped spline (the last two relations written with blue).*