Minimal DNF representation of false point interval Boolean functions

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1 Introduction

When we identify binary vectors with the numbers they express in binary form, we can define n-ary Boolean functions as functions from $\{0, \ldots, 2^n - 1\}$ to the Boolean domain. We continue to define *false point interval Boolean functions* as functions whose *false* points, when considered numbers, form a single interval.

In this text, I will show an efficient algorithm for construction of minimal DNF representation (i.e. one with the least clauses) of false point interval Boolean functions. Doing so, I will simplify the method shown in [1]. I will build on the results about true point interval Boolean functions shown in [2].

2 Definitions

Definition 2.1 (Interval Boolean function)**.** Let $a \leq b$ be n-bit numbers ($0 \leq a \leq b \leq 2^n - 1$). Then $f^n_{[a,b]} : \mathbf{B}^n \to \mathbf{B}$ is a function defined as follows:

$$f^n_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.2 (False point interval Boolean function)**.** Let $a \leq b$ be n-bit numbers ($0 \leq a \leq b \leq 2^n - 1$). Then $f^n_{\overline{[a,b]}} : \mathbf{B}^n \to \mathbf{B}$ is a function defined as follows:

$$f^n_{\overline{[a,b]}}(x) = \begin{cases} 1 & \text{if } x \notin [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Definitions of ternary vectors, spanning, bit extraction $a^{[i]}$, sequence extraction $a^{[i,j]}$, vector complement $\overline{a}$ etc. are shown in [2].

3 Spanning false point interval Boolean functions

Let $n \geq 0$ and $0 \leq a \leq b \leq 2^n - 1$. We need to find a set of ternary vectors of minimal cardinality that spans exactly $\overline{[a,b]} = [0, a-1] \cup [b+1, 2^n - 1]$.

Since we'll solve the problem recursively, let's first simply observe that the optimal spanning set of a 0-bit Boolean function ($n = 0$) is an empty set. From now on, let $n \geq 1$.

We'll consider two cases separately:

1. $a^{[1]} = b^{[1]}$

2. $a^{[1]} \neq b^{[1]}$

We'll deal with the first case by recursively solving the $(n-1)$-bit instance $\overline{[a^{[2,n]}, b^{[2,n]}]}$ and adding one extra ternary vector.

We'll transform the second case to an instance of spanning a true point interval Boolean function of equal arity, which can be solved optimally using a method introduced in [2].

3.1 $a^{[1]} = b^{[1]}$

Let $a^{[1]} = b^{[1]} = 0$. Note that the other case (MSB of a and b is 1) is symmetric so we'll only discuss the case of 0.

Let $a_2 = a^{[2,n]}$ and $b_2 = b^{[2,n]}$.

Let $\mathcal{T}_2$ be the $(n-1)$-bit minimal spanning set of $\overline{[a_2, b_2]}$. We can get such set by recursion.

Let $\mathcal{T}_0$ be the n-bit set we get by prepending each vector from $\mathcal{T}_2$ by 0.

Let $\mathcal{T}_1 = \{1\phi^{n-1}\}$.

Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$.

We claim that $\mathcal{T}$ spans exactly $\overline{[a, b]}$ and that $\mathcal{T}$ is minimal such.

Theorem 3.1.1. *$\mathcal{T}$ spans exactly $\overline{[a, b]}$.*

Proof. Part 1: $span(\mathcal{T}) \subseteq \overline{[a, b]}$

Let $c \in span(T)$, $T \in \mathcal{T}$. We need to show that $c \in \overline{[a, b]}$.

1. $c^{[1]} = 1$: Since $b^{[1]} = 0$, necessarily $c > b$, so $c \in \overline{[a, b]}$.

2. $c^{[1]} = 0$: Necessarily $T \in \mathcal{T}_0$. Let $T_2 = T^{[2,n]}$ and $c_2 = c^{[2,n]}$. Since T spans c, T_2 spans c_2. By construction of $\mathcal{T}_0$ necessarily $T_2 \in \mathcal{T}_2$, so T_2 doesn't span any number in $[a_2, b_2]$. Thus $c_2 \in \overline{[a_2, b_2]}$. Observe that prepending a 0 bit to c_2, a_2 and b_2 preserves the interval membership: $c_2 \in \overline{[a_2, b_2]} \Rightarrow 0c_2 \in \overline{[0a_2, 0b_2]}$. Since $0c_2 = c$, $0a_2 = a$ and $0b_2 = b$, $c \in \overline{[a, b]}$.

Part 2: $\overline{[a, b]} \subseteq span(\mathcal{T})$

Let $c \in \overline{[a, b]}$. We need a $T \in \mathcal{T}$ that spans c.

1. $c^{[1]} = 1$: $1\phi^{n-1} \in \mathcal{T}_1 \subseteq \mathcal{T}$ spans c.

2. $c^{[1]} = 0$: Similarly to a_2 and b_2, let $c_2 = c^{[2,n]}$. Since a, b and c start with a 0 bit, $c_2 \in \overline{[a_2, b_2]}$. Since $\mathcal{T}_2$ spans $\overline{[a_2, b_2]}$, $\exists T_2 \in \mathcal{T}_2 . T_2$ spans c_2. Prepending 0 preserves spanning, so $0T_2 \in \mathcal{T}_0 \subseteq \mathcal{T}$ spans c.

□

Theorem 3.1.2. *$\mathcal{T}$ is a minimal spanning set of $\overline{[a,b]}$.*

Proof. Clearly T_0 optimally spans $\overline{[a,b]} \cap [0^n, 01^{n-1}]$.

The optimal spanning set $\mathcal{T}_{opt}$ of $\overline{[a,b]}$ must span the binary vector $a' = 1a^{[2,n]} \in \overline{[a,b]}$. Let $T \in \mathcal{T}_{opt}$ be a ternary vector that spans a'. If T spanned any vector that starts with 0, then T would also span a, which is a false point. So T can't span any number that starts with 0.

This means we can't correctly generalize T_0 to span *all* the true points of $\overline{[a,b]}$ as we need to add at least one independent ternary vector to span the true point a', which is exactly what the algorithm does (the independent vector $1\phi^{n-1}$ spanning the whole $[10^{n-1}, 1^n]$). □

We have shown that in case $a^{[1]} = b^{[1]}$, our algorithm produces a sound and optimal result.

3.2 $a^{[1]} \neq b^{[1]}$

Let $a^{[1]} \neq b^{[1]}$. Since $a \leq b$, necessarily $a^{[1]} = 0$ and $b^{[1]} = 1$.

By flipping the first bit in the external boundary values $a-1$ and $b+1$, we'll effectively reduce this case to an instance of spanning a true point interval.

Let $a_1 = a - 1$ and $b_1 = b + 1$. Note that these are exactly the external boundary values of the interval $[a,b]$. Also note that $a_1^{[1]} = 0$ and $b_1^{[1]} = 1$.

Let $a'_1 = 1a_1^{[2,n]}$ and $b'_1 = 0b_1^{[2,n]}$ (flipping the first bits of a_1 and b_1). Note that $b'_1 \leq a'_1$.

Let $\mathcal{T}'_1$ be the minimal spanning set of the true point interval $[b'_1, a'_1]$. This set can be computed efficiently using the method introduced in [2].

Let's flip all the non-ϕ first bits of vectors in $\mathcal{T}'_1$, forming $\mathcal{T}$. Note that $|\mathcal{T}| = |\mathcal{T}'_1|$ and also that all the vectors in $\mathcal{T}'_1$ and $\mathcal{T}$ have the length n.

We claim that $\mathcal{T}$ is an optimal spanning set of $\overline{[a,b]}$.

Theorem 3.2.1. *$\mathcal{T}$ spans exactly $\overline{[a,b]}$.*

Proof. Part 1: $span(\mathcal{T}) \subseteq \overline{[a,b]}$

Let $c \in span(\mathcal{T})$. We need to show that $c \in \overline{[a,b]}$.

Let $T \in \mathcal{T}$ span c.

Let $T' = \overline{T^{[1]}}T^{[2,n]}$ (flipping the first bit).

By definition of $\mathcal{T}$, $T' \in \mathcal{T}'_1$.

Let $c' = \overline{c^{[1]}}c^{[2,n]}$ (flipping the first bit). Note that T' spans c'.

Since $\mathcal{T}'_1$ spans $[b'_1, a'_1]$ and $T' \in \mathcal{T}'_1$ spans c', $c' \in [b'_1, a'_1]$.

This means that $c' \geq b'_1$ and $c' \leq a'_1$.

Let $c'^{[1]} = 0$. Since $b_1'^{[1]} = 0$ (by definition), necessarily $c'^{[2,n]} \geq b_1'^{[2,n]}$.

Since $c = 1c'^{[2,n]}$ and $b_1 = 1b_1'^{[2,n]}$, also $c \geq b_1$.

Since $b_1 = b + 1$, $c > b$, so $c \in \overline{[a,b]}$.

A symmetric argument can be shown for $c'^{[1]} = 1$ using a'_1. □

Proof. Part 2: $\overline{[a,b]} \subseteq span(\mathcal{T})$

Let $c \in \overline{[a,b]}$. We need to show that $\mathcal{T}$ spans c.

Let $c' = \overline{c^{[1]}}c^{[2,n]}$ (flipping the first bit in c).

We'll first show that $c' \in [b'_1, a'_1]$ (see definition of b'_1 and a'_1 in the algorithm description).

1. $c^{[1]} = 0$

   From $b^{[1]} = 1$, $c \in \overline{[a,b]}$ and $c^{[1]} = 0$, we get $c < a$.

   Then trivially $c \leq a_1 = a - 1$.

   Since $a_1^{[1]} = 0 = c^{[1]}$, the inequality is preserved after flipping the first bit, i.e. $c' \leq a'_1$.

   Since $b_1'^{[1]} = 0$, we get $c' \geq b'_1$.

   Altogether, we get $c' \in [b'_1, a'_1]$.

2. $c^{[1]} = 1$

   This case is symmetric to the previous one. Using an analogous argumentation, we get $c' \in [b'_1, a'_1]$.

Let $T' \in \mathcal{T}'_1$ span c'. Such vector exists because $\mathcal{T}'_1$ spans $[b'_1, a'_1]$ by definition and $c' \in [b'_1, a'_1]$.

Let $T = \overline{T'^{[1]}}T'^{[2,n]}$. $T \in \mathcal{T}$ by definition. T spans c because T' spans c' and flipping a fixed bit preserves spanning.

Thus $\mathcal{T}$ spans c. $\square$

Theorem 3.2.2. *$\mathcal{T}$ is a minimal spanning set of $\overline{[a,b]}$.*

Proof. We'll prove the statement by contradiction. Let the size of the minimal spanning set of the interval $[b'_1, a'_1]$ be k. Then trivially $|\mathcal{T}| = k$, and $\mathcal{T}$ spans $\overline{[a,b]}$ as shown above.

Let a set $\mathcal{T}_{k-1}$ of size $k-1$ span $\overline{[a,b]}$. We'll use $\mathcal{T}_{k-1}$ to construct a set of size $k-1$ that spans $[b'_1, a'_1]$, reaching a contradiction and thus rejecting the existence of a smaller set spanning exactly $\overline{[a,b]}$.

Let $\mathcal{T}'_{k-1}$ be the result of flipping all the non-ϕ first bits of vectors in $\mathcal{T}_{k-1}$. Trivially $|\mathcal{T}'_{k-1}| = |\mathcal{T}_{k-1}| = k-1$. We claim that $\mathcal{T}'_{k-1}$ spans $[b'_1, a'_1]$.

The proof is very similar to the proof of theorem 3.2.1:

1. $span(\mathcal{T}'_{k-1}) \subseteq [b'_1, a'_1]$

   Let $c' \in span(\mathcal{T}'_{k-1})$. We'll show that $c' \in [b'_1, a'_1]$.

   Let $T' \in \mathcal{T}'_{k-1}$ span c'.

   Let $c = \overline{c'^{[1]}}c'^{[2,n]}$ and $T = \overline{T'^{[1]}}T'^{[2,n]}$.

   $T \in \mathcal{T}_{k-1}$, so $c \in \overline{[a,b]}$ by definition of $\mathcal{T}_{k-1}$.

(a) $c'^{[1]} = 0$:
$c^{[1]} = 1$.
Since $c \in \overline{[a,b]}$, then $c > b$.
Then also $c \geq b + 1 = b_1$ and $c' \geq b'_1$.
Since $c'^{[1]} = 0$ and $a'^{[1]}_1 = 1$, then $c' \leq a'_1$.
Altogether $c' \in [b'_1, a'_1]$.

(b) $c'^{[1]} = 1$: symmetric.

2. $[b'_1, a'_1] \subseteq span(\mathcal{T}'_{k-1})$
Let $c' \in [b'_1, a'_1]$.
Let $c = \overline{c'^{[1]}} c'^{[2,n]}$.
We'll first show that $c \in \overline{[a,b]}$.

(a) $c'^{[1]} = 1$:
$c' \leq a'_1$
$c \leq a_1$
$c < a$
$c \in \overline{[a,b]}$

(b) $c'^{[1]} = 0$: symmetric.

$\mathcal{T}_{k-1}$ spans c
$\mathcal{T}'_{k-1}$ spans c'

Thus we have shown that $\mathcal{T}'_{k-1}$ spans exactly $[b'_1, a'_1]$. Since $|\mathcal{T}'_{k-1}| = k - 1$, this contradicts the assumption that the size of the minimal spanning set of the interval $[b'_1, a'_1]$ is k. Thus we reject the existence of a set of size $k - 1$ spanning exactly $\overline{[a,b]}$, proving that $\mathcal{T}$ constructed by our algorithm ($|\mathcal{T}| = k$) is optimal. $\square$

4 Conclusions

We have shown an linear (in the arity of numbers) time algorithm for constructing minimal DNF representation of a false point interval Boolean function specified by a pair of boundary values.

References

[1] Jakub Dubovský. A construction of minimum DNF representations of 2-interval functions. Master's thesis, Charles University in Prague, 2012.

[2] Baruch Schieber, Daniel Geist, and Ayal Zaks. Computing the minimum DNF representation of boolean functions defined by intervals. *Discrete Applied Mathematics*, 149(1–3):154 – 173, 2005. Boolean and Pseudo-Boolean Functions Boolean and Pseudo-Boolean Functions.

Glossary

DNF disjunctive normal form. 1, 5

MSB most significant bit. 2