

## Discrete and Algorithmic Geometry

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### Sheet 2

due on Monday, November 19, 2018

WRITING (DUE NOVEMBER 19)

- (1) (a) Using Gale's Evenness Criterion, explicitly write down all subsets of vertices that make up a facet of the 4-dimensional cyclic polytope  $C_4(8)$ .
- (b) Draw the dual graph of  $C_4(8)$ , i.e. the graph that has the facets of  $C_4(8)$  as nodes and an edge between two nodes if the corresponding facets share a ridge. Try to make your drawing as legible as possible. *Hint:* Labeling the nodes with the corresponding vertex sets will help you to get organized.
- (2) Let  $P$  be the 24-cell,  $P = \text{conv}\{\pm e_i \pm e_j : 1 \leq i \neq j \leq 4\}$ , where  $(e_1, \dots, e_4)$  is the standard basis of  $\mathbb{R}^4$ . Also, consider the *simple roots of type  $F_4$* , namely the rows  $r_1, \dots, r_4$  of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and the linear hyperplanes  $H_1, \dots, H_4$  orthogonal to them.

- (a) Count the vertices of  $P$ .
- (b) The *root system of type  $F_4$*  arises from the four simple roots by reflecting them in the  $H_i$ , adding all new vectors (and the linear hyperplanes orthogonal to them) obtained in this way to the set of reflecting hyperplanes, and repeating the process until no new normal vectors/hyperplanes are found. Count the number of resulting hyperplanes. *Answer:* 48
- (c) Show that reflecting the vertices of  $P$  in the  $H_i$  leaves  $P$  invariant.
- (d) Find all vertices of  $P$  inside the *fundamental cone of  $F_4$* ,

$$C = \{x \in \mathbb{R}^4 : \langle r_i, x \rangle \geq 0 \text{ for } i = 1, \dots, 4\}.$$

- (e) Use this to efficiently describe and count the number of facets of  $P$ .