

Charles University in Prague
Faculty of Mathematics and Physics

MASTER THESIS



Filip Bártek

Minimum representations of Boolean functions defined by multiple intervals

Department of Theoretical Computer Science and Mathematical Logic

Supervisor of the master thesis: RNDr. Petr Kučera, Ph.D.

Study programme: Informatics

Study branch: Theoretical Computer Science

Prague 2015

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In date

signature

Title: Minimum representations of Boolean functions defined by multiple intervals

Author: Filip Bártěk

Department: Department of Theoretical Computer Science and Mathematical Logic

Supervisor: RNDr. Petr Kučera, Ph.D., Department of Theoretical Computer Science and Mathematical Logic

Abstract:

When we interpret the input vector of a Boolean function as a binary number, we define interval Boolean function $f_{[a,b]}^n$ so that $f_{[a,b]}^n(x) = 1$ if and only if $a \leq x \leq b$. Disjunctive normal form is a common way of representing Boolean functions. Minimizing DNF representation of an interval Boolean function can be performed in linear time.[3] The natural generalization to k -interval functions seems to be significantly harder to tackle. In this thesis, I discuss the difficulties with finding an optimal solution and introduce a $2k$ -approximation algorithm.

Keywords: Boolean minimization, disjunctive normal form, interval functions

Contents

1	Introduction	2
1.1	Definitions	2
1.1.1	Interval Boolean functions	4
2	1-interval functions	6
2.1	Trivial cases	6
2.1.1	$a = b$	6
2.1.2	$a = 0^{\{n\}}$ and $b = 1^{\{n\}}$	6
2.2	Prefix and suffix case	6
2.3	General case	7
2.3.1	$a^{[1,2]} = 01$ and $b^{[1,2]} = 10$	8
2.3.2	$a^{[1,2]} = 00$ and $b^{[1,2]} = 10$	8
2.3.3	$a^{[1,2]} = 01$ and $b^{[1,2]} = 11$	8
2.3.4	$a^{[1,2]} = 00$ and $b^{[1,2]} = 11$	8
3	2-interval functions	10
3.1	2-step functions	10
3.1.1	$b_1^{[1]} = a_2^{[1]}$	10
3.1.2	$b_1^{[1]} \neq a_2^{[1]}$	12
3.2	3-step functions	13
4	$2k$-approximation algorithm for minimizing DNF representation of k-interval Boolean functions	14
4.1	Introduction	14
4.2	Algorithm	14
4.2.1	Description	14
4.2.2	Correctness	14
4.2.3	Approximation ratio	14
	Conclusion	16
	Bibliography	17
	List of Tables	18
	Glossary	18

1. Introduction

1.1 Definitions

The basic terminology used in this thesis was introduced by Schieber et al. [3] and Dubovský [1].

Definition 1.1.1 (Binary vector). In this thesis, we'll use the term *binary vector* to denote a vector over the Boolean domain. That is, x is a binary vector if and only if $\exists n : x \in \{0, 1\}^n$. We call such x an *n -bit binary vector*.

Note that each binary vector represents a natural number. More specifically, n -bit binary vectors represent numbers between 0 and $2^n - 1$. We will not differentiate between a number and its binary representation. Especially, we'll use a lexicographic linear ordering on binary vectors – for n -bit vectors a and b , $a \leq b$ if and only if $a = b$ or the most significant bit in which a and b differ is 0 in a and 1 in b . Clearly, this ordering corresponds to the natural ordering on the represented numbers.

Definition 1.1.2 (Symbol extraction). Let v be a vector of length n and $1 \leq i \leq n$. Then $v^{[i]}$ is the i -th component of v .

Let $1 \leq a \leq b \leq n$. Then $v^{[a,b]}$ is the subvector of v that starts at a -th position and ends at b -th position ($v^{[a,b]} = (v^{[a]}, \dots, v^{[b]})$).

Notably, $v = (v^{[1]}, \dots, v^{[n]}) = v^{[1,n]}$.

Definition 1.1.3 (Symbol repetition). Let x be a symbol (for example 0 or 1) and $0 \leq n$. Then $x^{\{n\}}$ is the vector of length n each component of which is equal to x .

Definition 1.1.4 (Boolean function). $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is an *n -ary Boolean function*.

Since we will not be dealing with any non-Boolean functions, in the remainder of the text, I will use the terms “function” and “Boolean function” interchangeably.

Every n -ary Boolean function f can be expressed as a propositional formula \mathcal{F} on n variables x_1, \dots, x_n . There's a natural bijection between binary vectors of length n and valuations of n variables (1-bits in the binary vector correspond exactly to 1-valued variables in the valuation).

Definition 1.1.5 (Disjunctive normal form). A propositional logic formula \mathcal{F} is in *disjunctive normal form (DNF)* if and only if \mathcal{F} is a disjunction of terms, where *terms* are elementary¹ conjunctions of literals.

We'll often view terms as sets of literals and DNF formulas as sets of terms.

A DNF formula \mathcal{F} is a *DNF representation* of a Boolean function f if it expresses the same function, that is $\forall x \in \{0, 1\}^n : f(x) = 1 \iff \mathcal{F}(x) = 1$.

To make our work easier, we'll define a vector representation of a term:

¹In an elementary conjunction of literals, no variable appears in both polarities. A trivial example of a non-elementary conjunction is $x_1 \bar{x}_1$.

Definition 1.1.6 (Ternary vector). A *ternary vector* is a vector over the set $\{0, 1, \phi\}$.

There's a natural bijection between ternary vectors and terms (elementary conjunctions of literals). Each of the n components of a ternary vector T corresponds to the single occurrence (or its absence) of one of the n variables of a term C :

$T^{[i]}$	$C \cap \{x_i, \overline{x_i}\}$
ϕ	\emptyset
1	$\{x_i\}$
0	$\{\overline{x_i}\}$

Note that since C is elementary, it can not contain both x_i and $\overline{x_i}$.

Namely, a *binary* vector corresponds to a full term (that is one that contains every variable), and the ternary vector $\phi^{\{n\}}$ corresponds to the empty term (tautology).

It's easy to see that sets of ternary vectors of length n correspond to DNF *formulas* on n variables.

Definition 1.1.7 (Spanning). A ternary vector T of length n *spans* a binary vector x of length n if and only if $\forall i \in \{1, \dots, n\} : T^{[i]} = \phi$ or $T^{[i]} = x^{[i]}$.

The i -th symbol of a ternary vector constrains the i -th symbol of the spanned binary vector (or, equivalently, the valuation of the i -th variable). If $T^{[i]}$ is a “fixed bit” ($T^{[i]} \in \{0, 1\}$), the spanned binary vector x must have the same value in its i -th position, that is $x^{[i]} = T^{[i]}$. If $T^{[i]}$ is the “don't care symbol” ϕ , the spanned binary vector may have any value in its i -th position.

Definition 1.1.8 (Spanned set). Let T be a ternary vector of length n . Set of points spanned by T , denoted $span(T)$, is defined in the following way:

$$span(T) = \{x \in \{0, 1\}^n \mid T \text{ spans } x\}$$

Note that a ternary vector with m ϕ -positions spans 2^m binary vectors.

We'll also use a natural extension of spanning to sets of ternary vectors – if \mathcal{T} is a set of ternary vectors, then

$$span(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} span(T)$$

Definition 1.1.9 (True point). Let f be an n -ary Boolean function. An n -bit binary vector x is a *true point* of f if and only if $f(x) = 1$.

Equivalently, we define a *false point* of f as any point x such that $f(x) = 0$.

We'll denote the set of all true points of f as $TP(f)$ and the set of all false points as $FP(f)$.

Definition 1.1.10 (Minimum DNF representation). A DNF representation of a function is *minimum* if and only if there's no representation with fewer terms.

The focus of this thesis is minimizing the number of terms of DNF representations of certain Boolean functions.

Note that a function may have more than one minimum DNF representation.

Definition 1.1.11 (Spanning set). Let f be an n -ary Boolean function. Set \mathcal{T} of n -bit ternary vectors is a *spanning set* of f if and only if $\text{span}(\mathcal{T}) = TP(f)$.

In other words, a spanning set of a function spans all of its true points and none of its false points.

Note that spanning sets of a function correspond to DNF representations of the function. The correspondence preserves size (number of ternary vectors and terms, respectively), so a minimum spanning set corresponds to a minimum DNF representation.

Definition 1.1.12 (Disjoint spanning set). A spanning set \mathcal{T} is *disjoint* if and only if the spanned sets of its vectors do not overlap, that is

$$\forall T_i, T_j \in \mathcal{T} : T_i = T_j \text{ or } \text{span}(T_i) \cap \text{span}(T_j) = \emptyset$$

In terms of DNF, disjoint spanning sets correspond to DNF formulas such that every valuation satisfies at most one term.

We have introduced two equivalent representations of Boolean functions – one based on ternary vectors and the other being the DNF representation. Table 1.1 shows the corresponding terms side by side.

ternary vector	term
ternary vector set	DNF formula
binary vector	variable valuation
ternary vector <i>spans</i> a binary vector	valuation <i>satisfies</i> a term
ternary vector set <i>spans</i> a binary vector	valuation <i>satisfies</i> a DNF formula

Table 1.1: Corresponding terms used in Boolean function representations

1.1.1 Interval Boolean functions

Definition 1.1.13 (k -interval Boolean function). Let $a_1, b_1, \dots, a_k, b_k$ be n -bit binary vectors such that $0^{\{n\}} \leq a_1 \leq b_1 < a_2 \leq \dots < a_i \leq b_i < a_{i+1} \leq \dots < a_k \leq b_k \leq 1^{\{n\}}$. Then $f_{[a_1, b_1], \dots, [a_k, b_k]}^n : \{0, 1\}^n \rightarrow \{0, 1\}$ is a function defined as follows:

$$f_{[a_1, b_1], \dots, [a_k, b_k]}^n(x) = \begin{cases} 1 & \text{if } a_i \leq x \leq b_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

We call $f_{[a_1, b_1], \dots, [a_k, b_k]}^n$ a *k-interval Boolean function* and the vectors $a_1, b_1, \dots, a_k, b_k$ its *endpoints*.

Note that the inequalities ensure that the intervals $[a_i, b_i]$ are non-empty and don't intersect.

Definition 1.1.14 (Proper k -interval Boolean function). Let $f_{[a_1, b_1], \dots, [a_k, b_k]}^n$ be a k -interval Boolean function. If the endpoints satisfy the inequalities $b_1 < a_2 - 1, \dots, b_i < a_{i+1} - 1, \dots, b_{k-1} < a_k - 1$ (in other words, adjacent intervals are separated by at least one false point), we call $f_{[a_1, b_1], \dots, [a_k, b_k]}^n$ a *proper k-interval Boolean function*.

Especially, $f_{[a, b]}^n$ is a function whose true points form the interval $[a, b]$.

Table 1.1.1 shows examples of interval functions.

Interval function	Propositional formula	Description
$f_{[11,11]}^2$	$x_1 \wedge x_2$	Conjunction
$f_{[01,11]}^2$	$x_1 \vee x_2$	Disjunction
$f_{[0\{n\},1\{n\}]}^n$	1	Tautology
$f_{[0\{n-1\}1,1\{n-1\}0]}^n$	$\overline{x_1 \dots x_n} \wedge \overline{\overline{x_1} \dots \overline{x_n}}$	Non-equivalence
$f_{[0\{n\},0\{n\}], [1\{n\},1\{n\}]}^n$	$x_1 \dots x_n \vee \overline{x_1} \dots \overline{x_n}$	Equivalence

Table 1.2: Examples of interval functions

2. 1-interval functions

In this chapter, I will introduce an efficient algorithm for optimal spanning of single interval Boolean functions, originally shown by Schieber et al.[3]

In the chapters that follow, I will use this algorithm as a procedure in order to span multi-interval functions.

In the remainder of this chapter, a and b ($a \leq b$) will denote the endpoints of the spanned interval and n the number of input bits; we'll be spanning the function $f_{[a,b]}^n$.

The algorithm is recursive in input length.

2.1 Trivial cases

We will first deal with the trivial intervals.

2.1.1 $a = b$

We span the interval $[a, a]$ with the single ternary vector a .

From now on, let $a < b$.

2.1.2 $a = 0^{\{n\}}$ and $b = 1^{\{n\}}$

We span the interval $[0^{\{n\}}, 1^{\{n\}}]$ with the single ternary vector $\phi^{\{n\}}$.

From now on, let $a > 0^{\{n\}}$ or $b < 1^{\{n\}}$.

2.2 Prefix and suffix case

Since we have dealt with the trivial cases, we are now left with the situation $0^{\{n\}} \leq a < b \leq 1^{\{n\}}$ and $a > 0^{\{n\}}$ or $b < 1^{\{n\}}$.

Let's consider the prefix case, that is $[0^{\{n\}}, b]$ ($a = 0^{\{n\}}$), and the suffix case, that is $[a, 1^{\{n\}}]$ ($b = 1^{\{n\}}$).

Definition 2.2.1 (Complement). The complement of a binary symbol α ($\alpha \in \{0, 1\}$), denoted $\bar{\alpha}$, is $1 - \alpha$.

We get the complement of a binary vector x by flipping all of its bits, that is $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) = 1^{\{n\}} - x$.

The complement of the ϕ symbol is $\bar{\phi}$ ($\bar{\bar{\phi}} = \phi$).

It follows that we obtain the complement of a ternary vector by flipping all of its fixed bits.

Note that prefix and suffix cases are complementary – we may transform a suffix instance $[a, 1^{\{n\}}]$ to a prefix instance $[0^{\{n\}}, \bar{a}]$. Flipping the polarity of the resulting ternary vectors yields a solution of the initial suffix instance $[a, 1^{\{n\}}]$. This means it suffices to solve the prefix case, which is what we will do.

Let $a = 0^{\{n\}}$ and $b < 1^{\{n\}}$. Let c be the n -bit number $b + 1$. Since $b < 1^{\{n\}}$, we don't need more than n bits to encode c .

The algorithm produces one ternary vector for each 1-bit in c . If o is a position of a 1-bit in c ($c^{[o]} = 1$), then the corresponding ternary vector is $c^{[1,o-1]}0\phi^{\{n-o\}}$. Thus we get the following spanning set:

$$\mathcal{T} = \{c^{[1,o-1]}0\phi^{\{n-o\}} | c^{[o]} = 1\}$$

Theorem 2.2.0.1. \mathcal{T} spans exactly the interval $[0^{\{n\}}, b]$ (feasibility).

Proof. To see that every number spanned by \mathcal{T} is in $[0^{\{n\}}, b]$, note that all the numbers spanned by \mathcal{T} are smaller than c , since their most significant bit different from c must be a 0-bit and they must differ from c .

On the other hand, every number x smaller than c must differ from c , and the leftmost different bit must be 0 in x and 1 in c . Let o be its position. Then $c^{[1,o-1]}0\phi^{\{n-o\}} \in \mathcal{T}$ spans x . \square

Theorem 2.2.0.2. \mathcal{T} is minimal in size (optimality).

Proof. We'll construct a set V of $|\mathcal{T}|$ true points no pair of which can be spanned by a single ternary vector. Dubovský[1] calls such sets "orthogonal". Doing so, we'll show a lower bound $|\mathcal{T}|$ on the size of feasible solutions, proving optimality of \mathcal{T} .

Similarly to the spanning vectors, the orthogonal true points correspond to 1-bits of c :

$$V = \{c^{[1,o-1]}0c^{[o+1,n]} | c^{[o]} = 1\} \quad (2.1)$$

Clearly $|V| = |\mathcal{T}|$.

It's easy to see that any ternary vector that spans two different points in V must also span the false point c , so it can't be a part of the solution. Also note that all points in V are smaller than c , so they are true points.

If there was a feasible spanning set of size smaller than $|V|$, at least one of its vectors would need to span at least two points in V . As we have shown, such ternary vector would necessarily also span the false point c , leading to contradiction with the set's feasibility. \square

2.3 General case

Having solved the trivial and prefix and suffix cases, we are left with the situation $0^{\{n\}} < a < b < 1^{\{n\}}$.

If a and b have the same MSB, we recursively span $[a^{[2,n]}, b^{[2,n]}]$ and prepend the MSB to the solution.

Let $a^{[1]} \neq b^{[1]}$. Since $a \leq b$, necessarily $a^{[1]} = 0$ and $b^{[1]} = 1$. This restriction leaves us with four possible combinations of pairs of MSBs:

1. $a^{[1,2]} = 01, b^{[1,2]} = 10$
2. $a^{[1,2]} = 00, b^{[1,2]} = 10$
3. $a^{[1,2]} = 01, b^{[1,2]} = 11$
4. $a^{[1,2]} = 00, b^{[1,2]} = 11$

Following Schieber et al., [3] we'll deal with each of these cases separately. We won't prove feasibility nor optimality in this text. Please refer to the original article for the proofs.

2.3.1 $a^{[1,2]} = 01$ and $b^{[1,2]} = 10$

In this case, we'll span the two sub-intervals $[a, 01^{\{n-1\}}]$ and $[10^{\{n-1\}}, b]$ separately. Note that after leaving out the initial shared bit, they are prefix and suffix interval, respectively, so the algorithm from section 2.2 is used to span each of the sub-intervals.

2.3.2 $a^{[1,2]} = 00$ and $b^{[1,2]} = 10$

In this case, we divide the interval in three sub-intervals:

- $[a, 001^{\{n-2\}}]$
- $[010^{\{n-2\}}, 011^{\{n-2\}}]$
- $[100^{\{n-2\}}, b]$

The subinterval $[010^{\{n-2\}}, 011^{\{n-2\}}]$ is spanned by the single ternary vector $01\phi^{\{n-2\}}$.

The subintervals $[a, 001^{\{n-2\}}]$ and $[100^{\{n-2\}}, b]$ are spanned together as follows:

- Recursively solve the $(n-1)$ -bit instance $[0a^{[3,n]}, 1b^{[3,n]}] = [a^{[1]}a^{[3,n]}, b^{[1]}b^{[3,n]}]$
- Insert a 0-bit in the second position of the vectors from the resulting spanning set

2.3.3 $a^{[1,2]} = 01$ and $b^{[1,2]} = 11$

This case is complementary to case 2.3.2. As in suffix case, note that flipping the bits in the endpoints transforms this case to case 2.3.2 and flipping the fixed bits in the resulting spanning set preserves correctness.

2.3.4 $a^{[1,2]} = 00$ and $b^{[1,2]} = 11$

Let j be maximal such that $a^{[1,j]} = 0^{\{j\}}$ and $b^{[1,j]} = 1^{\{j\}}$. Since $a > 0^{\{n\}}$ (and $b < 1^{\{n\}}$), necessarily $j < n$.

j gives us three sub-intervals to span:

- $[a, 0^{\{j\}}1^{\{n-j\}}]$
- $[0^{\{j-1\}}10^{\{n-j\}}, 1^{\{j-1\}}01^{\{n-j\}}]$
- $[1^{\{j\}}0^{\{n-j\}}, b]$

Note that the second sub-interval contains exactly the numbers that don't start with either j zeros or j ones, and as such can be spanned by j ternary vectors. We construct T_1, \dots, T_j that span the second sub-interval by appending $\phi^{\{n-j\}}$ to each of the j cyclic shifts of $01\phi^{\{j-2\}}$. Thus, $T_1 = 01\phi^{\{j-2\}}\phi^{\{n-j\}}, \dots, T_i = \phi^{\{i-1\}}01\phi^{\{j-1-i\}}\phi^{\{n-j\}}, \dots, T_{j-1} = \phi^{\{j-2\}}01\phi^{\{n-j\}}, T_j = 1\phi^{\{j-2\}}0\phi^{\{n-j\}}$.

The other two sub-intervals are spanned recursively. Let $a'' = a^{[j,n]}$ and $b'' = b^{[j,n]}$. Note that $a''^{[1]} = 0$ and $b''^{[1]} = 1$. Note that $|a''| = |b''| = n - j + 1$. Since $j \geq 2$, $n - j + 1 < n$. Let \mathcal{T}'' be the spanning set of $[a'', b'']$ computed recursively.

The spanning set of $[a, b]$ will be computed from the vectors T_1, \dots, T_j and \mathcal{T}'' based on the relation between $(n - j)$ -bit suffixes of a and b . Let $a' = a^{[j+1,n]}$ and $b' = b^{[j+1,n]}$.

$$b' < a' - 1$$

In this case, the spanning set of $[a, b]$ consists of the vectors T_1, \dots, T_j and vectors obtained by prepending each vector from \mathcal{T}'' with $j - 1$ leading ϕ s.

$$b' \leq a' - 1$$

In this case, the spanning set of $[a, b]$ consists of the vectors T_1, \dots, T_{j-1} (omitting T_j) and a set \mathcal{T}' derived from \mathcal{T}'' in the following way:

$$\begin{aligned} \mathcal{T}' = & \{ \phi^{\{j-1\}}T | T \in \mathcal{T}'' \text{ and } T^{[1]} \in \{0, \phi\} \} \\ & \cup \{ 1\phi^{\{j-1\}}T^{[2,n-j+1]} | T \in \mathcal{T}'' \text{ and } T^{[1]} = 1 \} \end{aligned}$$

3. 2-interval functions

In this chapter we will revise the results specific to 2-interval functions. Dubovský's thesis [1] provides more detailed information.

Dubovský identified several classes of 2-interval functions [1, p. 5] which seem to require different approaches for spanning. I find Hušek's concept of "step" [2, p. 13] ("zlom" in Czech) useful to differentiate between these classes.

Definition 3.0.1 (*l*-step function). A Boolean function f is *l*-step if and only if there are exactly l input vectors x such that $f(x) \neq f(x + 1)$.

It is easy to see that prefix and suffix interval functions are 1-step.

Depending on $f(0)$, an *l*-step Boolean function f is $\lceil \frac{l}{2} \rceil$ -interval in case $f(0) = 0$, or $\lceil \frac{l+1}{2} \rceil$ -interval in case $f(0) = 1$.

Namely, a 2-step Boolean function such that $f(0) = 0$ is 1-interval.

3.1 2-step functions

Let f be a 2-interval 2-step Boolean function (necessarily $f(0^{\{n\}}) = f(1^{\{n\}}) = 1$).¹ We will show that f can be optimally spanned by reduction to a 2-step 1-interval Boolean function f' of at most the same arity, which can be spanned using the algorithm introduced in chapter 2.

Note that Dubovský introduced an optimization algorithm for this class of functions [1, section 3.2], following an argument similar to the one used by Schieber et al. to span 1-interval functions [3]. However, the reduction approach gives an interesting insight in the problem and seems easier to describe.

From the definition of f we know that it is 2-interval with the outer interval endpoints being $0^{\{n\}}$ and $1^{\{n\}}$:

$$f = f_{[0^{\{n\}}, b_1], [a_2, 1^{\{n\}}]}^n$$

Let's call the inner interval endpoints b_1 and a_2 . $0^{\{n\}} \leq b_1$, $b_1 < a_2 - 1$ (f is proper 2-interval – otherwise it would be a tautology and thus trivial to span), $a_2 \leq 1^{\{n\}}$.

3.1.1 $b_1^{[1]} = a_2^{[1]}$

In case b_1 and a_2 have the same MSBs, we recursively solve the $(n - 1)$ -bit case $[0^{\{n-1\}}, b_1^{[2,n]}], [a_2^{[2,n]}, 1^{\{n-1\}}]$, and add an extra ternary vector to span the remainder of the true points.

Without loss of generality, let $b_1^{[1]} = a_2^{[1]} = 0$. The complementary case is symmetric.

Let $b'_1 = b_1^{[2,n]}$ and $a'_2 = a_2^{[2,n]}$. Let \mathcal{T}'_0 be an optimal spanning set of $f' = f_{[0^{\{n-1\}}, b'_1], [a'_2, 1^{\{n-1\}}]}^{n-1}$. Note that f' is of the same type (that is 2-interval 2-step) and has a lower arity, so we can span it by recursion. Let \mathcal{T}_0 be the n -bit set we get by prepending the symbol 0 to each vector from \mathcal{T}'_0 ($\mathcal{T}_0 = \{0T | T \in \mathcal{T}'_0\}$). Let

¹This corresponds to class A_0 in Dubovský's classification [1, p. 5].

$\mathcal{T}_1 = \{1\phi^{\{n-1\}}\}$. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$. We claim that \mathcal{T} spans exactly $f_{[0^{\{n\}}, b_1], [a_2, 1^{\{n\}}]}^n$ and that \mathcal{T} is minimum such.

Theorem 3.1.1.1. \mathcal{T} spans exactly $f_{[0^{\{n\}}, b_1], [a_2, 1^{\{n\}}]}^n$ (*feasibility*).

Proof. 1. $\text{span}(\mathcal{T}) \subseteq TP(f)$

Let $c \in \text{span}(T)$, $T \in \mathcal{T}$. We need to show that $c \leq b_1$ or $c \geq a_2$.

- (a) $c^{[1]} = 1$: Since $a_2^{[1]} = 0$, necessarily $c \geq a_2$.
- (b) $c^{[1]} = 0$: Necessarily $T \in \mathcal{T}_0$. Let $T' = T^{[2, n]}$ and $c' = c^{[2, n]}$. Since T spans c , T' spans c' . By construction of \mathcal{T}_0 necessarily $T' \in \mathcal{T}'_0$, so T' does not span any number in $[b'_1 + 1, a'_2 - 1]$. Thus $c' \leq b'_1$ or $c' \geq a'_2$. Observe that prepending a 0 bit to c' , b'_1 and a'_2 preserves the inequalities:

$$c' \leq b'_1 \text{ or } c' \geq a'_2 \Rightarrow 0c' \leq 0b'_1 \text{ or } 0c' \geq 0a'_2$$

Since $0c' = c$, $0a' = a$ and $0b' = b$, we get $c \leq b_1$ or $c \geq a_2$.

2. $TP(f) \subseteq \text{span}(\mathcal{T})$

Let $c \leq b_1$ or $c \geq a_2$. We need a $T \in \mathcal{T}$ that spans c .

- (a) $c^{[1]} = 1$: $1\phi^{n-1} \in \mathcal{T}_1 \subseteq \mathcal{T}$ spans c .
- (b) $c^{[1]} = 0$: Similarly to b'_1 and a'_2 , let $c' = c^{[2, n]}$. Since b_1 , a_2 and c all start with a 0 bit, $c' \leq b'_1$ or $c' \geq a'_2$. Since \mathcal{T}'_0 spans $f_{[0^{\{n-1\}}, b'_1], [a'_2, 1^{\{n-1\}}]}^{n-1}$, there is $T \in \mathcal{T}'_0$ that spans c' . Prepending the symbol 0 to both T and c' preserves spanning, so $0T \in \mathcal{T}_0 \subseteq \mathcal{T}$ spans c .

□

Theorem 3.1.1.2. \mathcal{T} is a minimum spanning set of $f_{[0^{\{n\}}, b_1], [a_2, 1^{\{n\}}]}^n$ (*optimality*).

Proof. Let \mathcal{T}_{opt} be an optimal spanning set of $f_{[0^{\{n\}}, b_1], [a_2, 1^{\{n\}}]}^n$. We'll show that $|\mathcal{T}_{opt}| \geq |\mathcal{T}|$.

First recall that \mathcal{T}'_0 is an optimal spanning set of $f_{[0^{\{n-1\}}, b'_1], [a'_2, 1^{\{n-1\}}]}^{n-1}$ and $|\mathcal{T}| = |\mathcal{T}'_0| + 1$. Thus we need to show $|\mathcal{T}_{opt}| \geq |\mathcal{T}'_0| + 1$.

\mathcal{T}_{opt} must span the true point $\bar{b}_1 = 1(b_1 + 1)^{[2, n]}$. We don't need more than n bits for \bar{b}_1 because $b_1 < a_2 \leq 01^{\{n-1\}}$. Let $T_{\bar{b}_1} \in \mathcal{T}_{opt}$ be a ternary vector that spans \bar{b}_1 . If $T_{\bar{b}_1}$ spanned any vector that starts with a 0, then $T_{\bar{b}_1}$ would also span the vector $b_1 + 1$. However, since f is *proper* 2-interval, $b_1 + 1$ is a false point ($b_1 < b_1 + 1 < a_2$). So $T_{\bar{b}_1}$ can't span any vector that starts with a 0.

To get a contradiction, let's suppose that $|\mathcal{T}_{opt}| \leq |\mathcal{T}'_0|$. Since we need a dedicated vector $T_{\bar{b}_1} \in \mathcal{T}_{opt}$ to span \bar{b}_1 , there are at most $|\mathcal{T}_{opt}| - 1 \leq |\mathcal{T}'_0| - 1$ vectors in \mathcal{T}_{opt} that span a number that starts with 0 (so these vectors start with 0 or ϕ). Removing the first symbol of these vectors, we get a spanning set of $f_{[0^{\{n-1\}}, b'_1], [a'_2, 1^{\{n-1\}}]}^{n-1}$ of size at most $|\mathcal{T}'_0| - 1$, which contradicts \mathcal{T}'_0 's optimality.

We conclude that we need at least $|\mathcal{T}'_0| + 1$ vectors to span $f_{[0^{\{n\}}, b_1], [a_2, 1^{\{n\}}]}^n$, proving the lower bound of $|\mathcal{T}_{opt}|$ and the optimality of \mathcal{T} . □

We have shown that in case $b_1^{[1]} = a_2^{[1]}$, our algorithm produces a feasible and optimal spanning set.

3.1.2 $b_1^{[1]} \neq a_2^{[1]}$

Let $b_1^{[1]} \neq a_2^{[1]}$. Since $b_1 \leq a_2$, necessarily $b_1^{[1]} = 0$ and $a_2^{[1]} = 1$.

By flipping the leading bit in each of the endpoints b_1 and a_2 , we will reduce this case to an instance of spanning a single true point interval on n bits.

Let $b'_1 = 1b_1^{[2,n]}$ and $a'_2 = 0a_2^{[2,n]}$ (flipping the leading bits of b_1 and a_2). Note that $a'_2 \leq b'_1$.

Let \mathcal{T}' be a minimum spanning set of $f_{[a'_2, b'_1]}^n$. This set can be computed efficiently using the method shown in chapter 2.

Let's flip all the non- ϕ leading symbols of vectors in \mathcal{T}' , forming \mathcal{T} . Note that $|\mathcal{T}| = |\mathcal{T}'|$ and also that all the vectors in \mathcal{T}' and \mathcal{T} have the length n .

We claim that \mathcal{T} is a minimum spanning set of $f_{[0^{n\{n\}}, b_1], [a_2, 1^{n\{n\}}]}^n$.

The argument is based on a pair of lemmas.

Lemma 3.1.2.1. *Let c be an n -bit binary vector and a, b be $(n-1)$ -bit binary vectors. Let $c' = \overline{c^{[1]}}c^{[2,n]}$ (flipping the leading bit of c). Then $(c \leq 0b \text{ or } c \geq 1a)$ if and only if $(c' \geq 0a \text{ and } c' \leq 1b)$.*

Proof. 1. $c^{[1]} = 0$:

Let $c = 0c^{[2,n]}$ and $c' = 1c^{[2,n]}$.

Since $c^{[1]} = 0$, $c \not\geq 1a$. Thus $(c \leq 0b \text{ or } c \geq 1a)$ if and only if $c \leq 0b$.

Since $c^{[1]} = 0$, $c \leq 0b$ if and only if $c^{[2,n]} \leq b$.

Since $c'^{[1]} = 1$ and $c'^{[2,n]} = c^{[2,n]}$, $c'^{[2,n]} \leq b$ if and only if $c' \leq 1b$.

From $c'^{[1]} = 1$ we immediately get $c' \geq 0a$.

Chaining the equivalences together:

$$\begin{aligned} c \leq 0b \text{ or } c \geq 1a &\iff c \leq 0b \\ &\iff c^{[2,n]} \leq b \\ &\iff c' \leq 1b \\ &\iff c' \leq 1b \text{ and } c' \geq 0a \end{aligned}$$

2. $c^{[1]} = 1$:

This case is symmetric to the previous one. □

Lemma 3.1.2.2. *Let \mathcal{T} be a set of ternary vectors and $\mathcal{T}' = \{\overline{T^{[1]}}T^{[2,n]} | T \in \mathcal{T}\}$. Let a, b be any $(n-1)$ -bit binary vectors. Then \mathcal{T} spans $f_{[0^{n\{n\}}, 0b], [1a, 1^{n\{n\}}]}^n$ if and only if \mathcal{T}' spans $f_{[0a, 1b]}^n$.²*

Proof. 1. If \mathcal{T} spans $f_{[0^{n\{n\}}, 0b], [1a, 1^{n\{n\}}]}^n$, then \mathcal{T}' spans $f_{[0a, 1b]}^n$:

Let \mathcal{T} span $f_{[0^{n\{n\}}, 0b], [1a, 1^{n\{n\}}]}^n$.

Let c' be an n -bit binary vector. We need to show that \mathcal{T}' spans c' if and only if $c' \geq 0a$ and $c' \leq 1b$.

²Note that the statement holds even in the single case that the function is not proper 2-interval, that is $a = 1^{n-1}$ and $b = 0^{n-1}$.

Let $c = \overline{c'[1]}c'^{[2,n]}$. By definition of \mathcal{T}' , \mathcal{T}' spans c' if and only if \mathcal{T} spans c .

Since \mathcal{T} spans $f_{[0\{n\},0b],[1a,1\{n\}]}^n$, \mathcal{T} spans c if and only if $c \leq 0b$ or $c \geq 1a$.

From Lemma 3.1.2.1 we get $(c \leq 0b \text{ or } c \geq 1a)$ if and only if $(c' \geq 0a \text{ and } c' \leq 1b)$.

Chaining the equivalences together:

$$\begin{aligned} \mathcal{T}' \text{ spans } c' &\iff \mathcal{T} \text{ spans } c \\ &\iff c \leq 0b \text{ or } c \geq 1a \\ &\iff c' \geq 0a \text{ and } c' \leq 1b \\ &\iff f_{[0a,1b]}^n(c') = 1 \end{aligned}$$

2. If \mathcal{T}' spans $f_{[0a,1b]}^n$, then \mathcal{T} spans $f_{[0\{n\},0b],[1a,1\{n\}]}^n$:

Let \mathcal{T}' span $f_{[0a,1b]}^n$.

Let c be an n -bit binary vector. We need to show that \mathcal{T} spans c if and only if $c \leq 0b$ or $c \geq 1a$.

Let $c' = \overline{c[1]}c^{[2,n]}$. By definition of \mathcal{T}' , \mathcal{T} spans c if and only if \mathcal{T}' spans c' .

Since \mathcal{T}' spans $f_{[0a,1b]}^n$, \mathcal{T}' spans c' if and only if $c' \geq 0a$ and $c' \leq 1b$.

Using Lemma 3.1.2.1, we get $(c' \geq 0a \text{ and } c' \leq 1b)$ if and only if $(c \leq 0b \text{ or } c \geq 1a)$.

Chaining the equivalences together:

$$\begin{aligned} \mathcal{T} \text{ spans } c &\iff \mathcal{T}' \text{ spans } c' \\ &\iff c' \geq 0a \text{ and } c' \leq 1b \\ &\iff c \leq 0b \text{ or } c \geq 1a \\ &\iff f_{[0\{n\},0b],[1a,1\{n\}]}^n(c) = 1 \end{aligned}$$

□

Lemma 3.1.2.2 shows a bijection between the spanning sets of functions of form $f_{[0\{n\},0b],[1a,1\{n\}]}^n$ and $f_{[0a,1b]}^n$ for general a and b . Since b_1 starts with a 0 ($b_1 = 0b$), a_2 starts with a 1 ($a_2 = 1a$) and the set \mathcal{T} was constructed from an optimal spanning set \mathcal{T}' of $f_{[0a_2^{[2,n]},1b_1^{[2,n]}]}^n$ using the bijection shown in 3.1.2.2, that is flipping the leading symbols of all the ternary vectors, both feasibility and optimality of \mathcal{T} with respect to $f_{[0\{n\},b_1],[a_2,1\{n\}]}^n$ follow from the feasibility and optimality of \mathcal{T}' with respect to $f_{[0a_2^{[2,n]},1b_1^{[2,n]}]}^n$.

3.2 3-step functions

4. $2k$ -approximation algorithm for minimizing DNF representation of k -interval Boolean functions

4.1 Introduction

In this chapter, an algorithm will be shown that computes a small DNF representation of a Boolean function given as a set of k intervals. The input intervals are represented by pairs of endpoints (n -bit numbers). An approximation ratio of $2k$ will be proved.

4.2 Algorithm

4.2.1 Description

Input Numbers $a_1, b_1, \dots, a_k, b_k$ that satisfy the inequalities in definition 1.1.13

Output A set of ternary vectors

Procedure The algorithm goes through all the intervals $[a_i, b_i]$. For each i , the longest common prefix of a_i and b_i is computed. Let j be its length. Note that $a^{[j+1]} = 0$ and $b^{[j+1]} = 1$. Now let $a'' = a_i^{[j+2, n]}$ and $b'' = b_i^{[j+2, n]}$. Optimally span the suffix interval $[a'', 1^{n-j-1}]$ and the prefix interval $[0^{n-j-1}, b'']$ using the (linear time) algorithm introduced in [3]. Prepend $a_i^{[1, j+1]}$ and $b_i^{[1, j+1]}$ to the respective ternary vectors and add them to the output spanning set.

4.2.2 Correctness

Theorem 4.2.2.1. *The algorithm spans exactly $f_{[a_1, b_1], \dots, [a_k, b_k]}^n$.*

Proof. This is easy to see from the fact that the subintervals form a partition of the multi-interval (i.e. the set of all true points) and that each of them is spanned exactly by the suffix or prefix procedure. \square

4.2.3 Approximation ratio

Theorem 4.2.3.1. *Let \mathcal{T}_{opt} be an optimal spanning set of $f_{[a_1, b_1], \dots, [a_k, b_k]}^n$ and let \mathcal{T}_{approx} be the spanning set returned by the algorithm. We claim that:*

$$|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_{opt}| \quad (4.1)$$

Proof. Let \mathcal{T}_x be the largest (n -bit) spanning set of a "suffix" or "prefix" subinterval added in the algorithm. Without loss of generality, let the respective

subinterval be "prefix" $[b_i^{[1,j+1]}, b_i]$. From [1, p. 36] we know that there is an orthogonal set of $[b_i^{[1,j+1]}, b_i]$ of size $|\mathcal{T}_x|$, and moreover that its orthogonality only depends on the false point $b + 1$. Note, however, that $b + 1$ is also a false point in $f_{[a_1, b_1], \dots, [a_k, b_k]}^n$. Thus we obtain an orthogonal set of size $|\mathcal{T}_x|$ for the k -interval function, limiting the size of its optimal spanning set $|\mathcal{T}_{opt}| \geq |\mathcal{T}_x|$.

Since \mathcal{T}_x is the largest of the $2k$ partial sets used to span the function in the approximation algorithm, we know that $|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_x|$.

Joining the inequalities together we conclude: $|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_x| \leq 2k|\mathcal{T}_{opt}|$. \square

Theorem 4.2.3.2. *The approximation ratio of $2k$ is tight.*

Proof. For every k that is a power of 2 we'll show a k -interval function such that $|\mathcal{T}_{approx}| = 2k|\mathcal{T}_{opt}|$ (following the notation from Theorem 4.2.3.1).

Let $k = 2^{n_k}$. Let P be all the n_k -bit numbers, that is $P = \{0, 1\}^{n_k}$. Note that $|P| = k$.

For each $p \in P$, we define the interval $[a_p, b_p]$ by appending 2-bit suffixes to p :

- $a_p = p00$
- $b_p = p01$

The first appended bit (0 for both a_p and b_p) ensures that there is at least one false point between any pair of intervals defined this way. The second appended bit (0 for a_p and 1 for b_p) ensures that the interval has two points, so the approximation algorithm will use two vectors to span it. Thus we have k ($n_k + 2$)-bit intervals, none of which intersect or touch.

Since $P = \{0, 1\}^{n_k}$, we can span all the intervals by the single ternary vector $\phi^{\{n_k\}}0\phi$. Clearly $|\mathcal{T}_{opt}| = 1$.

However, the approximation algorithm uses $2k$ vectors to span the intervals, since it spans each of the intervals separately and uses two vectors for each interval.

We get $|\mathcal{T}_{approx}| = 2k|\mathcal{T}_{opt}|$.

Note that the optimal spanning set is disjoint, so the ratio is tight in disjoint case as well. \square

Conclusion

Bibliography

- [1] Jakub Dubovský. A construction of minimum DNF representations of 2-interval functions. Master's thesis, Charles University in Prague, 2012.
- [2] Radek Hušek. Properties of interval boolean functions. Master's thesis, Charles University in Prague, 2014. [In Czech].
- [3] Baruch Schieber, Daniel Geist, and Ayal Zaks. Computing the minimum DNF representation of boolean functions defined by intervals. *Discrete Applied Mathematics*, 149(1–3):154 – 173, 2005. Boolean and Pseudo-Boolean Functions.

List of Tables

1.1	Corresponding terms used in Boolean function representations . .	4
1.2	Examples of interval functions	5

Glossary

DNF disjunctive normal form. ii, 1–4, 14

MSB most significant bit. 2, 7, 10