Information of Chemical Reaction Networks

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***Index Terms*—component, formatting, style, styling, insert**

I. Introduction

- Definition of chemical reaction networks
- Definition of the (Chemical) Master Equation;
- Evaluation of statistical moments of the CME;
- Entropy analysis of the solution of the CME;
- Thermodynamical analysis of equilibrium and non-equilibrium systems

We define a chemical reaction network as a set of M reaction channels such that

$$\underline{s}_{\mu,1}\underline{S}_1 + ... + \underline{s}_{\mu,N_r}\underline{S}_{N_r} \xrightarrow{k_\mu} \overline{s}_{\mu,1}\overline{S}_1 + ... + \overline{s}_{\mu,N_r}\overline{S}_{N_r}. \quad (1)$$

The species S acts like a reactant $\underline{S}$ through a channel $\mu = 1, ..., M$, producing $\overline{S}$. The stoichiometric coefficients are indicated by $\underline{s}$ and $\overline{s}$ for the reactants and the products. We define a stoichiometric matrix $\nu \in \mathbb{Z}^{M\times N_r}$, such that

$$[\nu]_{ij} = \overline{s}_{i,j} - \underline{s}_{i,j}. \quad (2)$$

We also define $\alpha_\mu(x)$ as the probability reaction channel μ be activated [1], to the μ-th reaction to occur. Armed with this we define a chemical master equation as

$$\frac{\partial P(x,t|x_0,t_0)}{\partial t} = \sum_{\mu=1}^{M} \alpha_\mu(x-\nu_\mu)P(x-\nu_\mu|x_0,t_0) - \alpha_\mu(x)P(x,t|x_0,t_0), \quad (3)$$

with $\nu_\mu = [\nu]_{\mu,:}$. [Define x].

The chemical master equation is a set of differential equations for the evolution of the transition probability $P(x,t|x_0,t_0)$, between the states x and x_0. We are not able to find an analytical solution for $P(x,t|x_0,t_0)$ in general but for special cases[Jahnke]. To obtain solutions numerically, one needs to crop the infinite state space.[Cropping and out-flow]

The system is said *closed*, if there is no production or degradation of species. Otherwise the system is said *open*. In the closed case the system can only change its state according to $x = x_0 + \xi\nu, \xi \in \mathbb{Z}$. Thus, once defined the initial state x_0 of the system, the states in which the system can be are restricted to the transitions that the reactions allow.

The CME operator A (?) is *irreducible* if and only if the adjacency matrix of A is strongly connected [3]. Otherwise, the strongly connected components of A will form a set of the irreducible parts of the system. Each of them has its own unique stationary probability distribution. Thus, the system has a limiting distribution which is a linear combination of those stationary distributions.

A necessary condition for strong connection is that every state to be reachable from every other state, which for closed systems will be always wrong once the system is restricted to a subset of the states. This subset correspond to the states in which the number of species of the systems is conserved. This has another implications as the non-ergodicity of closed systems. The subset of possible states of the system will always be dependent on the initial condition once the later defines the [copy number] of the species. For instance, in reactions like

$$E + A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} EA \xrightarrow{k_2} E + B, \quad (4)$$

known as Michaelis-Menten enzyme mechanism reaction, a state which has zero copy number of enzyme molecules E can be reached, but the system will never transition from it. This state is said to absorb the probability through the evolution of the Markov chain, and is not strongly connected with the other states. Thus the system is not irreducible.

For open systems, the uniqueness of the stationary distribution is not assured. But for reactions like

$$\emptyset \to A, \quad A \to \emptyset \quad (5)$$

called production and degradation, all the states are connected given there will always be a reaction to drive the system the every state. Thus the system is irreducible. The system can be ergodic if all the states are aperiodic [citation needed]. For this, one needs to prove if the states have periodicity equals the unity.

II. Statistics of the solution of the CME

The solution of the ODE is

$$P(x,t|x_0,t_0) = e^{At}P(x,t_0|x_0,t_0), \quad P(x,t_0|x_0,t_0) = I. \quad (6)$$

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The evaluation of (6) depends on the computation of the matrix exponential e^{At}, which can be hard for a large A.

In possession of the probability vector $p(x,t) = P(x,t|x_0,t_0)p(x_0,t_0)$, we obtain the statistical moments of it.

$$\mu(x,t) = \sum_i i[p(x,t)]_i \tag{7}$$

$$\sigma^2(x,t) = \sum_i (i - \mathbb{E}[x,t])^2[p(x,t)]_i \tag{8}$$

$$sk(x,t) = \sum_i \left(\frac{i-\mu(x,t)}{\sigma(x,t)}\right)^3 [p(x,t)]_i \tag{9}$$

$$S(x,t) = \sum_i -\log([p(x,t)]_i)[p(x,t)]_i \tag{10}$$

A. Entropy balance

The Shannon's entropy can be defined in terms of the Boltzmann's entropy considering $k_B = 1$

$$S(t) = -\sum_i \log([p(x,t)]_i)[p(x,t)]_i. \tag{11}$$

Taking the derivative w.r.t. we obtain

$$\frac{dS}{dt} = -\sum_i \frac{\partial[p(x,t)]_i}{\partial t} \log([p(x,t)]_i) \tag{12}$$

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