Discrete and Algorithmic Geometry

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Sheet 2

due on Monday, November 19, 2018

Writing (due November 19)

(1) (a) Using Gale's Evenness Criterion, explicitly write down all subsets of vertices that make up a facet of the 4-dimensional cyclic polytope $C_4(8)$.
(b) Draw the dual graph of $C_4(8)$, i.e. the graph that has the facets of $C_4(8)$ as nodes and an edge between two nodes if the corresponding facets share a ridge. Try to make your drawing as legible as possible. *Hint:* Labeling the nodes with the corresponding vertex sets will help you to get organized.

(2) Let P be the *24-cell*, $P = \operatorname{conv}\{\pm e_i \pm e_j : 1 \le i \ne j \le 4\}$, where $(e_1, \dots, e_4)$ is the standard basis of $\mathbb{R}^4$. Also, consider the *simple roots of type* F_4, namely the rows $r_1, \dots, r_4$ of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and the linear hyperplanes $H_1, \dots, H_4$ orthogonal to them.
(a) Count the vertices of P.
(b) The *root system of type* F_4 arises from the four simple roots by reflecting them in the H_i, adding all new vectors (and the linear hyperplanes orthogonal to them) obtained in this way to the set of reflecting hyperplanes, and repeating the process until no new normal vectors/hyperplanes are found. Count the number of resulting hyperplanes. *Answer:* 48
(c) Show that reflecting the vertices of P in the H_i leaves P invariant.
(d) Find all vertices of P inside the *fundamental cone of* F_4,

$$C = \{x \in \mathbb{R}^4 : \langle r_i, x\rangle \ge 0 \text{ for } i = 1, \dots, 4\}.$$

(e) Use this to efficiently describe and count the number of facets of P.