

Ecumenical Modal Logic

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A world where classical and intuitionistic logicians live in harmony...

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Roadmap:

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Roadmap:

1. The challenge of constructive modal logic
 - ▶ constructive and beyond

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2. Ecumenism from an outsider
 - ▶ more in the Q&A

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1. The challenge of constructive modal logic
 - ▶ constructive and beyond
2. Ecumenism from an outsider
 - ▶ more in the Q&A
3. Ecumenical modal logic
 - ▶ the first steps

The challenge of constructive modal logic

Classical Modal Logic

- ▶ Formulas: $A ::= p \mid \perp \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$
- ▶ **Duality** by De Morgan laws and $\neg\Box A = \Diamond\neg A$
- ▶ Axioms: **classical** propositional logic and

$$k: \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

- ▶ Rules: modus ponens: $\frac{A \quad A \rightarrow B}{B}$ necessitation: $\frac{A}{\Box A}$
- ▶ Semantics: Relational structures (W, R)
 - a non-empty set W of *worlds*;
 - a binary relation $R \subseteq W \times W$;

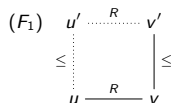
Intuitionistic Modal Logic

- ▶ Formulas: $A ::= p \mid \perp \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$
- ▶ **Independence** of the modalities
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a non-empty set W of *worlds*;
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and a **preorder** \leq on W .



Intuitionistic Modal Logic

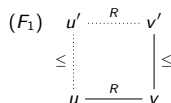
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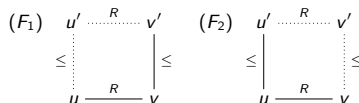
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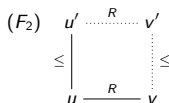
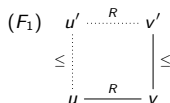
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$$x \models \Box A \Leftrightarrow \exists y, z. x \leq y \ \& \ y R z \ \& \ z \models A$$

Axioms: classical propositional logic and

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Sequent system: classical sequent calculus and

$$k_{\Box} \frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A}$$

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Problem? k_4 is not derivable.

- ▶ not a problem for modal type theory...

Intuitionistic modal proof theory

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Labelled sequent system: (Simpson 1994)

$$\begin{array}{c} \Box_L \frac{xRy, \Gamma, x : \Box A, y : A \Rightarrow z : B}{xRy, \Gamma, x : \Box A \Rightarrow z : B} \quad \Box_R \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A} \text{ } y \text{ is fresh} \\ \\ \Diamond_L \frac{xRy, \Gamma, y : A \Rightarrow z : B}{\Gamma, x : \Diamond A \Rightarrow z : B} \text{ } y \text{ is fresh} \quad \Diamond_R \frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond A} \end{array}$$

Ecumenism from an outsider

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But why (and where) do they disagree?

$$\frac{\frac{\overline{A \vdash A} \text{ init}}{\vdash A, \neg A} \neg R}{\vdash A \vee \neg A} \vee R \qquad \frac{\frac{\frac{?}{A \vdash \perp}}{\vdash \neg A} \neg R}{\vdash A \vee \neg A} \vee R_2$$

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A solution: They are not talking about the same connective(s) (Prawitz 2015)

$$\frac{\Gamma, A, \neg B \vdash \perp}{\Gamma \vdash A \rightarrow_c B} \rightarrow_c R$$

$$\frac{\Gamma, \neg A, \neg B \vdash \perp}{\Gamma \vdash A \vee_c B} \vee_c R$$

$$\frac{\Gamma, \forall x. \neg A \vdash \perp}{\Gamma \vdash \exists_c x. A} \exists_c R$$

Classical

$$\overline{\Gamma, \perp \vdash C} \perp L$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R$$

$$\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x. A} \forall R$$

Shared

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R$$

$$\frac{\Gamma \vdash A_j}{\Gamma \vdash A_1 \vee_i A_2} \vee_i R_j$$

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Intuitionistic

(Pimentel, Pereira, de Paiva 2020)

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In this work: extend Prawitz' Ecumenical idea to modalities.

Ecumenical modal logic

Ecumenical standard translation:

$$[\Box A]_x^e = \forall y (R(x, y) \rightarrow_i [A]_y^e)$$

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- ▶ $\Diamond_c A \leftrightarrow_i \neg \Box \neg A$ but $\Diamond_i A \not\leftrightarrow_i \neg \Box \neg A$.
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Labelled modal rules:

$$\frac{x : \Box \neg A, \Gamma \vdash x : \perp}{\Gamma \vdash x : \Diamond_c A} \Diamond_c R$$

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- ▶ A labelled sequent system for EK.

- ▶ Axiomatization:

$$k_1 : \Box(A \rightarrow_i B) \rightarrow_i (\Box A \rightarrow_i \Box B)$$

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$$k_5 : \Diamond_i \perp \rightarrow_i \perp$$

- ▶ Ecumenical birelational models.
- ▶ Extensions.
- ▶ Cut-elimination.

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Important remark:

It is true that we can prove $(A \vee_c B) \equiv \neg(\neg A \wedge \neg B)$ in the ecumenical system, but this analysis relies on having three different operators, \neg , \vee_c and \wedge .

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What can we say about *modal* Ecumenical systems?

- ▶ constructive modal logic and beyond;
- ▶ what would be the meaning of classical possibility with “purer” rules;
- ▶ explore some relations between general results on translations and Ecumenical systems, expanding this discussion to modalities.