4 Examples for Lecture 11 (Nonlinear equations in $\mathbb{R}$)

For theory see Course 11.

Example 4.1 *Consider the equation $x^3 - 2x^2 = 5$. Give the next two iterations for approximating the solution of this equation using:*

1. Newton's method starting with $x_0 = 2$.
   Considering the equation $f(x) = 0$ (in our case we will have $f(x) = x^3 - 2x^2 - 5$), for Newton's method the next iteration is obtained from
   $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
   for a starting value x_0. This method is a *one-step method*, i.e., to obtain an approximation for x_k we need only the previous approximation x_{k-1}, in particular we only need one starting value.
   $$f'(x) = (x^3 - 2x^2 - 5)' = 3x^2 - 4x$$
   The next two iterations are:
   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-5}{4} = \frac{13}{4} = 3.25$$
   and
   $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.25 - \frac{f(3.25)}{f'(3.25)} = 3.25 - \frac{8.2}{18.68} \approx 2.81.$$
   Remark 4.2 See another example on slide 20 and the theory on slides 8-19.

2. secant method starting with $x_0 = 1$ and $x_1 = 3$.
   The secant method is a two-step method (we need two previous approximations to get a new one) and it has the formula
   $$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k)$$
   for two starting values x_0 and x_1. In our case we have:
   $$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{(3-1)f(3)}{f(3) - f(1)} = 3 - \frac{2 \cdot 4}{4 - (-6)} = 3 - 0.8 = 2.2$$
   and the second approximation
   $$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2) = 2.2 - \frac{(2.2-3)f(2.2)}{f(2.2) - f(3)} = 2.2 - \frac{-0.8 \cdot (-4.03)}{-4.03 - 4} = 2.2 + \frac{0.8 \cdot 4.03}{8.03} \approx 2.60.$$
   Remark 4.3 See another example on slide 28 and the theory on slides 22-27.

3. bisection method starting with $a_0 = 1$ and $b_0 = 3$.
   The bisection method is based on the following algorithm: Supposing $f(a) \cdot f(b) < 0$ with f continuous on $[a, b]$, then there is at least one root of f in (a, b). First we compute the middle of the interval $[a, b]$, i.e., $c = \frac{a+b}{2}$ and we check:
   - if $f(c) \cdot f(b) < 0$, then the root is in the interval $[c, b]$ and we consider $a_1 = c$ and $b_1 = b$;
   - otherwise (if $f(c) \cdot f(b) > 0$),then the root is in $[a, c]$ and we consider $a_1 = a$ and $b_1 = c$.

   We apply then the same steps on the new interval $[a_1, b_1]$.
   In our case, we have :
   $$f(a_0) \cdot f(b_0) = f(1) \cdot f(3) = (1^3 - 2 \cdot 1^2 - 5) \cdot (3^3 - 2 \cdot 3^2 - 5) = (-6) \cdot 4 = -24 < 0,$$

so

$$c_0 = \frac{a_0 + b_0}{2} = \frac{1+3}{2} = 2.$$

Next, we check:

$$f(c_0) \cdot f(b_0) = f(2) \cdot f(3) = (-5) \cdot 4 = -20 < 0$$

and the root must be in the interval $[c_0, b_0] = [2, 3]$. This is why we set $a_1 = c_0$ and $b_1 = b_0$ and we move in the interval $[a_1, b_1] = [2, 3]$. This time we have

$$c_1 = \frac{a_1 + b_1}{2} = \frac{2+3}{2} = 2.5$$

$$f(c_1) \cdot f(b_1) = f(2.5) \cdot f(3) = (-1.875) \cdot 4 < 0,$$

so the root must be in the interval $[c_1, b_1] = [2.5, 3]$. We set $a_2 = c_1$ and $b_2 = b_1$ and we move in the interval $[a_2, b_2] = [2.5, 3]$, with $c_2 = \frac{2.5+3}{2} = 2.75$.

Remark 4.4 See another example on slides 31-32 and the theory on slides 29-31.

4. false position method starting with $a_0 = 1$ and $b_0 = 3$.
   This method is similar to a combination between the secant and bisection methods.
   Since $f(a_0) \cdot f(b_0) < 0$, we compute

$$c_0 = \frac{f(b_0)a_0 - f(a_0)b_0}{f(b_0) - f(a_0)} = \frac{f(3) - 3f(1)}{f(3) - f(1)} = \frac{4 - 3 \cdot (-6)}{4 - (-6)} = 2.2$$

$f(c_0) \cdot f(b_0) = -4.03 \cdot 4 < 0$, so the root must be in the interval $[c_0, b_0] = [2.2, 3]$. We set $a_1 = c_0$ and $b_1 = b_0$ and we have

$$c_1 = \frac{f(b_1)a_1 - f(a_1)b_1}{f(b_1) - f(a_1)} = \frac{f(3) \cdot 2.2 - f(2.2) \cdot 3}{f(3) - f(2.2)} \approx 2.6$$

We check $f(c_1) \cdot f(b_1) = f(2.6) \cdot f(3) = (-0.94) \cdot 4 < 0$, so we have to move in the interval $[c_1, b_1] = [2.6, 3]$. We set $a_2 = c_1$ and $b_2 = b_1$ and obtain

$$c_2 = \frac{f(b_2)a_2 - f(a_2)b_2}{f(b_2) - f(a_2)} = \frac{f(3) \cdot 2.6 - f(2.6) \cdot 3}{f(3) - f(2.6)} = \frac{4 \cdot 2.6 - (-0.94) \cdot 3}{4 - (-0.94)} = \frac{4 \cdot 2.6 + 0.94 \cdot 3}{4.94} \approx 2.67.$$

Remark 4.5 See another example on slides 35-36 and the theory on slides 33-34.

Example 4.6 *Approximate $\sqrt{10}$ using two iterations of the Newton's method.*
If we let $x = \sqrt{10}$, then $x^2 = 10$ and $x^2 - 10 = 0$ so we can consider the equation $f(x) = 0$ with $f(x) = x^2 - 10$ and $f'(x) = 2x$. Let the first approximation be 4, so $x_0 = 4$. We have then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \frac{6}{8} = \frac{26}{8} = 3.25$$

and

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.25 - \frac{0.5625}{6.5} \approx 3.16.$$