Focusing, Axioms, and Synthetic Inference Rules

Elaine Pimentel
UFRN/Brazil
&
Sonia Marin
University College London/UK

Joint work with Dale Miller and Marco Volpe

IJCAR 2020

July 2nd, 2020

The axioms-as-rules problem

How to incorporate inference rules encoding axioms into existing proof systems for classical and intuitionistic logics?

A fresh view to an old problem:
The combination of bipolars and focusing provides simple rules for atomic formulas.

Motivation

Object

Reasoning

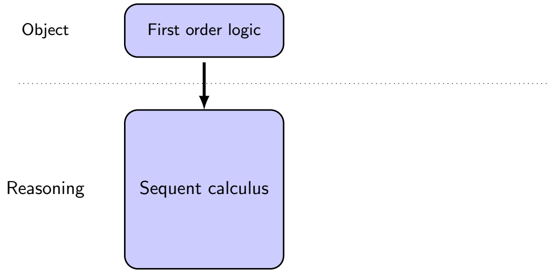
Motivation

Object

First order logic

Reasoning

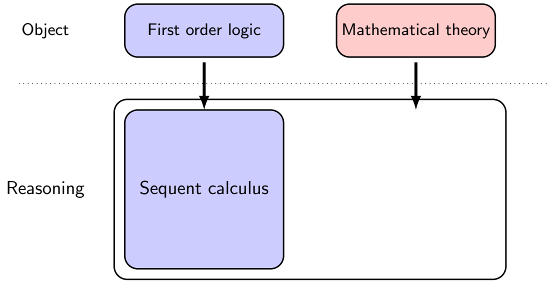
Motivation



Advantages of sequent systems [Gen35b] as frameworks

- simple calculi;
- good proof theoretical properties (cut-elimination, consistency);
- can be easily implemented (λ-Prolog, rewriting).

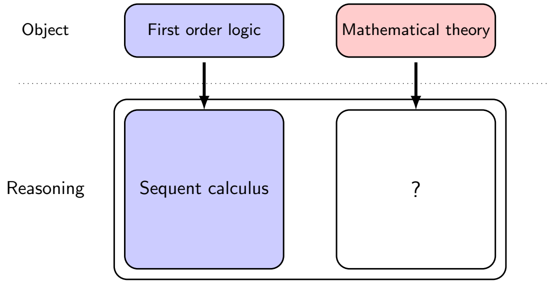
Motivation



Nice idea:

Add mathematical theories to first order logics and reason about them using all the machinery already built for the sequent framework.

Motivation



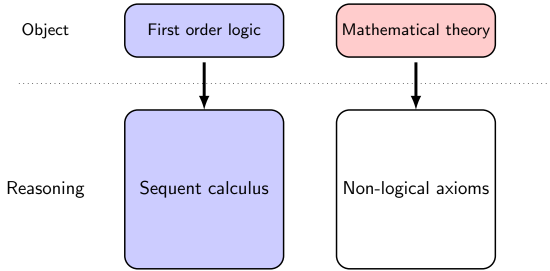
Nice idea:

Add mathematical theories to first order logics and reason about them using all the machinery already built for the sequent framework.

So nice that many (nice ☺) people pursued it:

- ⋆ Sara Negri, Jan von Plato, and Roy Dyckhoff, in first-order logic [NvP98, DN15];
- ⋆ as well as, Alex Simpson [Sim94], Luca Viganò [Vig00], Agata Ciabattoni [CGT08], in fragments of first-order logic such as modal and substructural logics;
- ⋆ and Gilles Dowek [DW05], in Deduction Modulo Theories, to only name a few.

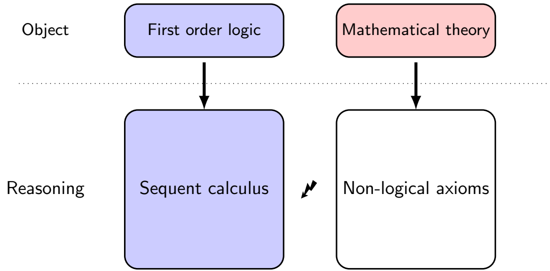
Motivation



Add non-logical axioms [NvP98]: assume $\vdash P \supset Q$ and $\vdash P$. Then

$$\dfrac{\dfrac{}{\vdash P} \qquad \dfrac{\dfrac{}{\vdash P \supset Q} \qquad \dfrac{\dfrac{}{P \vdash P} \quad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q}\,\supset_l}{P \vdash Q}\,cut}{\vdash Q}\,cut$$

Motivation

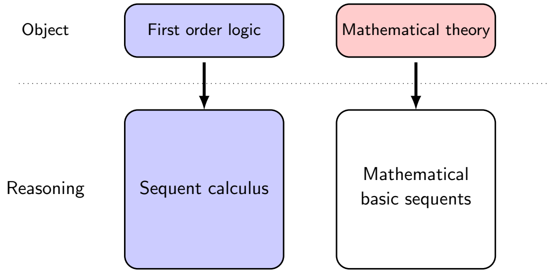


Add non-logical axioms [NvP98]: assume $\vdash P \supset Q$ and $\vdash P$. Then

$$\dfrac{\dfrac{}{\vdash P} \qquad \dfrac{\dfrac{}{\vdash P \supset Q} \qquad \dfrac{\dfrac{}{P \vdash P} \quad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q}\,\supset_l}{P \vdash Q}\,\mathit{cut}}{\vdash Q}\,\mathit{cut}$$

Girard: The *Hauptsatz* fails for systems with proper axioms. [Gir87]

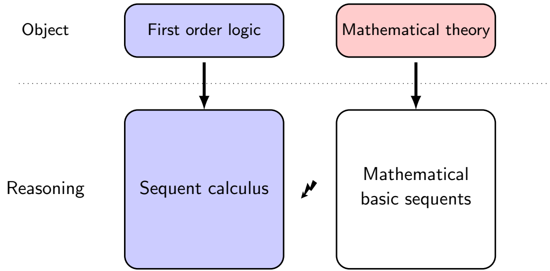
Motivation



Add mathematical basic sequents [NvP98]: assume $P \vdash Q$ and $\vdash P$. Then

$$\dfrac{\overline{\vdash P} \quad \overline{P \vdash Q}}{\vdash Q}\ cut$$

Motivation

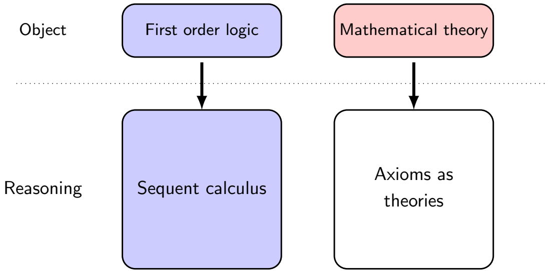


Add mathematical basic sequents [NvP98]: assume $P \vdash Q$ and $\vdash P$. Then

$$\dfrac{\overline{\vdash P} \quad \overline{P \vdash Q}}{\vdash Q}\ \mathit{cut}$$

Gentzen: *Hauptsatz* doesn't extend to basic sequents as premises. [Gen38]

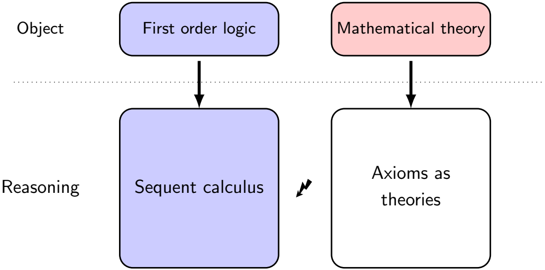
Motivation



Add axioms as theories [NvP98]:

$$\dfrac{\overline{P \vdash P} \quad \overline{Q \vdash Q}}{P, P \supset Q \vdash Q} \supset_l$$

Motivation

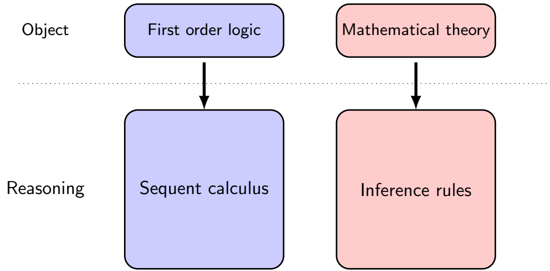


Add axioms as theories [NvP98]:

$$\dfrac{\overline{P \vdash P} \quad \overline{Q \vdash Q}}{P, P \supset Q \vdash Q} \supset_l$$

Gentzen's consistency proof of elementary arithmetic. [Gen35a]

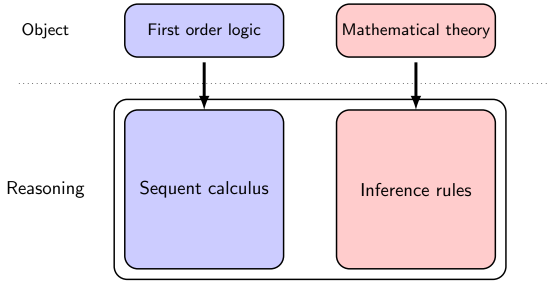
Motivation



Add non-logical rules of inference [Sim94, NvP98]:

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\ P \supset Q \qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\ P$$

Motivation



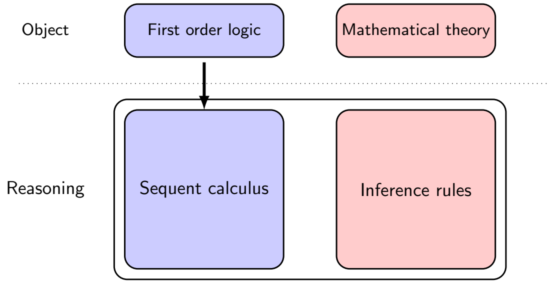
Add non-logical rules of inference [Sim94, NvP98]:

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\; P \supset Q \qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\; P$$

The sequent $\vdash Q$ now has the (cut-free) proof

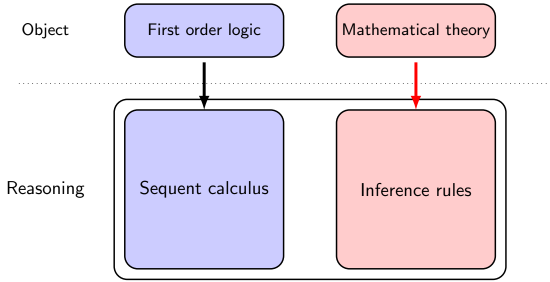
$$\cfrac{\cfrac{\overline{Q \vdash Q}}{P \vdash Q}\; P \supset Q}{\vdash Q}\; P$$

In this talk



A fresh view to an old problem:

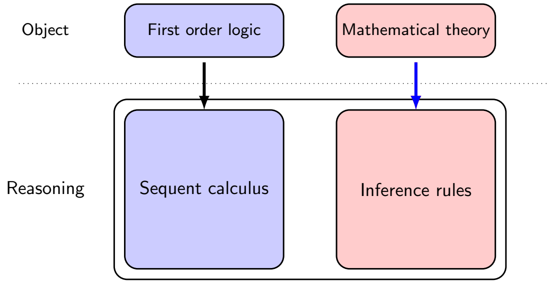
In this talk



Which ones and why?

A fresh view to an old problem:

In this talk

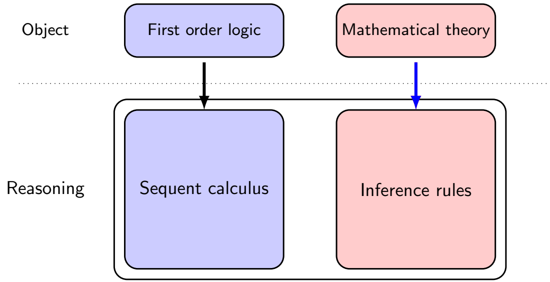


Which ones and why?

bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:

In this talk

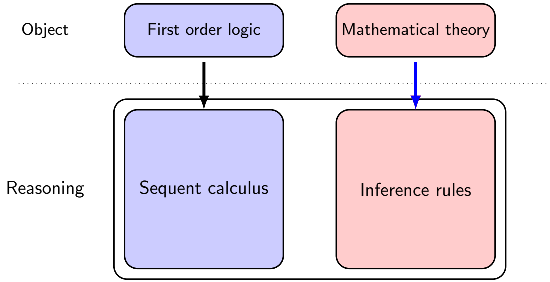


Which ones and why?

bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:
Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08])

In this talk

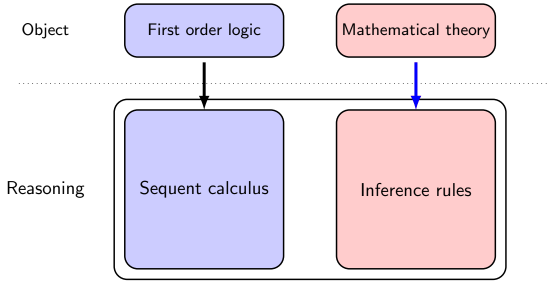


Which ones and why?

bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:
Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08]) with a systematic construction of inference rules from axioms using focusing [And92],

In this talk

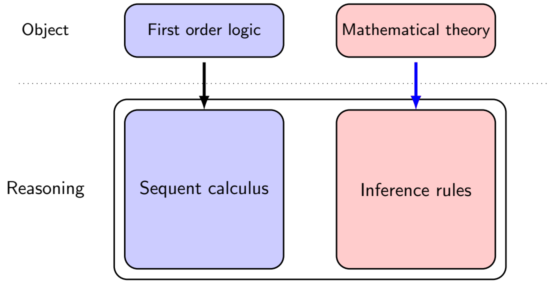


Which ones and why?

bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:
Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08]) with a systematic construction of inference rules from axioms using focusing [And92], justifies the introduction of the class of bipolar axioms.

In this talk



Which ones and why?

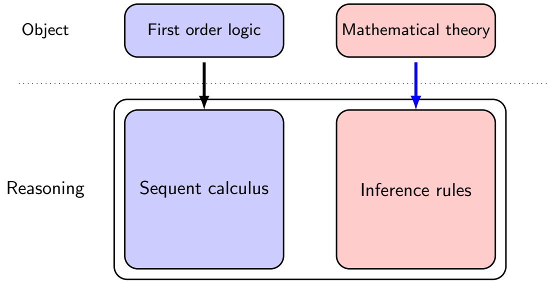
bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:

Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08]) with a systematic construction of inference rules from axioms using focusing [And92], justifies the introduction of the class of bipolar axioms.

- Systematically compute inference rules from bipolar axioms (λ-Prolog prototype);

In this talk



Which ones and why?

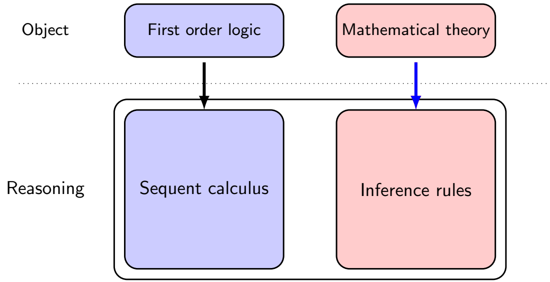
bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:

Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08]) with a systematic construction of inference rules from axioms using focusing [And92], justifies the introduction of the class of bipolar axioms.

- Systematically compute inference rules from bipolar axioms (λ-Prolog prototype);
- Uniform presentation for classical and intuitionistic first order systems;

In this talk



Which ones and why?

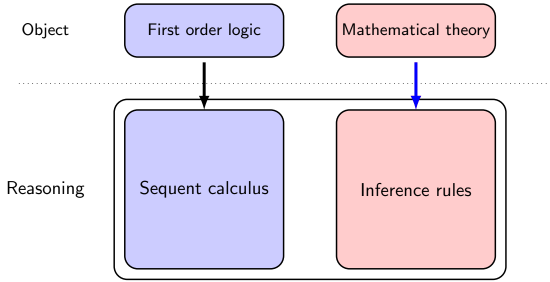
bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:

Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08]) with a systematic construction of inference rules from axioms using focusing [And92], justifies the introduction of the class of bipolar axioms.

- Systematically compute inference rules from bipolar axioms (λ-Prolog prototype);
- Uniform presentation for classical and intuitionistic first order systems;
- Generalization of the literature (e.g. on geometric theories [Neg03, NvP11, Neg16, CMS13] and [Vig00]);

In this talk



Which ones and why?

bipolars + focusing
=
synthetic inference rules
for atoms

A fresh view to an old problem:

Combining the classification of axioms into a polarities' hierarchy (inspired by [CGT08]) with a systematic construction of inference rules from axioms using focusing [And92], justifies the introduction of the class of bipolar axioms.

- Systematically compute inference rules from bipolar axioms (λ-Prolog prototype);
- Uniform presentation for classical and intuitionistic first order systems;
- Generalization of the literature (e.g. on geometric theories [Neg03, NvP11, Neg16, CMS13] and [Vig00]);
- Cut-elimination guaranteed for the system with the new inferences, via focusing.

Outline

Outline

Polarities of connectives

First-order classical and intuitionistic language:

$$A ::= P(x) \mid A \wedge A \mid t \mid A \vee A \mid f \mid A \supset A \mid \exists x\, A \mid \forall x\, A$$

Polarized connectives:

- In classical logic
  - positive and negative versions of the logical connectives and constants:
    $$\wedge^-, \wedge^+, t^-, t^+, \vee^-, \vee^+, f^-, f^+$$
  - first-order quantifiers: $\forall$ negative and $\exists$ positive.
- In intuitionistic logic
  - polarized classical connectives and constants where $f^-, \vee^-$ do not occur;
  - negative implication: $\supset$.

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\frac{\Gamma \vdash A_1 \quad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\frac{\Gamma \vdash A_1 \quad \overline{\Gamma, A_0 \vdash B}}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Negative protocol: The right branch is trivial: $A_0 = B$. Continue with $\Gamma \vdash A_1$.

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\dfrac{\Gamma \vdash A_1 \qquad \overline{\Gamma, A_0 \vdash B}}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Negative protocol: The right branch is trivial: $A_0 = B$. Continue with $\Gamma \vdash A_1$.

$$\dfrac{\Gamma \vdash A_1 \qquad \dfrac{\Gamma \vdash A_2 \qquad \dfrac{\Gamma \vdash A_3 \qquad \dfrac{\Gamma \vdash A_4 \qquad \dfrac{B = A_0}{\Gamma, A_0 \vdash B}}{\Gamma, A_4 \supset A_0 \vdash B}}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\ L\supset$$

Back-chaining!

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\dfrac{\overline{\Gamma \vdash A_1} \quad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Negative protocol: The right branch is trivial: $A_0 = B$. Continue with $\Gamma \vdash A_1$.

Positive protocol: The left branch is trivial: $\Gamma = \Gamma', A_1$. Continue with $\Gamma', A_1, A_0 \vdash B$.

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\frac{\overline{\Gamma \vdash A_1} \quad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Negative protocol: The right branch is trivial: $A_0 = B$. Continue with $\Gamma \vdash A_1$.

Positive protocol: The left branch is trivial: $\Gamma = \Gamma', A_1$. Continue with $\Gamma', A_1, A_0 \vdash B$.

$$\dfrac{\dfrac{A_1 \in \Gamma}{\Gamma \vdash A_1} \quad \dfrac{\dfrac{A_2 \in \Gamma}{\Gamma \vdash A_2} \quad \dfrac{\dfrac{A_3 \in \Gamma}{\Gamma \vdash A_3} \quad \dfrac{\dfrac{A_4 \in \Gamma}{\Gamma \vdash A_4} \quad \Gamma, A_0 \vdash B}{\Gamma, A_4 \supset A_0 \vdash B}}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\ L\supset$$

Forward-chaining!

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\dfrac{\Gamma \vdash A_1 \quad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset \qquad \dfrac{\overline{\Gamma \vdash A_1} \quad \overline{\Gamma, A_0 \vdash B}}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Negative protocol: The right branch is trivial: $A_0 = B$. Continue with $\Gamma \vdash A_1$.

Positive protocol: The left branch is trivial: $\Gamma = \Gamma', A_1$. Continue with $\Gamma', A_1, A_0 \vdash B$.

Mixed protocol:

Even atomic formulas are polarized [DJS95]

Take A_i atomic, B a formula and Γ a multiset of formulas.

$$\frac{\Gamma \vdash A_1 \quad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset \qquad \frac{\overline{\Gamma \vdash A_1} \quad \overline{\Gamma, A_0 \vdash B}}{\Gamma, A_1 \supset A_0 \vdash B}\ L\supset$$

Negative protocol: The right branch is trivial: $A_0 = B$. Continue with $\Gamma \vdash A_1$.

Positive protocol: The left branch is trivial: $\Gamma = \Gamma', A_1$. Continue with $\Gamma', A_1, A_0 \vdash B$.

Mixed protocol:

Mixing them, e.g., A_i positive for i odd and A_i negative for i even:

$$\frac{\dfrac{A_1 \in \Gamma}{\Gamma \vdash A_1} \quad \dfrac{\Gamma \vdash A_2 \quad \dfrac{\dfrac{A_3 \in \Gamma}{\Gamma \vdash A_3} \quad \dfrac{\Gamma \vdash A_4 \quad \dfrac{A_0 = B}{\Gamma, A_0 \vdash B}}{\Gamma, A_4 \supset A_0 \vdash B}}{\Gamma, A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\ L\supset$$

Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

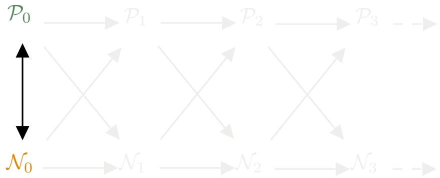
$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$$

$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$$

Q

R



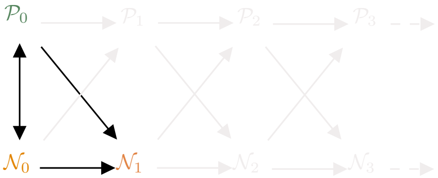
Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$$
$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$$

$Q_1 \wedge^- Q_2$

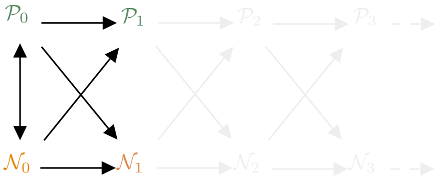


Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and
$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$
$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$

$R_1 \vee^+ R_2$



Polarity-based hierarchy

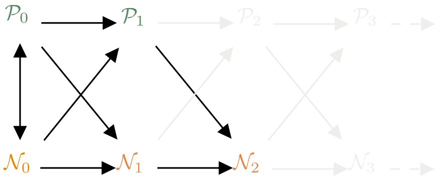
Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$

$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$

$(Q_1 \wedge^- Q_2) \supset (R_1 \vee^+ R_2)$



Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$

$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r, dashed] & {} \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r, dashed] & {}
\end{tikzcd}$$

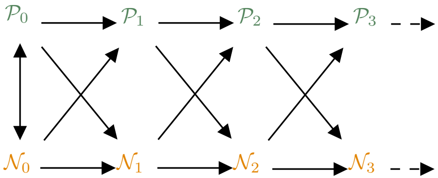
Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

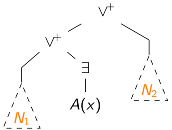
$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$$

$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$$



$(N_1 \vee^+ \exists x A(x)) \vee^+ N_2$

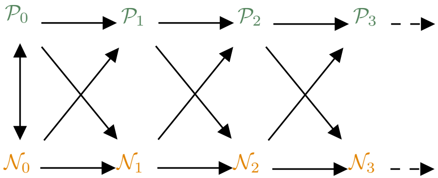


Polarity-based hierarchy

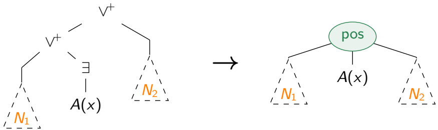
Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$$
$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$$



$(N_1 \vee^+ \exists x A(x)) \vee^+ N_2$



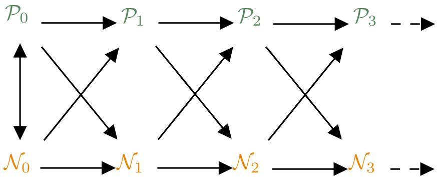
Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

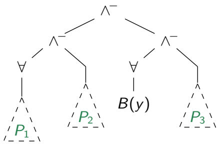
$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and

$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$

$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$



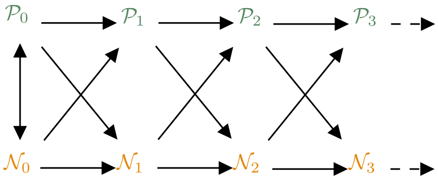
$(\forall x P_1 \wedge^- P_2) \wedge^- (\forall y B(y) \wedge^- P_3)$



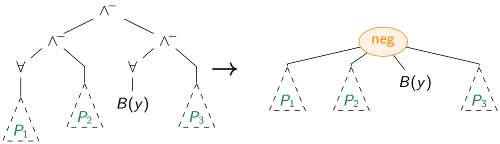
Polarity-based hierarchy

Hierarchy of negative and positive classical formulas: inspired by [CGT08, CST09]

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms and
$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge^- \mathcal{N}_{n+1} \mid t^- \mid \mathcal{N}_{n+1} \vee^- \mathcal{N}_{n+1} \mid f^- \mid \forall x \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1}$
$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge^+ \mathcal{P}_{n+1} \mid t^+ \mid \mathcal{P}_{n+1} \vee^+ \mathcal{P}_{n+1} \mid f^+ \mid \exists x \mathcal{P}_{n+1} \mid$



$(\forall x P_1 \wedge^- P_2) \wedge^- (\forall y B(y) \wedge^- P_3)$



Bipolar formulas

The hierarchy can be specified for intuitionistic or classical formulas.

Any formula in the class $\mathcal{N}_2^C$ / $\mathcal{N}_2^I$ is a classical/ intuitionistic bipolar formula.

Bipolar formulas

The hierarchy can be specified for intuitionistic or classical formulas.

Any formula in the class $\mathcal{N}_2^C$ / $\mathcal{N}_2^I$ is a classical/ intuitionistic bipolar formula.

Aside: How to polarize a formula?

- atomic formulas are labeled either positive or negative
- replace all occurrences of constants and connectives with a polarized variant.
  - in intuitionistic logic: always rename false and disjunction as f^+ and $\vee^+$!

Bipolar formulas

The hierarchy can be specified for intuitionistic or classical formulas.

Any formula in the class $\mathcal{N}_2^C$ / $\mathcal{N}_2^I$ is a classical/ intuitionistic bipolar formula.

Aside: How to polarize a formula?

- atomic formulas are labeled either positive or negative
- replace all occurrences of constants and connectives with a polarized variant.
  - in intuitionistic logic: always rename false and disjunction as f^+ and $\vee^+$!

Example. $(P_1 \supset P_2) \vee (Q_1 \supset Q_2) \rightsquigarrow$ classical bipolar $(P_1 \supset P_2) \vee^- (Q_1 \supset Q_2)$.
No polarization yields an intuitionistic bipolar formula.

Outline

What is focusing?

Consider again the sequent

$$\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B$$

with A_i atomic, B a formula and Γ a multiset of formulas.

What is focusing?

Consider again the sequent

$$\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B$$

with A_i atomic, B a formula and Γ a multiset of formulas.

How to prove it?

What is focusing?

Consider again the sequent

$$\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B$$

with A_i atomic, B a formula and Γ a multiset of formulas.

How to prove it?

Many ways to proceed!

What is focusing?

Consider again the sequent

$$\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B$$

with A_i atomic, B a formula and Γ a multiset of formulas.

How to prove it?

Many ways to proceed!

Focused rule application [And92]:
commit to repeat the $L{\supset}$ rule on the right premise until the atomic formula A_0 results:

$$\cfrac{\Gamma \vdash A_1 \quad \cfrac{\Gamma \vdash A_2 \quad \cfrac{\Gamma \vdash A_3 \quad \cfrac{\Gamma \vdash A_4 \quad \Gamma, A_0 \vdash B}{\Gamma, A_4 \supset A_0 \vdash B}\,L{\supset}}{\Gamma, A_3 \supset \cdots \supset A_n \supset A_0 \vdash B}\,L{\supset}}{\Gamma, A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\,L{\supset}}{\Gamma, A_1 \supset A_2 \supset A_3 \supset A_4 \supset A_0 \vdash B}\,L{\supset}$$

An organisational tool

Focusing provides a way to restrict the proof search space while remaining complete.

An organisational tool

Focusing provides a way to restrict the proof search space while remaining complete.

- Identify and always apply invertible introduction rules;
- Chain together the other rules (non-invertible/consuming external information).

An organisational tool

Focusing provides a way to restrict the proof search space while remaining complete.

- Identify and always apply invertible introduction rules;
- Chain together the other rules (non-invertible/consuming external information).

$\Rightarrow$ Maximal chaining of the decomposition.

An organisational tool

Focusing provides a way to restrict the proof search space while remaining complete.

- Identify and always apply invertible introduction rules;
- Chain together the other rules (non-invertible/consuming external information).

$\Rightarrow$ Maximal chaining of the decomposition.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, \neg A} \quad \dfrac{}{A, B, \neg B}}{A, B, \neg A \wedge \neg B}\ \wedge}{A, B \vee C, \neg A \wedge \neg B}\ \vee}{\exists x.A, B \vee C, \neg A \wedge \neg B}\ \exists}{\exists x.A, \exists y.(B \vee C), \neg A \wedge \neg B}\ \exists}{\exists x.A, \exists y.(B \vee C), \forall z.(\neg A \wedge \neg B)}\ \forall
$$

$\boxed{\text{Unfocused}}$

An organisational tool

Focusing provides a way to restrict the proof search space while remaining complete.

- Identify and always apply invertible introduction rules;
- Chain together the other rules (non-invertible/consuming external information).

$\Rightarrow$ Maximal chaining of the decomposition.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A, B, \neg A} \qquad \overline{A, B, \neg B}}{A, B, \neg A \wedge \neg B}\,\wedge}{A, B \vee C, \neg A \wedge \neg B}\,\vee}{\exists x.A, B \vee C, \neg A \wedge \neg B}\,\exists}{\exists x.A, \exists y.(B \vee C), \neg A \wedge \neg B}\,\exists}{\exists x.A, \exists y.(B \vee C), \forall z.(\neg A \wedge \neg B)}\,\forall$$

$$\dfrac{\dfrac{\dfrac{\overline{A, \exists y.(B \vee C), \neg A}}{\exists x.A, \exists y.(B \vee C), \neg A}\,\exists \qquad \dfrac{\dfrac{\overline{\exists x.A, B, \neg B}}{\exists x.A, B \vee C, \neg B}\,\vee}{\exists x.A, \exists y.(B \vee C), \neg B}\,\exists}{\exists x.A, \exists y.(B \vee C), \neg A \wedge \neg B}\,\wedge}{\exists x.A, \exists y.(B \vee C), \forall z.(\neg A \wedge \neg B)}\,\forall$$

Unfocused $\longrightarrow$ Focused

LKF and LJF [LM07, LM09]

Two kinds of focused sequents

- $\Downarrow$ sequents to decompose the formula under focus

$$\Gamma \Downarrow B \vdash \Delta \text{ with a left focus on } B$$
$$\Gamma \vdash B \Downarrow \Delta \text{ with a right focus on } B$$

When the conclusion of an introduction rule, then that rule introduced B.

- $\Uparrow$ sequents for invertible introduction rules

$$\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2$$

LKF and LJF [LM07, LM09]

Two kinds of focused sequents

- $\Downarrow$ sequents to decompose the formula under focus

$$\Gamma \Downarrow B \vdash \Delta \text{ with a left focus on } B$$
$$\Gamma \vdash B \Downarrow \Delta \text{ with a right focus on } B$$

  When the conclusion of an introduction rule, then that rule introduced B.

- $\Uparrow$ sequents for invertible introduction rules

$$\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2$$

Example of rules:

$$\frac{\Gamma \vdash B_1 \Downarrow \Delta \quad \Gamma \Downarrow B_2 \vdash \Delta}{\Gamma \Downarrow B_1 \supset B_2 \vdash \Delta}$$

non-invertible

$$\frac{\Gamma_1 \Uparrow \Gamma_2, B_1 \vdash B_2 \Uparrow \Delta}{\Gamma_1 \Uparrow \Gamma_2 \vdash B_1 \supset B_2 \Uparrow \Delta}$$

invertible

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\, D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\, D_r$$

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\, D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\, D_r$$

- Focus is transferred from conclusion to premises until

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

  Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\, D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\, D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\, R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\, R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\, I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\, I_r$$

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\ I_r$$

- Once the focus is released, the formula is eagerly decomposed into subformulas,

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta} D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta} D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta} R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta} R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta} I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta} I_r$$

- Once the focus is released, the formula is eagerly decomposed into subformulas, which are ultimately stored in the context.

  Store: $$\frac{\Gamma_1, P \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, P \vdash \Delta_1 \Uparrow \Delta_2} S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \Delta_1 \Uparrow N, \Delta_2}{\Gamma \Uparrow \cdot \vdash N, \Delta_1 \Uparrow \Delta_2} S_r$$

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $\dfrac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \dfrac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $\dfrac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \dfrac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$

  - or the derivation ends

    Initial: $\dfrac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\ I_l \qquad \dfrac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\ I_r$

- Once the focus is released, the formula is eagerly decomposed into subformulas, which are ultimately stored in the context.

  Store: $\dfrac{\Gamma_1, P \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, P \vdash \Delta_1 \Uparrow \Delta_2}\ S_l \qquad \dfrac{\Gamma \Uparrow \cdot \vdash \Delta_1 \Uparrow N, \Delta_2}{\Gamma \Uparrow \cdot \vdash N, \Delta_1 \Uparrow \Delta_2}\ S_r$

$\Rightarrow$ Sequent derivations are organized into synchronous/asynchronous phases

LKF and LJF [LM07, LM09]

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\ D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\ R_r$$

  - or the derivation ends

    Initial: $$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N, \Delta}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow \Delta}\ I_r$$

- Once the focus is released, the formula is eagerly decomposed into subformulas, which are ultimately stored in the context.

  Store: $$\frac{\Gamma_1, P \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, P \vdash \Delta_1 \Uparrow \Delta_2}\ S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \Delta_1 \Uparrow N, \Delta_2}{\Gamma \Uparrow \cdot \vdash N, \Delta_1 \Uparrow \Delta_2}\ S_r$$

$\Rightarrow$ Sequent derivations are organized into synchronous/asynchronous phases

$\Rightarrow$ Synthetic rules result from looking only at border sequents: $\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta$ [Cha08]

Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;

Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;
2. in which no synchronous rule occurs above an asynchronous rule;

Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;
2. in which no synchronous rule occurs above an asynchronous rule;
3. and only atomic formulas are stored.

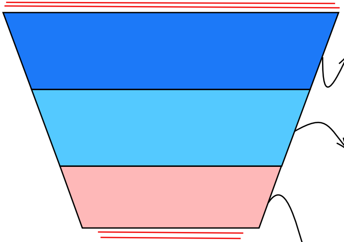
Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;
2. in which no synchronous rule occurs above an asynchronous rule;
3. and only atomic formulas are stored.

$$\Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta_1 \quad \dots \quad \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow \Delta_n$$



$$\dfrac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l$$

Atomic storage

Atoms are stored

$$\dfrac{C, \Gamma \Uparrow \Theta \vdash \Omega \Uparrow \Delta}{\Gamma \Uparrow C, \Theta \vdash \Omega \Uparrow \Delta}\ \mathsf{s}_l \qquad \dfrac{\Gamma \Uparrow \cdot \vdash \Omega \Uparrow D, \Delta}{\Gamma \Uparrow \cdot \vdash D, \Omega \Uparrow \Delta}\ \mathsf{s}_r$$

Asynchronous phase

Invertible rules are applied eagerly

$$\dfrac{\Gamma \Uparrow \ \vdash A, \Omega \Uparrow \Delta \quad \Gamma \Uparrow \ \vdash B, \Omega \Uparrow \Delta}{\Gamma \Uparrow \ \vdash A \wedge^- B, \Omega \Uparrow \Delta}\ \wedge_r^-$$

Synchronous phase

Focusing persists

$$\dfrac{\Gamma \vdash A \Downarrow \Delta \quad \Gamma \vdash B \Downarrow \Delta}{\Gamma \vdash A \wedge^+ B \Downarrow \Delta}\ \wedge_r^+$$

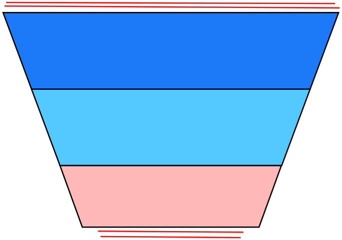
Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

A bipole for B is a synthetic inference rule corresponding to a derivation (in LKF or LJF)

1. starting with a decide on B;
2. in which no synchronous rule occurs above an asynchronous rule;
3. and only atomic formulas are stored.

$$\Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta_1 \quad \ldots \quad \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow \Delta_n$$



$$\dfrac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l$$

Corresponding synthetic rule

(in LK or LJ)

$$\dfrac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

Outline

1st result: Bipolar $\longleftrightarrow$ Bipole

Let B be a polarized negative (classical or intuitionistic) formula.

Theorem:

- If B is bipolar, then any synthetic inference rule for B is a bipole.
- If every synthetic inference rule for B is a bipole then B is bipolar.

Prototype implementation:
λProlog [MN12, NM88] executable specification of a predicate that relates a bipolar formula to its various bipoles.

$\Rightarrow$ compact given the nature of λProlog

$\Rightarrow$ explicit about the scope of bindings for schematic variables and eigenvariables.

2nd result: Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas.

$\mathsf{LK}\langle\mathcal{T}\rangle/\mathsf{LJ}\langle\mathcal{T}\rangle$ denotes the extension of LK/LJ with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

Theorem: The cut rule is admissible for the proof systems $\mathsf{LK}\langle\mathcal{T}\rangle/\mathsf{LJ}\langle\mathcal{T}\rangle$.

Note: the proof is simple!

It is a direct consequence of (polarized) cut admissibility in LKF/LJF.

$$\frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \Delta \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ \mathit{Cut}$$

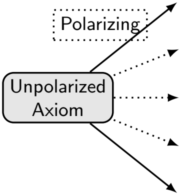
Rules from axioms

Rules from axioms

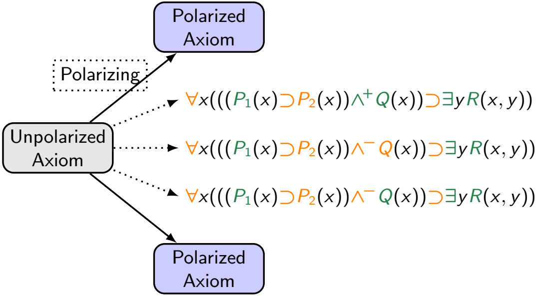
Unpolarized Axiom

$$\forall x(((P_1(x) \supset P_2(x)) \wedge Q(x)) \supset \exists y R(x, y))$$

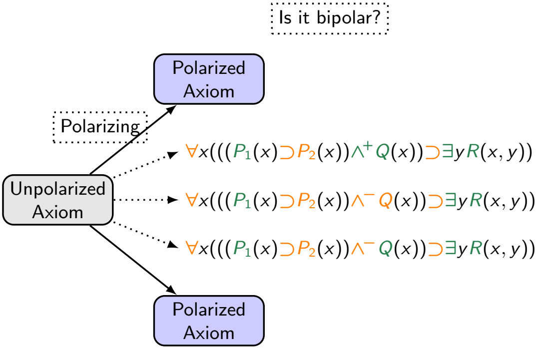
Rules from axioms



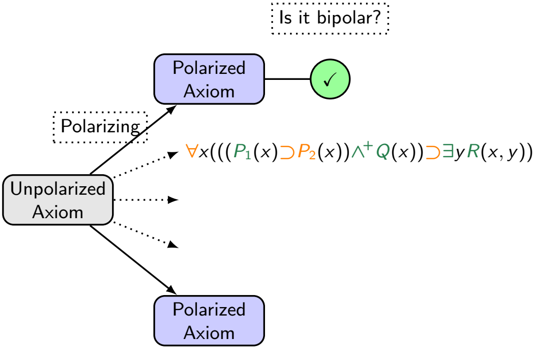
Rules from axioms



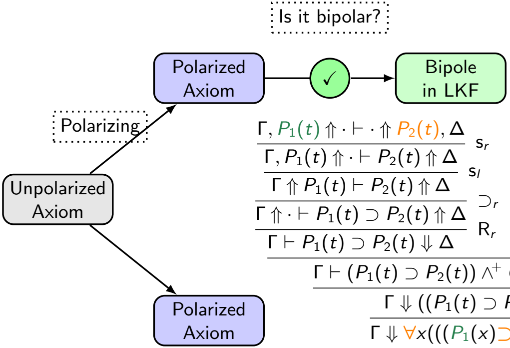
Rules from axioms



Rules from axioms

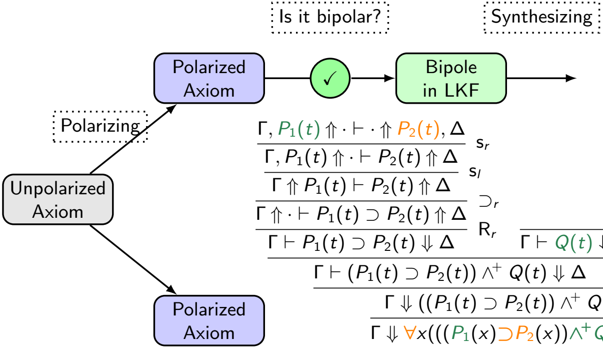


Rules from axioms



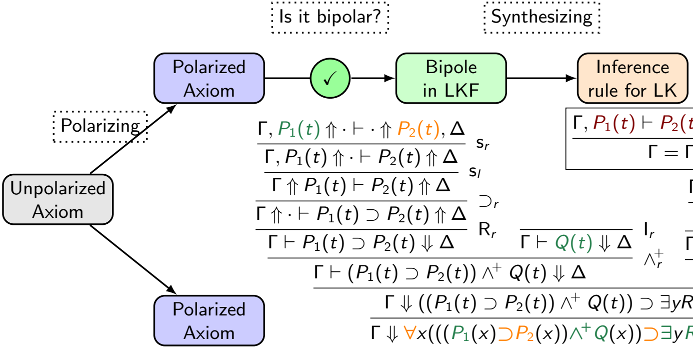
$$
\begin{prooftree}
\AxiomC{$\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta$}
\RightLabel{s_r}
\UnaryInfC{$\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta$}
\RightLabel{s_l}
\UnaryInfC{$\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta$}
\RightLabel{$\supset_r$}
\UnaryInfC{$\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta$}
\RightLabel{R_r}
\UnaryInfC{$\Gamma \vdash P_1(t) \supset P_2(t) \Downarrow \Delta$}
\AxiomC{}
\RightLabel{I_r}
\UnaryInfC{$\Gamma \vdash Q(t) \Downarrow \Delta$}
\RightLabel{$\wedge^+_r$}
\BinaryInfC{$\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^+ Q(t) \Downarrow \Delta$}
\AxiomC{$\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta$}
\RightLabel{s_l}
\UnaryInfC{$\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta$}
\RightLabel{$\exists_l$}
\UnaryInfC{$\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta$}
\RightLabel{R_l}
\UnaryInfC{$\Gamma \Downarrow \exists y R(t,y) \vdash \Delta$}
\RightLabel{$\supset_l$}
\BinaryInfC{$\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^+ Q(t)) \supset \exists y R(t,y) \vdash \Delta$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\Gamma \Downarrow \forall x (((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y)) \vdash \Delta$}
\end{prooftree}
$$

Rules from axioms



$$
\begin{prooftree}
\AxiomC{$\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta$}
\RightLabel{s_r}
\UnaryInfC{$\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta$}
\RightLabel{s_l}
\UnaryInfC{$\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta$}
\RightLabel{$\supset_r$}
\UnaryInfC{$\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta$}
\RightLabel{R_r}
\UnaryInfC{$\Gamma \vdash P_1(t) \supset P_2(t) \Downarrow \Delta$}
\AxiomC{}
\RightLabel{I_r}
\UnaryInfC{$\Gamma \vdash Q(t) \Downarrow \Delta$}
\RightLabel{$\wedge_r^+$}
\BinaryInfC{$\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^+ Q(t) \Downarrow \Delta$}
\AxiomC{$\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta$}
\RightLabel{s_l}
\UnaryInfC{$\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta$}
\RightLabel{$\exists_l$}
\UnaryInfC{$\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta$}
\RightLabel{R_l}
\UnaryInfC{$\Gamma \Downarrow \exists y R(t,y) \vdash \Delta$}
\RightLabel{$\supset_l$}
\BinaryInfC{$\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^+ Q(t)) \supset \exists y R(t,y) \vdash \Delta$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y)) \vdash \Delta$}
\end{prooftree}
$$

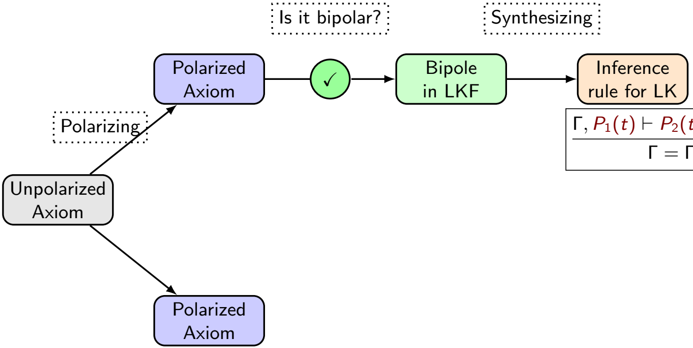
Rules from axioms



$$\dfrac{\Gamma, P_2(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

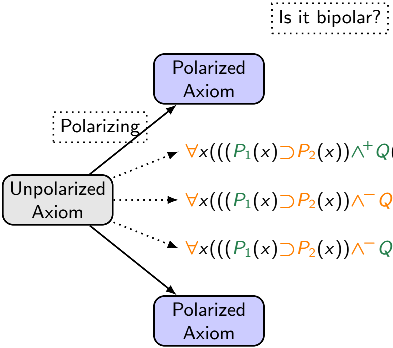
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\, s_r}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta}\, \supset_r}{\Gamma \vdash P_1(t) \supset P_2(t) \Downarrow \Delta}\, R_r \quad \dfrac{}{\Gamma \vdash Q(t) \Downarrow \Delta}\, I_r}{\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^+ Q(t) \Downarrow \Delta}\, \wedge^+_r \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\, s_l}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\, \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\, R_l}{\dfrac{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^+ Q(t)) \supset \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x (((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\, \forall_l}\, \supset_l$$

Rules from axioms



$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

Rules from axioms

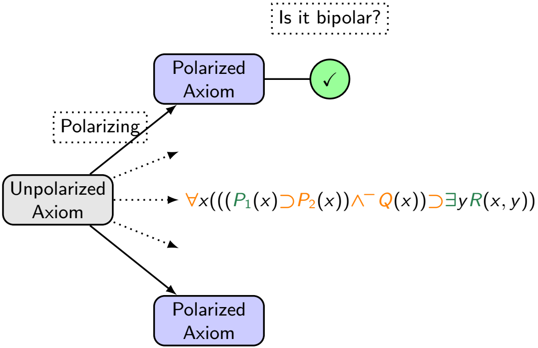


$$\forall x(((P_1(x) \supset P_2(x)) \wedge^+ Q(x)) \supset \exists y R(x,y))$$

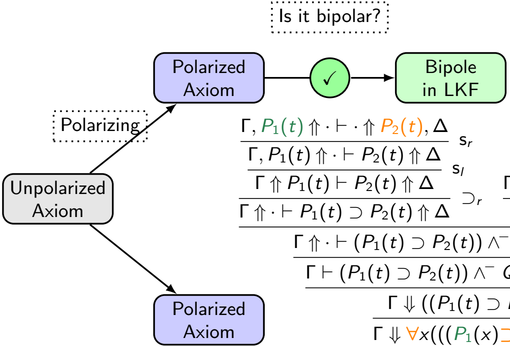
$$\forall x(((P_1(x) \supset P_2(x)) \wedge^- Q(x)) \supset \exists y R(x,y))$$

$$\forall x(((P_1(x) \supset P_2(x)) \wedge^- Q(x)) \supset \exists y R(x,y))$$

Rules from axioms

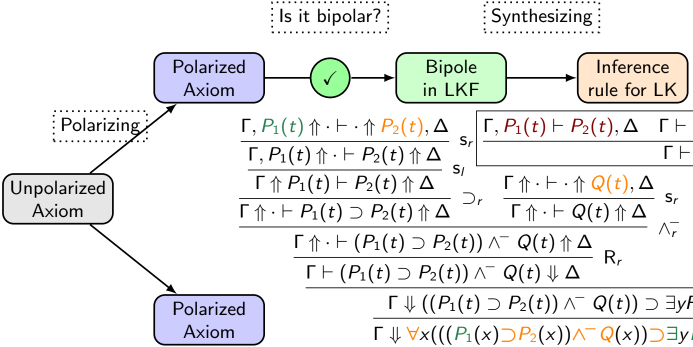


Rules from axioms



$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\,\mathsf{s}_r
      }{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\,\mathsf{s}_l
    }{\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta}\,\supset_r
    \quad
    \dfrac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t), \Delta}{\Gamma \Uparrow \cdot \vdash Q(t) \Uparrow \Delta}\,\mathsf{s}_r
  }{
    \dfrac{\Gamma \Uparrow \cdot \vdash (P_1(t) \supset P_2(t)) \wedge^- Q(t) \Uparrow \Delta}{\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^- Q(t) \Downarrow \Delta}\,\mathsf{R}_r
  }\,\wedge^-_r
  \quad
  \dfrac{
    \dfrac{
      \dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\,\mathsf{s}_l
    }{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\,\exists_l
  }{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\,\mathsf{R}_l
}{
  \dfrac{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^- Q(t)) \supset \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \wedge^- Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\,\forall_l
}\,\supset_l
$$

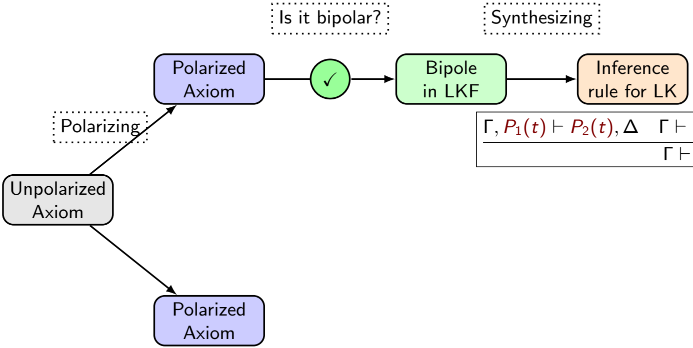
Rules from axioms



$$\boxed{\dfrac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}}$$

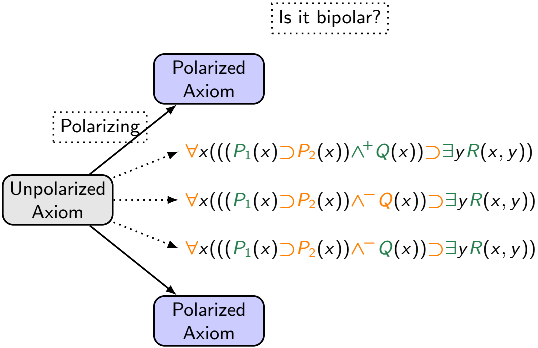
$$\begin{prooftree}
\AxiomC{$\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta$}
\RightLabel{s_r}
\UnaryInfC{$\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta$}
\RightLabel{s_l}
\UnaryInfC{$\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta$}
\RightLabel{$\supset_r$}
\UnaryInfC{$\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta$}
\AxiomC{$\Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t), \Delta$}
\RightLabel{s_r}
\UnaryInfC{$\Gamma \Uparrow \cdot \vdash Q(t) \Uparrow \Delta$}
\RightLabel{$\wedge^-_r$}
\BinaryInfC{$\Gamma \Uparrow \cdot \vdash (P_1(t) \supset P_2(t)) \wedge^- Q(t) \Uparrow \Delta$}
\RightLabel{R_r}
\UnaryInfC{$\Gamma \vdash (P_1(t) \supset P_2(t)) \wedge^- Q(t) \Downarrow \Delta$}
\AxiomC{$\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta$}
\RightLabel{s_l}
\UnaryInfC{$\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta$}
\RightLabel{$\exists_l$}
\UnaryInfC{$\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta$}
\RightLabel{R_l}
\UnaryInfC{$\Gamma \Downarrow \exists y R(t,y) \vdash \Delta$}
\RightLabel{$\supset_l$}
\BinaryInfC{$\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \wedge^- Q(t)) \supset \exists y R(t,y) \vdash \Delta$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \wedge^- Q(x)) \supset \exists y R(x,y)) \vdash \Delta$}
\end{prooftree}$$

Rules from axioms

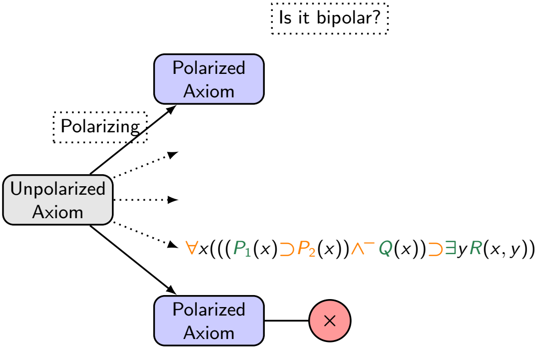


$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

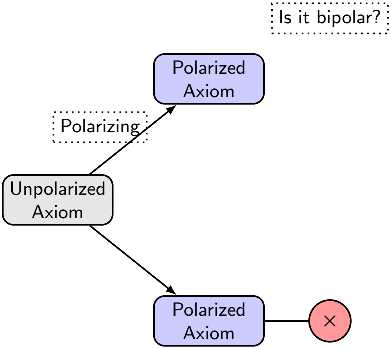
Rules from axioms



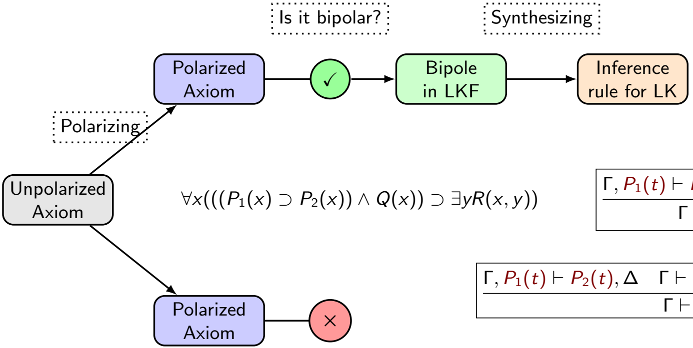
Rules from axioms



Rules from axioms



Rules from axioms

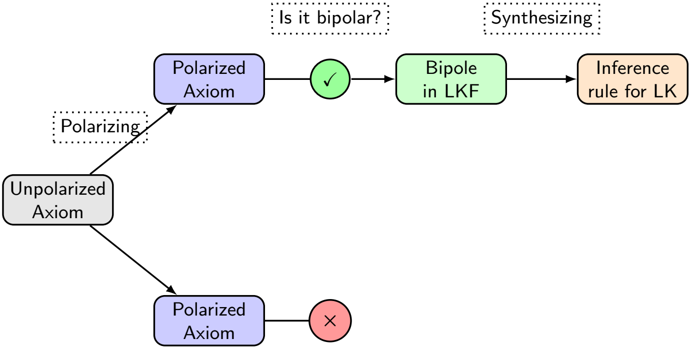


$$\forall x(((P_1(x) \supset P_2(x)) \wedge Q(x)) \supset \exists y R(x,y))$$

$$\boxed{\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}}$$

$$\boxed{\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}}$$

Rules from axioms



Outline

Geometric axioms as bipoles

Geometric implication:

$$\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \supset \exists \overline{x}_1 M_1 \vee \ldots \vee \exists \overline{x}_n M_n)$$

- P_i atomic;
- $M_j = Q_{j_1} \wedge \ldots \wedge Q_{j_{k_j}}$, Q_{j_k} atomic;
- none of the variables in the vectors $\overline{x}_j$ are free in P_i.

Geometric axioms as bipoles

Polarized geometric implication:

$$\forall \overline{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \supset \exists \overline{x}_1 \hat{M}_1 \vee^{\pm} \ldots \vee^{\pm} \exists \overline{x}_n \hat{M}_n)$$

- P_i^+, P_i^- atomic;
- $\hat{M}_j = Q_{j_1}^{\pm} \wedge^+ \ldots \wedge^+ Q_{j_{k_j}}^{\pm}$, $Q_{j_k}^{\pm}$ atomic;
- none of the variables in the vectors $\overline{x}_j$ are free in P_i.

Geometric axioms as bipoles

Polarized geometric implication:

$$\forall \overline{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset \exists \overline{x}_1 \hat{M}_1 \vee^\pm \ldots \vee^\pm \exists \overline{x}_n \hat{M}_n)\,,$$

Corresponding bipole:

$$\frac{\overline{Q}_1[\overline{y}_1/\overline{x}_1], \Gamma \Uparrow \vdash \Uparrow \Delta \quad \ldots \quad \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Gamma \Uparrow \vdash \Uparrow \Delta}{\overline{P}, \Gamma' \Uparrow \vdash \Uparrow \Delta}$$

with $\overline{P} = \{P_i^+\}, \overline{Q_j} = \{Q_{j_k}^\pm\}$.

Corresponding LK rule:

$$\frac{\overline{Q}_1[\overline{y}_1/\overline{x}_1], \Gamma \vdash \Delta \quad \ldots \quad \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Gamma \vdash \Delta}{\overline{P}, \Gamma' \vdash \Delta}\ GRS$$

Geometric axioms as bipoles

Polarized geometric implication:

$$\forall \overline{z}(P_1^- \wedge^{\pm} \dots \wedge^{\pm} P_m^- \supset \exists \overline{x}_1 \hat{M}_1 \vee^{\pm} \dots \vee^{\pm} \exists \overline{x}_n \hat{M}_n)\,,$$

Corresponding bipole:

$$\frac{\overline{Q}_j[\overline{y}_j/\overline{x}_j], \Gamma \Uparrow \vdash \Uparrow \Delta \quad \dots \quad \Gamma \Uparrow \vdash \Uparrow P_i, \Delta}{\Gamma \Uparrow \vdash \Uparrow \Delta} \; m+n \text{ premises}$$

with $\overline{Q_j} = \{Q_{j_k}\}$.

Corresponding LK rule:

$$\frac{\overline{Q}_j[\overline{y}_j/\overline{x}_j], \Gamma \vdash \Delta \quad \dots \quad \Gamma \vdash P_i, \Delta}{\Gamma \vdash \Delta} \; m+n \text{ premises}$$

Co-geometric axioms as bipoles

Polarized co-geometric implication:

$$\forall \overline{z}(\forall \overline{x}_1 \hat{M}_1 \wedge^{\pm} \ldots \wedge^{\pm} \forall \overline{x}_n \hat{M}_n \supset P_1^{-} \vee^{-} \ldots \vee^{-} P_m^{-}),$$

with $\hat{M}_j = Q_{j_1}^{\pm} \vee^{-} \ldots \vee^{-} Q_{j_{k_j}}^{\pm}$.

Corresponding bipole:

$$\frac{\Gamma \Uparrow \vdash \Uparrow \overline{Q}_1[\overline{y}_1/\overline{x}_1], \Delta \quad \ldots \quad \Gamma \Uparrow \vdash \Uparrow \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Delta}{\Gamma \Uparrow \vdash \Uparrow \overline{P}, \Delta'}$$

Corresponding LK rule:

$$\frac{\Gamma \vdash \overline{Q}_1[\overline{y}_1/\overline{x}_1], \Delta \quad \ldots \quad \Gamma \vdash \overline{Q}_n[\overline{y}_n/\overline{x}_n], \Delta}{\Gamma \vdash \overline{P}, \Delta'}\ co-GRS_c$$

Co-geometric axioms as bipoles

Polarized co-geometric implication:

$$\forall\overline{z}(\forall\overline{x}_1 \hat{M}_1 \wedge^{\pm} \ldots \wedge^{\pm} \forall\overline{x}_n \hat{M}_n \supset P_1^{+} \vee^{\pm} \ldots \vee^{\pm} P_m^{+}),$$

with $\hat{M}_j = Q_{j_1}^{\pm} \vee^{-} \ldots \vee^{-} Q_{j_{k_j}}^{\pm}$.

Corresponding bipole:

$$\frac{\Gamma \Uparrow \ \vdash \ \Uparrow \overline{Q}_j[\overline{y}_j/\overline{x}_j], \Delta \quad \ldots \quad \Gamma, P_i \Uparrow \ \vdash \ \Uparrow \Delta}{\Gamma \Uparrow \ \vdash \ \Uparrow \Delta} \; m+n \text{ premises}$$

Corresponding LK rule:

$$\frac{\Gamma \vdash \overline{Q}_j[\overline{y}_j/\overline{x}_j], \Delta \quad \ldots \quad \Gamma, P_i \vdash \Delta}{\Gamma \vdash \Delta} \; m+n \text{ premises}$$

Universal axioms as bipoles

$$\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \supset Q_1 \vee \ldots \vee Q_n)$$

Universal axioms as bipoles

$$\forall \bar{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \supset Q_1^{\pm} \vee^{\pm} \ldots \vee^{\pm} Q_n^{\pm})$$

More choices in the selection of polarities while still remaining bipolar formulas.

Universal axioms as bipoles

$$\forall\overline{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset Q_1^\pm \vee^+ \ldots \vee^+ Q_n^\pm)$$

More choices in the selection of polarities while still remaining bipolar formulas.

$$\frac{Q_1, \Gamma \Uparrow\ \vdash\ \Uparrow \Delta \quad \ldots \quad Q_n, \Gamma \Uparrow\ \vdash\ \Uparrow \Delta}{\overline{P}, \Gamma' \Uparrow\ \vdash\ \Uparrow \Delta}\ FRL_c$$

Universal axioms as bipoles

$$\forall \overline{z}(P_1^{\pm} \wedge^- \ldots \wedge^- P_m^{\pm} \supset Q_1^- \vee^- \ldots \vee^- Q_n^-)$$

More choices in the selection of polarities while still remaining bipolar formulas.

$$\frac{\Gamma \Uparrow \vdash \Uparrow P_1, \Delta \quad \ldots \quad \Gamma \Uparrow \vdash \Uparrow P_m, \Delta}{\Gamma \Uparrow \vdash \Uparrow \overline{Q}, \Delta'} \ FRR_c$$

Horn clauses as bipoles

$$\forall \bar{z}(P_1 \wedge \ldots \wedge P_m \supset Q)$$

Horn clauses as bipoles

$$\forall \bar{z}(P_1^{\pm} \wedge^{\pm} \ldots \wedge^{\pm} P_m^{\pm} \supset Q^{\pm})$$

Even more choices in the selection of polarities while still remaining bipolar formulas!

Horn clauses as bipoles

$$\forall \bar{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset Q^+)$$

Even more choices in the selection of polarities while still remaining bipolar formulas!

$$\frac{Q, \Gamma \vdash \Delta}{\overline{P}, \Gamma' \vdash \Delta}\ FC$$

Forward-chaining
[Sim94, NvP98, CMS13]

Horn clauses as bipoles

$$\forall \bar{z}(P_1^- \wedge^- \ldots \wedge^- P_m^- \supset Q^-)$$

Even more choices in the selection of polarities while still remaining bipolar formulas!

$$\frac{\Gamma \vdash P_1, \Delta \quad \ldots \quad \Gamma \vdash P_m, \Delta}{\Gamma \vdash Q, \Delta'}\ BC$$

Back-chaining
[Vig00]

Implementation – Part I [MMPV20]

Formula

$\forall u \forall v \forall w (adj\ u\ v \supset (path\ v\ w \supset path\ u\ w))$ Positive atoms.

λProlog encoding

```
(all u\ all v\ all w\ imp (atm (adj u v))
                        (imp (atm (path v w)) (atm (path u w)))),
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{adj\ X\ Z, path\ Z\ Y, path\ X\ Y, L \vdash B}{adj\ X\ Z, path\ Z\ Y, L \vdash B}$$

Implementation – Part I [MMPV20]

Formula

$\forall u \forall v \forall w(adj\ u\ v \supset (path\ v\ w \supset path\ u\ w))$ Negative atoms.

λProlog encoding

```
(all u\ all v\ all w\ imp (atm (adj u v))
                        (imp (atm (path v w)) (atm (path u w)))),
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{\Gamma \vdash adj\ X\ Y \quad \Gamma \vdash path\ Y\ Z}{\Gamma \vdash path\ X\ Z}$$

Implementation – Part II [MMPV20]

Formula

$\forall u \forall v(\forall w(\mathtt{in}\ w\ u \supset \mathtt{in}\ w\ v) \supset \mathtt{subset}\ u\ v)$ Positive atoms.

λProlog encoding

```
(all u\ all v\ imp (all w\ imp (atm (in w u)) (atm (in w v)))
                    (atm (subset u v))).
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{\mathtt{in}\ w\ X, \Gamma \vdash \mathtt{in}\ w\ Y \quad \mathtt{subset}\ X\ Y, \Gamma \vdash B}{\Gamma \vdash B}$$

Implementation – Part II [MMPV20]

Formula

$\forall u \forall v (\forall w (\mathtt{in}\ w\ u \supset \mathtt{in}\ w\ v) \supset \mathtt{subset}\ u\ v)$ Negative atoms.

λProlog encoding

```
(all u\ all v\ imp (all w\ imp (atm (in w u)) (atm (in w v)))
                    (atm (subset u v))).
```

Goal

```
reduce (syncL Gamma F (atm B)) Premises.
```

Inference rule

$$\frac{\Gamma,\ \mathtt{in}\ w\ X \vdash \mathtt{in}\ w\ Y}{\Gamma \vdash \mathtt{subset}\ X\ Y}$$

Outline

Some discussion

- Dyckhoff and Negri: every (classical) first-order theory has a geometric conservative extension.
- In the literature: *ad hoc*, propositional, structural, labeled.
- Modal systems.
- Beyond bipolars.

To conclude

- ⋆ Synthetic inference rules generated using polarization and focusing provide inference rules that capture certain classes of axioms.
- ⋆ In particular: bipolar formulas correspond to inference rules for atoms.
- ⋆ As geometric formulas are examples of bipolar formulas, polarized versions of such formulas yield well known inference systems derived from geometric formulas.
- ⋆ Polarization of subsets of geometric formulas explain the forward-chaining and backward-chaining variants of their synthetic inference rules.
- ⋆ Direct proof of cut-elimination for the proof systems that arise from incorporating synthetic inference rules based on polarized formulas.
- ⋆ Additionally, all of these results work equally well in both classical and intuitionistic logics using the corresponding LKF and LJF focused proof systems.
- ⋆ Future work: beyond focusing-bipoles.

Thank you!

Obrigada!!!

Merci!!!

☺

References I

Jean-Marc Andreoli.
Logic programming with focusing proofs in linear logic.
J. Log. Comput., 2(3):297–347, 1992.

Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui.
From axioms to analytic rules in nonclassical logics.
In *Proceedings of the Twenty-Third Annual IEEE Symposium on Logic in Computer Science, LICS 2008, 24-27 June 2008, Pittsburgh, PA, USA*, pages 229–240, 2008.

Kaustuv Chaudhuri.
Focusing strategies in the sequent calculus of synthetic connectives.
In Iliano Cervesato, Helmut Veith, and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, 15th International Conference, LPAR 2008, Doha, Qatar, November 22-27, 2008. Proceedings*, volume 5330 of *Lecture Notes in Computer Science*, pages 467–481. Springer, 2008.

Agata Ciabattoni, Paolo Maffezioli, and Lara Spendier.
Hypersequent and labelled calculi for intermediate logics.
In Didier Galmiche and Dominique Larchey-Wendling, editors, *Automated Reasoning with Analytic Tableaux and Related Methods - 22th International Conference, TABLEAUX 2013, Nancy, France, September 16-19, 2013. Proceedings*, volume 8123 of *Lecture Notes in Computer Science*, pages 81–96. Springer, 2013.

References II

Agata Ciabattoni, Lutz Straßburger, and Kazushige Terui.
Expanding the realm of systematic proof theory.
In Erich Grädel and Reinhard Kahle, editors, *Computer Science Logic, 23rd international Workshop, CSL 2009, 18th Annual Conference of the EACSL, Coimbra, Portugal, September 7-11, 2009. Proceedings*, volume 5771 of *Lecture Notes in Computer Science*, pages 163–178. Springer, 2009.

V. Danos, J.-B. Joinet, and H. Schellinx.
LKT and LKQ: sequent calculi for second order logic based upon dual linear decompositions of classical implication.
In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, number 222 in London Mathematical Society Lecture Note Series, pages 211–224. Cambridge University Press, 1995.

Roy Dyckhoff and Sara Negri.
Geometrisation of first-order logic.
The Bulletin of Symbolic Logic, 21(2):123–163, 2015.

References III

Gilles Dowek and Benjamin Werner.
Arithmetic as a theory modulo.
In Jürgen Giesl, editor, *Term Rewriting and Applications, 16th International Conference, RTA 2005, Nara, Japan, April 19-21, 2005, Proceedings*, volume 3467 of *Lecture Notes in Computer Science*, pages 423–437. Springer, 2005.

Gerard Gentzen.
Untersuchungen über das logische Schliessen.
Mathematische Zeitschrift, 39:176–210, 1935.

Gerhard Gentzen.
Investigations into logical deduction.
In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1935.

Gerard Gentzen.
Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie.
Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, 4:19–44, 1938.

References IV

Jean-Yves Girard.
Proof Theory and Logical Complexity, volume I of *Studies in Proof Theory*.
Bibliopolis, edizioni di filosofia e scienze, Napoli, 1987.

Chuck Liang and Dale Miller.
Focusing and polarization in intuitionistic logic.
In J. Duparc and T. A. Henzinger, editors, *CSL 2007: Computer Science Logic*, volume 4646 of *LNCS*, pages 451–465. Springer, 2007.

Chuck Liang and Dale Miller.
Focusing and polarization in linear, intuitionistic, and classical logics.
Theor. Comput. Sci., 410(46):4747–4768, 2009.

Sonia Marin, Dale Miller, Elaine Pimentel, and Marco Volpe.
From axioms to synthetic inference rules via focusing.
Available at `https://drive.google.com/file/d/1_gNtKjvmxyH7T7VwpUD0QZtXARei8t5K/view`, 2020.

Dale Miller and Gopalan Nadathur.
Programming with Higher-Order Logic.
Cambridge University Press, June 2012.

References V

Sara Negri.
Contraction-free sequent calculi for geometric theories with an application to Barr's theorem.
Arch. Math. Log., 42(4):389–401, 2003.

Sara Negri.
Proof analysis beyond geometric theories: from rule systems to systems of rules.
J. Log. Comput., 26(2):513–537, 2016.

Gopalan Nadathur and Dale Miller.
An Overview of λProlog.
In *Fifth International Logic Programming Conference*, pages 810–827, Seattle, August 1988. MIT Press.

Sara Negri and Jan von Plato.
Cut elimination in the presence of axioms.
Bulletin of Symbolic Logic, 4(4):418–435, 1998.

Sara Negri and Jan von Plato.
Proof Analysis - a contribution to Hilbert's last problem.
Cambridge University Press, 2011.

References VI

Alex K. Simpson.
The Proof Theory and Semantics of Intuitionistic Modal Logic.
PhD thesis, College of Science and Engineering, School of Informatics, University of Edinburgh, 1994.

Luca Viganò.
Labelled Non-Classical Logics.
Kluwer Academic Publishers, 2000.