3 Examples for Lectures 9 – 10 (Numerical methods for solving linear systems)

3.1 Direct methods

Example 3.1 *Solve the system*

$$\begin{cases} 2x_1 + 4x_3 + x_4 = 7 \\ 2x_2 + 4x_3 + x_4 = 7 \\ 2x_1 + 4x_2 + 3x_3 = 9 \\ x_1 + 2x_2 + 2x_4 = 5 \end{cases}$$

using the Gauss method with partial pivoting.

We start by writing the matrix A that contains the coefficients of the unknowns (x_1, x_2, x_3, x_4). We also write $\overline{A}$ which contains also the column vector b (the result of each equation), since the modifications should be performed on this column too.

$$A = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \overline{A} = \begin{pmatrix} \mathbf{2} & 0 & 4 & 1 & | & 7 \\ 0 & 2 & 4 & 1 & | & 7 \\ 2 & 4 & 3 & 0 & | & 9 \\ 1 & 2 & 0 & 2 & | & 5 \end{pmatrix}.$$

On the first column of $\overline{A}$, the pivot (maximum element in absolute value) is $a_{11} = 2$, so we do not interchange any rows. $a_{21} = 0$ so we let it the same, and to obtain $a_{31} = 0$ and $a_{41} = 0$, we have to perform $L_3 - L_1$ and $L_4 - \frac{1}{2}L_1$. (Don't forget to change also the column of free term b!)

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & 2 & 4 & 1 & | & 7 \\ 0 & 4 & -1 & -1 & | & 2 \\ 0 & 2 & -2 & \frac{3}{2} & | & \frac{3}{2} \end{pmatrix}$$

On the second column (below the main diagonal - and including it), the maximum element in absolute value is $a_{32} = 4$, so we interchange L_2 and L_3.

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & \mathbf{4} & -1 & -1 & | & 2 \\ 0 & 2 & 4 & 1 & | & 7 \\ 0 & 2 & -2 & \frac{3}{2} & | & \frac{3}{2} \end{pmatrix}$$

The pivot is now $a_{22} = 4$. Now, to obtain 0 below the main diagonal on the second column, we need $a_{32} = 0$ and $a_{42} = 0$, so we perform $L_3 - \frac{1}{2}L_2$ and $L_4 - \frac{1}{2}L_2$, obtaining

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & 4 & -1 & -1 & | & 2 \\ 0 & 0 & \boldsymbol{\frac{9}{2}} & \frac{3}{2} & | & 6 \\ 0 & 0 & -\frac{3}{2} & 2 & | & \frac{1}{2} \end{pmatrix}$$

On the third column, the maximum element in absolute value below the main diagonal (including it) is $a_{33} = \frac{9}{2}$, so we don't interchange anything. To obtain 0 below the main diagonal, we need $a_{43} = 0$, so we have to compute $L_4 + \frac{1}{3}L_3$, obtaining

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & 4 & -1 & -1 & | & 2 \\ 0 & 0 & \frac{9}{2} & \frac{3}{2} & | & 6 \\ 0 & 0 & 0 & \frac{5}{2} & | & \frac{5}{2} \end{pmatrix}$$

Now, using **backward substitution**, we obtain

$$\begin{aligned}
\frac{5}{2}x_4 = \frac{5}{2} &\implies x_4 = 1\\
\frac{9}{2}x_3 + \frac{3}{2}\cdot 1 = 6 &\implies x_3 = 1\\
4x_2 - 1\cdot 1 - 1\cdot 1 = 2 &\implies x_2 = 1\\
2x_1 + 0\cdot 1 + 4\cdot 1 + 1\cdot 1 = 7 &\implies x_1 = 1
\end{aligned}$$

Remark 3.2 The **theory** can be found in Course 9, slides 17–20. Other **examples** using the **Gauss elimination with partial pivoting** are in Course 9, slides 21–25.

Example 3.3 *Solve the system*

$$\begin{cases} x_1 + x_2 - 3x_3 = -9\\ 4x_1 + x_2 + 2x_3 = 9\\ 2x_1 + 4x_2 - x_3 = -5\end{cases}$$

using the Gauss-Jordan method.

The idea here is to make zeros below and above the main diagonal, reducing the system to something of the form

$$\begin{pmatrix}\hat{a}_{11} & 0 & \dots & 0\\ 0 & \hat{a}_{22} & \dots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \dots & \hat{a}_{nn}\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}\hat{b}_1\\\hat{b}_2\\\vdots\\\hat{b}_n\end{pmatrix} \text{ such that } x_i = \frac{\hat{b}_i}{\hat{a}_{ii}},\ i = \overline{1,n}.$$

$$\overline{A} = \begin{pmatrix}1 & 1 & -3 & | & -9\\ 4 & 1 & 2 & | & 9\\ 2 & 4 & -1 & | & -5\end{pmatrix} \overset{\substack{L_2-4L_1\\L_3-2L_1}}{\sim} \begin{pmatrix}1 & 1 & -3 & | & -9\\ 0 & -3 & 14 & | & 45\\ 0 & 2 & 5 & | & 13\end{pmatrix} \overset{L_3+\frac{2}{3}L_2}{\sim} \begin{pmatrix}1 & 1 & -3 & | & -9\\ 0 & -3 & 14 & | & 45\\ 0 & 0 & \frac{43}{3} & | & 43\end{pmatrix} \overset{\substack{L_1+\frac{1}{3}L_2\\ \frac{1}{43}L_3}}{\sim}$$

$$\begin{pmatrix}1 & 0 & \frac{5}{3} & | & 6\\ 0 & -3 & 14 & | & 45\\ 0 & 0 & \frac{1}{3} & | & 1\end{pmatrix} \overset{\substack{L_1-5L_3\\L_2-3\cdot 14L_3}}{\sim} \begin{pmatrix}1 & 0 & 0 & | & 1\\ 0 & -3 & 0 & | & 3\\ 0 & 0 & \frac{1}{3} & | & 1\end{pmatrix} \implies \begin{cases}x_1 = 1\\ x_2 = -1\\ x_3 = 3\end{cases}.$$

Remark 3.4 *In this case we did not use the partial pivoting, we have only solved this system with the usual Gauss-Jordan method.*

Remark 3.5 *For the LU factorization (modified forms of Gauss elimination), see Course 10, slides 1–3 for theory and slides 4–6 for an example.*

3.2 Iterative methods

Example 3.6 *Determine the approximate solution for the system*

$$\begin{cases}5x_1 + x_2 - x_3 = 7\\ x_1 + 5x_2 + x_3 = 7\\ x_1 + x_2 + 5x_3 = 7\end{cases}$$

with the initial approximation $x^{(0)} = (0,0,0)^T$ *using*

a) Jacobi method in 3 steps ;
We can see that the matrix

$$A = \begin{pmatrix}5 & 1 & -1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{pmatrix}$$

is strictly diagonally dominant since

$$
\begin{aligned}
|a_{11}| = |5| > |a_{12}| + |a_{13}| = |1| + |-1| = 2 \\
|a_{22}| = |5| > |a_{21}| + |a_{23}| = |1| + |1| = 2 \\
|a_{3}| = |5| > |a_{31}| + |a_{32}| = |1| + |1| = 2
\end{aligned}
$$

hence all the three methods will converge (no matter what the initial approximation $x^{(0)}$ is).
To apply the methods, we have to express the unknown x_k from the equation k with respect to the other unknowns. So, we have

$$
\begin{cases}
x_1 = \dfrac{7 - x_2 + x_3}{5} \\
x_2 = \dfrac{7 - x_1 - x_3}{5} \\
x_3 = \dfrac{7 - x_1 - x_2}{5}
\end{cases}
\tag{3.1}
$$

Now, the Jacobi method consists in expressing $x^{(k)}$ (the unknown x at step k) using the previous approximations $x^{(k-1)}$. We have:

$$
\begin{cases}
x_1^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\
x_2^{(1)} = \dfrac{7 - x_1^{(0)} - x_3^{(0)}}{5} = \dfrac{7 - 0 - 0}{5} = \dfrac{7}{5} = 1.4 \\
x_3^{(1)} = \dfrac{7 - x_1^{(0)} - x_2^{(0)}}{5} = \dfrac{7 - 0 - 0}{5} = \dfrac{7}{5} = 1.4
\end{cases}
$$

Next, on the second iteration we have:

$$
\begin{cases}
x_1^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - \frac{7}{5} + \frac{7}{5}}{5} = \dfrac{7}{5} = 1.4 \\
x_2^{(2)} = \dfrac{7 - x_1^{(1)} - x_3^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{7}{5}}{5} = \dfrac{21}{25} = 0.84 \\
x_3^{(2)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{7}{5}}{5} = \dfrac{21}{25} = 0.84
\end{cases}
$$

and the last one:

$$
\begin{cases}
x_1^{(3)} = \dfrac{7 - x_2^{(2)} + x_3^{(2)}}{5} = \dfrac{7 - \frac{21}{25} + \frac{21}{25}}{5} = \dfrac{7}{5} = 1.4 \\
x_2^{(3)} = \dfrac{7 - x_1^{(2)} - x_3^{(2)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{21}{25}}{5} = \dfrac{119}{125} = 0.952 \cdot \\
x_3^{(3)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{21}{25}}{5} = \dfrac{119}{125} = 0.952
\end{cases}
$$

b) Gauss-Seidel method in 2 steps ;
The difference between Jacobi and Gauss-Seidel is that in this case, we have to replace the unknowns with their most recent approximations. So, if we are at the step k, when we compute $x_3^{(k)}$, we won't use x_1 and x_2 from the previous step ($x_1^{(k-1)}$, $x_2^{(k-1)}$), but instead we will use their values from the current step, since we have already determined them. Using again (3.1), we obtain

$$
\begin{cases}
x_1^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\
x_2^{(1)} = \dfrac{7 - x_1^{(1)} - x_3^{(0)}}{5} = \dfrac{7 - \frac{7}{5} - 0}{5} = \dfrac{28}{25} = 1.12 \\
x_3^{(1)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{28}{25}}{5} = \dfrac{112}{125} = 0.896
\end{cases}
$$

Next, we have:

$$\begin{cases} x_1^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - \frac{28}{25} + \frac{112}{125}}{5} = \dfrac{847}{625} = 1.3552 \\ x_2^{(2)} = \dfrac{7 - x_1^{(2)} - x_3^{(1)}}{5} = \dfrac{7 - \frac{847}{625} - \frac{112}{125}}{5} = \dfrac{2968}{3125} = 0.94976 \\ x_3^{(2)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7 - \frac{847}{625} - \frac{2968}{3125}}{5} = \dfrac{14672}{15625} = 0.939008 \end{cases}$$

c) SOR method for $\omega = \frac{1}{2}$ in 2 steps .

It is similar to Gauss-Seidel method. First, we compute an intermediary point $\tilde{x}^{(k)}$ as in Gauss-Seidel and then $x^{(k)} = \omega\tilde{x}^{(k)} + (1-\omega)x^{(k-1)}$. So, for (3.1), we have:

$$\begin{cases} \tilde{x_1}^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\ x_1^{(1)} = \omega\tilde{x_1}^{(1)} + (1-\omega)x_1^{(0)} = \dfrac{1}{2}\cdot 1.4 + \dfrac{1}{2}\cdot 0 = 0.7 \\ \tilde{x_2}^{(1)} = \dfrac{7 - x_1^{(1)} - x_3^{(0)}}{5} = \dfrac{7 - 0.7 - 0}{5} = 1.26 \\ x_2^{(1)} = \omega\tilde{x_2}^{(1)} + (1-\omega)x_2^{(0)} = \dfrac{1}{2}\cdot 1.26 + \dfrac{1}{2}\cdot 0 = 0.63 \\ \tilde{x_3}^{(1)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - 0.7 - 0.63}{5} = 1.134 \\ x_3^{(1)} = \omega\tilde{x_3}^{(1)} + (1-\omega)x_3^{(0)} = \dfrac{1}{2}\cdot 1.134 + \dfrac{1}{2}\cdot 0 = 0.567 \end{cases}$$

And the second iteration is

$$\begin{cases} \tilde{x_1}^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - 0.63 + 0.567}{5} = 1.3874 \\ x_1^{(2)} = \omega\tilde{x_1}^{(2)} + (1-\omega)x_1^{(1)} = \dfrac{1}{2}\cdot 1.3874 + \dfrac{1}{2}\cdot 0.7 = 1.0437 \\ \tilde{x_2}^{(2)} = \dfrac{7 - x_1^{(2)} - x_3^{(1)}}{5} = \dfrac{7 - 1.0437 - 0.567}{5} = 1.07786 \\ x_2^{(2)} = \omega\tilde{x_2}^{(2)} + (1-\omega)x_2^{(1)} = \dfrac{1}{2}\cdot 1.07786 + \dfrac{1}{2}\cdot 0.63 = 0.85393 \\ \tilde{x_3}^{(2)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7 - 1.0437 - 0.85393}{5} = 1.020474 \\ x_3^{(2)} = \omega\tilde{x_3}^{(2)} + (1-\omega)x_3^{(1)} = \dfrac{1}{2}\cdot 1.020474 + \dfrac{1}{2}\cdot 0.567 = 0.793737 \end{cases}$$

Remark 3.7 The exact solution is (1.4; 0.9(3); 0.9(3)).

Remark 3.8 *See the theory and other examples for the three Iterative methods in Course 10, slides 7–22.*