Widening the Razor-Thin Edge of Chaos Into a Musical Highway: Connecting Chaotic Maps to Digital Waveguides

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ABSTRACT

For the purpose of creating new musical instruments, chaotic dynamical systems can be simulated in real time to synthesize complex sounds. This work investigates a series of discrete-time chaotic maps, which have the potential to generate intriguing sounds when they are adjusted to be on the edge of chaos. With these chaotic maps as studied historically, the edge of chaos tends to be razor-thin, which can make it difficult to employ them for making new musical instruments. The authors therefore suggest connecting chaotic maps with digital waveguides, which (1) make it easier to synthesize harmonic tones and (2) make it harder to fall off of the edge of chaos while playing a musical instrument. The authors argue therefore that this technique widens the razor-thin edge of chaos into a musical highway.

Author Keywords

Chaos, edge of chaos, nonlinear, digital waveguide, chaotic maps, sound synthesis

CCS Concepts

•Applied computing → Performing arts; Sound and music computing; •Mathematics of computing → *Nonlinear equations;*

1. INTRODUCTION

1.1 Background

Since the late 80s and early 90s, computer music researchers have studied chaos [9]. Chaotic dynamical systems have been explored in greater detail by di Scipio and Truax [14, 4, 5]. Essl reinvestigated this idea in the following framework [7, 6]. For example, with a simple chaotic map, the next sample $\mathbf{v}_n$ is computed from the prior sample $\mathbf{v}_{n-1}$ by way of a nonlinear function ϕ:

$$\mathbf{v}_n = \phi(\mathbf{v}_{n-1}). \tag{1}$$

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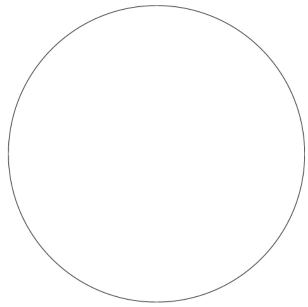

Figure 1: A sinusoidal oscillator traces out a series of points that form a circle in (x, y)-space.

2. TWO-DIMENSIONAL MAPS

2.1 Introduction

With two-dimensional chaotic maps, the state vector $\mathbf{v}_n$ can be decomposed into x_n and y_n where $\mathbf{v}_n = [x_n\ y_n]$. One nice thing about these maps is that the points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), ... \tag{2}$$

can be plotted in phase space to observe the behavior.

The sinusoidal oscillator can serve as a nice introductory example. It is not a chaotic map, but its state can be represented in (x, y)-space. As shown in Figure 1, the state traces out a series of points that form a circle, with the x-coordinate or the y-coordinate used alone to generate a sinusoidal output.

2.2 Peter de Jong Chaotic Map

In contrast, chaotic maps trace out much more complex series of points. Consider the Peter de Jong chaotic map as specified by (3) and (4):

$$x_n = \sin(ay_{n-1}) - \cos(bx_{n-1}) \tag{3}$$

$$y_n = \sin(cx_{n-1}) - \cos(dy_{n-1}). \tag{4}$$

The series of points traced out by the Peter de Jong chaotic map contain a lot of detailed structure and depend in detail on the a, b, c, and d constants. One example shape for $a = 1.641$, $b = 1.902$, $c = 0.316$, and $d = 1.525$ is shown in Figure 2.[1]

[1]See `http://paulbourke.net/fractals/peterdejong/`

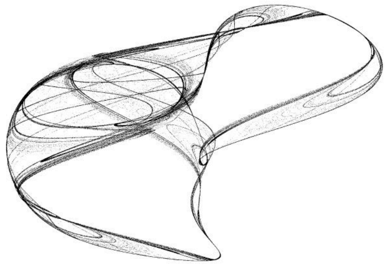

Figure 2: The Peter de Jong chaotic map traces out complicated shapes in (x, y)-space, with the exact details depending on the particular values of a, b, c, and d.

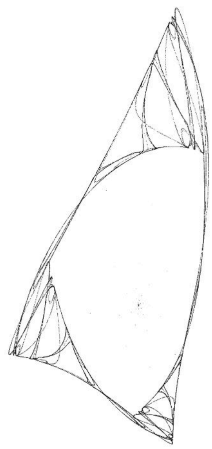

Figure 3: The Tinkerbell chaotic map can generate a complex shape such as this one when its state vector is plotted in (x, y)-space over time.

2.3 Tinkerbell Map

The Tinkerbell map can also be represented using two recursive equations with four parameters a, b, c, and d as follows:

$$x_n = x_{n-1}^2 - y_{n-1}^2 + ax_{n-1} + by_{n-1} \tag{5}$$

$$y_n = 2x_{n-1}y_{n-1} + cx_{n-1} + dy_{n-1}. \tag{6}$$

The Tinkerbell map also generates a lot of structure. One example series of points for $a = -0.3$, $b = -0.6$, $c = 2.0$, and $d = -0.27$ [2] is shown in Figure 3.

2.4 Circle Map

The circle map is a one-dimensional map in which the state variable θ_n represents the phase of an oscillator [7, 6]. For this reason, each time the next state is calculated, it is taken modulo 2π:

$$x_n = \Big(x_{n-1} + \Omega - \frac{K}{2\pi}\sin(x_{n-1})\Big) \ \%\ 2\pi, \tag{7}$$

and the output signal r_n is calculated from the phase

$$r_n = \sin(x_n). \tag{8}$$

Therefore, the coefficient K adjusts the nonlinear qualities. If $K = 0$, this term vanishes and then Ω directly controls the linear advancement of the phase for a sinusoidal oscillator with fundamental frequency $\Omega f_S/(2\pi)$, where f_S is the sampling rate. Otherwise, if $K \neq 0$, then Ω still tends to affect the pitch, but in a more complicated way. For instance, if $K \approx 0$, then the oscillator is tonal and has a bright sound. In contrast, if K is much larger, then the circle map sounds noisy, as the phase advances in a seemingly random (but still deterministic) way. For intermediate values of K, the algorithm may get stuck at a certain phase angle.

Because the circle map only has one state variable, its trajectories are not displayed for comparison in this paper. More details can be found in papers by Essl [7, 6].

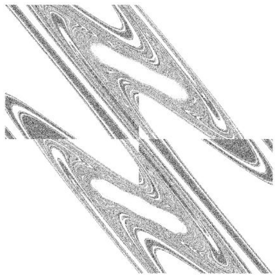

Figure 4: The Standard map can generate more densely populated shapes when its state vector is plotted as a series of points in (p, θ)-space.

2.5 Standard Map

The standard map is another map that was investigated. For a given parameter K, the standard map traverses a wide range of points in phase space, but the structure is oddly specific as shown in Figure 4. The chaotic map is specified by the following equations [3]:

$$p_n = \Big(p_{n-1} + K\sin(\theta_{n-1})\Big) \ \%\ 2\pi \tag{9}$$

$$\theta_n = \Big(\theta_{n-1} + p_n\Big) \ \%\ 2\pi. \tag{10}$$

2.6 Summary

With such models, the sound tends to be most interesting right on the *edge of chaos*, just as the parameters change the dynamic behavior from being very predictable to being hard for humans to predict intuitively. Through gradual adjustment of the parameters describing the chaotic map function, one can explore timbres that evolve in complex ways. However, since the edge of chaos tends to be razor thin, it can be challenging to avoid falling off of it when making music.

It can also be hard to achieve tones that bear much resemblance to traditional musical tones. This is because low-order models generally do not have enough degrees of freedom/memory to realize a series of resonance frequencies.

Rodet recognized this drawback in his suggestion that chaotic models of acoustic musical instruments [8] would involve a Chua-style nonlinearity [10] connected to a delay line [11]. However, he did not describe a specific extension for integrating a delay line with chaotic maps. This is the concept explored by the present paper.

3. INTERFACING DIGITAL WAVEGUIDES WITH CHAOTIC MAPS

With great ease, any chaotic map in the form of (1) can be interfaced with a digital waveguide [13]. The composite structure still has the potential to oscillate chaotically, and it retains many of the same behaviors as with (1), yet it also has the potential to readily make harmonic tones.

To interface a digital waveguide with a chaotic map, one can simply make (1) refer to an earlier time step $\mathbf{v}_{n-L}$ as

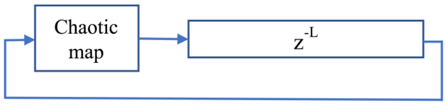

Figure 5: Block diagram illustrating a chaotic map interfaced with a digital waveguide as in (11).

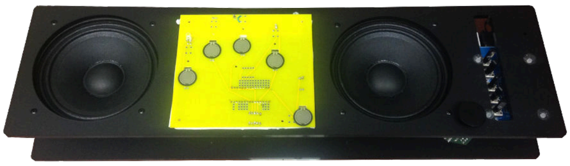

Figure 6: An embedded acoustic instrument for controlling the Peter De Jong chaotic map as extended with a digital waveguide.

follows:

$$\mathbf{v}_n = \phi(\mathbf{v}_{n-L}). \tag{11}$$

In other words, the chaotic map is applied through a delay line instead through a single-sample delay.

Then, for certain configurations, (11) has the potential to produce harmonic tones. Often the tones will tend to be centered around f_S/L (or subharmonics of this if the state vector is multidimensional), as is the case with many related digital waveguide structures [13].

Exploring timbre spaces with (11) is quite simple but does require a concerted effort. One chooses a chaotic map,[2] feeds its output into a delay line, and feeds the output of the delay line back into the input of the chaotic map as in Figure 5. Then, one finds a suitable way to adjust the parameters of ϕ in order to control the timbre.

3.1 Peter de Jong Map

For example, the Peter De Jong chaotic map from Section 2.2 can be fed back into itself (actually through two parallel delay lines instead of one) by way of four parameters a, b, c, and d:

$$x_n = \sin(ay_{n-L}) - \cos(bx_{n-L}) \tag{12}$$

$$y_n = \sin(cx_{n-L}) - \cos(dy_{n-L}). \tag{13}$$

If a, b, c, and d are controlled in real time by four pressure sensors, and if the delay line length L is adjustable according to a knob, a musical instrument can be realized such as shown in Figure 6. The sound of this instrument is presented in the demo video provided with this paper: `https://goo.gl/ZevfrW` In addition, another electroacoustic miniature made with this instrument entitled *Romp in Chaos* is presented in audio and score formats at the following links: `goo.gl/hGkt9F` and `goo.gl/1U5rge`

3.2 Tinkerbell Map

The Tinkerbell map from Section 2.3 can also be connected to two digital waveguides by way of four parameters a, b, c, and d. This map is sensitive to parameter changes in real-time, so a clipping nonlinearity can be introduced to prevent instability. In addition, a unit variance noise signal N_n can be scaled and added in to help prevent the system from getting stuck in a fixed point at the origin:

$$x_n = clip\Big(-1, 1, lowpass(x_{n-L}^2 - y_{n-L}^2 + ax_{n-L} + by_{n-L})\Big) + 0.000001N_n$$

[2] `https://en.wikipedia.org/wiki/List_of_chaotic_maps`

Figure 7: An embedded acoustic instrument for controlling a pair of coupled circle maps as extended with a digital waveguide.

$$y_n = clip\Big(-1, 1, lowpass(2x_{n-L}y_{n-L} + cx_{n-L} + dy_{n-L})\Big). + 0.000001N_n$$

Like the De Jong example, a, b, c, and d are controlled by pressure sensors. A fifth pressure sensor is mapped to the cutoff frequency of a lowpass filter in the feedback loop, which provides additional timbral variation.

In this test implementation, a, b, c, and d parameters start at non-zero values and are mapped inversely to pressure sensor values. That is, increased pressure on the pressure sensors move all parameters closer to zero. This yields a different interaction paradigm, one in which the instrument is self-actuating and judicious touch gestures are used for damping purposes [1].

The sound of this instrument is presented in the demo audio provided with this paper, and it can also be listened to via the following link: `https://goo.gl/ysQouw`

3.3 Coupled Circle Maps

Because the basic circle map from Section 2.4 is relatively simple, it was not investigated in detail. However, it was interesting to couple two circle maps together, each with a separately controlled delay line length L_1/L_2 and cross-coupling parameters K_{crossx} and K_{crossy}:

$$x_n = \Big(x_{n-L} + \Omega - \frac{K}{2\pi}\sin(x_{n-L_1}) - \frac{K_{crossx}}{2\pi}\sin(y_{n-L_2})\Big)\%2\pi$$

$$y_n = \Big(y_{n-L} + \Omega - \frac{K}{2\pi}\sin(y_{n-L_2}) - \frac{K_{crossy}}{2\pi}\sin(x_{n-L_1})\Big)\%2\pi.$$

The output signals r_n and s_n were calculated from the phase using $r_n = \sin(x_n)$ and $s_n = \sin(y_n)$.

For intermediately tuned values, K_{crossx} caused the oscillator x_n to start to lock onto the oscillator y_n, and vice versa for K_{crossy}. An electro-acoustic miniature entitled *A Sound Walk Through Chaos Forest* was composed for two coupled circle maps. It was performed using the embedded instrument shown in Figure 7. The audio and score are available here: `goo.gl/DQkHZR` and `goo.gl/txKPPF`

3.4 "Double Standard" Map

Relative to the other chaotic maps selected, it seemed to be more challenging to use the Standard map from Section 2.5 musically. It tended to have a noisy character, as suggested by the fact that Figure 4 appears to be "more random," in the sense that its points more densely populate the phase plane (see Figure 4) than for the other chaotic maps described in this paper. Therefore, in order to try to use the Standard map musically, the authors decided to pull out all of the stops. Besides using *two* Standard maps, each augmented with a waveguide, the p-variables of these Standard maps were coupled together using a waveguide junction parameter K_{mix}:

$$p_{1,n} = \Big(p_{1,n-L_{p1}} + K_1\sin(\theta_{1,n-L_{\theta 1}}) - K_{mix}(p_{1,n-L_{p1}} + p_{2,n-L_{p2}})\Big)\ \%\ 2\pi,$$

$$p_{2,n} = \Big(p_{2,n-L_{p2}} + K_2\sin(\theta_{2,n-L_{\theta 2}}) - K_{mix}(p_{1,n-L_{p1}} + p_{2,n-L_{p2}})\Big)\%2\pi,$$

$$\theta_{1,n} = \Big(\theta_{1,n-L_{\theta 1}} + p_{1,n}\Big)\%2\pi,\ \theta_{2,n} = \Big(\theta_{2,n-L_{\theta 2}} + p_{2,n}\Big)\%2\pi.$$

Initial tests showed that adjusting the K_1 and K_2 parameters higher than 0.1 radians began to cause the resulting timbre to start to gradually start to sound somewhat noisy. Pushing the K_1 or K_2 parameter into the range of whole-numbers generated more and more significant noise. Chordal drones could be produced through creative adjustment of the delay line lengths L_{p1}, L_{p2}, $L_{\theta 1}$, and $L_{\theta 2}$. For example, setting L_{p1} and $L_{\theta 1}$ at a difference of three to five milliseconds from each other could produce a gradually evolving beating effect. Integer multiples of a time value set across waveguide delay lengths produced octave drones, while non-integer multiples produced more inharmonic pitched results.

An alternate approach could be taken to use long delay lengths to create rhythmic pulsing patterns—for example, this could be achieved by setting K_1 and K_2 to the range between 0.01 and 0.05 and making any of the delay lines longer than 100 milliseconds. Generally speaking, the K_{mix} parameter was especially fruitful. It could be set to zero to allow the two Standard maps to evolve separately, or it could be increased away from zero to cause their sounds to intermix with each other, which was how the "Double Standard" name for this algorithm was selected.

4. PRACTICAL CONSIDERATIONS

4.1 Avoiding Instability

For some chaotic maps, configurations may be discovered that can asymptotically go unstable. One way to prevent this is if ϕ is bounded [12]. For example, the output of (12) and (13) can never be greater than 2 nor less than -2. This is because the outputs of sin and cos are bounded between -1 and 1. Accordingly, x_n or y_n for the de Jong chaotic map are prevented from growing without bound and becoming too loud to listen to. Therefore, it is recommend that, when selecting a function for ϕ, one strongly considers choosing ϕ that are bounded to avoid instability.

In some cases, one might insert a clipping function (or other limiting nonlinearity) into the feedback loop somewhere to help prevent the energy from spiraling out of control. This worked well with the Tinkerbell map described in Section 3.2.

4.2 Potential of Getting Stuck

In this work, it was noticed only in the case of the Tinkerbell map that the algorithm had the potential to get stuck at a fixed point. This was observed to happen when all parameters were briefly set to zero simultaneously. Therefore, in Section 3.2 above, a small amount of noise was added in the recursion to help prevent the Tinkerbell map from getting stuck at the origin. As more and more chaotic maps are explored, this the issue of getting stuck will likely resurface; however, it appears to not be quite as widespread as the authors originally believed.

5. CONCLUSIONS

The authors argue that (11) is taking the razor-thin edge of chaos and transforming it into a musical highway. More experiments are forthcoming, but the results so far suggest that a wild world of complex dynamical behaviors are waiting to be discovered and used for music composition. The following points have already been demonstrated:

- By using music controllers to adjust the algorithm parameters in real-time, intriguing musical instruments can be created.
- Through the introduction of the digital waveguides in (11), it becomes easier to generate harmonic tones using a chaotic sound synthesis technique.
- By gradually adjusting model parameters to interpolate between the periodic and noisy/chaotic regimes, the *edge of chaos* can be explored more broadly.

Future theoretical work will describe in more detail how a chaotic map as in (11) can be interpreted as a digital waveguide termination, particularly if the wave impedance of the digital waveguide is specified [13]. Future work will also address the stability properties of (11) in more detail.

As one reviewer noted, it could also be interesting to someday consider using chaotic dynamical systems not to construct oscillators themselves as in this paper, but rather to control the amplitude envelope, pitch or other parameters of various unit generators. This could be another fruitful area for future investigation.

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