Coxeter matroid polytopes

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Abstract. If Δ is a polytope in real affine space, each edge of Δ determines a reflection in the perpendicular bisector of the edge. The exchange group $W(\Delta)$ is the group generated by these reflections, and Δ is a (Coxeter) matroid polytope if this group is finite. This simple concept of matroid polytope turns out to be an equivalent way to define Coxeter matroids. The Gelfand- Serganova Theorem and the structure of the exchange group both give us information about the matroid polytope. We then specialise this information to the case of ordinary matroids to learn more about the classical matroid polytope already familiar to matroid theorists. Definitions and notation mostly follow [**H**] for reflection groups and Coxeter groups, [**O**, **Wel**, **Wh**] for matroids, [**BG**, **BR**, **BGW1**] for Coxeter matroids.

Keywords: matroid polytope, Coxeter matroid, exchange group.

1 Coxeter matroids and polytopes

Matroid polytopes. The purpose of the present paper is to offer a very elementary approach to Coxeter matroids (or WP-matroids as they were called in the original papers [**GS1**, **GS2**] by Gelfand and Serganova).

Let Δ be a convex polytope in the real affine Euclidean space $\mathbb{AR}^n$. For any two adjacent (i.e. connected by an edge) vertices α and β of Δ we can consider the reflection $s_{\alpha\beta}$ in the mirror of symmetry of the edge $[\alpha\beta]$. All these reflections generate a group $W(\Delta)$ of affine isometries of the space $\mathbb{AR}^n$. We say that Δ is a (*Coxeter*)*matroid polytope* if the group $W(\Delta)$ is finite.

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Examples of matroid polytopes are abundant. Obviously, Platonic solids (as well as most regular and semi-regular polytopes) are matroid polytopes. The classical matroid polytopes from ordinary matroid theory are another class of examples, as we shall see.

If Δ is a matroid polytope, the group $W = W(\Delta)$ will be called the *exchange group* of Δ. Being a finite group, W fixes the baricenter of each of its (finite) orbits, so we can assume without loss of generality that W fixes the origin O of the vector space $\mathbb{R}^n$ and hence is a linear group. Moreover it is a finite reflection group and hence a Coxeter group. By definition of W all vertices of Δ belong to one W-orbit. It follows from [**H**, Proposition 1.15] that the stabilizer P of a vertex δ of Δ is a parabolic subgroup of W. We can choose a system of simple reflections $r_1, \dots, r_n$ (r_i are generators of W as a Coxeter group) in such a way that P is a standard parabolic subgroup, i.e. is generated by some r_i's. Therefore the set of vertices of Δ can be identified with some subset $\mathcal{M}$ of the factor set W/P.

Theorem 1 *If Δ is a matroid polytope then $\mathcal{M}$ is a Coxeter matroid for W and P.*

The definition [**BG**] of a *Coxeter matroid* for a Coxeter group W and a standard parabolic subgroup $P < W$ is a subset $\mathcal{M} \subseteq W/P$ which satisfies the *Maximality Property*: for every $w \in W$ the set $\mathcal{M}$ contains a unique coset A maximal in $\mathcal{M}$ with respect to the w-shifted Bruhat ordering on W/P:

$$w^{-1}B \leqslant w^{-1}A$$

for all $B \in \mathcal{M}$. The classical concept of matroid constitutes a special case of a Coxeter matroid, for the Coxeter group $W = Sym_n$ and P the stabiliser of a k-subset in $[n] = \{1, \dots, n\}$.

The proof of Theorem 1 is based on Theorem 2 below, a version of the Gelfand-Serganova Theorem [**GS2**, Theorem 8.1] proven in [**SVZ**, **ZS**]. Theorem 2 also provides the converse to Theorem 1: every Coxeter matroid gives rise to a matroid polytope. In view of these two results, the very simple and elementary concept of matroid polytope can be taken as an alternate definition of a Coxeter matroid.

The Gelfand-Serganova Theorem. Let W be a finite Coxeter group and V the space of the reflection representation for W. We shall identify W and the image of its reflection representation. Take a point $\delta \in V$ in general position with respect to W (i.e. the stabilizer of δ in W is trivial). For a subset $X \subset W$, denote by δ_X the baricenter of the points $w\delta$ for $w \in X$.

Now let P be a parabolic subgroup in W and $\mathcal{M}$ a subset of the factor set W/P. Denote by $\Delta_{\mathcal{M}}$ the convex hull of the point δ_A for $A \in \mathcal{M}$. The following result is a slightly generalised version of the Gelfand-Serganova Theorem [**GS2**, Theorem 8.1].

Theorem 2 [**SVZ**, **ZS**] *A subset $\mathcal{M} \subseteq W/P$ is a Coxeter matroid for W and P if and only if $W(\Delta_{\mathcal{M}}) \leqslant W$.*

Proof of Theorem 1. We use notation of the beginning of this section. If we succeed in representing the vertex δ as the baricenter of the orbit $P{\cdot}\delta_1$ for some point δ_1 in general position then our polytope Δ will have the form $\Delta = \Delta_{\mathcal{M}}$. Since, by the construction of the exchange group W, we have $W = W(\Delta)$, we can apply the Gelfand-Serganova Theorem and conclude that $\mathcal{M}$ is a Coxeter matroid for W and P.

Therefore it will be enough to find a point δ_1 in general position such that δ is the baricenter of the orbit $P \cdot \delta_1$.

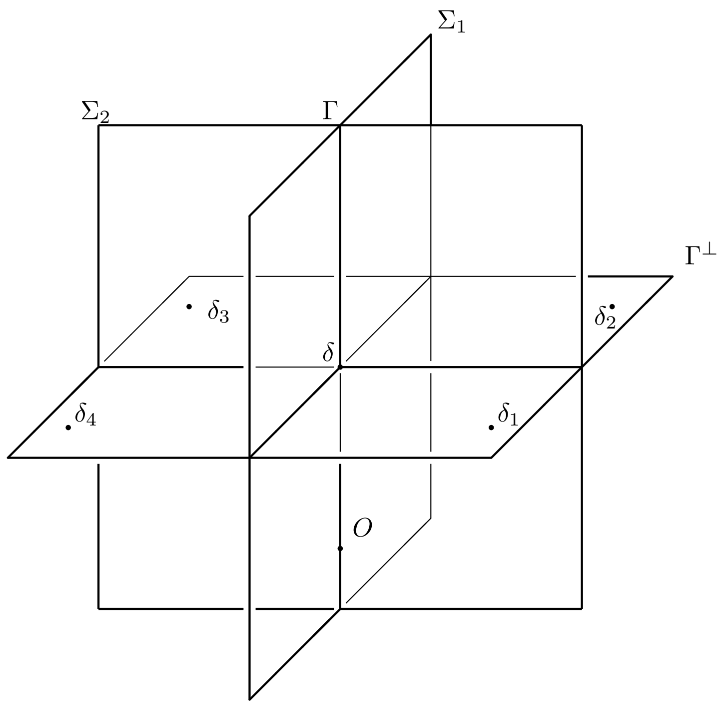

Figure 1: The point δ is the baricenter of δ_i's.

Our choice of δ_1 is illustrated by Figure 1. Let Γ be the fixed point space of P and $\Gamma^\perp$ the affine subspace perpendicular to Γ and passing through the point $\delta \in \Gamma$. Let $r_{i_1}, \dots, r_{i_k}$ be the standard generators of W belonging to P and $\Sigma_1, \dots, \Sigma_k$ their mirrors. Then $\Gamma = \Sigma_1 \cap \cdots \cap \Sigma_k$. Let δ_1 be an arbitrary point in $\Gamma^\perp$. The baricenter of the orbit $P \cdot \delta_1 = \{ \delta_1, \delta_2, \dots \}$ is P-invariant and therefore belongs to $\Gamma \cap \Gamma^\perp = \{ \delta \}$.

It remains to prove that δ_1 can be chosen in general posiiton with respect to W. If not, then every point of $\Gamma^\perp \smallsetminus \{ \delta \}$ is fixed by some non-identity element $w \in W$. But the fixed point sets are affine subspaces of $\mathbb{AR}^n$, and if finitely many of them cover $\Gamma^\perp \smallsetminus \{ \delta \}$ then the set $\Gamma^\perp \smallsetminus \{ \delta \}$ is covered already by the fixed point set of one element $w \in W$. But then w fixes δ, so $w \in P$ and w fixes all points in Γ. This means that w acts trivially on the entire space $\mathbb{AR}^n$, hence that w is the identity. $\square$

2 Examples

The theorems above provide a simple and elementary definition of Coxeter matroids which does not refer to the concept of the Bruhat ordering. It is even more important that the concept of the exchange group of a matroid polytope sheds some light on interactions between different types of Coxeter matroids.

Consider, for example, *symplectic matroids* as they were introduced in [**BGW3**].

Example 1: symplectic matroids. Let $[n] = \{ 1, 2, \dots, n \}$, $[n]^* = \{ 1^*, 2^*, \dots, n^* \}$ and $J = [n] \sqcup [n]^*$. Define the map $* : J \to J$ by $i \mapsto i^*$ and $i^* \mapsto i$. Then $*$ is an involutive permutation of the set J. A subset $K \subset J$ is *admissible* if and only if $K \cap K^* = \emptyset$. The set of all admissible k-subsets in J is denoted J_k.

Let W be the group of all permutations of the set J which permute with the involution *, i.e. a permutation w belongs to W if and only if $w(i^*) = w(i)^*$ for all $i \in J$. Then W is the *hyperoctahedral group* BC_n and is isomorphic to the group of symmetries of the n-cube $[-1,1]^n$ in the n-dimensional real Euclidean space $\mathbb{R}^n$, and this is the reflection representation of W as a Coxeter group. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ be the standard orthonormal basis in $\mathbb{R}^n$ and set $\epsilon_{i^*} = -\epsilon_i$. The action of the group W on $\mathbb{R}^n$ is given by $w\epsilon_i = \epsilon_{w(i)}$ for $i \in [n]$.

The involutions

$$r_1 = (1,2)(1^*,2^*),\ \ldots,\ r_{n-1} = (n-1,n)((n-1)^*,n^*),\ r_n = (n,n^*)$$

form a set of standard generators for W as a Coxeter group. Denote by P_k, $k = 1, \ldots, n$, the maximal parabolic subgroup in W generated by all r_j with $j \neq k$. Then the set J_k of admissible k-subsets in J can be identified with the factor set W/P_k. A subset $\mathcal{B} \subseteq J_k$ is called a *symplectic matroid of rank* k if, being considered as a subset of W/P_k, it is a Coxeter matroid for W and P_k.

Symplectic matroids of maximal rank $k = n$, i.e. Coxeter matroids for BC_n and P_n (they are called *Lagrangian* in [**BGW3**]) have been first introduced, under the name of *symmetric* matroids, by A. Bouchet [**Bou**]; W. Wenzel [**Wen**, Proposition 1.15] proved that symmetric matroids are Coxeter matroids.

It is shown in [**BGW3**] that symplectic matroids arise naturally from symplectic geometry, in much the same way that ordinary matroids arise from projective geometry. The construction starts with a *standard symplectic space*, a vector space V with a basis $E = \{e_1, e_2, \ldots, e_n, e_{1^*}, e_{2^*}, \ldots, e_{n^*}\}$ endowed with an anti-symmetric bilinear form $(.,.)$ such that $(e_i, e_j) = 0$ for all $i, j \in J, i \neq j^*$, whereas $(e_i, e_{i^*}) = 1$ for $i \in [n]$. A *totally isotropic subspace* of V is a subspace U such that $(u,v) = 0$ for all $u, v \in U$. Let U be a totally isotropic subspace of V of dimension k. Now choose a basis $\{u_1, u_2, \ldots, u_k\}$ of U, and expand each of these vectors in terms of the basis E: $u_i = \sum_{j=1}^n a_{i,j} e_j + \sum_{j=1}^n b_{i,j} e_{j^*}$. Thus we have represented the totally isotropic subspace U by a $k \times 2n$ matrix $C = (A, B)$, $A = (a_{i,j}), B = (b_{i,j})$, with the columns of C indexed by the elements of E.

Now, define a family $\mathcal{B} \subseteq J_k$ by saying $X \in \mathcal{B}$ if $X \in J_k$ and the $k \times k$ minor formed by taking the j-th column of C for all $j \in X$ is non-zero. Obviously the set $\mathcal{B}$ does not depend on choice of a basis in U.

Theorem 3 [**BGW3**] *If U is totally isotropic, then $\mathcal{B}$ is a symplectic matroid.*

For an admissible set $A \in J_k$ define the point $\delta_A \in \mathbb{R}^n$ as

$$\delta_A = \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_k} \text{ where } A = \{i_1, i_2, \ldots, i_k\}.$$

If $\mathcal{B}$ is a subset of J_k then $\Delta_{\mathcal{B}}$ is the convex hull of δ_B for $B \in \mathcal{B}$.

The Gelfand-Serganova Theorem claims that $\mathcal{B}$ is a symplectic matroid if and only if $W(\Delta_{\mathcal{B}})$ is a subgroup of $W = BC_n$, or, equivalently, the edges of $\Delta_{\mathcal{B}}$ are perpendicular to the mirrors of reflections in BC_n.

For a more concrete example, consider the isotropic subspace in 6-dimensional symplectic space spanned by the vectors $u = e_1 + e_2 - 2e_3$ and $w = e_{1^*} + e_{2^*} + e_{3^*}$. Then

$$\mathcal{B} = \{\, 12^*, 13^*, 1^*2, 23^*, 1^*3, 2^*3 \,\}$$

and the resulting polytope $\Delta = \Delta_{\mathcal{B}}$ is shown on Figure 2. Obviously $W(\Delta) \simeq Sym_3$ is the Coxeter group of type A_2 and $\mathcal{B}$ can be identified with the set of all 6 permutations in Sym_3. Thus $\mathcal{B}$ is a Coxeter matroid for the Coxeter group Sym_3 and its trivial parabolic subgroup. This, in turn, means (see [**BGW4**]) that $\mathcal{B}$ may be realized as a flag of ordinary matroids, in this case, the uniform matroids of ranks 1 and 2 on [3].

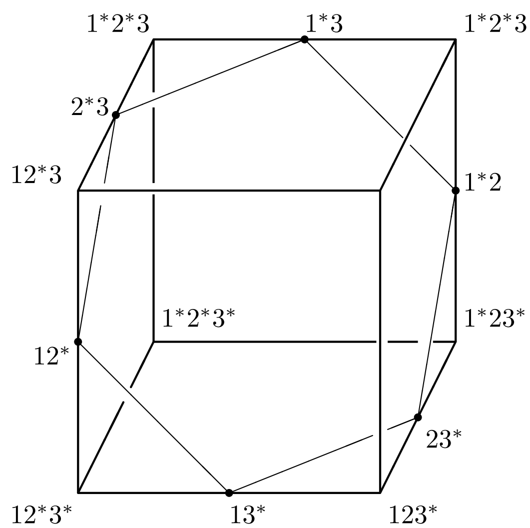

Figure 2: Matroid polytope of the symplectic matroid of Example 1

Example 2: ordinary matroids. If we are given an ordinary matroid of rank k on $[n]$, then each of its independent sets may be encoded as a vector of 0's and 1's in $\mathbb{R}^n$. The classical matroid polytope is defined as the convex hull of these vectors. It has a face, defined by the equation $\Sigma_{i=1}^n x_i = k$, which is the convex hull of the vectors encoding the bases of the matroid. Both of these polytopes are Coxeter matroid polytopes. To avoid confusion, we will refer to them as the *independent-set matroid polytope* and the *basis matroid polytope*, respectively. For our purposes, the latter is the more interesting one, since it corresponds to the usual way of regarding an ordinary matroid as a Coxeter matroid.

Theorem 4 *The independent-set matroid polytope of an ordinary matroid is a Coxeter matroid polytope.*

Proof. Bouchet ([**Bou**, Cor. 7.3]) shows that the collection $\mathcal{I}$ of independent sets of a matroid can be realized as the unstarred subsets of a collection of admissible sets forming a Lagrangian matroid $\mathcal{B}_\mathcal{I}$. The independent-set matroid polytope is just the polytope $\Delta_{\mathcal{B}_\mathcal{I}}$ from Example 1 above, up to an obvious affine transformation (replacing $0, 1$ vectors by $-1, 1$ vectors). The group $W(\Delta_{\mathcal{B}_\mathcal{I}})$ is a subgroup of BC_n. $\square$

Corollary 5 *The basis matroid polytope is a Coxeter matroid polytope.*

Consider now a matroid of rank 2 on [4] given by the collection of its bases $\mathcal{B} = \{12, 14, 23, 34\}$. The corresponding Coxeter group $A_3 = Sym_4$ acts, in its reflection representation, as the group of symmetries of the regular tetrahedron in $\mathbb{R}^3$. If we denote the vertices of the tetrahedron by $1, 2, 3, 4$, then a basis $\{i, j\}$ in $\mathcal{B}$ is represented by the midpoint of the edge ij. Again up to affine transformation, the basis matroid polytope is just the convex hull of these vertices, as shown on Figure 3.

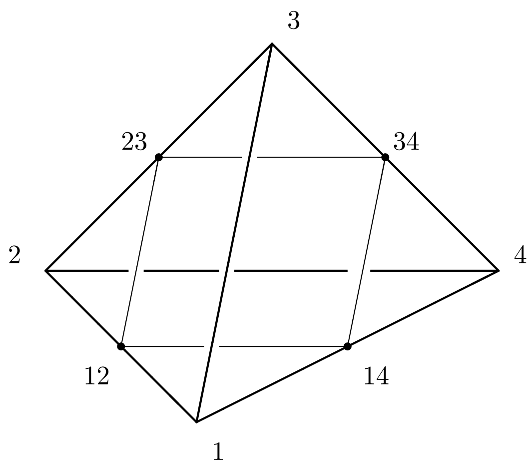

Figure 3: The matroid polytope of Example 2.

Obviously $W(\Delta_{\mathcal{B}}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq Sym_2 \times Sym_2$ is the Klein four-group though originally $\mathcal{B}$, being a matroid of rank 2 on $[4]$, is a Coxeter matroid for Sym_4 and its parabolic subgroup $Sym_2 \times Sym_2$. We shall see in Section 4 that, in general, the exchange group of a matroid on $[n]$ equals the direct product $Sym_{n_1} \times \cdots \times Sym_{n_d}$ for some n_i with $n_1 + \cdots + n_d \leqslant n$, where these direct product factors correspond to the non-trivial components of the matroid (not a loop or isthmus).

3 Exchange groups of Coxeter matroids

Let $\mathcal{M}$ be a Coxeter matroid for W and P. Consider a matroid polytope $\Delta = \Delta_{\mathcal{M}}$. In its construction we started with an arbitrary point in general position δ in the space V of the reflection representation for W. The following theorem shows that adjacency of the vertices δ_A and δ_B for two cosets $A, B \in \mathcal{M}$ is determined entirely by $\mathcal{M}$ and does not depend on choice of δ.

Theorem 6 (A. Borovik and A. Vince [**BV**]) *Let δ_A and δ_B be two vertices of Δ. Then δ_B is adjacent to δ_A if and only if there is an element $w \in W$ with the property that the coset A is w-maximal in $\mathcal{M}$ and the coset B immediately preceeds A in $\mathcal{M}$ with respect to the ordering $\leqslant^w$: $B <^w A$, and there is no coset $C \in \mathcal{M}$ with $B <^w C <^w A$.*

It is now clear, in view of Theorem 6, that the exchange group $W_0 = W(\Delta)$ and its imbedding into W depend only on the Coxeter matroid $\mathcal{M} \subseteq W/P$ and do not depend on choice of the point δ. Therefore we can call W_0 the *exchange group* of the Coxeter matroid $\mathcal{M}$ and denote it $W(\mathcal{M})$.

Moreover, denote by P_0 the stabiliser in W_0 of some vertex δ_A of Δ. Then we can identify $\mathcal{M}$ with some Coxeter matroid $\mathcal{M}_0$ for W_0 and P_0. Indeed, denote by V_0 the subspace orthogonal to the space V_1 of fixed points of W_0 in V. The group W_0 faithfully acts on V_0 as a reflection group. In particular W_0 is a Coxeter group and $P_0 = W_0 \cap P$ is a parabolic subgroup of W_0. We can chose a system $r_1', \dots, r_k'$ of Coxeter generators in W_0 in such a way that P_0 becomes a standard parabolic subgroup, i.e. is generated by some r_i''s. Denote by Δ_0 the orthogonal projection of Δ into V_0. Then Δ_0 is isometric to Δ and Δ_0 is a matroid polytope for a Coxeter matroid for W_0 and P_0. We shall denote $P_0 = P(\mathcal{M})$; this group is obviously defined up to conjugacy in $W(\mathcal{M})$. Denote by $\mathcal{M}^\circ$ the matroid for W_0 and P_0 determined by Δ_0. Obviously $W(\mathcal{M}^\circ) = W(\mathcal{M})$.

It is natural to call two Coxeter matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ *isomorphic* if and there is an isomorphism from $W_1 = W(\mathcal{M}_1)$ onto $W_2 = W(\mathcal{M}_2)$ which sends $P_1 = P(\mathcal{M}_1)$ onto $P_2 = P(\mathcal{M}_2)$ and the induced map $W_1/P_1 \longrightarrow W_2/P_2$ maps $\mathcal{M}_1^\circ$ onto $\mathcal{M}_2^\circ$.

Dimension of the matroid polytope. One useful application of the concept of the exchange group of a Coxeter matroid is the following simple formula for the dimension of the matroid polytope.

Recall that the *rank* of a finite reflection group W is defined as the dimension of the vector subspace generated by the roots of W [**H**, p. 9]. Every finite Coxeter group W is a finite reflection group [**H**, Theorem 6.4] and its rank equals to the number of Coxeter generators of W (see, for example, Theorem 1.9 and Proposition 5.6 in [**H**]).

Theorem 7 *Let $\mathcal{M}$ be a Coxeter matroid and Δ its matroid polytope. Then* $\dim \Delta(\mathcal{M})$ *equals the rank of $W(\mathcal{M})$ as a Coxeter group.*

Proof. Let V be a vector space containing Δ. As has been discussed at the beginning of the paper, we may assume that W acts on V as reflection group. Consider the set of all mirrors of reflections in $W = W(\mathcal{M})$ and take for every mirror Σ two vectors of length 1 normal to Σ. The resulting system Φ of vectors is a root system for the reflection group W (see [**H**, Section 1.2]). Consider now, for any two adjacent vertices α and β of Δ, the mirror $\Sigma_{\alpha,\beta}$ of symmetry of the edge $[\alpha, \beta]$. Let Y denote the intersection of all hyperplanes $\Sigma_{\alpha,\beta}$. Then Y is the orthogonal complement in V of the vector subspace X spanned by the edges of Δ. Obviously $\dim \Delta = \dim X$ and $\dim Y = \dim V - \dim X = \dim V - \dim \Delta$.

On the other hand, since the group W is generated by the reflections $s_{\alpha,\beta}$ in the mirrors $\Sigma_{\alpha,\beta}$, Y is the fixed point subset of W and is contained in all mirrors of reflections in W. Therefore $Y = R^\perp$ is the othogonal complement to the subspace R spanned by the root system Φ. We obviously have $\dim Y = \dim V - \dim R$ and, as the result, $\dim \Delta = \dim R = \mathrm{rank} W$. □

4 Exchange groups of ordinary matroids

Now we wish to apply the concept of the exchange group of a Coxeter matroid in the classical situation of ordinary matroids. In this case adjacency of vertices of the matroid polytope (Theorem 6) has a very elementary interpretation.

Matroids and their polytopes. Let $\mathcal{B}$ be the collection of bases of a matroid of rank k on the set $[n]$. The Basis Exchange Axiom for matroids is that for any two distinct bases A and B in $\mathcal{B}$ and an element $a \in A \smallsetminus B$ there is an element $b \in B \smallsetminus A$ such that the set $A \smallsetminus \{a\} \cup \{b\}$ belongs to $\mathcal{B}$. This axiom is is one of the many equivalent definitions of matroid, see any book on matroid theory or [**BGW2**]. We say that the basis $A \setminus \{a\} \cup \{b\}$ is obtained from A by a transposition (or *elementary exchange*) (a,b).

The matroid $\mathcal{B}$ is a Coxeter matroid for the group $W = Sym_n$ and its maximal parabolic subgroup P_k, the stabilizer of $[k]$ in W. It is known that any Coxeter matroid has a matroid polytope, Δ [**GS2**, **SVZ**, **ZS**]. Now we will show that any matroid polytope Δ with $W(\Delta) = Sym_n$ and the stabiliser of a vertex conjugate to P_k must be the basis matroid polytope, up to affine transformation. Let V be the real Euclidean space containing Δ and U the space of fixed points of $W = W(\Delta)$. The orbit of W in V containing the vertices ofΔ is mapped isometrically by the canonical orthogonal projection of V onto $U^\perp$, so we can assume without loss of generality that $U = 0$ and V is the space of canonical reflection representation for W. In particular, we can imbed V into the space $\mathbb{R}^n$ of the permutation representation of Sym_n (Sym_n acts on $\mathbb{R}^n$ by permuting the basis vectors $\epsilon_1, \dots, \epsilon_n$). We can assume without loss of generality that P_k fixes some vertex δ of Δ. Since P_k acts transitively on the sets $\{1, \dots, k\}$ and $\{k+1, \dots, n\}$, δ has the coordinates $(x, \dots, x, y, \dots, y)$ where x occurs k times and y occurs $n-k$ times. We can translate Δ through the vector $-y(\epsilon_{k+1} + \cdots \epsilon_n)$ and replace the basis vectors ϵ_i by $x\epsilon_i$. After these transformations the point δ has the form $(1, \dots, 1, 0, \dots, 0)$. Since the involutions in $W(\Delta)$ completely determine the matroid, it is clear that, up to perhaps further permutation of coordinate axes (within $[k]$ and $[n] \setminus [k]$), Δ is now the basis matroid polytope.

Historically the following theorem was one of the first results in the theory of Coxeter matroids. It is due to I. M. Gelfand, M. Goresky, R. MacPherson and V. V. Serganova [**GGMS**]. It is a special case of Theorem 6.

Theorem 8 [**GGMS**, Theorem 4.1] *The points δ_A, $A \in \mathcal{B}$, form the vertex set of Δ. Two vertices δ_A, δ_B are adjacent if and only if the bases A and B of the matroid can be obtained from each other by a transposition.*

Transposition graph. Let T be an arbitrary set of transpositions in the symmetric group Sym_n. Denote by $W(T)$ the group generated by T. We shall associate with T the graph $\Gamma = \Gamma(T)$ constructed as follows. Vertices of Γ are elements in $[n]$ and two vertices a and b are connected by a (non-oriented) edge if and only if the transposition (a,b) belongs to T. Now let $[n] = S_1 \sqcup \cdots \sqcup S_d$ be the partition of $[n]$ into the vertex sets of connected components of Γ and $T = T_1 \sqcup \cdots \sqcup T_d$ the corresponding partition of the set of edges.

Lemma 9 *If the graph Γ is connected then the set of transpositions T generates the symmetric group Sym_n.*

Proof. We shall use induction on n, the result being trivial for $n = 1$ or 2. Assume now that $n > 2$. After an appropriate renumbering of elements in $[n]$ we can assume that $\{n-1, n\}$ is an edge of Γ and that the restriction Γ' of the graph Γ to the set $[n-1]$ is connected. Let T' be the set of transpositions in T corresponding to edges in Γ'.

By the inductive assumption, the set T' generates the symetric group Sym_{n-1}. Therefore the group $W(T)$ contains the transpositions $(1,2), \dots, (n-2, n-1)$, and, since $W(T)$ also contains the transposition $(n-1, n)$, $W(T) = Sym_n$. $\square$

As an immediate corollary, we have the following description of groups generated by transpositions.

Theorem 10 *In the notation above, the group $W(T)$ generated by a set of transpositions T equals*

$$W(T) = Sym(S_1) \times \cdots \times Sym(S_d).$$

Components of matroids and the transposition graph. Let $\mathcal{B}_i$ be matroids on the pairwise disjoint finite sets E_i, $i = 1, 2, \ldots, d$. The collection of subsets

$$\mathcal{B} = \{\, B_1 \cup \cdots \cup B_d \mid B_i \in \mathcal{B}_i \,\}$$

is obviously a matroid on the set $E = E_1 \cup \cdots \cup E_d$. It is called the *direct sum* of the matroids $\mathcal{B}_i$ and denoted

$$\mathcal{B} = \mathcal{B}_i \oplus \cdots \oplus \mathcal{B}_d.$$

If the direct summands $\mathcal{B}_i$ are themselves *connected*, or indecomposable under direct sum, then the subsets E_i of E are called the *components* of the matroid $\mathcal{B}$. It is well-known in matroid theory that the components are uniquely determined by the matroid, and are, in fact, the equivalence classes of the equivalence relation $\sim$ defined on E by $x \sim y$ if and only if x and y are equal or lie in a common circuit of $\mathcal{B}$, see, for example, [**Wh**], Sections 3.4-3.5.

Given a matroid $\mathcal{B}$ on $[n]$, consider the set $T = T(\mathcal{B})$ of all elementary exchanges in $\mathcal{B}$. Denote by $W(\mathcal{B})$ the subgroup $W(T)$ of Sym_n generated by T and by $\Gamma(\mathcal{B})$ the graph $\Gamma(T)$. The graph $\Gamma(\mathcal{B})$ will be called the *transposition graph*, and the group $W(\mathcal{B})$ the *exchange group* of the matroid $\mathcal{B}$.

Theorem 11 *The vertex sets of the connected components of the transposition graph $\Gamma(\mathcal{B})$ are precisely the components of the matroid $\mathcal{B}$.*

Proof. It suffices to prove that the equivalence classes of $\sim$ are the connected components of the transposition graph of $\Gamma(\mathcal{B})$. In fact, we will prove a stronger statement: for $x, y \in E = [n]$, $x \sim y$, with $x \neq y$, if and only if $\{x, y\}$ is an edge of $\Gamma(\mathcal{B})$. Let $x \sim y, x \neq y$. Then there exists a circuit C containing both x and y. Extend the independent set $C \setminus \{x\}$ to a basis B. The unique circuit contained in $B \cup \{x\}$ is obviously C, and it follows that $B \setminus \{y\} \cup \{x\}$ contains no circuit, and hence is also a basis. Thus x and y are related by an elementary exchange, and therefore joined by an edge in $\Gamma(\mathcal{B})$. Conversely, if x and y are related by an elementary exchange, say between B and $B \setminus \{y\} \cup \{x\}$, then y must be in the unique circuit contained in $B \cup \{x\}$, showing that $x \sim y$. □

Corollary 12 *A matroid is connected if and only if its transposition graph is connected.*

Corollary 13 *Each component of the transposition graph of a matroid is a complete graph.*

Corollary 14 *If n_i is the cardinality of the i-th component of the matroid $\mathcal{B}$, then*

$$W(\mathcal{B}) \cong Sym_{n_1} \times \cdots \times Sym_{n_d}.$$

Corollary 15 *Let $\mathcal{B}$ be a matroid on the set $[n]$ and d the number of connected components of $\mathcal{B}$. Then the dimension of the basis matroid polytope associated with $\mathcal{B}$ is given by the formula*

$$\dim \Delta_{\mathcal{B}} = n - d.$$

Proof. We know that $W(\mathcal{B}) = Sym_{n_1} \times \cdots \times Sym_{n_d}$ where n_i is the cardinality of the i-th component of the matroid $\mathcal{B}$. The symmetric group Sym_k is generated, as a Coxeter group, by $k-1$ transpositions, hence $W(\mathcal{B})$ is generated, as a Coxeter group, by $n_1 - 1 + \cdots + n_d - 1 = n - d$ transpositions. By Theorem 7, $\dim \Delta_{\mathcal{B}}$ is the rank of $W(\mathcal{B})$, which is $n - d$. □

Note that a trivial component (loop or isthmus) contributes only trivially to 14, and does not affect $\Delta_{\mathcal{B}}$, up to isometry.

2-dimensional faces of matroid polytopes. The portion of the following result which relates to the basis matroid polytope is implicit in [**M**], and somewhat more explicit in [**BKL**]. The full proof, by a different method, was also given in [**C**].

Theorem 16 *Let Δ be an independent-set matroid polytope, and Δ' a basis matroid polytope. The 2-dimensional faces of Δ are equilateral triangles, isoceles right triangles, squares, or rectangles whose side lengths are in the ratio of $\sqrt{2}$. The 2-dimensional faces of Δ' are equilateral triangles or squares.*

Proof. Notice first that it immediately follows from Theorem 8 that all edges of Δ' are orthogonal to a mirror of reflection which transposes two standard basis elements, say ϵ_i and ϵ_j, where the two bases involved are related by an elementary exchange of i and j. Hence the edge is parallel to the root $\epsilon_i - \epsilon_j$ of A_n and has length $\sqrt{2}$. Similarly, using 4 and the Gelfand-Serganova Theorem for symplectic matroids, all edges of Δ are orthogonal to a mirror of reflectional symmetry of the n-cube, hence parallel to a root $\epsilon_i - \epsilon_j$ or ϵ_l of BC_n. This means that two independent sets I_1 and I_2 correspond to vertices joined by an edge of Δ only if they are related by an elementary exchange as above (except that I_1 and I_2 need not be bases now; they may possibly be equicardinal independent sets of lower cardinality), or I_1 and I_2 are related by containment with cardinality differing by exactly 1. These two types of edges have lengths $\sqrt{2}$ and 1, respectively. Let Γ be a 2-dimensional face of Δ. Consideration of the scalar products of all possible vectors of the forms $\epsilon_i - \epsilon_j$ and ϵ_l shows that the possible angles between the edges of Γ are $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, or $3\pi/4$. Furthermore, $\pi/4$ or $3\pi/4$ occur only between an edge of length 1 and one of length $\sqrt{2}$, whereas $\pi/3$ or $2\pi/3$ occur only between two edges of length $\sqrt{2}$. However, $2\pi/3$ and $3\pi/4$ cannot actually occur. Suppose that consecutive vertices of Γ corresponding to independent sets I_1, I_2, I_3 determine an angle of $2\pi/3$. But the correct scalar product for this angle occurs only when I_1 and I_2 are related by an elementary exchange, say (i, j), and I_2 and I_3 by (j, l), with exactly one element common to the two transpositions. But if $j \in I_2$, then $I_1 = I_2 \setminus \{j\} \cup \{i\}$ and $I_3 = I_2 \setminus \{j\} \cup \{l\}$, and otherwise, $I_1 = I_2 \setminus \{i\} \cup \{j\}$ and $I_3 = I_2 \setminus \{l\} \cup \{j\}$. But in either case, I_1 and I_3 are now also related by an elementary exchange, hence the angle must have been $\pi/3$ instead of $2\pi/3$. A similar argument rules out $3\pi/4$; instead of two elementary exchanges, one elementary exchange and one containment must be involved.

Therefore, only the following possibilities remain for Γ: an isoceles right triangle, an equilateral triangle, a rectangle, and a square. For the basis matroid polytope, only the equilateral triangle and square remain.

□

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