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MASTER THESIS

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Minimum representations of boolean functions defined by multiple intervals

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Abstract:

When we interpret the input vector of a Boolean function as a binary number, we define interval Boolean function $f_{[a,b]}$ so that $f_{[a,b]}(x) = 1$ if and only if $a \leq x \leq b$. Disjunctive normal form is a common way of representing Boolean functions. Minimizing DNF representation of an interval Boolean function can be performed in linear time.[2] The natural generalization to k-interval functions seems to be significantly harder to tackle. In this thesis, I discuss the difficulties with finding an optimal solution and introduce a $2k$-approximation algorithm.

Keywords: Boolean minimization, disjunctive normal form, interval functions

Contents

Introduction

I will build on the results about interval Boolean functions shown by Schieber et al.[2]

1. Single interval functions

In this chapter, I will introduce an efficient algorithm for optimal spanning of single interval Boolean functions, originally shown by Schieber et al.[2]

In the following chapters, I will use this algorithm as a procedure in order to span multi-interval functions.

In the remainder of this chapter, a and b ($a \leq b$) will denote the endpoints of the spanned interval and n the number of input bits. Thus, we'll be spanning the function $f^n_{[a,b]}$.

1.1 Prefix and suffix case

Let's first consider the prefix case, that is $[0^{\{n\}}, b]$ ($a = 0^{\{n\}}$), and the suffix case, that is $[a, 1^{\{n\}}]$ ($b = 1^{\{n\}}$).

If both $a = 0^{\{n\}}$ and $b = 1^{\{n\}}$, the optimal spanning set trivially consists of the single ternary vector $\phi^{\{n\}}$, corresponding to the trivial formula 1.

From now on, let $a > 0^{\{n\}}$ or $b < 1^{\{n\}}$.

Note that prefix and suffix cases are complementary – we may transform a suffix instance $[a, 1^{\{n\}}]$ to a prefix instance $[0^{\{n\}}, \overline{a}]$. Flipping the polarity of the resulting ternary vectors yields a solution for the initial suffix instance $[a, 1^{\{n\}}]$. This means it's enough to solve the prefix case, which is what we'll do.

Let $a = 0^{\{n\}}$ and $b < 1^{\{n\}}$. Let c be the n-bit number $b + 1$. Since $b < 1^{\{n\}}$, we don't need more than n bits to encode c.

The algorithm produces one ternary vector for each 1-bit in c. If o is a position of a 1-bit in c ($c^{[o]} = 1$), then the corresponding ternary vector is $c^{[1,o-1]}0\phi^{\{n-o\}}$. Thus we get the following spanning set:

$$\mathcal{T} = \{c^{[1,o-1]}0\phi^{\{n-o\}} | c^{[o]} = 1\}$$

Theorem 1.1.0.1. *$\mathcal{T}$ spans exactly the interval $[0^{\{n\}}, b]$ (feasibility).*

Proof. To see that every number spanned by $\mathcal{T}$ is in $[0^{\{n\}}, b]$, note that all the numbers spanned by $\mathcal{T}$ are smaller than c, since their most significant bit different from c must be a 0-bit and they must differ from c.

On the other hand, every number x smaller than c must differ from c, and the leftmost different bit must be 0. Let o be its position. Then $c^{[1,o-1]}0\phi^{\{n-o\}} \in \mathcal{T}$ spans x. □

Theorem 1.1.0.2. *$\mathcal{T}$ is minimal in size (optimality).*

Proof. We'll construct a set of $|\mathcal{T}|$ true points no pair of which can be spanned by a single ternary vector. Doing so, we'll show a lower bound $|\mathcal{T}|$ on the size of feasible solutions, proving optimality of $\mathcal{T}$.

Once more, the orthogonal true points correspond to 1-bits of c:

$$V = \{c^{[1,o-1]}0c^{[o+1,n]} | c^{[o]} = 1\} \tag{1.1}$$

Clearly $|V| = |\mathcal{T}|$.

It's easy to see that any ternary vector that spans two different points in V must also span the false point c, so it can't be a part of the solution. Also note that all points in V are smaller than c, so they are true points.

If there was a feasible spanning set of size smaller than $|V|$, at least one of its vectors would need to span at least two points in V. As we have shown, such ternary vector would necessarily also span the false point c, leading to contradiction with its feasibility. $\square$

2. $2k$-approximation algorithm for minimizing DNF representation of k-interval Boolean functions

2.1 Introduction

In this chapter, an algorithm will be shown that computes a small DNF representation of a Boolean function given as a set of k intervals. The input intervals are represented by pairs of endpoints (n-bit numbers). An approximation ratio of $2k$ will be proved.

2.2 Definitions

Definition 2.2.1 (k-interval Boolean function)**.** Let $a_1, b_1, \dots, a_k, b_k$ be n-bit numbers such that $0 \leq a_1$, $a_1 \leq b_1$, $b_1 \leq a_2 - 2$, …, $b_k \leq 2^n - 1$. Then $f^n_{[a_1,b_1],\dots,[a_k,b_k]} : \mathbf{B}^n \to \mathbf{B}$ is a function defined as follows:

$$f^n_{[a_1,b_1],\dots,[a_k,b_k]}(x) = \begin{cases} 1 & \text{if } x \in [a_i, b_i] \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

Note that the adjacent intervals are required to be separated by at least one false point.

2.3 Algorithm

2.3.1 Description

Input Numbers $a_1, b_1, \dots, a_k, b_k$ that satisfy the inequalities in definition 2.2.1

Output A set of ternary vectors

Procedure The algorithm goes through all the intervals $[a_i, b_i]$. For each i, the longest common prefix of a_i and b_i is computed. Let j be its length. Note that $a^{[j+1]} = 0$ and $b^{[j+1]} = 1$. Now let $a'' = a_i^{[j+2,n]}$ and $b'' = b_i^{[j+2,n]}$. Optimally span the suffix interval $[a'', 1^{n-j-1}]$ and the prefix interval $[0^{n-j-1}, b'']$ using the (linear time) algorithm introduced in [2]. Prepend $a_i^{[1,j+1]}$ and $b_i^{[1,j+1]}$ to the respective ternary vectors and add them to the output spanning set.

2.3.2 Correctness

Theorem 2.3.2.1. *The algorithm spans exactly* $f^n_{[a_1,b_1],\dots,[a_k,b_k]}$.

Proof. This is easy to see from the fact that the subintervals form a partition of the multi-interval (i.e. the set of all true points) and that each of them is spanned exactly by the suffix or prefix procedure. $\square$

2.3.3 Approximation ratio

Theorem 2.3.3.1. *Let $\mathcal{T}_{opt}$ be an optimal spanning set of $f^n_{[a_1,b_1],\ldots,[a_k,b_k]}$ and let $\mathcal{T}_{approx}$ be the spanning set returned by the algorithm. We claim that:*

$$|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_{opt}| \tag{2.1}$$

Proof. Let $\mathcal{T}_x$ be the largest (n-bit) spanning set of a "suffix" or "prefix" subinterval added in the algorithm. Without loss of generality, let the respective subinterval be "prefix" $[b_i^{[1,j+1]}, b_i]$. From [1, p. 36] we know that there is an orthogonal set of $[b_i^{[1,j+1]}, b_i]$ of size $|\mathcal{T}_x|$, and moreover that its orthogonality only depends on the false point $b+1$. Note, however, that $b+1$ is also a false point in $f^n_{[a_1,b_1],\ldots,[a_k,b_k]}$. Thus we obtain an orthogonal set of size $|\mathcal{T}_x|$ for the k-interval function, limiting the size of its optimal spanning set $|\mathcal{T}_{opt}| \geq |\mathcal{T}_x|$.

Since $\mathcal{T}_x$ is the largest of the $2k$ partial sets used to span the function in the approximation algorithm, we know that $|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_x|$.

Joining the inequalities together we conclude: $|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_x| \leq 2k|\mathcal{T}_{opt}|$. $\square$

Theorem 2.3.3.2. *The approximation ratio of $2k$ is tight.*

Proof. For every k that is a power of 2 we'll show a k-interval function such that $|\mathcal{T}_{approx}| = 2k|\mathcal{T}_{opt}|$ (following the notation from Theorem 2.3.3.1.

Let $k = 2^{n_k}$. Let P be all the n_k-bit numbers, that is $P = \mathbf{B}^{n_k}$. Note that $|P| = k$.

For each $p \in P$, we define the interval $[a_p, b_p]$ by appending 2-bit suffixes to p:

- $a_p = p00$
- $b_p = p01$

The first appended bit (0 for both a_p and b_p) ensures that there is at least one false point between any pair of intervals defined this way. The second appended bit (0 for a_p and 1 for b_p) ensures that the interval has two points, so the approximation algorithm will use two vectors to span it. Thus we have k $(n_k + 2)$-bit intervals, none of which intersect or touch.

Since $P = \mathbf{B}^{n_k}$, we can span all the intervals by the single ternary vector $\phi^{\{n_k\}}0\phi$. Clearly $|\mathcal{T}_{opt}| = 1$.

However, the approximation algorithm uses $2k$ vectors to span the intervals, since it spans each of the intervals separately and uses two vectors for each interval.

We get $|\mathcal{T}_{approx}| = 2k|\mathcal{T}_{opt}|$.

Note that the optimal spanning set is disjoint, so the ratio is tight in disjoint case as well. $\square$

Conclusion

Bibliography

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Glossary

DNF disjunctive normal form. ii, 1, 4