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MASTER THESIS

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Minimum representations of boolean functions defined by multiple intervals

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Abstract:

When we interpret the input vector of a Boolean function as a binary number, we define interval Boolean function $f_{[a,b]}$ so that $f_{[a,b]}(x) = 1$ if and only if $a \leq x \leq b$. Disjunctive normal form is a common way of representing Boolean functions. Minimizing DNF representation of an interval Boolean function can be performed in linear time.[2] The natural generalization to k-interval functions seems to be significantly harder to tackle. In this thesis, I discuss the difficulties with finding an optimal solution and introduce a $2k$-approximation algorithm.

Keywords: Boolean minimization, disjunctive normal form, interval functions

Contents

Introduction

I will build on the results about interval Boolean functions shown by Schieber et al.[2]

1. $2k$-approximation algorithm for minimizing DNF representation of k-interval Boolean functions

1.1 Introduction

In this chapter, an algorithm will be shown that computes a small DNF representation of a Boolean function given as a set of k intervals. The input intervals are represented by pairs of endpoints (n-bit numbers). An approximation ratio of $2k$ will be proved.

1.2 Definitions

Definition 1.2.1 (k-interval Boolean function)**.** Let $a_1, b_1, \ldots, a_k, b_k$ be n-bit numbers such that $0 \leq a_1$, $a_1 \leq b_1$, $b_1 \leq a_2 - 2$, …, $b_k \leq 2^n - 1$. Then $f^n_{[a_1,b_1],\ldots,[a_k,b_k]} : \mathbf{B}^n \to \mathbf{B}$ is a function defined as follows:

$$f^n_{[a_1,b_1],\ldots,[a_k,b_k]}(x) = \begin{cases} 1 & \text{if } x \in [a_i, b_i] \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

Note that the adjacent intervals are required to be separated by at least one false point.

1.3 Algorithm

1.3.1 Description

Input Numbers $a_1, b_1, \ldots, a_k, b_k$ that satisfy the inequalities in definition 1.2.1

Output A set of ternary vectors

Procedure The algorithm goes through all the intervals $[a_i, b_i]$. For each i, the longest common prefix of a_i and b_i is computed. Let j be its length. Note that $a^{[j+1]} = 0$ and $b^{[j+1]} = 1$. Now let $a'' = a_i^{[j+2,n]}$ and $b'' = b_i^{[j+2,n]}$. Optimally span the suffix interval $[a'', 1^{n-j-1}]$ and the prefix interval $[0^{n-j-1}, b'']$ using the (linear time) algorithm introduced in [2]. Prepend $a_i^{[1,j+1]}$ and $b_i^{[1,j+1]}$ to the respective ternary vectors and add them to the output spanning set.

1.3.2 Correctness

Theorem 1.3.2.1. *The algorithm spans exactly* $f^n_{[a_1,b_1],\ldots,[a_k,b_k]}$.

Proof. This is easy to see from the fact that the subintervals form a partition of the multi-interval (i.e. the set of all true points) and that each of them is spanned exactly by the suffix or prefix procedure. □

1.3.3 Approximation ratio

Theorem 1.3.3.1. *Let $\mathcal{T}_{opt}$ be an optimal spanning set of $f^n_{[a_1,b_1],\ldots,[a_k,b_k]}$ and let $\mathcal{T}_{approx}$ be the spanning set returned by the algorithm. We claim that:*

$$|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_{opt}| \tag{1.1}$$

Proof. Let $\mathcal{T}_x$ be the largest (n-bit) spanning set of a "suffix" or "prefix" subinterval added in the algorithm. Without loss of generality, let the respective subinterval be "prefix" $[b_i^{[1,j+1]}, b_i]$. From [1, p. 36] we know that there is an orthogonal set of $[b_i^{[1,j+1]}, b_i]$ of size $|\mathcal{T}_x|$, and moreover that its orthogonality only depends on the false point $b+1$. Note, however, that $b+1$ is also a false point in $f^n_{[a_1,b_1],\ldots,[a_k,b_k]}$. Thus we obtain an orthogonal set of size $|\mathcal{T}_x|$ for the k-interval function, limiting the size of its optimal spanning set $|\mathcal{T}_{opt}| \geq |\mathcal{T}_x|$.

Since $\mathcal{T}_x$ is the largest of the $2k$ partial sets used to span the function in the approximation algorithm, we know that $|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_x|$.

Joining the inequalities together we conclude: $|\mathcal{T}_{approx}| \leq 2k|\mathcal{T}_x| \leq 2k|\mathcal{T}_{opt}|$. □

Theorem 1.3.3.2. *The approximation ratio of $2k$ is tight.*

Proof. For every k that is a power of 2 we'll show a k-interval function such that $|\mathcal{T}_{approx}| = 2k|\mathcal{T}_{opt}|$ (following the notation from Theorem 1.3.3.1.

Let $k = 2^{n_k}$. Let P be all the n_k-bit numbers, that is $P = \mathbf{B}^{n_k}$. Note that $|P| = k$.

For each $p \in P$, we define the interval $[a_p, b_p]$ by appending 2-bit suffixes to p:

- $a_p = p00$
- $b_p = p01$

The first appended bit (0 for both a_p and b_p) ensures that there is at least one false point between any pair of intervals defined this way. The second appended bit (0 for a_p and 1 for b_p) ensures that the interval has two points, so the approximation algorithm will use two vectors to span it. Thus we have k (n_k+2)-bit intervals, none of which intersect or touch.

Let $n = n_k + 2$.

Since $P = \mathbf{B}^{n_k}$, we can span all the intervals by the single ternary vector $\phi^{\{n_k\}}0\phi$. Clearly $|\mathcal{T}_{opt}| = 1$.

However, the approximation algorithm uses $2k$ vectors to span the intervals, since it spans each of the intervals separately and uses two vectors for each interval.

We get $|\mathcal{T}_{approx}| = 2k|\mathcal{T}_{opt}|$.

Note that the optimal spanning set is disjoint, so the ratio is tight in disjoint case as well. □

Conclusion

Bibliography

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Glossary

DNF disjunctive normal form. ii, 1, 5