Modal logic and the polynomial hierarchy: from QBFs to K and back

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Ladner's seminal work [7] showed that a large number of modal logics between K and $S4$ are **PSPACE**-complete. Adding further axioms, such as 5, can simplify the underlying complexity of the validity problem, with $S5$ being *co***NP**-complete. Indeed, the 'gap' between *co***NP**-complete and **PSPACE**-complete normal modal logics has formed the subject of several works in recent years. That said, as far as we know, attempts to characterise fragments of modal logics corresponding to levels of the polynomial hierarchy (**PH**) have not appeared in the literature. On the other hand, translations from QBFs to modal logic now comprise a fundamental benchmark in modal satisfiability solving.

There are many known translations from QBFs to the basic normal modal logic K. However, despite the considerable literature relating QBFs and modal logic, their commonly employed complexity measures do not match up. In modal solving the key measure is that of *modal depth*. For QBFs the key measure is quantifier complexity, i.e. the number of alternations of $\exists$ and $\forall$ in a (prenex) QBF. While it is well-known that the alternation of quantifiers in QBFs corresponds precisely with the levels of the polynomial hierarchy, Halpern has showed in [6] that the validity problem for K (with any number of agents) for formulas with modal depth bounded by some constant $d \geq 2$ is in fact only *co***NP**-complete.

Contribution. It is this shortcoming of 'modal depth' that forms our principal motivation: can we identify a measure for modal formulas that coincides with quantifier complexity for QBFs, formally? In other words, can we find fragments of the modal logic K complete for each level of the polynomial hierarchy?

We answer this question positively in the present work by designing an *inverse* translation from true QBFs back into K. Our key idea is to encode modal provability itself as an alternating predicate, and analyse the alternation between 'invertible' and 'non-invertible' rules during proof search so as to delineate theorems according to the polynomial hierarchy. For this to work, we must first carefully devise a particular translation from QBFs to modal logics that is compatible with alternation in the aforementioned proof search predicate. In particular, composing our two translations yields an automorphism on (true) QBFs that *preserves* quantifier complexity.

Related work and methodology. The idea of encoding proof search to obtain upper bounds for modal validity or satisfiability is not new. However no previous works on complexity of modal logics (as far as we know) give refinements of **PSPACE**-completeness to levels of the polynomial hierarchy, regardless of the method employed.

This work builds on recent work achieving similar delineations for multiplicative additive linear/affine logic [4, 5] and fragments of intuitionistic logic [3], also well-known **PSPACE**-complete logics. Those works leveraged (alternative presentations of) *focussed* systems from structural proof theory (see, e.g., [1]), which elegantly control the alternation of invertible and non-invertible rules during proof search, the principal contributor to quantifier alternation in a proof search predicate.

(Normal) modal logics such as K (and, indeed, the entire 'modal $S5$ cube') have also recently received focussed treatments in the setting of *labelled* sequents [8] and *nested* sequents [2]. However nested and labelled systems, while admitting an elegant proof theory, do not enjoy terminating proof search per se, and thus are not adequate for obtaining alternating time bounds. Instead we work with a standard cut-free sequent calculus for K and give a bespoke analysis of the proof search space according to invertible and non-invertible rules that suffices for our purposes.

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