INSTITUTO SUPERIOR TÉCNICO

PROJECTO MEFT

# Linearized General Relativity in Hyperboloidal Coordinates

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# 1 Introduction

When studying gravitational waves, one has to keep in mind that gravitational wave sources (like black hole mergers, neutron stars, etc.) are usually located billions of light years away from us. From the point of view of an observer on Earth, this distance is so vast that it can be approximated as being at infinity. This distance, however, is not sufficiently far away to require accounting for the cosmological constant, allowing us to model spacetime as asymptotically flat. Thus, we are interested in studying the behavior of the gravitational fields at null infinity, $\mathscr{I}$.

To do so, we must perform a conformal compactification of the spacetime, which brings $\mathscr{I}$ to a finite distance on our computational grid. This is done by working in the hyperboloidal coordinate system.

## 1.1 Hyperboloidal Coordinates

The hyperboloidal coordinate system, as previously mentioned, maps our previously unbonded domain to a finite one. This is done by introducing new time and radial coordinates $(t, r)$, which are related to the spherical coordinates of Minkowski spacetime $(T, R)$ by the following transformations:

$$t = T - H(R) \quad R = \tfrac{r}{\Omega(r)} \ , \tag{1}$$

where $H(R)$ is called the height function and $\Omega(r)$ is called the compress function. Throughout this work, we will use the following choices for these functions:

$$H(R) = \sqrt{S^2 + R^2} \quad \Omega(r) = \tfrac{1}{2}\left(1 - \tfrac{r^2}{S^2}\right) \ , \tag{2}$$

where $S$ is a constant that determines the size of the compactified domain. The choice of $S$ is arbitrary as it simply defines what point $\mathscr{I}$ will be maped to. In this work, we will use $S = 1$.

This coordinate transformation gives rize to the following Jacobian matrix:

$$\left(J^{Hyp}\right)_{\alpha'}^{\ \beta} = \begin{pmatrix} 1 & -H'(r) & 0 & 0 \\ 0 & \frac{L(r)}{\Omega^2(r)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{3}$$

where $H'(r)$ denotes the derivative of the height function with respect to $R$ written as a function of $r$, and $L(r)$ is defined as

$$L(r) \equiv \Omega(r) - r\,\partial_r\Omega(r) \ . \tag{4}$$

For the previously chosen height and compress functions, we have

$$H'(r) = \tfrac{2\,r\,S}{S^2+r^2} \quad L(r) = \tfrac{1}{2}\left(1 + \tfrac{r^2}{S^2}\right) \ . \tag{5}$$

## 1.2 Computational Setup

This work is a continuation of my previous work on numerical relativity. As such, the framework used here is the same as the one used in that work, with the adition of truncation error matching for the derivatives, interpolation at the boundaries and the Evans Method for regularization at the origin. The description of the base code can be found in [], and this section will elaborate on the upgrades made.

### 1.2.1 Truncation Error Matching

Truncation error matching is a technique used to improve the accuracy of the numerical solution by matching the truncation error of the finite difference scheme used on the boundaries to the truncation error of the one used in the interior of our computational domain. This is done by using a one sided finite difference scheme on the boundaries such that the leading order error term is the same as the one used in the interior.

In our framework, we use the following second order finite difference scheme for the first derivative of a field $\psi$ at an interior point $i$ (where the leading order error term was written explicitly):

$$\psi_i' = \frac{\psi_{i+1} - \psi_{i-1}}{2h} - \frac{h^2}{6}\psi_i''' + \dots \, , \tag{6}$$

where $\psi_{i+1}$ and $\psi_{i-1}$ are the values of the field $\psi$ at the points $i+1$ and $i-1$ respectively, and $h$ is the grid spacing.

To match this leading order term of the error, we use the following one sided finite difference scheme for the derivative of $f$ at the left and right boundary points respectively []:

$$\psi_i' = \frac{\psi_{i+3} - 4\psi_{i+2} + 7\psi_{i+1} - 4\psi_i}{2h} - \frac{h^2}{6}\psi_i''' + \dots \tag{7}$$

$$\psi_i' = \frac{4\psi_i - 7\psi_{i-1} + 4\psi_{i-2} - \psi_{i-3}}{2h} - \frac{h^2}{6}\psi_i''' + \dots \tag{8}$$

### 1.2.2 Interpolation at the Boundaries

Since we are interested in evolving the fields at null infinity, we must choose a boundary condition that allows the fields to freely propagate outwards. To do so, we use interpolation at the outer boundaries of our computational domain to fill the ghost points in those regions. This is done by using the following interpolation scheme:

$$\psi_i = 4\psi_{i-1} - 6\psi_{i-2} + 4\psi_{i-3} - \psi_{i-4} \, , \tag{9}$$

where $\psi_i$ is the value of the field at the ghost point $i$, and $\psi_{i-1}$, $\psi_{i-2}$, $\psi_{i-3}$ and $\psi_{i-4}$ are the values of the field at the points $i-1$, $i-2$, $i-3$ and $i-4$ respectively.

### 1.2.3 Evans Method

When dealing with operators like the Laplacian in spherical coordinates, we find some formal singularities which need to be removed in order for our code to work. To remove those singularities, we can apply the Evans Method. This method consists of rewriting the singular terms as a different differential operator, called the Evans operator, which can be evaluated at the grid points. The Evans operator is defined as

$$\partial_r\psi + \frac{p}{r}\psi = (p+1)\frac{d(r^p\psi)}{dr^{p+1}} \, , \tag{10}$$

where $p$ is a constant. This operator can be expressed in terms of the grid points as

$$(p+1)\frac{d(r^p\psi)}{dr^{p+1}} = (\tilde{D}\psi)_i = (p+1)\frac{r_{i+1}^p\psi_{i+1} - r_{i-1}^p\psi_{i-1}}{r_{i+1}^{p+1} - r_{i-1}^{p+1}} \, , \tag{11}$$

where the subscripts $i+1$ and $i-1$ denote the grid points $i+1$ and $i-1$ respectively.

## 2 Wave Equation in 1+1 Dimensions

We start by considering the wave equation in 1+1 dimensions

$$\Box\psi \equiv -\partial_T^2\psi + \partial_X^2\psi = 0\,. \tag{12}$$

From my previous work on the subject [], we concluded that we should use systems of equations that are first order both in space and in time, since the second order in space and first order in time scheme proved to be problematic. Thus, we proceed to do a first order reduction of the wave equation by defining $\Pi \equiv -\partial_T\psi$, obtaining

$$\begin{cases} \partial_T\psi = -\Pi \\ \partial_T\Pi = -\partial_{\underline{i}}\partial^{\underline{i}}\psi \end{cases}. \tag{13}$$

We can add an additional constraint $C_{\underline{i}} = \partial_{\underline{i}}\psi - \Phi_{\underline{i}} \overset{!}{=} 0$ to our system of equations, to which small violations could be allowed. Using the time derivative of this constraint as an evolution equation for $\Phi$ and expressing the small violation of this constraint as $\gamma_2 C_{\underline{i}}$, where $\gamma_2$ is a small parameter corresponding to the allowed violation, we get

$$\begin{cases} \partial_T\psi = -\Pi \\ \partial_T\Phi = -\partial_X\Pi + \gamma_2\,\partial_X\psi - \gamma_2\,\Phi \\ \partial_T\Pi = -\partial_X\Phi \end{cases}. \tag{14}$$

We then proceed to make a coordinate change from inertial Minkowski coordinates to hyperboloidal coordinates. Addicionally, even though we could allow for small violations of our constraint, we will not. Thus, we set $\gamma_2 = 0$, obtaining the final form of our system of equations:

$$\begin{cases} \partial_t\psi = -\Pi \\ \partial_t\Phi = \mathcal{A}\,\left(H'\,\partial_x\Phi + \partial_x\Pi\right) \\ \partial_t\Pi = \mathcal{A}\,\left(H'\,\partial_x\Pi + \partial_x\Phi\right) \end{cases}, \tag{15}$$

where we defined $\mathcal{A}(x) = \frac{\Omega^2(x)}{L(x)(H'^2(x)-1)}$, and dropped the explicit dependences on $x$ to simplify the notation.

We can now solve this system of equations using the aforementioned code, using truncation error matching for the derivatives on the boundaries and turning off the artificial dissipation on those points. Using that framework and giving the initial conditions

$$\psi(0,x) = A\,e^{-C\,x^2/2}\,,\quad \Phi(0,x) = -A\,C\,x\,\frac{\Omega^2(x)}{L(x)}\,e^{-C\,x^2/2}\quad \text{and}\quad \Pi(0,x) = 0\,, \tag{16}$$

with $A = 1.0$ and $C = 100$, we obtain the evolution present in figure 1. In that evolution, it is noticeable that after the wave leaves the grid, we are left with a permanent negative displacement on our field. Even though this result is counterintuitive, it was proven in [] to be correct.

Doing a norm convergence test (using the $L^2$ norm), we obtain a clean second order convergence during the whole evolution, as can be seen for $\psi$ in the left of figure 2. We can also see that these results stay promising up until $\mathscr{I}$ through the pointwise convergence at that point, represented in the right of figure 2.

## 3 Wave Equation in 3+1 Dimensions With Spherical Symmetry

Now, let us consider the wave equation in 3+1 dimensions with spherical symmetry

$$\Box\psi \equiv -\partial_T^2\psi + \frac{1}{R^2}\partial_R\left(R^2\partial_R\psi\right) = 0\,. \tag{17}$$

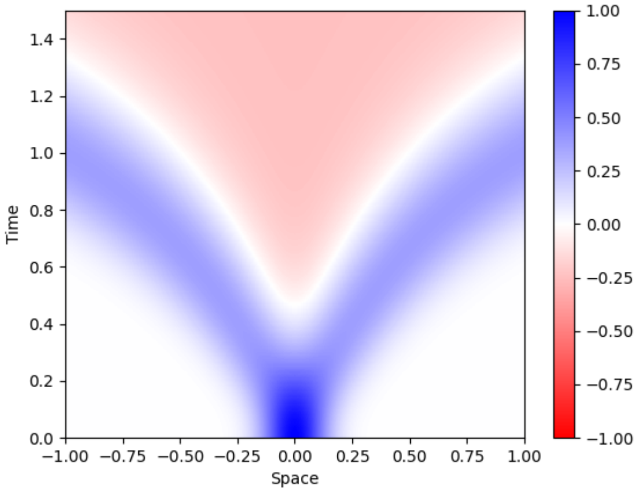


FIGURE 1: Evolution of the wave equation in 1+1 dimensions using hyperboloidal coordinates with the initial conditions given in equation (16).

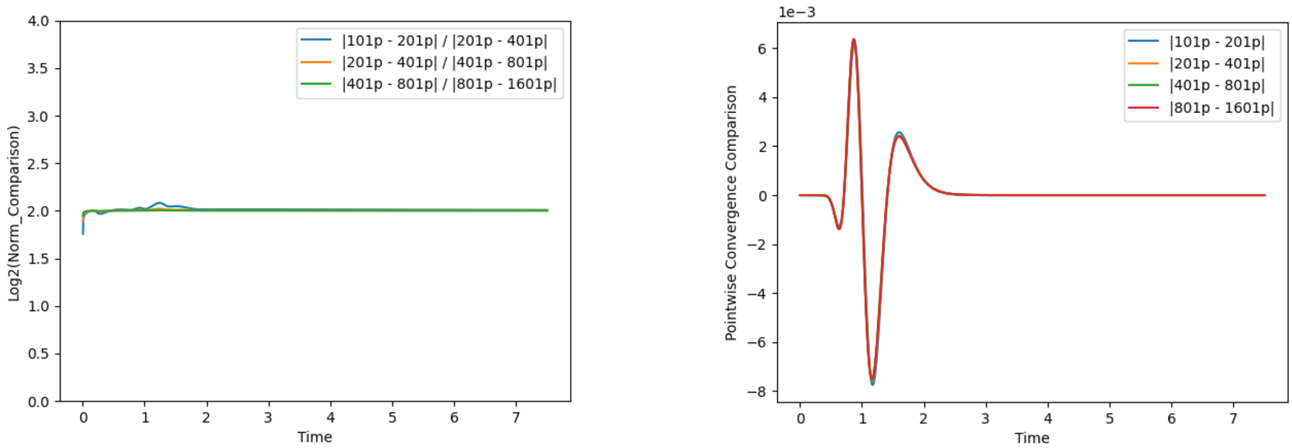


FIGURE 2: Main caption for all figures

It is important to note that, since $\psi$ is a solution to the wave equation in 3+1 dimensions with spherical symmetry, this field will decay at a rate of $1/R$. Since we want to know how our field behaves at $\mathscr{I}$, we will need to perform a rescaling of this field. For this, we define a new field $\Psi$ such that

$$\Psi \equiv \chi \psi \,,$$

where $\chi = \sqrt{1+R^2}$.

Applying that transformation, our wave equation becomes

$$\partial_T^2 \Psi = \partial_R^2 \Psi + \frac{2}{R(R^2+1)} \partial_R \Psi - \frac{3}{(R^2+1)^2} \Psi \,, \tag{18}$$

to which we can apply a first order reduction, obtaining

$$\begin{cases} \partial_T \Psi = -\Pi \\ \partial_T \Phi = -\partial_R \Pi + \gamma_2 \partial_R \Psi - \gamma_2 \Phi \\ \partial_T \Pi = -\partial_R \Phi - \frac{2}{R(R^2+1)} \Phi + \frac{3}{(R^2+1)^2} \Psi \end{cases} . \tag{19}$$

Doing a coordinate change from inertial Minkowski coordinates to hyperboloidal coordinates and setting $\gamma_2 = 0$, we get

$$\partial_T = \partial_t \quad \text{and} \quad \partial_R = -H'\partial_t + \tfrac{\Omega^2}{L}\partial_r \quad , \tag{20}$$

which is very similar to the results we obtained before.

Our equation then becomes

$$\begin{cases} \partial_T \Psi = -\Pi \\ \partial_T \Phi = \mathcal{B}\left((r^2+\Omega^2)^2 \left(H'\partial_r\Phi + \partial_r\Pi\right) + H'L\Omega\left(2r\Phi - 3\Omega\Psi + 2r^{-1}\Omega^2\Phi\right)\right) \\ \partial_T \Pi = \mathcal{B}\left((r^2+\Omega^2)^2 \left(\partial_r\Phi + H'\partial_r\Pi\right) + L\Omega\left(2r\Phi - 3\Omega\Psi + 2r^{-1}\Omega^2\Phi\right)\right) \end{cases} \quad , \tag{21}$$

where we defined $\mathcal{B} = \frac{\Omega^2}{L(H'^2-1)(r^2+\Omega^2)^2}$.

We can see that in our evolution equations for $\Phi$ and for $\Pi$, we have a term that is formally singular, to which we will need to apply Evans method. For that, we rewrite those evolution equations, obtaining

$$\begin{cases} \partial_T \Psi = -\Pi \\ \partial_T \Phi = \mathcal{B}\left((r^2+\Omega^2)^2 \left(H'\partial_r\Phi + \partial_r\Pi\right) + H'L\Omega\left(2r\Phi - 3\Omega\Psi - \Omega^2\partial_r\Phi\right) + H'L\Omega^3\left(\partial_r\Phi + 2r^{-1}\Phi\right)\right) \\ \partial_T \Pi = \mathcal{B}\left((r^2+\Omega^2)^2 \left(\partial_r\Phi + H'\partial_r\Pi\right) + L\Omega\left(2r\Phi - 3\Omega\Psi - \Omega^2\partial_r\Phi\right) + L\Omega^3\left(\partial_r\Phi + 2r^{-1}\Phi\right)\right) \end{cases} \quad , \tag{22}$$

where it is easy to see that we can apply the Evans method to the last terms in parenthesis.

Choosing the height and compress functions as

$$H = \sqrt{S^2 + R^2} \quad \text{and} \quad \Omega = \tfrac{1}{2}\left(1 - \tfrac{r^2}{S^2}\right) \quad , \tag{23}$$

we can calculate the boost function to be

$$H' = \frac{2rS}{S^2 + r^2} \ . \tag{24}$$

Additionally, our auxiliary function $L$ becomes

$$L = \frac{1}{2}\left(1 + \frac{r^2}{S^2}\right) \ . \tag{25}$$

Differently from before, this time we impose the parity of each field at the origin and do extrapolation at $\mathscr{I}$ (instead of truncation error matching). Using as initial data the following fields

$$\psi(0,r) = A\, e^{-C\,r^2/2} \ , \quad \Phi(0,r) = -A\,C\,r\,\tfrac{\Omega^2(r)}{L(r)}\, e^{-C\,r^2/2} \quad \text{and} \quad \Pi(0,r) = 0 \quad , \tag{26}$$

with $A = 1.0$ and $C = 100$, we obtain the evolution represented in the following figure:

Similarly to before, we obtain a clean second order convergence during the whole evolution, as can be seen in the norm convergence (represented in the following figure for $\psi$).

It is also important to note that we still have very good norm convergence at $\mathscr{I}$, as can be seen in the following figure:

These results could be further improved by doing truncation error matching of the Evans method at $\mathscr{I}$.

## 4 Cubic Wave Equation in 3+1 Dimensions With Spherical Symmetry

## 5 Conclusions