Ecumenical Modal Logic

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Enter the utopia

A world where classical and intuitionistic logicians live in harmony...

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Roadmap:

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Roadmap:

1. The challenge of constructive modal logic
   - constructive and beyond

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2. Ecumenism from an outsider
   - more in the Q&A

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Roadmap:

1. The challenge of constructive modal logic
   - constructive and beyond
2. Ecumenism from an outsider
   - more in the Q&A
3. Ecumenical modal logic
   - the first steps

The challenge of constructive modal logic

Classical Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$
- **Duality** by De Morgan laws and $\neg\Box A = \Diamond\neg A$

- Axioms: **classical** propositional logic and

$$\text{k}: \ \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Relational structures (W, R)

  a non-empty set W of *worlds*;

  a binary relation $R \subseteq W \times W$;

Intuitionistic Modal Logic

- Formulas: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A$
- Independence of the modalities

- Axioms: intuitionistic propositional logic and

$$\mathsf{k}\colon\ \Box(A \to B) \to (\Box A \to \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Bi relational structures $(W, R, \leq)$

  a non-empty set W of *worlds*;

  a binary relation $R \subseteq W \times W$;

  and a preorder $\leq$ on W.

$$(F_1)\quad \begin{tikzcd} u' \arrow[r, dotted, dash, "R"] & v' \\ u \arrow[u, dotted, dash, "\leq"] \arrow[r, dash, "R"] & v \arrow[u, dash, "\leq"'] \end{tikzcd}$$

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\mathsf{k}_4: & (\Diamond A \to \Box B) \to \Box(A \to B) \\
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$$x \vDash \Box A \Leftrightarrow \exists y, z. x \leq y \;\&\; y R z \;\&\; z \vDash A$$

Axioms: classical propositional logic and

$$\mathsf{k}:\ \Box(A \to B) \to (\Box A \to \Box B)$$

Sequent system: classical sequent calculus and

$$\mathsf{k}_{\Box}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

Intuitionistic modal proof theory

Axioms: intuitionistic propositional logic and

$$\mathsf{k}\colon\ \Box(A \to B) \to (\Box A \to \Box B)$$

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Problem? k_4 is not derivable.

- not a problem for modal type theory...

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Labelled sequent system: (Simpson 1994)

$$\Box_\mathrm{L} \ \frac{xRy, \Gamma, x : \Box A, y : A \Rightarrow z : B}{xRy, \Gamma, x : \Box A \Rightarrow z : B} \qquad \Box_\mathrm{R} \ \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A} \ y \text{ is fresh}$$

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Ecumenism from an outsider

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But why (and where) do they disagree?

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{init}}{\vdash A, \neg A}\ \neg R}{\vdash A \vee \neg A}\ \vee R \qquad \dfrac{\dfrac{\overset{?}{A \vdash \bot}}{\vdash \neg A}\ \neg R}{\vdash A \vee \neg A}\ \vee R_2$$

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A solution: They are not talking about the same connective(s) (Prawitz 2015)

Ecumenical connectives and rules

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \rightarrow_c B} \rightarrow_c R \qquad \frac{}{\Gamma, \bot \vdash C} \bot L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R$$

$$\frac{\Gamma, \neg A, \neg B \vdash \bot}{\Gamma \vdash A \vee_c B} \vee_c R \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma \vdash A_j}{\Gamma \vdash A_1 \vee_i A_2} \vee_i R_j$$

$$\frac{\Gamma, \forall x.\neg A \vdash \bot}{\Gamma \vdash \exists_c x.A} \exists_c R \qquad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x.A} \forall R \qquad \frac{\Gamma \vdash A[a/x]}{\Gamma \vdash \exists_i x.A} \exists_i R$$

Classical **Shared** **Intuitionistic**

(Pimentel, Pereira, de Paiva 2020)

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Classical

$$\frac{}{\Gamma, \bot \vdash C} \bot L$$

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Intuitionistic

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Ecumenism:

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In this work: extend Prawitz' Ecumenical idea to modalities.

Ecumenical modal logic

Ecumenical standard translation:

$$[\Box A]^e_x = \forall y(R(x,y) \to_i [A]^e_y)$$

$$[\Diamond_i A]^e_x = \exists_i y(R(x,y) \wedge [A]^e_y) \qquad [\Diamond_c A]^e_x = \exists_c y(R(x,y) \wedge [A]^e_y)$$

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- $\Diamond_c A \leftrightarrow_i \neg\Box\neg A$ but $\Diamond_i A \not\leftrightarrow_i \neg\Box\neg A$.
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Labelled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R$$

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- A labelled sequent system for EK.
- Axiomatization:

$$
\begin{aligned}
\mathsf{k}_1 &: \Box(A \to_i B) \to_i (\Box A \to_i \Box B)\\
\mathsf{k}_2 &: \Box(A \to_i B) \to_i (\Diamond_i A \to_i \Diamond_i B)\\
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\mathsf{k}_4 &: (\Diamond_i A \to_i \Box B) \to_i \Box(A \to_i B)\\
\mathsf{k}_5 &: \Diamond_i \bot \to_i \bot
\end{aligned}
$$

- Ecumenical birelational models.
- Extensions.
- Cut-elimination.

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Important remark:
It is true that we can prove $(A \vee_c B) \equiv \neg(\neg A \wedge \neg B)$ in the ecumenical system, but this analysis relies on having three different operators, $\neg$, $\vee_c$ and $\wedge$.

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Ecumenical systems may help us to have a better understanding of the relation between classical logic and intuitionistic logic.

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What can we say about *modal* Ecumenical systems?

- constructive modal logic and beyond;
- what would be the meaning of classical possibility with "purer" rules;
- explore some relations between general results on translations and Ecumenical systems, expanding this discussion to modalities.