A fully labelled proof system for intuitionistic modal logics

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Abstract

In this paper we present a labelled sequent system for intuitionistic modal logics such that there is not only one, but two relation symbols appearing in sequents: one for the accessibility relation associated with the Kripke semantics for normal modal logics and one for the preorder relation associated with the Kripke semantics for intuitionistic logic. This puts our system in close correspondence with the standard birelational Kripke semantics for intuitionistic modal logics. As a consequence it can encompass a wider range of intuitionistic modal logics than existing labelled systems. We also show an internal cut elimination proof for our system.

1 Introduction

Since their introduction in the 1980s by Gabbay [Gab96], *labelled proof calculi* have been widely used by proof theorist to give sound, complete and cut-free deductive systems to a broad range of logics. Unlike so-called *internal calculi*, like hypersequents [Avr96], nested sequents [Kas94, Brü09, Pog09], 2-sequents [Mas92], or linear nested sequents [Lel15], labelled calculi have the advantage of being more uniform and being able to accommode a larger class of logics.

Standard labelled sequent calculi attach to every formla A a label x, witten as $x{:}A$, and additionally use *relational atoms* of the form xRy where R is a binary relation symbol. These calculi work best for logics with standard Kripke semantics, as in this case the relation R is used to encode the accessibility relation in the Kripke models, and the frame conditions corresponding to the desired logic can be directly encoded as inference rules. Prominent examples are classical modal logics and intuitionistic propositional logic, where, e.g., the frame condition of transitivity ($\forall xyz.\, xRy \wedge yRz \supset xRz$), can be straightforwardly translated into the inference rule

$$\mathsf{trans}\ \frac{\mathcal{R}, xRy, yRz, xRz, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, yRz, \Gamma \Longrightarrow \Delta} \tag{1}$$

where $\mathcal{R}$ stands for a set of relational atoms, and Γ and Δ for multi-sets of labelled formulas.

However, in this paper we are concerned with *intuitionistic modal logics*, whose Kripke semantics is based on *birelational* frames, i.e., they have two binary relations instead of one: one relation R that corresponds to the accessibility relation in Kripke frames for modal logics, and a relation $\leq$ that corresponds to the preorder relation in Kripke frames for intuitionistic logic. Consequently, standard labelled systems for these logics have certain shortcomings:

1. The transitivity rule in (1) can be axiomatised by the conjunction of the two versions of the $\mathsf{4}$-axiom

$$\Box A \supset \Box\Box A \qquad \text{and} \qquad \Diamond\Diamond A \supset \Diamond A \quad , \tag{2}$$

which are equivalent in classical modal logic. However, in intuitionistic modal logic they are not equivalent, and even though the logic $\mathsf{IK4}$, i.e., the intuitionistic version of the modal logic $\mathsf{K4}$, contains both axioms, they can also be added independently to the logic IK (an intuitionistic version of K). The proof theory of these new logics has not been studied before; no existing labelled (or label-free) proof system can handle them, even though the corresponding frame conditions

$$\forall xyz.\, xRy \wedge yRz \supset (\exists x'.\, x \leq x' \wedge x'Rz) \quad \text{and} \quad \forall xyz.\, xRy \wedge yRz \supset (\exists z'.\, z \leq z' \wedge xRz')\ , \tag{3}$$

$$\begin{tikzcd} x' \arrow[rrd, dotted, "R"] & & \\ x \arrow[u, dotted, "\leq"] \arrow[r, "R"'] & y \arrow[r, "R"'] & z \end{tikzcd} \qquad \begin{tikzcd} & & z'' \\ x \arrow[rru, dotted, "R"] \arrow[r, "R"'] & y \arrow[r, "R"'] & z \arrow[u, dotted, "\leq"'] \end{tikzcd}$$

respectively, have already been studied in [PS86].

2. The correspondence between the syntax and the semantics is not as clean as one would expect. As only the R-relation (and not the $\leq$-relation) of the frame is visible in an ordinary labelled sequent, we only have that a sequent Γ is provable if and only if is satisfied in all *graph-consistent*[1] models, as already observed in by Simpson in his PhD thesis [Sim94] and considered as an inelegant solution (see also [MS17]).

In order to address these two concerns we propose here to enrich usual labelled sequent by allowing both, relational atoms of the form $x \leq y$ and of the form xRy. Consequently, we can easily translate the frame conditions in (3) into inference rules:

$$4^{\Box}\ \frac{\mathcal{R}, xRy, yRz, x'Rz, x \leq x', \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, yRz, \Gamma \Longrightarrow \Delta}\ x' \text{ fresh} \qquad \text{and} \qquad 4^{\Diamond}\ \frac{\mathcal{R}, xRy, yRz, xRz', z \leq z', \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, yRz, \Gamma \Longrightarrow \Delta}\ z' \text{ fresh} \tag{4}$$

This allows us to define cut-free deductive systems for a wide range of logics that could not be treated before. Furthermore, the relation between syntax and semantics is as one would expect: A sequent is provable in our system if and only if it is valid in all models.

Besides that, there is another pleasant observation to make about our system. Ordinary labelled sequent systems for intuitionistic modal logic are single-conclusion [Sim94]. The same is true for the corresponding nested sequent systems [Str13, MS14]. It is possible to express Maehara style multiple-conclusion systems in nested sequents [SK19], and therefore also in ordinary labelled sequents. However, also in these systems there are rules ($\supset_{\mathsf{R}}$ and $\Box_{\mathsf{R}}$) that force a single-conclusion premise, even though this is not the case in labelled systems [Neg05] or nested sequents [Fit83] for intuitionistic logic. Maffezioli, Naibo and Negri have considered in [MNN13] a labelled system for intuitionistic bimodal epistemic logic which is multi-conclusion. Our system uses the same principle, both labelled and multi-conclusion sequents, but we use in a more general setting and extend it to a framework for many intuitionistic modal logics. This eliminates the undesired discrepancy as, consequently, every rule in our system is invertible, i.e. we never delete information in a bottom-up proof search.

This paper is organized as follows, In the next section (Section 2) we recall the standard syntax and semantics of intuitionistic modal logics. Then, in Section 3 we present our system for the intuitionistic modal logic IK. In Sections 5 and 4, we show soundness and completeness of the system with cut. The cut elimination theorem, proved in Section 6, then entails soundness and completeness for the cut-free system. Finally, in Section 7 we discuss the possible extension to the system to capture other intuitionistic modal logics.

2 Intuitionistic modal logics

The language of intuitionisitic modal logic is the one of intuitionistic propositional logic with the modal operators $\Box$ and $\Diamond$, standing most generally for *necessity* and *possibility*. Starting with a set $\mathcal{A}$ of atomic propositions, denoted by lower case letters $a, b, c, \ldots$, modal formulas, denoted by capital letters $A, B, C, \ldots$, are constructed from the grammar:

$$A ::= a \mid A \wedge A \mid A \vee A \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A$$

Obtaining the intuitionistic variant of K is more involved than the classical variant. Lacking De Morgan duality, there are many variants of the *distributivity axiom* k that are classically but not intuitionistically equivalent. Five axioms have been considered as primitives in the literature. An intuitionistic variant of the modal logic K can then be obtained from ordinary intuitionistic propositional logic IPL by adding:

- the *necessitation rule*: if A is a theorem then $\Box A$ is also a theorem; and
- the following five variants of k:

$$\begin{array}{lll} \mathsf{k}_1:\ \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3:\ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) & \mathsf{k}_5:\ \Diamond\bot \supset \bot \\ \mathsf{k}_2:\ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4:\ (\Diamond A \supset \Box B) \supset \Box(A \supset B) & \end{array}$$

The idea is that intuitionistic propositional logic does not allow the principle of *Excluded Middle*, so the modalities $\Box$ and $\Diamond$ are not de Morgan duals any more, but one can choose to design the axiomatisation in order to relate them in different ways. The most basic intuitionistic modal system one can think of would be to consider only the $\Box$ modality as regulated by the k axiom (or as called here k_1), which gives the system $\mathsf{IPL} + \mathsf{nec} + \mathsf{k}_1$. However this would give strictly no information on the behaviour of the $\Diamond$ modality. It seems that Fitch [Fit48] was the first one to propose a way to treat $\Diamond$ in an intuitionistic system by considering the system $\mathsf{IPL} + \mathsf{nec} + \mathsf{k}_1 + \mathsf{k}_2$, which is now sometimes called CK for *constructive modal logic*. Wijekesera [Wij90] also considered the axiom k_5, which states that $\Diamond$ distributes over 0-ary disjunctions, but did not assume that it would always distribute over binary disjunctions; the system he proposed was therefore $\mathsf{IPL} + \mathsf{nec} + \mathsf{k}_1 +$

[1] This means that every layer in the model can be lifted to any future of any world in that layer. See [Sim94] and [MS17] for a formal definition and discussion.

$\mathsf{k}_2 + \mathsf{k}_5$. In these systems, however, the addition of the *Excluded Middle* principle to it does not yield classical modal logic K, that is, it is not possible to retrieve the De Morgan duality of $\Box$ and $\Diamond$ in this case.

The axiomatisation that is now generally accepted as *intuitionistic modal logic* denoted by IK was given by Plotkin and Stirling [PS86] and is equivalent to the one proposed by Fischer-Servi [Ser84] and by Ewald [Ewa86] in the case of intuitionistic tense logic. It is taken to be $\mathsf{IPL} + \mathsf{nec} + \mathsf{k}_1 + \mathsf{k}_2 + \mathsf{k}_3 + \mathsf{k}_4 + \mathsf{k}_5$.

The Kripke semantics for IK was first defined by Fischer-Servi [Ser84]. It combines the Kripke semantics for intuitionistic propositional logic and the one for classical modal logic, using two distinct relations on the set of worlds.

Definition 2.1. A *bi-relational frame* $\mathcal{F}$ is a triple $\langle W, R, \leq\rangle$ of a set of worlds W equipped with an accessibility relation R and a preorder $\leq$ (*i.e.* a reflexive and transitive relation) satisfying:

(F_1) For all $u, v, v' \in W$, if uRv and $v \leq v'$, there exists u' s.t. $u \leq u'$ and $u'Rv'$.

$$\begin{tikzcd} u' \arrow[r, dotted, "R"] & v' \\ u \arrow[u, dotted, "\leq"] \arrow[r, "R"] & v \arrow[u, "\leq"'] \end{tikzcd}$$

(F_2) For all $u', u, v \in W$, if $u \leq v$, there exists v' s.t. $u'Rv'$ and $v \leq v'$.

$$\begin{tikzcd} u' \arrow[r, dotted, "R"] & v' \\ u \arrow[u, "\leq"] \arrow[r, "R"] & v \arrow[u, dotted, "\leq"'] \end{tikzcd}$$

Definition 2.2. A *bi-relational model* $\mathcal{M}$ is a quadruple $\langle W, R, \leq, V\rangle$ with $\langle W, R, \leq\rangle$ a bi-relational frame and $V\colon W \to 2^{\mathcal{A}}$ a monotone valuation function, that is, a function mapping each world w to the subset of propositional atoms true at w, additionally subject to: if $w \leq w'$ then $V(w) \subseteq V(w')$.

We write $\mathcal{M}, w \Vdash a$ if $a \in V(w)$, and inductively extend the $\Vdash$ relation to all formulas, following the rules for both intuitionistic and modal Kripke models:

$$\begin{array}{rcl} \mathcal{M}, w \Vdash A \wedge B & \text{iff} & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\ \mathcal{M}, w \Vdash A \vee B & \text{iff} & \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \\ \mathcal{M}, w \Vdash A \supset B & \text{iff} & \text{for all } w' \text{ with } w \leq w',\ \text{if } \mathcal{M}, w' \Vdash A \text{ then } \mathcal{M}, w' \Vdash B \\ \mathcal{M}, w \Vdash \Box A & \text{iff} & \text{for all } w' \text{ and } u \text{ with } w \leq w' \text{ and } w'Ru, \mathcal{M}, u \Vdash A \\ \mathcal{M}, w \Vdash \Diamond A & \text{iff} & \text{there exists a } u \text{ such that } wRu \text{ and } \mathcal{M}, u \Vdash A. \end{array} \tag{5}$$

Observe that we never have that $\mathcal{M}, w \Vdash \bot$. We write $\mathcal{M}, w \nVdash A$ if it is not the case that $\mathcal{M}, w \Vdash A$, but contrarily to the classical case, we do not have $\mathcal{M}, w \Vdash \neg A$ iff $\mathcal{M}, w \nVdash A$ (since $\neg A$ is defined as $A \supset \bot$).

From the monotonicity of the valuation function V, we get a monotonicity property for the relation:

Proposition 2.3. *(Monotonicity) For any formula A and for $w, w' \in W$, if $w \leq w'$ and $\mathcal{M}, w \Vdash A$, then $\mathcal{M}, w' \Vdash A$.*

Definition 2.4. A formula A is *satisfied* in a model $\mathcal{M} = \langle W, R, \leq, V\rangle$, if for all $w \in W$ we have $\mathcal{M}, w \Vdash A$. A formula A is *valid* in a frame $\mathcal{F} = \langle W, R, \leq\rangle$, if for all valuations V, the formula A is satisfied in $\langle W, R, \leq, V\rangle$.

Similarly to the classical case, the correspondence between syntax and semantics for IK can be stated as follows.

Theorem 2.5 ([Ser84, PS86])**.** *A formula A is a theorem of IK if and only if A is valid in every bi-relational frame.*

3 The system

In this section we present our fully labelled sequent proof system for intuitionistic modal logics. The starting point is the notion of a *labelled formula* which is a pair $x{:}A$ of a label x and a formula A. A *relation atom* is either an expression xRy or $x \leq y$ where x and y are labels. A *(labelled) sequent* is a triple $\mathcal{R}, \Gamma \Longrightarrow \Delta$, where $\mathcal{R}$ is a set of relational atoms and Γ and Δ are multi-sets of labelled formulas, all written as lists, separated by commas.

$$\mathsf{id}\ \frac{}{\mathcal{R}, x \leq y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a} \qquad \bot_{\mathsf{L}}\ \frac{}{\mathcal{R}, \Gamma, x{:}\bot \Longrightarrow \Delta}$$

$$\wedge_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}A \wedge B \Longrightarrow \Delta} \qquad \wedge_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma, x{:}A \Longrightarrow \Delta \quad \mathcal{R}, \Gamma, x{:}B \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}A \vee B \Longrightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \vee B}$$

$$\supset_{\mathsf{R}}\ \frac{\mathcal{R}, x \leq y, \Gamma, y{:}A \Longrightarrow \Delta, y{:}B}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B}\ y \text{ fresh}$$

$$\supset_{\mathsf{L}}\ \frac{\mathcal{R}, x \leq y, x{:}A \supset B, \Gamma \Longrightarrow \Delta, y{:}A \quad \mathcal{R}, x \leq y, \Gamma, y{:}B \Longrightarrow \Delta}{\mathcal{R}, x \leq y, \Gamma, x{:}A \supset B \Longrightarrow \Delta}$$

$$\Box_{\mathsf{L}}\ \frac{\mathcal{R}, x \leq y, yRz, \Gamma, x{:}\Box A, z{:}A \Longrightarrow \Delta}{\mathcal{R}, x \leq y, yRz, \Gamma, x{:}\Box A \Longrightarrow \Delta} \qquad \Box_{\mathsf{R}}\ \frac{\mathcal{R}, x \leq y, yRz, \Gamma \Longrightarrow \Delta, z{:}A}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A}\ y, z \text{ fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{\mathcal{R}, xRy, \Gamma, y{:}A \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta}\ y \text{ fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, y{:}A}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A}$$

. . . .

$$\mathsf{refl}\ \frac{\mathcal{R}, x \leq x, \Gamma \Longrightarrow \Delta}{\mathcal{R}, \Gamma \Longrightarrow \Delta} \qquad \mathsf{trans}\ \frac{\mathcal{R}, x \leq y, y \leq z, x \leq z, \Gamma \Longrightarrow \Delta}{\mathcal{R}, x \leq y, y \leq z, \Gamma \Longrightarrow \Delta}$$

$$\mathsf{F}_1\ \frac{\mathcal{R}, xRy, y \leq z, x \leq u, uRz, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, y \leq z, \Gamma \Longrightarrow \Delta}\ u \text{ fresh}$$

$$\mathsf{F}_2\ \frac{\mathcal{R}, xRy, x \leq z, y \leq u, zRu, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xRy, x \leq z, \Gamma \Longrightarrow \Delta}\ u \text{ fresh}$$

Figure 1: System $\mathsf{labIK}_{\leq}$

Now we can present the inference rules for *system* $\mathsf{labIK}_{\leq}$ for the logic IK. We obtained this system, shown in Figure 1, as follows. Our starting point was the multiple-conlusion nested sequent system *à la* Maehara (as presented in [SK19]), which can straightforwardly translated into the labelled setting, and which yields the rules $\bot_{\mathsf{L}}$, $\wedge_{\mathsf{L}}$, $\wedge_{\mathsf{R}}$, $\vee_{\mathsf{L}}$, $\vee_{\mathsf{R}}$, $\Diamond_{\mathsf{L}}$, and $\Diamond_{\mathsf{R}}$ as we show them in Figure 1. However, this naive translation would also yield the rules id', $\supset_{\mathsf{L}}'$, and $\Box_{\mathsf{L}}'$:

$$\mathsf{id}'\ \frac{}{\mathcal{R}, \Gamma, x{:}a \Longrightarrow \Delta, x{:}a} \qquad \supset_{\mathsf{L}}'\ \frac{\mathcal{R}, x{:}A \supset B, \Gamma \Longrightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma, x{:}B \Longrightarrow \Delta}{\mathcal{R}, \Gamma, x{:}A \supset B \Longrightarrow \Delta} \qquad \Box_{\mathsf{L}}\ \frac{\mathcal{R}, xRz, \Gamma, x{:}\Box A, z{:}A \Longrightarrow \Delta}{\mathcal{R}, xRz, \Gamma, x{:}\Box A \Longrightarrow \Delta} \tag{6}$$

that are not sufficient for a complete system, and we will see below why. Before, let us first look at the rules $\supset_{\mathsf{R}}$ and $\Box_{\mathsf{R}}$. In the multiple-conlusion nested sequent system of [SK19], these are the two rules that force single-conclusion. In our system, this phenomenon is replaced by a re-positioning of the considered formulas to a fresh label. In the Kripke-semantics in (5) the two connectives $\supset$ and $\Box$ are the ones that make use of the pre-order relation $\leq$. This relation is reflexive and transitive. In order to capture that in the proof system, we need to add the rules refl and trans. These can be obtained by applying the axioms-as-rules methodology as in [MNN13].

Finally, in the semantics, the two relations R and $\leq$ are strongly connected through the two conditions F_1 and F_2. These need to be reflected at the level of the proof system, which is done by the two rules F_1 and F_2. However, these two rules create new labels, and in order to be complete, the system needs the *monotonicity* rule $\mathsf{mon}_{\mathsf{L}}$, shown on the left below.

$$\mathsf{mon}_{\mathsf{L}}\ \frac{\mathcal{R}, x \leq y, \Gamma, x{:}A, y{:}A \Longrightarrow \Delta}{\mathcal{R}, x \leq y, \Gamma, x{:}A \Longrightarrow \Delta} \qquad \mathsf{mon}_{\mathsf{R}}\ \frac{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta, x{:}A, y{:}A}{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta, y{:}A} \tag{7}$$

Since this rule is a form of contraction, it would cause the same problems as contraction in a cut elimination proof. Hence, it is preferable to have a system in which this rule is admissible. This is the reason why we have monotonicity incorporated in the rules id, $\supset_\mathsf{L}$ and $\Box_\mathsf{L}$ in Figure 1, instead of using the rules in (6). Then, not only $\mathsf{mon_L}$ but also its right-hand side version $\mathsf{mon_R}$, shown on the right in (7) above become admissible.

Proposition 3.1. *The rules* $\mathsf{mon_L}$ *and* $\mathsf{mon_R}$ *are admissible for* $\mathsf{labIK}_\leq$.

One can prove this proposition in the same way as one usually proves admissibity of contraction in a sequent calculus, by induction on the height of the derivation, which in fact would yield a stronger result, namely that $\mathsf{mon_L}$ and $\mathsf{mon_R}$ are *height preserving* admissible for $\mathsf{labIK}_\leq$. However, we do not need this result in this paper, and therefore we leave it to the interested reader. Nonetheless, we will give a short proof of Proposition 3.1 at the end of this section.

Before, let us give another indication of the fact that $\mathsf{labIK}_\leq$ is well-designed, namely that the general identity axiom is admissible.

Proposition 3.2. *The following general identity axiom* $\mathsf{id_g}\,\dfrac{}{\mathcal{R}, x \leq y, \Gamma, x{:}A \Longrightarrow \Delta, y{:}A}$ *is admissible for* $\mathsf{labIK}_\leq$.

Proof. As standard, we proceed by structural induction on A. The two base cases $A = a$ and $A = \bot$ are trivial. The inductive cases are shown below.

- $A \wedge B$

$$\dfrac{\dfrac{\dfrac{}{\mathcal{R}, x \leq y, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta, y{:}A}\mathsf{id_g} \quad \dfrac{}{\mathcal{R}, x \leq y, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta, y{:}B}\mathsf{id_g}}{\mathcal{R}, x \leq y, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta, y{:}A \wedge B}\wedge_\mathsf{R}}{\mathcal{R}, x \leq y, \Gamma, x{:}A \wedge B \Longrightarrow \Delta, y{:}A \wedge B}\wedge_\mathsf{L}$$

- $A \vee B$

$$\dfrac{\dfrac{\dfrac{}{\mathcal{R}, x \leq y, \Gamma, x{:}A \Longrightarrow \Delta, y{:}A}\mathsf{id_g}}{\mathcal{R}, x \leq y, \Gamma, x{:}A \Longrightarrow \Delta, y{:}A \vee B}\vee_\mathsf{R} \quad \dfrac{\dfrac{}{\mathcal{R}, x \leq y, \Gamma, x{:}B \Longrightarrow \Delta, y{:}B}\mathsf{id_g}}{\mathcal{R}, x \leq y, \Gamma, x{:}B \Longrightarrow \Delta, y{:}A \vee B}\vee_\mathsf{R}}{\mathcal{R}, x \leq y, \Gamma, x{:}A \vee B \Longrightarrow \Delta, y{:}A \vee B}\vee_\mathsf{L}$$

- $A \supset B$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{R}, x \leq y, y \leq z, x \leq z, z \leq z, \Gamma, x{:}A \supset B, z{:}A \Longrightarrow \Delta, z{:}B, z{:}A}\mathsf{id_g}}{\mathcal{R}, x \leq y, y \leq z, x \leq z, \Gamma, x{:}A \supset B, z{:}A \Longrightarrow \Delta, z{:}B, z{:}A}\mathsf{refl} \quad \dfrac{\dfrac{}{\mathcal{R}, x \leq y, y \leq z, x \leq z, z \leq z, \Gamma, z{:}B, z{:}A \Longrightarrow \Delta, z{:}B}\mathsf{id_g}}{\mathcal{R}, x \leq y, y \leq z, x \leq z, \Gamma, z{:}B, z{:}A \Longrightarrow \Delta, z{:}B}\mathsf{refl}}{\mathcal{R}, x \leq y, y \leq z, x \leq z, \Gamma, x{:}A \supset B, z{:}A \Longrightarrow \Delta, z{:}B}\supset_\mathsf{L}}{\mathcal{R}, x \leq y, y \leq z, \Gamma, x{:}A \supset B, z{:}A \Longrightarrow \Delta, z{:}B}\mathsf{trans}}{\mathcal{R}, x \leq y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, y{:}A \supset B}\supset_\mathsf{R}\ z \text{ fresh}$$

- $\Box A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{R}, x \leq y, y \leq z, x \leq z, zRw, w \leq w, \Gamma, z{:}\Box A, w{:}A \Longrightarrow \Delta, w{:}A}\mathsf{id_g}}{\mathcal{R}, x \leq y, y \leq z, x \leq z, zRw; \Gamma, z{:}\Box A, w{:}A \Longrightarrow \Delta, w{:}A}\mathsf{refl}}{\mathcal{R}, x \leq y, y \leq z, x \leq z, zRw, \Gamma, x{:}\Box A \Longrightarrow \Delta, w{:}A}\Box_\mathsf{L}}{\mathcal{R}, x \leq y, y \leq z, zRw, \Gamma, x{:}\Box A \Longrightarrow \Delta, w{:}A}\mathsf{trans}}{\mathcal{R}, x \leq y, \Gamma, x{:}\Box A \Longrightarrow \Delta, y{:}\Box A}\Box_\mathsf{R}\ z, w \text{ fresh}$$

- $\Diamond A$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{R}, x \leq y, xRz, z \leq u, yRu, \Gamma, z{:}A \Longrightarrow \Delta, y{:}\Diamond A, u{:}A}\mathsf{id_g}}{\mathcal{R}, x \leq y, xRz, z \leq u, yRu, \Gamma, z{:}A \Longrightarrow \Delta, y{:}\Diamond A}\Diamond_\mathsf{R}}{\mathcal{R}, x \leq y, xRz, \Gamma, z{:}A \Longrightarrow \Delta, y{:}\Diamond A}\mathsf{F_2}\ u \text{ fresh}}{\mathcal{R}, x \leq y, \Gamma, x{:}\Diamond A \Longrightarrow \Delta, y{:}\Diamond A}\Diamond_\mathsf{L}\ z \text{ fresh}$$

∎

In the following sections, we will show that the system $\mathsf{labIK}_\leq$ is sound and complete. For the completeness proof we proceed via cut elimination. The cut rule has the following shape:

$$\mathsf{cut}\,\dfrac{\mathcal{R}, \Gamma \Longrightarrow \Delta, z{:}C \qquad \mathcal{R}, \Gamma, z{:}C \Longrightarrow \Delta}{\mathcal{R}, \Gamma \Longrightarrow \Delta} \tag{8}$$

Then we can summarize soundness, completeness, and cut admissibility of $\mathsf{labIK}_\leq$ in the following Theorem:

Theorem 3.3. *For any formula A, the following are equivalent.*

1. *A is a theorem of* IK.
2. *A is provable in* $\mathsf{labIK}_{\leq} + \mathsf{cut}$.
3. *A is provable in* $\mathsf{labIK}_{\leq}$.
4. *A is valid in every birelational frame.*

The proof of this theorem is the topic of the following sections. The equivalence of 1 and 4 has already been stated in Theorem 2.5 [Ser84, PS86]. The implication $1 \Longrightarrow 2$ is shown in Section 4, the implication $2 \Longrightarrow 3$ is shown in Section 6, and finally, the implication $3 \Longrightarrow 4$ is shown in Section 5.

Once we have shown cut elimination (the implication $2 \Longrightarrow 3$ of Theorem 3.3), the proof of Proposition 3.1 becomes trivial.

Proof of Proposition 3.1. The rule $\mathsf{mon_L}$ can be derived using the general identity and cut:

$$\mathsf{cut}\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{R}, x \leq y, \Gamma, x{:}A \Longrightarrow \Delta, y{:}A} \qquad \mathcal{R}, x \leq y, \Gamma, x{:}A, y{:}A \Longrightarrow \Delta}{\mathcal{R}, x \leq y, \Gamma, x{:}A \Longrightarrow \Delta}$$

and both these rules are admissible by Proposition 3.2 and Theorem 3.3. The case for $\mathsf{mon_R}$ is similar. ☐

4 Completeness

In this section we show our system at work, as most of the section consists of derivations of axioms of IK in $\mathsf{labIK}_{\leq}$. More precisely, we prove completeness of $\mathsf{labIK}_{\leq} + \mathsf{cut}$, i.e., the implication $1 \Longrightarrow 2$ of Theorem 3.3, which is stated again below:

Theorem 4.1. *For any formula A. If A is a theorem of* IK *then A is provable in* $\mathsf{labIK}_{\leq} + \mathsf{cut}$.

Remark 4.2. We have seen already in the proof of Proposition 3.2 the need of the rule $\mathsf{F_2}$. In the following proof of Theorem 4.1 we also see the need of the rules $\mathsf{F_1}$, refl, and trans.

Proof of Theorem 4.1. We begin by showing how the axioms $\mathsf{k_1}$–$\mathsf{k_5}$ are proved in system $\mathsf{labIK}_{\leq}$.

- $\mathsf{k_1}$:

$$\supset_{\mathsf{R}}\dfrac{\supset_{\mathsf{R}}\dfrac{\Box_{\mathsf{R}}\dfrac{\Box_{\mathsf{L}}\dfrac{\mathsf{trans}\dfrac{\Box_{\mathsf{L}}\dfrac{\mathsf{refl}\dfrac{\supset_{\mathsf{L}}\dfrac{\mathsf{id_g}\dfrac{}{\mathcal{R}, y{:}\Box(A \supset B), z{:}\Box A, u{:}A, u{:}A \supset B \Longrightarrow u{:}B, u{:}A} \quad \mathsf{id_g}\dfrac{}{\mathcal{R}, y{:}\Box(A \supset B), z{:}\Box A, u{:}A, u{:}A \supset B, u{:}B \Longrightarrow u{:}B}}{x \leq y, y \leq z, z \leq w, y \leq w, u \leq u, wRu, y{:}\Box(A \supset B), z{:}\Box A, u{:}A, u{:}A \supset B \Longrightarrow u{:}B}}{x \leq y, y \leq z, z \leq w, y \leq w, wRu, y{:}\Box(A \supset B), z{:}\Box A, u{:}A, u{:}A \supset B \Longrightarrow u{:}B}}{x \leq y, y \leq z, z \leq w, y \leq w, wRu, y{:}\Box(A \supset B), z{:}\Box A, u{:}A \Longrightarrow u{:}B}}{x \leq y, y \leq z, z \leq w, wRu, y{:}\Box(A \supset B), z{:}\Box A, u{:}A \Longrightarrow u{:}B}}{x \leq y, y \leq z, z \leq w, wRu, y{:}\Box(A \supset B), z{:}\Box A \Longrightarrow u{:}B}\; w, u \text{ fresh}}{x \leq y, y \leq z, y{:}\Box(A \supset B), z{:}\Box A \Longrightarrow z{:}\Box B}\; z \text{ fresh}}{x \leq y, y{:}\Box(A \supset B) \Longrightarrow y{:}\Box A \supset \Box B}\; y \text{ fresh}}{\Longrightarrow x{:}\Box(A \supset B) \supset (\Box A \supset \Box B)}$$

  where $\mathcal{R}$ is equal to: $x \leq y, y \leq z, z \leq w, y \leq w, u \leq u, wRu$.

- $\mathsf{k_2}$:

$$\supset_{\mathsf{R}}\dfrac{\supset_{\mathsf{R}}\dfrac{\Diamond_{\mathsf{L}}\dfrac{\Diamond_{\mathsf{R}}\dfrac{\Box_{\mathsf{L}}\dfrac{\mathsf{refl}\dfrac{\supset_{\mathsf{L}}\dfrac{\mathsf{id_g}\dfrac{}{\mathcal{R}, y{:}\Box(A \supset B), u{:}A, u{:}A \supset B \Longrightarrow z{:}\Diamond B, u{:}B, u{:}A} \quad \mathsf{id_g}\dfrac{}{\mathcal{R}, y{:}\Box(A \supset B), u{:}A, u{:}A \supset B, u{:}B \Longrightarrow z{:}\Diamond B, u{:}B}}{x \leq y, y \leq z, zRu, u \leq u, y{:}\Box(A \supset B), u{:}A, u{:}A \supset B \Longrightarrow z{:}\Diamond B, u{:}B}}{x \leq y, y \leq z, zRu, y{:}\Box(A \supset B), u{:}A, u{:}A \supset B \Longrightarrow z{:}\Diamond B, u{:}B}}{x \leq y, y \leq z, zRu, y{:}\Box(A \supset B), u{:}A \Longrightarrow z{:}\Diamond B, u{:}B}}{x \leq y, y \leq z, zRu, y{:}\Box(A \supset B), u{:}A \Longrightarrow z{:}\Diamond B}\; u \text{ fresh}}{x \leq y, y \leq z, y{:}\Box(A \supset B), z{:}\Diamond A \Longrightarrow z{:}\Diamond B}\; z \text{ fresh}}{x \leq y, y{:}\Box(A \supset B) \Longrightarrow y{:}(\Diamond A \supset \Diamond B)}\; y \text{ fresh}}{\Longrightarrow x{:}\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)}$$

  where $\mathcal{R}$ is equal to $x \leq y, y \leq z, zRu, u \leq u$.

- k_3:

$$
\mathsf{\supset_R}\,\dfrac{\mathsf{\Diamond_L}\,\dfrac{\mathsf{\vee_L}\,\dfrac{\mathsf{\vee_R}\,\dfrac{\mathsf{\Diamond_R}\,\dfrac{\mathsf{refl}\,\dfrac{\mathsf{id_g}\,\dfrac{}{x \le y, z \le z, yRz, z{:}A \Longrightarrow y{:}\Diamond A, z{:}A, y{:}\Diamond B}}{x \le y, yRz, z{:}A \Longrightarrow y{:}\Diamond A, z{:}A, y{:}\Diamond B}}{x \le y, yRz, z{:}A \Longrightarrow y{:}\Diamond A, y{:}\Diamond B}}{x \le y, yRz, z{:}A \Longrightarrow y{:}\Diamond A \vee \Diamond B} \quad \mathsf{\vee_R}\,\dfrac{\mathsf{\Diamond_R}\,\dfrac{\mathsf{refl}\,\dfrac{\mathsf{id_g}\,\dfrac{}{x \le y, z \le z, yRz, z{:}B \Longrightarrow y{:}\Diamond A, y{:}\Diamond B, z{:}B}}{x \le y, yRz, z{:}B \Longrightarrow y{:}\Diamond A, y{:}\Diamond B, z{:}B}}{x \le y, yRz, z{:}B \Longrightarrow y{:}\Diamond A, y{:}\Diamond B}}{x \le y, yRz, z{:}B \Longrightarrow y{:}\Diamond A \vee \Diamond B}}{x \le y, yRz, z{:}A \vee B \Longrightarrow y{:}\Diamond A \vee \Diamond B}}{x \le y, y{:}\Diamond(A \vee B) \Longrightarrow y{:}\Diamond A \vee \Diamond B}\; z \text{ fresh}}{\Longrightarrow x{:}\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)}\; y \text{ fresh}
$$

- k_4:

$$
\mathsf{\supset_R}\,\dfrac{\mathsf{\Box_R}\,\dfrac{\mathsf{\supset_R}\,\dfrac{\mathsf{F_1}\,\dfrac{\mathsf{trans}\,\dfrac{\mathsf{\supset_L}\,\dfrac{\mathsf{\Diamond_R}\,\dfrac{\mathsf{refl}\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{R}, u \le u, y{:}\Diamond A \supset \Box B, u{:}A \Longrightarrow u{:}B, t{:}\Diamond A, u{:}A}}{\mathcal{R}, y{:}\Diamond A \supset \Box B, u{:}A \Longrightarrow u{:}B, t{:}\Diamond A, u{:}A}}{\mathcal{R}, y{:}\Diamond A \supset \Box B, u{:}A \Longrightarrow u{:}B, t{:}\Diamond A} \quad \mathsf{refl}\,\dfrac{\mathsf{\Box_L}\,\dfrac{\mathsf{refl}\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{R}, t \le t, u \le u, y{:}\Diamond A \supset \Box B, u{:}A, t{:}\Box B, u{:}B \Longrightarrow u{:}B}}{\mathcal{R}, t \le t, y{:}\Diamond A \supset \Box B, u{:}A, t{:}\Box B, u{:}B \Longrightarrow u{:}B}}{\mathcal{R}, t \le t, y{:}\Diamond A \supset \Box B, u{:}A, t{:}\Box B \Longrightarrow u{:}B}}{\mathcal{R}, y{:}\Diamond A \supset \Box B, u{:}A, t{:}\Box B \Longrightarrow u{:}B}}{x \le y, y \le z, w \le u, z \le t, y \le t, zRw, tRu, y{:}\Diamond A \supset \Box B, u{:}A \Longrightarrow u{:}B}}{x \le y, y \le z, w \le u, z \le t, zRw, tRu, y{:}\Diamond A \supset \Box B, u{:}A \Longrightarrow u{:}B}\; t \text{ fresh}}{x \le y, y \le z, w \le u, zRw, y{:}\Diamond A \supset \Box B, u{:}A \Longrightarrow u{:}B}\; u \text{ fresh}}{x \le y, y \le z, zRw, y{:}\Diamond A \supset \Box B \Longrightarrow w{:}A \supset B}\; z, w \text{ fresh}}{x \le y, y{:}\Diamond A \supset \Box B \Longrightarrow y{:}\Box(A \supset B)}\; y \text{ fresh}
$$

$$
\Longrightarrow x{:}(\Diamond A \supset \Box B) \supset \Box(A \supset B)
$$

where $\mathcal{R}$ is equal to $x \le y, y \le z, w \le u, z \le t, y \le t, zRw, tRu$.

- k_5:

$$
\mathsf{\supset_R}\,\dfrac{\mathsf{\Diamond_L}\,\dfrac{\mathsf{\bot_L}\,\dfrac{}{x \le y, yRz, z{:}\bot \Longrightarrow y{:}\bot}}{x \le y, y{:}\Diamond\bot \Longrightarrow y{:}\bot}\; z \text{ fresh}}{\Longrightarrow x{:}\Diamond\bot \supset \bot}\; y \text{ fresh}
$$

Next, we have to prove all axioms of intuitionistic propositional logic can be shown in $\mathsf{labIK}_{\le}$. We do this only for $A \wedge B \supset B$ and leave the rest to the reader:

$$
\mathsf{\supset_R}\,\dfrac{\mathsf{\wedge_L}\,\dfrac{\mathsf{refl}\,\dfrac{\mathsf{id_g}\,\dfrac{}{x \le y, y \le y, y{:}A, y{:}B \Longrightarrow y{:}B}}{x \le y, y{:}A, y{:}B \Longrightarrow y{:}B}}{x \le y, y{:}A \wedge B \Longrightarrow y{:}B}}{\Longrightarrow x{:}A \wedge B \supset B}\; y \text{ fresh}
$$

Finally, we have to show how the rules of modus ponens and necessitation can be simulated in our system. For modus ponens, this is standard using the cut rule and for necessitation, we can transform a proof of A into a proof of $\Box A$ as follows. A proof of A is in fact a proof $\mathcal{D}$ of the sequent $\Longrightarrow z{:}A$ for some label z. If x and y are fresh labels, we can transform $\mathcal{D}$ into a proof $\mathcal{D}'$ of the sequent $x \le y, yRz \Longrightarrow z{:}A$ by adding $x \le y, yRz$ to every line. We can now apply the $\Box_{\mathsf{R}}$-rule to obtain a proof of $\Longrightarrow x{:}\Box A$.

This completes the proof Theorem 4.1. $\square$

5 Soundness

In order to prove the implication 3 $\Longrightarrow$ 4 from Theorem 3.3 we need to show that each sequent rule of our system $\mathsf{labIK}_{\le}$ is sound. To make precise what that actually means, we have to extend the relation $\Vdash$, defined in Section 2 from formulas to sequents. This is the purpose of the following definitions.

Definition 5.1. Let $\mathcal{M} = \langle W, R_{\mathcal{M}}, \le_{\mathcal{M}}, V\rangle$ be a model, and let $\mathcal{G}$ be the sequent $\mathcal{R}, \Gamma \Longrightarrow \Delta$. A *$\mathcal{G}$-interpretation in $\mathcal{M}$* is a mapping $[\![\cdot]\!]$ from the labels in $\mathcal{G}$ to the set W of worlds in $\mathcal{M}$, such that whenever xRy in $\mathcal{R}$, then $[\![x]\!] R_{\mathcal{M}} [\![y]\!]$, and whenever $x \le y$ in $\mathcal{R}$, then $[\![x]\!] \le_{\mathcal{M}} [\![y]\!]$. Now we can define

$$
\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{G} \qquad \text{iff} \qquad \begin{array}{l} \text{if for all } x{:}A \in \Gamma, \text{ we have } \mathcal{M}, [\![x]\!] \Vdash A, \text{ then} \\ \text{there exists } z{:}B \in \Delta \text{ such that } \mathcal{M}, [\![z]\!] \Vdash B. \end{array} \tag{9}
$$

Definition 5.2. A sequent $\mathcal{G}$ is *satisfied* in $\mathcal{M} = \langle W, R, \leq, V \rangle$ iff for all $\mathcal{G}$-interpretations $[\![\cdot]\!]$ we have $\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{G}$. A sequent $\mathcal{G}$ is *valid* in a frame $\mathcal{F} = \langle W, R, \leq \rangle$, if for all valuations V, the sequent $\mathcal{G}$ is satisfied in $\langle W, R, \leq, V \rangle$.

We are now ready to state the main theorem of this section, of which the implication $3 \Longrightarrow 4$ in Theorem 3.3 is an immediate consequence.

Theorem 5.3. *If a sequent $\mathcal{G}$ is provable in* $\mathsf{labIK}_{\leq}$*, then it is valid in every birelational frame.*

Proof. We proceed by induction on the height of the derivation of $\mathcal{G}$, and we show for all rules in $\mathsf{labIK}_{\leq}$

$$\mathsf{r}\,\frac{\mathcal{G}_1 \quad \cdots \quad \mathcal{G}_n}{\mathcal{G}}$$

for $n \in \{0, 1, 2\}$, that whenever $\mathcal{G}_1, \ldots, \mathcal{G}_n$ are valid in all birelational frame, then so is $\mathcal{G}$. It follows a case analysis on r:

- $\bot_{\mathsf{R}}$: This is trivial because $\bot$ is never forced.
- id: This follows immediately from Proposition 3.1.
- $\Box_{\mathsf{L}}$: By way of contradiction, assume that $\mathcal{R}, x \leq y, yRz, \Gamma, x{:}\Box A, z{:}A \Longrightarrow \Delta$ is valid in all birelational frames, but $\mathcal{R}, x \leq y, yRz, \Gamma, x{:}\Box A \Longrightarrow \Delta$ is not. This means that we have a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \mathcal{R}, x \leq y, yRz, \Gamma, x{:}\Box A \Longrightarrow \Delta$, i.e., $[\![x]\!] \leq_{\mathcal{M}} [\![y]\!]$ and $[\![y]\!] R_{\mathcal{M}} [\![z]\!]$ and $\mathcal{M}, x \Vdash \Box A$ and $\mathcal{M}, w \nVdash B$ for all $w{:}B \in \Delta$. However, by the definition of forcing in (5) we also have $\mathcal{M}, z \Vdash A$, and consequently $\mathcal{M}, [\![\cdot]\!] \nVdash \mathcal{R}, x \leq y, yRz, \Gamma, x{:}\Box A, z{:}A \Longrightarrow \Delta$. Contradiction.
- $\Box_{\mathsf{R}}$: By way of contradiction, assume that $\mathcal{R}, x \leq y, yRz, \Gamma \Longrightarrow \Delta, z{:}A$ is valid in all birelational frames, but $\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A$ is not, where y and z do not occur in $\mathcal{R}$ or Γ or Δ. This means that we have a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A$. So in particular, there are worlds y' and z' in $\mathcal{M}$ such that $[\![x]\!] \leq_{\mathcal{M}} y'$ and $y' R_{\mathcal{M}} z'$ and $\mathcal{M}, z' \nVdash A$. Now we let $[\![\cdot]\!]'$ be the extension of $[\![\cdot]\!]$ such that $[\![y]\!]' = y'$ and $[\![z]\!]' = z'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$ on all other labels. Then $\mathcal{M}, [\![\cdot]\!]' \nVdash \mathcal{R}, x \leq y, yRz, \Gamma \Longrightarrow \Delta, z{:}A$. Contradiction.
- $\supset_{\mathsf{R}}$: By way of contradiction, assume that $\mathcal{R}, x \leq y, \Gamma, y{:}A \Longrightarrow \Delta, y{:}B$ is valid in all birelational frames, but $\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B$ is not, where y does not occur in $\mathcal{R}$ or Γ or Δ. This means that we have a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B$. So there exists a world y' in $\mathcal{M}$ such that $[\![x]\!] \leq_{\mathcal{M}} y'$ and $\mathcal{M}, y' \Vdash A$ but $\mathcal{M}, y' \nVdash B$. Now we let $[\![\cdot]\!]'$ be the extension of $[\![\cdot]\!]$ such that $[\![y]\!]' = y'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$ on all other labels. Then $\mathcal{M}, [\![\cdot]\!]' \nVdash \mathcal{R}, x \leq y, \Gamma, y{:}A \Longrightarrow \Delta, y{:}B$. Contradiction.
- $\Diamond_{\mathsf{L}}$: By way of contradiction, assume that $\mathcal{R}, xRy, \Gamma, y{:}A \Longrightarrow \Delta$ is valid in all birelational frames, but $\mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta$ is not, where y does not occur in $\mathcal{R}$ or Γ or Δ. This means that we have a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta$, i.e. $\mathcal{M}, x \Vdash \Diamond A$. This means that there exists world y' in $\mathcal{M}$ such that $[\![x]\!] R_{\mathcal{M}} y'$ and $\mathcal{M}, y' \Vdash A$. Now we let $[\![\cdot]\!]'$ be the extension of $[\![\cdot]\!]$ such that $[\![y]\!]' = y'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$ on all other labels. Then $\mathcal{M}, [\![\cdot]\!]' \nVdash \mathcal{R}, xRy, \Gamma, y{:}A \Longrightarrow \Delta$. Contradiction.

The other cases are similar (and simpler), and we leave them to the reader. In particular, note that the cases for the rules refl, trans, F_1 and F_2 are trivial, as all birelations frames have to obey the corresponding conditions. $\square$

6 Cut Admissibility

In this section we are going to prove the admissibility of cut for $\mathsf{labIK}_{\leq}$.

Theorem 6.1. *The* cut *rule is admissible for* $\mathsf{labIK}_{\leq}$*.*

This theorem directly entails the implication $2 \Longrightarrow 3$ of Theorem 3.3. But before we can prove it, we need a series of auxiliary lemmas.

The *height* of a derivation $\mathcal{D}$, denoted by $|\mathcal{D}|$, is the height of D when seen as a tree, i.e., the length of the longest path in the tree from its root to one of its leaves.

We say that a rule is *height-preserving admissible* if for every derivation $\mathcal{D}$ of its premise(s) there is a derivation $\mathcal{D}'$ of its conclusion such that $|\mathcal{D}'| \leq |\mathcal{D}|$. A rule is *height-preserving invertible* if for every derivation of the conclusion of the rule there are derivations for each of its premises with at most the same height.

The first lemma is the height-preserving admissibility of weakening on both relational atoms and labelled formulas.

Lemma 6.2. *The weakening rule* $\mathsf{weak}\,\dfrac{\mathcal{R}, \Gamma \Longrightarrow \Delta}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Longrightarrow \Delta, \Delta'}$ *is height-preserving admissible for* $\mathsf{labIK}_{\leq}$*.*

Proof. By a straightforward induction on the height of the derivation, we can transform any derivation

$$\dfrac{\mathcal{D}\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta} \qquad \text{into} \qquad \dfrac{\mathcal{D}^{\mathsf{w}}\,\Vert}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Longrightarrow \Delta, \Delta'}$$

of the same (or smaller) height. □

The next lemma looks like a special case of Proposition 3.1, but it is not. First, we need to preserve the height, and second, we cannot prove it using cut rule as we are trying to eliminate it from derivations.

Lemma 6.3. *The atomic version of* $\mathsf{mon_L}$

$$\mathsf{mon_a}\ \frac{\mathcal{R}, x \leq x', \Gamma, x{:}a, x'{:}a \Longrightarrow \Delta}{\mathcal{R}, x \leq x', \Gamma, x{:}a \Longrightarrow \Delta}$$

is height-preserving admissible for $\mathsf{labIK}_{\leq}$.

Proof. By induction on the height of $\mathcal{D}$, we prove that for any proof of $\mathcal{R}, x \leq x', \Gamma, x{:}a, x'{:}a \Longrightarrow \Delta$, there exists a proof of $\mathcal{R}, x \leq x', \Gamma, x{:}a \Longrightarrow \Delta$ of the same (or smaller) height. The inductive step is straightforward by permutation of rules. The base cases are obtained as follows:

$$\mathsf{mon_a}\ \dfrac{\mathsf{id}\ \overline{\mathcal{R}, x \leq x', x' \leq x'', \Gamma, x{:}a, x'{:}a \Longrightarrow \Delta, x''{:}a}}{\mathcal{R}, x \leq x', x' \leq x'', \Gamma, x{:}a \Longrightarrow \Delta, x''{:}a} \rightsquigarrow \mathsf{trans}\ \dfrac{\mathsf{id}\ \overline{\mathcal{R}, x \leq x', x' \leq x'', x \leq x'', \Gamma, x{:}a \Longrightarrow \Delta, x''{:}a}}{\mathcal{R}, x \leq x', x' \leq x'', \Gamma, x{:}a \Longrightarrow \Delta, x''{:}a}$$

$$\mathsf{mon_a}\ \dfrac{\mathsf{id}\ \overline{\mathcal{R}, x \leq x', x{:}a, x'{:}a \Longrightarrow \Delta, x{:}a}}{\mathcal{R}, x \leq x', \Gamma, x{:}a \Longrightarrow \Delta, x{:}a} \rightsquigarrow \mathsf{refl}\ \dfrac{\mathsf{id}\ \overline{\mathcal{R}, x \leq x', x \leq x, x{:}a \Longrightarrow \Delta, x{:}a}}{\mathcal{R}, x \leq x', \Gamma, x{:}a \Longrightarrow \Delta, x{:}a}$$

$$\mathsf{mon_a}\ \dfrac{\mathsf{id}\ \overline{\mathcal{R}, x \leq x', x{:}a, x'{:}a \Longrightarrow \Delta, x'{:}a}}{\mathcal{R}, x \leq x', \Gamma, x{:}a \Longrightarrow \Delta, x'{:}a} \rightsquigarrow \mathsf{id}\ \overline{\mathcal{R}, x \leq x', \Gamma, x{:}a \Longrightarrow \Delta, x'{:}a}$$

□

The next lemma shows that all the rules in our system are invertible, as already mentioned in the introduction.

Lemma 6.4. *All single-premise rules of* $\mathsf{labIK}_{\leq}$ *are height-preserving invertible. Furthermore, the rules* $\vee_\mathsf{L}$ *and* $\wedge_\mathsf{R}$ *are invertible on both premises, and the rule* $\supset_\mathsf{L}$ *is invertible on the right premise.*

Proof. For each rule r, we need to show that if there exists a proof $\mathcal{D}$ of the conclusion, there exists a proof $\mathcal{D}^{\mathsf{r}_i}$ of the i-th premise, of the same (or smaller) height. For $\wedge_\mathsf{R}$, $\wedge_\mathsf{L}$, $\vee_\mathsf{R}$, $\vee_\mathsf{L}$, and the right premise of $\supset_\mathsf{L}$, we use a standard induction on the height of $\mathcal{D}$. For $\supset_\mathsf{R}$, $\Box_\mathsf{R}$, $\Diamond_\mathsf{L}$ as well, but we need to make sure that the obtained derivation uses a fresh label by using substitution inside $\mathcal{D}^{\mathsf{r}}$ when necessary. The other rules can be shown invertible using Lemma 6.2. □

The next lemma is the central ingredient of our cut elimination proof.

Lemma 6.5. *Given a derivation of shape*

$$\mathsf{cut}\ \dfrac{\dfrac{\mathcal{D}_1\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta, z{:}C} \qquad \dfrac{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma, z{:}C \Longrightarrow \Delta}}{\mathcal{R}, \Gamma \Longrightarrow \Delta}$$

where $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are both cut-free, there is a cut-free derivation of* $\mathcal{R}, \Gamma \Longrightarrow \Delta$

Proof. The proof is by a lexicographic induction on the complexity of the cut-formula C and the sum of the heights $|\mathcal{D}_1| + |\mathcal{D}_2|$. We perform a case analysis on the last rule used in $\mathcal{D}_1$ above the cut and whether it applies to the cut-formula or not. In case it does not, we are in a *commutative* case; in case it does, we have to perform a similar analysis on $\mathcal{D}_2$ to end up in a *key* case.

Base cases: When the complexity of the cut-formula is 0, i.e. C is atomic, or in general when the height of $\mathcal{D}_1$ or $\mathcal{D}_2$ is 0, we can produce a cut-free derivation of the conclusion. In the first two cases below, we appeal to Lemma 6.3, to use atomic monotonicity rules freely.

- $$\mathsf{cut}\;\dfrac{\mathsf{id}\;\dfrac{}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a} \qquad \begin{array}{c}\mathcal{D}_2 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}a, y{:}a \Longrightarrow \Delta\end{array}}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta} \quad \rightsquigarrow \quad \mathsf{mon_L}\;\dfrac{\begin{array}{c}\mathcal{D}_2^{\mathsf{w}} \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}a, y{:}a \Longrightarrow \Delta\end{array}}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta}$$

- $$\mathsf{cut}\;\dfrac{\begin{array}{c}\mathcal{D}_1 \,\Vert \\ \mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}a, y{:}a\end{array} \qquad \mathsf{id}\;\dfrac{}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a}}{\mathcal{R}, x \le y, \Gamma \Longrightarrow \Delta, y{:}a} \quad \rightsquigarrow \quad \mathsf{mon_R}\;\dfrac{\begin{array}{c}\mathcal{D}_1^{\mathsf{w}} \,\Vert \\ \mathcal{R}, x \le y, \Gamma \Longrightarrow \Delta, x{:}a, y{:}a\end{array}}{\mathcal{R}, x \le y, \Gamma \Longrightarrow \Delta, y{:}a}$$

- $$\mathsf{cut}\;\dfrac{\mathsf{id}\;\dfrac{}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a, z{:}C} \qquad \begin{array}{c}\mathcal{D}_2 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}a, z{:}C \Longrightarrow \Delta, y{:}a\end{array}}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a} \quad \rightsquigarrow \quad \mathsf{id}\;\dfrac{}{\mathcal{R}, x \le y, \Gamma, x{:}a \Longrightarrow \Delta, y{:}a}$$

Commutative cases: In such a case, the complexity of the cut-formula stays constant, but the height of the derivation above the cut decreases.

- $\supset_{\mathsf{L}}$:

$$\mathsf{cut}\;\dfrac{\supset_{\mathsf{L}}\;\dfrac{\begin{array}{c}\mathcal{D}_1 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, z{:}C, y{:}A\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, y{:}B \Longrightarrow \Delta, z{:}C\end{array}}{\mathcal{R}, x \le y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, z{:}C} \qquad \begin{array}{c}\mathcal{D}_3 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}A \supset B, z{:}C \Longrightarrow \Delta\end{array}}{\mathcal{R}, x \le y, \Gamma, x{:}A \supset B \Longrightarrow \Delta} \quad \rightsquigarrow$$

$$\supset_{\mathsf{L}}\;\dfrac{\mathsf{cut}\;\dfrac{\begin{array}{c}\mathcal{D}_1 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, z{:}C, y{:}A\end{array} \qquad \begin{array}{c}\mathcal{D}_3^{\mathsf{w}} \,\Vert \\ \mathcal{R}, x \le y, \Gamma, x{:}A \supset B, z{:}C \Longrightarrow \Delta, y{:}A\end{array}}{\mathcal{R}, x \le y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, y{:}A} \qquad \mathsf{cut}\;\dfrac{\begin{array}{c}\mathcal{D}_2 \,\Vert \\ \mathcal{R}, x \le y, \Gamma, y{:}B \Longrightarrow \Delta, z{:}C\end{array} \qquad \begin{array}{c}\mathcal{D}_3^{\supset_{\mathsf{L}}} \,\Vert \\ \mathcal{R}, x \le y, \Gamma, y{:}B, z{:}C \Longrightarrow \Delta\end{array}}{\mathcal{R}, x \le y, \Gamma, y{:}B \Longrightarrow \Delta}}{\mathcal{R}, x \le y, \Gamma, x{:}A \supset B \Longrightarrow \Delta}$$

where $\mathcal{D}_3^{\mathsf{w}}$ is obtained using Lemma 6.2 and $\mathcal{D}_3^{\supset_{\mathsf{L}}}$ is obtained using Lemma 6.4. We use the same naming scheme in the following cases.

- $\supset_{\mathsf{R}}$:

$$\mathsf{cut}\;\dfrac{\supset_{\mathsf{R}}\;\dfrac{\begin{array}{c}\mathcal{D}_1 \,\Vert \\ \mathcal{R}, x \le x', \Gamma, x'{:}A \Longrightarrow \Delta, x'{:}B, z{:}C\end{array}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B, z{:}C}\; x' \text{ fresh} \qquad \begin{array}{c}\mathcal{D}_2 \,\Vert \\ \mathcal{R}, \Gamma, z{:}C \Longrightarrow \Delta, x{:}A \supset B\end{array}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B}$$

$$\rightsquigarrow \quad \supset_{\mathsf{R}}\;\dfrac{\mathsf{cut}\;\dfrac{\begin{array}{c}\mathcal{D}_1[x''/x'] \,\Vert \\ \mathcal{R}, x \le x'', \Gamma, x''{:}A \Longrightarrow \Delta, x''{:}B, z{:}C\end{array} \qquad \begin{array}{c}\mathcal{D}_2^{\supset_{\mathsf{R}}} \,\Vert \\ \mathcal{R}, x \le x'', \Gamma, z{:}C, x''{:}A \Longrightarrow \Delta, x''{:}B\end{array}}{\mathcal{R}, x \le x'', \Gamma, x''{:}A \Longrightarrow \Delta, x''{:}B}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \supset B}\; x'' \text{ fresh (also in } \mathcal{D}_2)$$

- $\Box_{\mathsf{L}}$:

$$\mathsf{cut}\;\dfrac{\Box_{\mathsf{L}}\;\dfrac{\begin{array}{c}\mathcal{D}_1 \,\Vert \\ \mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A, v{:}A \Longrightarrow \Delta, z{:}C\end{array}}{\mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box \Longrightarrow \Delta, z{:}C} \qquad \begin{array}{c}\mathcal{D}_2 \,\Vert \\ \mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A, z{:}C \Longrightarrow \Delta\end{array}}{\mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A \Longrightarrow \Delta}$$

$$\rightsquigarrow \quad \Box_{\mathsf{L}}\;\dfrac{\mathsf{cut}\;\dfrac{\begin{array}{c}\mathcal{D}_1 \,\Vert \\ \mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A, v{:}A \Longrightarrow \Delta, z{:}C\end{array} \qquad \begin{array}{c}\mathcal{D}_2^{\mathsf{w}} \,\Vert \\ \mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A, v{:}A, z{:}C \Longrightarrow \Delta\end{array}}{\mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A, v{:}A \Longrightarrow \Delta}}{\mathcal{R}, x \le u, uRv, \Gamma, x{:}\Box A \Longrightarrow \Delta}$$

- $\Box_{\mathsf{R}}$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, x \leq x', x'Ry', \Gamma \Longrightarrow \Delta, y'{:}A, z{:}C}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A, z{:}C}\;{\scriptstyle \Box_{\mathsf{R}}}\; x', y' \text{ fresh} \qquad \overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma, z{:}C \Longrightarrow \Delta, x{:}\Box A}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A}\;{\scriptstyle \mathsf{cut}}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, x \leq u, uRv, \Gamma \Longrightarrow \Delta, v{:}A, z{:}C} \qquad \overset{\mathcal{D}_2^{\Box_{\mathsf{R}}}\,\Vert}{\mathcal{R}, x \leq u, uRv, \Gamma, z{:}C \Longrightarrow \Delta, v{:}A}}{\mathcal{R}, x \leq u, uRv, \Gamma \Longrightarrow \Delta, v{:}A}\;{\scriptstyle \mathsf{cut}}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}\Box A}\;{\scriptstyle \Box_{\mathsf{R}}}\; u, v \text{ fresh (also in } \mathcal{D}_2)$$

- $\Diamond_{\mathsf{L}}$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, xRy', \Gamma, y'{:}A \Longrightarrow \Delta, z{:}C}}{\mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta, z{:}C}\;{\scriptstyle \Diamond_{\mathsf{L}}}\; y' \text{ fresh} \qquad \overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma, x{:}\Diamond A, z{:}C \Longrightarrow \Delta}}{\mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}_1[y''/y']\,\Vert}{\mathcal{R}, xRy'', \Gamma, y''{:}A \Longrightarrow \Delta, z{:}C} \qquad \overset{\mathcal{D}_2^{\Diamond_{\mathsf{L}}}\,\Vert}{\mathcal{R}, xRy'', \Gamma, y''{:}A, z{:}C \Longrightarrow \Delta}}{\mathcal{R}, xRy'', \Gamma, y''{:}A \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}}}{\mathcal{R}, \Gamma, x{:}\Diamond A \Longrightarrow \Delta}\;{\scriptstyle \Diamond_{\mathsf{L}}}\; y'' \text{ fresh (also in } \mathcal{D}_2)$$

- $\Diamond_{\mathsf{R}}$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, y{:}A, z{:}C}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, z{:}C}\;{\scriptstyle \Diamond_{\mathsf{R}}} \qquad \overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma, z{:}C \Longrightarrow \Delta, x{:}\Diamond A}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A}\;{\scriptstyle \mathsf{cut}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, y{:}A, z{:}C} \qquad \overset{\mathcal{D}_2^{\mathsf{w}}\,\Vert}{\mathcal{R}, \Gamma, z{:}C \Longrightarrow \Delta, x{:}\Diamond A, y{:}A}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, y{:}A}\;{\scriptstyle \mathsf{cut}}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A}\;{\scriptstyle \Diamond_{\mathsf{R}}}$$

Key cases: In the case when the last rule in $\mathcal{D}_1$ and in $\mathcal{D}_2$ applies to the cut-formula, it is the complexity of the cut-formula that is the decresing inductive measure.

- $C = A \wedge B$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A} \qquad \overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}B}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \wedge B}\;{\scriptstyle \wedge_{\mathsf{R}}} \qquad \dfrac{\overset{\mathcal{D}_3\,\Vert}{\mathcal{R}, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta}}{\mathcal{R}, \Gamma, x{:}A \wedge B \Longrightarrow \Delta}\;{\scriptstyle \wedge_{\mathsf{L}}}}{\mathcal{R}, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}} \quad \rightsquigarrow \quad \dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A} \qquad \dfrac{\overset{\mathcal{D}_2^{\mathsf{w}}\,\Vert}{\mathcal{R}, \Gamma, x{:}A \Longrightarrow \Delta, x{:}B} \qquad \overset{\mathcal{D}_3\,\Vert}{\mathcal{R}, \Gamma, x{:}A, x{:}B \Longrightarrow \Delta}}{\mathcal{R}, \Gamma, x{:}A \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}}}{\mathcal{R}, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}}$$

- $C = A \vee B$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A, x{:}B}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A \vee B}\;{\scriptstyle \vee_{\mathsf{R}}} \qquad \dfrac{\overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma, x{:}A \Longrightarrow \Delta} \qquad \overset{\mathcal{D}_3\,\Vert}{\mathcal{R}, \Gamma, x{:}B \Longrightarrow \Delta}}{\mathcal{R}, \Gamma, x{:}A \vee B \Longrightarrow \Delta}\;{\scriptstyle \vee_{\mathsf{L}}}}{\mathcal{R}, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A, x{:}B} \qquad \overset{\mathcal{D}_3^{\mathsf{w}}\,\Vert}{\mathcal{R}, \Gamma, x{:}B \Longrightarrow \Delta, x{:}A}}{\mathcal{R}, \Gamma \Longrightarrow \Delta, x{:}A}\;{\scriptstyle \mathsf{cut}} \qquad \overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, \Gamma, x{:}A \Longrightarrow \Delta}}{\mathcal{R}, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}}$$

- $C = A \supset B$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, x \leq y, x \leq x', \Gamma, x'{:}A \Longrightarrow \Delta, x'{:}B}}{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta, x{:}A \supset B}\;{\scriptstyle \supset_{\mathsf{R}}} \qquad \dfrac{\overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, x \leq y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, y{:}A} \qquad \overset{\mathcal{D}_3\,\Vert}{\mathcal{R}, x \leq y, \Gamma, y{:}B \Longrightarrow \Delta}}{\mathcal{R}, x \leq y, \Gamma, x{:}A \supset B \Longrightarrow \Delta}\;{\scriptstyle \supset_{\mathsf{L}}}}{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1^{\mathsf{w}}\,\Vert}{\mathcal{R}, x \leq y, x \leq x', \Gamma, x'{:}A \Longrightarrow \Delta, x'{:}B, y{:}A}}{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta, x{:}A \supset B, y{:}A}\;{\scriptstyle \supset_{\mathsf{R}}} \qquad \overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, x \leq y, \Gamma, x{:}A \supset B \Longrightarrow \Delta, y{:}A}}{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta, y{:}A}\;{\scriptstyle \mathsf{cut}} \qquad \dfrac{\overset{\mathcal{D}_1[y/x']\,\Vert}{\mathcal{R}, x \leq y, \Gamma, y{:}A \Longrightarrow \Delta, y{:}B} \qquad \overset{\mathcal{D}_3\,\Vert}{\mathcal{R}, x \leq y, \Gamma, y{:}B \Longrightarrow \Delta}}{\mathcal{R}, x \leq y, \Gamma, y{:}A \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}}}{\mathcal{R}, x \leq y, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}}$$

- $C = \Box A$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1\,\Vert}{\mathcal{R}, x \leq u, uRv, x \leq x', x'Ry', \Gamma \Longrightarrow \Delta, y'{:}A}}{\mathcal{R}, x \leq u, uRv, \Gamma \Longrightarrow \Delta, x{:}\Box A}\;{\scriptstyle \Box_{\mathsf{R}}} \qquad \dfrac{\overset{\mathcal{D}_2\,\Vert}{\mathcal{R}, x \leq u, uRv, \Gamma, x{:}\Box A, v{:}A \Longrightarrow \Delta}}{\mathcal{R}, x \leq u, uRv, \Gamma, x{:}\Box A \Longrightarrow \Delta}\;{\scriptstyle \Box_{\mathsf{L}}}}{\mathcal{R}, x \leq u, uRv, \Gamma \Longrightarrow \Delta}\;{\scriptstyle \mathsf{cut}} \quad \rightsquigarrow$$

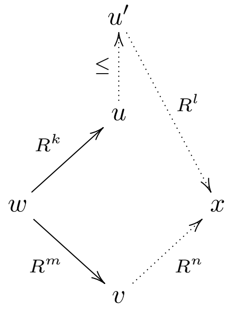

Figure 2: The intuitionistic klmn-incestuality condition

$$\mathsf{cut}\ \dfrac{\begin{matrix}\mathcal{D}_1[u/x',v/y'] \\ \mathcal{R}, x \leq u, uRv, \Gamma \Longrightarrow \Delta, v{:}A\end{matrix} \qquad \mathsf{cut}\ \dfrac{\Box_{\mathsf{R}}\ \dfrac{\begin{matrix}\mathcal{D}_1^{\mathsf{w}} \\ \mathcal{R}, x \leq u, uRv, x \leq x', x'Ry', \Gamma, v{:}A \Longrightarrow \Delta, x{:}\Box A, y'{:}A\end{matrix}}{\mathcal{R}, x \leq u, uRv, \Gamma, v{:}A \Longrightarrow \Delta, x{:}\Box A} \qquad \begin{matrix}\mathcal{D}_2 \\ \mathcal{R}, x \leq u, uRv, \Gamma, x{:}\Box A, v{:}A \Longrightarrow \Delta\end{matrix}}{\mathcal{R}, x \leq u, uRv, \Gamma, v{:}A \Longrightarrow \Delta}}{\mathcal{R}, x \leq u, uRv, \Gamma \Longrightarrow \Delta}$$

- $C = \Diamond A$:

$$\mathsf{cut}\ \dfrac{\Diamond_{\mathsf{R}}\ \dfrac{\begin{matrix}\mathcal{D}_1 \\ \mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A, y{:}A\end{matrix}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, x{:}\Diamond A} \qquad \Diamond_{\mathsf{L}}\ \dfrac{\begin{matrix}\mathcal{D}_2 \\ \mathcal{R}, xRy, xRy', \Gamma, y'{:}A \Longrightarrow \Delta\end{matrix}}{\mathcal{R}, xRy, \Gamma, x{:}\Diamond A \Longrightarrow \Delta}\ y' \text{ is fresh}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta}$$

$$\mathsf{cut}\ \dfrac{\mathsf{cut}\ \dfrac{\begin{matrix}\mathcal{D}_1 \\ \mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, y{:}A, x{:}\Diamond A\end{matrix} \qquad \Diamond_{\mathsf{L}}\ \dfrac{\begin{matrix}\mathcal{D}_2^{\mathsf{w}} \\ \mathcal{R}, xRy, xRy', \Gamma, y'{:}A \Longrightarrow \Delta, y{:}A\end{matrix}}{\mathcal{R}, xRy, \Gamma, x{:}\Diamond A \Longrightarrow \Delta, y{:}A}\ y' \text{ is fresh}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta, y{:}A} \qquad \begin{matrix}\mathcal{D}_2[y/y'] \\ \mathcal{R}, xRy, \Gamma, y{:}A \Longrightarrow \Delta\end{matrix}}{\mathcal{R}, xRy, \Gamma \Longrightarrow \Delta}$$

□

We can now complete the proof of Theorem 6.1.

Proof of Theorem 6.1. By induction on number of cut rules, always applying Lemma 6.5 to the leftmost topmost cut. □

7 Extensions

The main goal of this section is to generate stronger logics adding new axioms to our system. We say *stronger logic* to refer to the fact that we are restricting the class of frames we want to consider, imposing some restrictions on the accessibility relation.

In the *fully labelled* framework, we are able to consider the logics defined by *one-sided intuitionistic Scott-Lemmon axioms*:

$$\Diamond^k \Box^l A \supset \Box^m \Diamond^n A \tag{10}$$

for any natural numbers k, l, m, n.

Indeed, they are known to obey a strong correspondence with the class of frames satisfying the condition illustrated on Figure 2, which we call by analogy to the classical case, *intuitionistic klmn-incestuality condition.*

Theorem 7.1 ([PS86])**.** *An intuitionistic modal frame $\langle W, R, \leq \rangle$ validates $\Diamond^k \Box^l A \supset \Box^m \Diamond^n A$ if and only if the frame satisfies:*

if $wR^k u$ and $wR^m v$ then there exists u' such that $u \leq u'$ and there exists x such that $u'R^l x$ and $vR^n x$.

Following again the axiom-as-rule idea, to have a sound and complete system adding the axiom $\mathsf{g_{klmn}}$ to the system $\mathsf{labIK}_{\leq}$, we introduce the $\boxtimes_{\mathsf{gklmn}}$ rule, for any natural numbers k, l, m, n.

$$\boxtimes_{\mathsf{gklmn}}\ \dfrac{\mathcal{R}, y \leq y', xR^k y, xR^m z, y'R^l u, zR^n u, \Gamma \Longrightarrow \Delta}{\mathcal{R}, xR^k y, xR^m z, \Gamma \Longrightarrow \Delta}\ y', u \text{ fresh}$$

We can then show that Theorem 3.3 generalises nicely to $\mathsf{labIK}_{\leq}$ with any $\boxtimes_{\mathsf{gklmn}}$ rule to provide a sound and cut-free complete system for this family of logics.

Theorem 7.2. *For any formula A, the following are equivalent.*

1. *A is a theorem of* $\mathsf{IK} + \Diamond^k\Box^l A \supset \Box^m\Diamond^n A$.
2. *A is provable in* $\mathsf{labIK}_{\leq} + \boxtimes_{\mathsf{gklmn}} + \mathsf{cut}$.
3. *A is provable in* $\mathsf{labIK}_{\leq} + \boxtimes_{\mathsf{gklmn}}$.
4. *A is valid in every birelational frame satisfying the klmn-incestuality property.*

Proof. The proof is similar to the one of Theorem 3.3.

- $1 \implies 2$: Same as Thm. 4.1 with the additional derivation of $\Diamond^k\Box^l A \supset \Box^m\Diamond^n A$:

$$
\begin{array}{rc}
\mathsf{id} & \hline
y_k{:}\Box^l A, w{:}A \Longrightarrow x_m{:}\Diamond^n A, w{:}A \\
\Box_{\mathsf{L}} & \hline\hline
y_k \leq y_k'', \ldots, w \leq w, y_k{:}\Box^l A \Longrightarrow x_m{:}\Diamond^n A, w{:}A \\
\mathsf{trans{+}refl} & \hline
y_k{:}\Box^l A \Longrightarrow x_m{:}\Diamond^n A, w{:}A \\
\Diamond_{\mathsf{R}} & \hline\hline
y_k' \leq y_k'', y_k'' R^l w, x_m R^n w, y_k{:}\Box^l A \Longrightarrow x_m{:}\Diamond^n A \\
\boxtimes_{\mathsf{gklmn}} & \hline
x_0'' R y_1', \{y_j' R y_{j+1}'\}_{1\leq j\leq k-1}, y_k{:}\Box^l A \Longrightarrow x_m{:}\Diamond^n A \\
\mathsf{F}_2 & \hline\hline
x_0 R y_1, \{y_j R y_{j+1}\}_{1\leq j\leq k-1}, y_k{:}\Box^l A \Longrightarrow x_m{:}\Diamond^n A \\
\Diamond_{\mathsf{L}} & \hline\hline
x_{m-2}' \leq x_{m-2}'', x_{m-2}'' R x_{m-1}', \{x_i' \leq x_i'', x_i'' R x_{i+1}''\}_{0\leq i\leq m-2}, x_0{:}\Diamond^k\Box^l A \Longrightarrow x_m{:}\Diamond^n A \\
\mathsf{F}_1 & \hline\hline
\{x_i \leq x_i', x_i' R x_{i+1}\}_{0\leq i\leq m-1}, x_0{:}\Diamond^k\Box^l A \Longrightarrow x_m{:}\Diamond^n A \\
\Box_{\mathsf{R}} & \hline\hline
x \leq x_0, x_0{:}\Diamond^k\Box^l A \Longrightarrow x_0{:}\Box^m\Diamond^n A \\
\supset_{\mathsf{R}} & \hline
\Longrightarrow x{:}\Diamond^k\Box^l A \supset \Box^m\Diamond^n A
\end{array}
$$

  where we omit the accumulated relational context for space reason.

- $2 \implies 3$: To prove that the rule cut is admissible for $\mathsf{labIK}_{\leq} + \boxtimes_{\mathsf{gklmn}}$, it is enough to insert a case for the rule $\boxtimes_{\mathsf{gklmn}}$ in the proof of Theorem 3.3, which is straightforward as the $\boxtimes_{\mathsf{gklmn}}$ rule only manipulates the relational context.

- $3 \implies 4$: As we already proved the rules of $\mathsf{labIK}_{\leq}$ sound in Theorem 5.3, we only need to prove that $\boxtimes_{\mathsf{gklmn}}$ is sound. By way of contradiction, assume that $\mathcal{R}, y \leq y', xR^k y, xR^m z, y'R^l u, zR^n u, \Gamma \Longrightarrow \Delta$ is valid in any klmn-incestuous frame, but that there is such a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \not\Vdash \mathcal{R}, xR^k y, xR^m z, \Gamma \Longrightarrow \Delta$. That means, $[\![x]\!] R^k_{\mathcal{M}} [\![y]\!]$, $[\![x]\!] R^m_{\mathcal{M}} [\![z]\!]$, for all $x{:}A \in \Gamma$, $\mathcal{M}, x \Vdash A$ and for all $w{:}B \in \Delta$, $\mathcal{M}, w \not\Vdash B$. Since $\mathcal{M}$ is klmn-incestuous, there exists $v, w \in W_{\mathcal{M}}$, such that $[\![y]\!] \leq_{\mathcal{M}} v$, $vR^l_{\mathcal{M}} w$, and $[\![z]\!] R^n_{\mathcal{M}} w$. Now let $[\![\cdot]\!]^*$ be the extension of $[\![\cdot]\!]$ such that $[\![y']\!]^* = v$, $[\![u]\!]^* = w$, and $[\![\cdot]\!]^* = [\![\cdot]\!]$ otherwise. Then, $\mathcal{M}, [\![\cdot]\!]^* \not\Vdash \mathcal{R}, y \leq y', xR^k y, xR^m z, y'R^l u, zR^n u, \Gamma \Longrightarrow \Delta$. Contradiction.

The proof is completed by appealing to Theorem 7.1 used as $4 \implies 1$ to close the equivalence. $\square$

Remark 7.3. As an illustration of our system, we reconsider an example that as problematic in previous approaches to the logic $\mathsf{IK} + \mathsf{g}_{1111}$. Indeed, the formula $\Diamond(\Box(a \vee b) \wedge \Diamond a) \wedge \Diamond(\Box(a \vee b) \wedge \Diamond b) \supset \Diamond(\Diamond a \wedge \Diamond b)$ is not a theorem of this logic, but becomes provable if we directly add the rule corresponding to the directedness condition $(\forall xyz.\exists u.((xRy \wedge xRz) \supset (yRu \wedge zRu)))$ to our system. By adding the correct rule g_{1111} as defined above, we can mimic the birelational semantics precisely and we cannot derive this formula, as illustrated by the representation of the failed proof search below:

$$
\begin{array}{rc}
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
 & \mathcal{R}, z' \leq z'', z''Rv'', y'Rv'', y'{:}\Box(a\vee b), u''{:}b, y'{:}\Diamond a, z'{:}\Box(a\vee b), v''{:}a, z'{:}\Diamond b \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b, v''{:}b, z'{:}\Diamond a, u''{:}a \\
\vee_{\mathsf{L}}{+}\mathsf{id} & \hline
\mathcal{R}, z' \leq z'', z''Rv'', y'Rv'', y'{:}\Box(a\vee b), u''{:}b, y'{:}\Diamond a, z'{:}\Box(a\vee b), v''{:}a\vee b, z'{:}\Diamond b \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b, v''{:}b, z'{:}\Diamond a, u''{:}a \\
g1111,\Box_{\mathsf{L}},\Diamond_{\mathsf{R}} & \hline
x \leq x', x'Ry', x'Rz', y'Ry'', y''Ru'', z'Ru'', y'{:}\Box(a\vee b), u''{:}b, y'{:}\Diamond a, z'{:}\Box(a\vee b), z'{:}\Diamond b \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b, z'{:}\Diamond a, u''{:}a \\
\vee_{\mathsf{L}}{+}\mathsf{id} & \hline
x \leq x', x'Ry', x'Rz', y'Ry'', y''Ru'', z'Ru'', y'{:}\Box(a\vee b), u''{:}a\vee b, y'{:}\Diamond a, z'{:}\Box(a\vee b), z'{:}\Diamond b \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b, z'{:}\Diamond a, u''{:}a \\
g1111,\Box_{\mathsf{L}},\Diamond_{\mathsf{R}} & \hline
x \leq x', x'Ry', x'Rz', y'{:}\Box(a\vee b), y'{:}\Diamond a, z'{:}\Box(a\vee b), z'{:}\Diamond b \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b, z'{:}\Diamond a \\
\wedge_{\mathsf{R}}{+}\mathsf{id} & \hline
x \leq x', x'Ry', x'Rz', y'{:}\Box(a\vee b), y'{:}\Diamond a, z'{:}\Box(a\vee b), z'{:}\Diamond b \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b, z'{:}\Diamond a \wedge \Diamond b \\
\Diamond_{\mathsf{L}},\wedge_{\mathsf{L}},\Diamond_{\mathsf{R}} & \hline
x \leq x', x'Ry', y'{:}\Box(a\vee b), y'{:}\Diamond a, x'{:}\Diamond(\Box(a\vee b) \wedge \Diamond b) \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond b \\
\wedge_{\mathsf{R}}{+}\mathsf{id} & \hline
x \leq x', x'Ry', y'{:}\Box(a\vee b), y'{:}\Diamond a, x'{:}\Diamond(\Box(a\vee b) \wedge \Diamond b) \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b), y'{:}\Diamond a \wedge \Diamond b \\
\Diamond_{\mathsf{L}},\wedge_{\mathsf{L}},\Diamond_{\mathsf{R}} & \hline
x \leq x', x'{:}\Diamond(\Box(a\vee b) \wedge \Diamond a), x'{:}\Diamond(\Box(a\vee b) \wedge \Diamond b) \Longrightarrow x'{:}\Diamond(\Diamond a \wedge \Diamond b) \\
\supset_{\mathsf{R}},\wedge_{\mathsf{L}} & \hline
\Longrightarrow x{:}(\Diamond(\Box(a\vee b) \wedge \Diamond a) \wedge \Diamond(\Box(a\vee b) \wedge \Diamond b)) \supset \Diamond(\Diamond a \wedge \Diamond b)
\end{array}
$$

where $\mathcal{R} = x \leq x', x'Ry', x'Rz', y'Ry'', y''Ru'', z'Ru''$.

8 Conclusion

In this paper we embrace the fully labelled approach to intuitionistic modal logic as pioneered by [MNN13] and generalise it to the class of logics defined by (one-sided intuitionistic) Scott-Lemmon axioms. We establish that it is a valid approach to intuitionistic modal logic by proving soundness and completeness of our system, via a reductive cut-elimination argument.

For a restricted class of logics defined by so-called *path axioms*: $(\Diamond^k \Box A \supset \Box^m A) \wedge (\Diamond^k A \supset \Box^m \Diamond A)$ the standard labelled framework with one relation R was enough for Simpson to get a strong connection between the sequent system, the axiomatisation and the birelational semantics [Sim94]. We believe that the framework presented here might be the more appropriate way to treat logics outside of the path axioms definable fragment.

However, we have not showed that our system satisfies Simpson's 6th requirement, that is, "there is an intuitionistically comprehensible explanation of the meaning of the modalities relative to which [our system] is sound and complete". To make sure that his system satisfies this requirement, Simpson chose to depart from the direct correspondence with modal axioms and their corresponding class of Kripke frames, and to study intuitionistic purely as a fragment of intuitionistic first-order logic. We instead took the way of a direct correspondence of our system with the class of frames defined by Scott-Lemmon axioms as uncovered by [PS86], but as this class of logics seems to be rather well-behaved, we believe it shoud be possible to prove the satisfaction of Simpson's 6th requirement too.

As for more general future work, there is a real necessity of a global view on intutionistic modal logics. The work of [DGO19] is a great first step in understanding them in the context of non-normal modalities and neighbourhood semantics. It would be interesting to know how and where the class of logics we considered can be included in their framework.

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