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Entropy production fluctuations of finite Markov chains

Da-Quan Jiang,[a] Min Qian, and Fu-Xi Zhang[b]
LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

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For almost every trajectory segment over a finite time span of a finite Markov chain with any given initial distribution, the logarithm of the ratio of its probability to that of its time-reversal converges exponentially to the entropy production rate of the Markov chain. The large deviation rate function has a symmetry of Gallavotti–Cohen type, which is called the fluctuation theorem. Moreover, similar symmetries also hold for the rate functions of the joint distributions of general observables and the logarithmic probability ratio. © *2003 American Institute of Physics.*
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I. INTRODUCTION

Stationary nonequilibrium states play an important role in statistical physics and have been studied since Boltzmann's time. In 1993, Evans, Cohen, and Morriss[1] found in computer simulations that the natural invariant measure of a stationary nonequilibrium system has a symmetry which, being called later the fluctuation theorem by Gallavotti and Cohen,[2–5] gives a general formula for the probability ratio of observing trajectories that satisfy or violate the second law of thermodynamics. Since then there have been many derivations and generalizations of the fluctuation theorem. Motivated by the result in Ref. 1, Gallavotti and Cohen[6] gave the first mathematical presentation of the fluctuation theorem for stationary nonequilibrium systems modelled by hyperbolic dynamical systems: Provided that the dynamics is invariant under time reversal and is sufficiently chaotic, the probability distributions of the phase space contraction averaged over large time spans have a large deviation property and the large deviation rate function has a symmetry. Evans and Searles[7–13] considered transient, rather than stationary, nonequilibrium systems and employed a known equilibrium state (such as the Liouville measure) as the initial distribution to derive a transient fluctuation theorem. Gallavotti[14] and Evans *et al.*[13,15] proposed a local version of the fluctuation theorem. Kurchan[16] pointed out that the fluctuation theorem also holds for certain diffusion processes. Lebowitz and Spohn[17] extended Kurchan's results to general Markov processes, and Maes[18] thought of the fluctuation theorem as a property of space–time Gibbs measures. Searles and Evans[19] derived the transient fluctuation theorem for non-stationary stochastic systems.

For systems close to equilibrium, the distribution of trajectories over a finite time interval has little difference from that of their time reversals, and the fluctuation theorem yields the well-known Green–Kubo formula and the Onsager reciprocity relations,[3,17,18,20,21] i.e., the symmetry of the transport coefficients matrix which relate thermodynamic ''forces'' and ''fluxes.'' Surprisingly, the fluctuation theorem is also valid for systems in the nonlinear response regime far from equilibrium. In this sense, the fluctuation theorem can be thought of as an extension of the fluctuation-dissipation theorem, which holds for systems in the linear response regime close to equilibrium.

The concept of entropy production was first put forward in nonequilibrium statistical physics to describe how far a specific state of a system is away from its equilibrium state.[22–24] In Refs. 25–27, a measure-theoretic definition of entropy production rate is proposed for stochastic processes, unifying different entropy production formulas in various concrete cases. Suppose that ξ

[a]Electronic mail: jiangdq@math.pku.edu.cn
[b]Electronic mail: zhangfxi@math.pku.edu.cn

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$=\{\xi_n\}_{n\in\mathbf{Z}}$ is a stationary, irreducible and positive recurrent Markov chain with finite state space S, transition probability matrix $P=(p_{ij})_{i,j\in S}$, and invariant distribution $\Pi=\{\pi_i\}_{i\in S}$. Let $\mathbf{P}$ and $\mathbf{P}^-$ be the distributions of the Markov chain and its time-reversal respectively, and denote their restrictions on $\mathcal{F}_0^n=\sigma(\xi_i,0\leqslant i\leqslant n)$ by $\mathbf{P}_{[0,n]}$ and $\mathbf{P}^-_{[0,n]}$. The entropy production rate of ξ is defined as

$$e_p \stackrel{\text{def}}{=} \lim_{n\to+\infty}\frac{1}{n}H(\mathbf{P}_{[0,n]},\mathbf{P}^-_{[0,n]}),$$

where $H(\mathbf{P}_{[0,n]},\mathbf{P}^-_{[0,n]})$ is the relative entropy of $\mathbf{P}_{[0,n]}$ with respect to $\mathbf{P}^-_{[0,n]}$. A sufficient and necessary condition for $\mathbf{P}_{[0,n]}$ and $\mathbf{P}^-_{[0,n]}$ being mutually absolutely continuous is that $p_{ij}>0\Leftrightarrow p_{ji}>0$ for any i,j. Under this assumption,

$$e_p=E^{\mathbf{P}_{[0,n]}}\log\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}=\frac{1}{2}\sum_{i,j\in S}(\pi_i p_{ij}-\pi_j p_{ji})\log\frac{\pi_i p_{ij}}{\pi_j p_{ji}}. \tag{1}$$

For a stationary Markov process, its entropy production rate can be defined similarly. In Refs. 25 and 28–30, the entropy production rate e_p of a stationary finite Markov chain is expressed in terms of cycles, which occur along almost all sample paths, and their weights. Recently, Jiang *et al.*[31] gave a measure-theoretic exposition of the entropy production rate for hyperbolic dynamical systems, which was defined by Ruelle[32] from the physical point of view. Maes *et al.*[33] presented a definition of entropy production rate for some classes of deterministic and stochastic dynamics in the context of Gibbs measures.

In this paper, we prove the following strong limit theorem for a stationary irreducible finite Markov chain with discrete or continuous time parameter:

$$\lim_{t\to+\infty}\frac{1}{t}\log\frac{d\mathbf{P}_{[0,t)}}{d\mathbf{P}^-_{[0,t)}}(\omega)=e_p,\quad \mathbf{P}-\text{a.s.}$$

Furthermore, the convergence of the corresponding distributions is shown to be exponential, and the large deviation rate function $I(z)$ satisfies a symmetry: $I(z)=I(-z)-z$ for any $z\in\mathbf{R}$; this is actually the fluctuation theorem of Gallavotti–Cohen type for finite Markov chains. The proof is based on the well-known Perron–Frobenius theorem.[34–37] The statement of the theorem appeared in the pioneering paper Lebowitz and Spohn,[17] which also contained a proof. Here we present a mathematically strict proof. Part of the idea and techniques in our paper comes from Ref. 17, to which it is sincerely acknowledged. Moreover, we give a strict but very simple proof to the fluctuation theorem for the logarithmic probability ratios and for the joint distributions of them with general observables. We also discuss the transient fluctuation theorem for nonstationary Markov chains with discrete or continuous time parameter.

II. FINITE MARKOV CHAINS WITH DISCRETE TIME PARAMETER

To bring the main ideas into eminence, in this section we first treat the simplest case. $\xi=\{\xi_n:n\in\mathbf{Z}\}$ will be from now on a stationary irreducible discrete time Markov chain on probability space $(\Omega,\mathcal{F},\mathbf{P})$ with finite state space $S=\{1,2,\dots,d\}$, transition probability matrix $P=(p_{ij})_{i,j\in S}$, and invariant probability distribution $\Pi=\{\pi_i\}_{i\in S}$. Without loss of generality, we assume that ξ is a coordinate process on its canonical trajectory space $(\Omega,\mathcal{F},\mathbf{P})$. That is, $\Omega=S^{\mathbf{Z}}$, $\mathcal{F}=\sigma\{\xi_n:n\in\mathbf{Z}\}$ and $\mathbf{P}$ is the distribution of ξ. Now we introduce two transformations: the time-reversal transformation

$$r:\Omega\to\Omega,\quad \xi_n(r\omega)=\xi_{-n}(\omega),\quad \forall n\in\mathbf{Z};$$

and the shift operator

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$$\theta:(\Omega,\mathcal{F})\to(\Omega,\mathcal{F}),\quad \xi_n(\theta\omega)=\xi_{n+1}(\omega),\quad \forall n\in\mathbf{Z}.$$

It is easy to see that r and θ are $\mathcal{F}$-measurable and invertible with $r^{-1}=r$. Since ξ is stationary, $\theta^n\mathbf{P}=\mathbf{P}$, which yields $\theta^n\mathbf{P}^-=\mathbf{P}^-$ because $r\theta=\theta^{-1}r$, where $\mathbf{P}^-=r\mathbf{P}$ is the distribution of ξ's time reversal. The stationary Markov chain ξ is said to be reversible if $\mathbf{P}=\mathbf{P}^-$. As is well known, ξ is reversible if and only if the entropy production rate e_p of ξ vanishes, or iff $\pi_i p_{ij}=\pi_j p_{ji}$ for any $i,j\in S$, i.e. ξ is in detailed balance.[25–27]

For each $n\in\mathbf{Z}$ and $k\in\mathbf{N}$, denote by $\mathbf{P}_{[n,n+k]}$ and $\mathbf{P}^-_{[n,n+k]}$, respectively, the restrictions of $\mathbf{P}$ and $\mathbf{P}^-$ on $\mathcal{F}_n^{n+k}=\sigma(\xi_m:n\leqslant m\leqslant n+k)$. We assume that the transition matrix P satisfies the condition

$$p_{ij}>0\Leftrightarrow p_{ji}>0,\quad \forall i,j\in S. \tag{2}$$

Otherwise, $\mathbf{P}_{[0,n]}$ is not absolutely continuous with respect to $\mathbf{P}^-_{[0,n]}$, and by the definition of relative entropy, $H(\mathbf{P}_{[0,n]},\mathbf{P}^-_{[0,n]})$ is infinite for all $n\in\mathbf{N}$, hence $e_p=+\infty$. This is a trivial case.

Proposition 2.1: Under the condition (2), $\mathbf{P}_{[n,n+k]}$ *and* $\mathbf{P}^-_{[n,n+k]}$ *are absolutely continuous with respect to each other, and the Radon–Nikodym derivative is given by*

$$\frac{\mathrm{d}\mathbf{P}_{[n,n+k]}}{\mathrm{d}\mathbf{P}^-_{[n,n+k]}}(\omega)=\frac{\pi_{\xi_n(\omega)}p_{\xi_n(\omega)\xi_{n+1}(\omega)}\cdots p_{\xi_{n+k-1}(\omega)\xi_{n+k}(\omega)}}{\pi_{\xi_{n+k}(\omega)}p_{\xi_{n+k}(\omega)\xi_{n+k-1}(\omega)}\cdots p_{\xi_{n+1}(\omega)\xi_n(\omega)}},\quad \mathbf{P}-\text{a.s.}$$

Notice that $(1/n)\,E^{\mathbf{P}}\log(\mathrm{d}\mathbf{P}_{[0,n]}/\mathrm{d}\mathbf{P}^-_{[0,n]})$ converges to the entropy production rate e_p of ξ, which is given in (1). In fact, we can get a stronger result.

Proposition 2.2: Under the condition (2),

$$\lim_{n\to+\infty}\frac{1}{n}\log\frac{\mathrm{d}\mathbf{P}_{[0,n]}}{\mathrm{d}\mathbf{P}^-_{[0,n]}}(\omega)=e_p,\quad \mathbf{P}-\text{a.s. or } L^1(\mathrm{d}\mathbf{P}).$$

Proof: Let

$$f(\omega)=\log\frac{\pi_{\xi_0(\omega)}p_{\xi_0(\omega)\xi_1(\omega)}}{\pi_{\xi_1(\omega)}p_{\xi_1(\omega)\xi_0(\omega)}},$$

then

$$\begin{aligned}\frac{1}{n}\log\frac{\mathrm{d}\mathbf{P}_{[0,n]}}{\mathrm{d}\mathbf{P}^-_{[0,n]}}(\omega)&=\frac{1}{n}\log\frac{\pi_{\xi_0(\omega)}p_{\xi_0(\omega)\xi_1(\omega)}\cdots p_{\xi_{n-1}(\omega)\xi_n(\omega)}}{\pi_{\xi_n(\omega)}p_{\xi_n(\omega)\xi_{n-1}(\omega)}\cdots p_{\xi_1(\omega)\xi_0(\omega)}}\\&=\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{\pi_{\xi_k(\omega)}p_{\xi_k(\omega)\xi_{k+1}(\omega)}}{\pi_{\xi_{k+1}(\omega)}p_{\xi_{k+1}(\omega)\xi_k(\omega)}}\\&=\frac{1}{n}\sum_{k=0}^{n-1}f(\theta^k\omega),\quad \mathbf{P}-\text{a.s.}\end{aligned}$$

By the Birkhoff ergodic theorem,[38]

$$\lim_{n\to+\infty}\frac{1}{n}\log\frac{\mathrm{d}\mathbf{P}_{[0,n]}}{\mathrm{d}\mathbf{P}^-_{[0,n]}}(\omega)=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f(\theta^k\omega)=\int f\,\mathrm{d}\mathbf{P}=e_p,\quad \mathbf{P}-\text{a.s. or } L^1(\mathrm{d}\mathbf{P}).\qquad\square$$

There will naturally arise the question what is the convergence rate. The large deviation theory provides us a way to calculate this rate. Let $W_n(\omega)=\log(\mathrm{d}\mathbf{P}_{[0,n]}/\mathrm{d}\mathbf{P}^-_{[0,n]})(\omega)$ and $c_n(\lambda)=(\log Ee^{\lambda W_n})/n$. It is not difficult to see that W_n/n takes finitely many values and $e^{\lambda W_n(\omega)}>0$,

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$\mathbf{P}-$a.s. Thus, according to Theorem II.6.1 in Ellis,[39] in order to verify $\{\mu_n : n\in\mathbf{Z}^+\}$, where μ_n is the distribution of W_n/n, has a large deviation property, we only need to verify the free energy function $c(\lambda)\overset{\text{def}}{=}\lim_{n\to+\infty} c_n(\lambda)$ of $\{W_n : n\in\mathbf{Z}^+\}$ exists and is differentiable.

Theorem 2.3: *For all* $\lambda\in\mathbf{R}$, $\lim_{n\to+\infty} c_n(\lambda)$ *exists and the free energy function* $c(\lambda)$ *of* $\{W_n : n\in\mathbf{Z}^+\}$ *is differentiable.*

Proof: From Proposition 2.1,

$$Ee^{\lambda W_n}=\sum_{\substack{i_0,i_1,\cdots,i_n:\\ p_{i_0i_1}\cdots p_{i_{n-1}i_n}>0}} \pi_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n}\left(\frac{\pi_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n}}{\pi_{i_n}p_{i_1i_0}\cdots p_{i_ni_{n-1}}}\right)^{\lambda}$$

$$=\sum_{i_0,i_1,\cdots,i_n} \pi_{i_0}a_{i_0i_1}(\lambda)\cdots a_{i_{n-1}i_n}(\lambda)\left(\frac{\pi_{i_0}}{\pi_{i_n}}\right)^{\lambda},$$

where

$$a_{ij}(\lambda)=\begin{cases} p_{ij}^{1+\lambda}p_{ji}^{-\lambda}, & p_{ij}>0,\\ 0, & p_{ij}=0.\end{cases}$$

It is obvious that $p_{ij}>0\Leftrightarrow a_{ij}(\lambda)>0$. Hence $\mathbf{A}(\lambda)=(a_{ij}(\lambda))$ is an irreducible nonnegative matrix. By the Perron–Frobenius theorem, the spectral radius $e(\lambda)$ of $\mathbf{A}(\lambda)$ is a positive eigenvalue of $\mathbf{A}(\lambda)$ with one-dimensional eigenspace $\{k\vec{\alpha}:k\in\mathbf{R}\}$, where $\vec{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_d)^T$ and $\alpha_i>0$ for all $i\in S$.

For any given $\lambda>0$, denote

$$C_0=\max_{i,j}\left(\frac{\pi_i}{\pi_j}\right)^{\lambda}, \quad \alpha_{\min}=\min_i \alpha_i, \quad \alpha_{\max}=\max_i \alpha_i.$$

Then

$$\frac{1}{C_0\alpha_{\max}}\vec{\pi}\mathbf{A}(\lambda)^n\vec{\alpha}\leqslant Ee^{\lambda W_n}\leqslant\frac{C_0}{\alpha_{\min}}\vec{\pi}\mathbf{A}(\lambda)^n\vec{\alpha},$$

where $\vec{\pi}=(\pi_1,\pi_2,\ldots,\pi_d)$. Hence

$$\lim_{n\to+\infty}\frac{1}{n}\log Ee^{\lambda W_n}=\lim_{n\to+\infty}\frac{1}{n}\log\vec{\pi}\mathbf{A}(\lambda)^n\vec{\alpha}=\log e(\lambda),$$

where $e(\lambda)$ is differentiable because it is the simple eigenvalue of a differentiable matrix $\mathbf{A}(\lambda)$ (see Ref. 37 for details). □

By now, we have verified the fact that $\{\mu_n : n\in\mathbf{Z}^+\}$ has a large deviation property with rate function $I(z)=\sup_{\lambda\in\mathbf{R}}\{\lambda z-c(\lambda)\}$. Since $c_n(\cdot)$, $n\in\mathbf{N}$, and $c(\cdot)$ are all finite, and $c(\cdot)$ is differentiable at $\lambda=0$, by Theorem II.6.3 in Ellis,[39] W_n/n converges exponentially to the constant $c'(0)$. From Proposition 2.2, $c'(0)$ equals the entropy production rate e_p of the stationary Markov chain ξ. Furthermore, the innate symmetry of $\{W_n\}$ implies the symmetry of its free energy function and rate function.

Theorem 2.4: *(Fluctuation theorem) The free energy function* $c(\lambda)$ *and the large deviation rate function* $I(z)$ *of* $\{W_n : n\in\mathbf{Z}^+\}$ *have the following properties:*

$$c(\lambda)=c(-(1+\lambda)), \quad \forall\lambda\in\mathbf{R}, \quad I(z)=I(-z)-z, \quad \forall z\in\mathbf{R}.$$

Proof: Since $\mathbf{P}$ and $\mathbf{P}^-$ are reciprocal under the time-reversal transformation,

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$$\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(r\omega)=\frac{d\mathbf{P}^-_{[-n,0]}}{d\mathbf{P}_{[-n,0]}}(\omega)=\frac{d\mathbf{P}^-_{[0,n]}}{d\mathbf{P}_{[0,n]}}(\theta^{-n}\omega)=\left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(\theta^{-n}\omega)\right)^{-1}.$$

Thus

$$\begin{aligned}Ee^{\lambda W_n}&=\int\left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(\omega)\right)^{\lambda}d\mathbf{P}(\omega)\\&=\int\left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(r\omega)\right)^{\lambda}d\mathbf{P}^-(\omega)\\&=\int\left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(\theta^{-n}\omega)\right)^{-\lambda}d\mathbf{P}^-(\omega)\\&=\int\left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(\omega)\right)^{-(1+\lambda)}d\mathbf{P}(\omega)=Ee^{-(1+\lambda)W_n}.\end{aligned}$$

This yields $c(\lambda)=c(-(1+\lambda))$, and hence

$$\begin{aligned}I(z)&=\sup_{\lambda\in\mathbf{R}}\{\lambda z-c(\lambda)\}=\sup_{\lambda\in\mathbf{R}}\{\lambda z-c(-(1+\lambda))\}\\&=\sup_{\lambda\in\mathbf{R}}\{-(1+\lambda)z-c(\lambda)\}=\sup_{\lambda\in\mathbf{R}}\{\lambda\cdot(-z)-c(\lambda)\}-z=I(-z)-z.\end{aligned}$$ □

We could regard $W_n(\omega)/n=(1/n)\log(d\mathbf{P}_{[0,n]}/d\mathbf{P}^-_{[0,n]})(\omega)$ as the time-averaged entropy production rate of the sample trajectory ω of the stochastic system modelled by the Markov chain ξ. Roughly speaking, the fluctuation theorem gives a formula for the probability ratio that the sample entropy production rate W_n/n takes a value z to that of $-z$, and the ratio is roughly e^{nz}. If the Markov chain ξ is reversible, $I(0)=0$ and $I(z)=+\infty$ for any $z\neq 0$. In this case the fluctuation theorem gives a trivial result. However, if the Markov chain ξ is not reversible, for $z>0$ in a certain range, the sample entropy production rate W_n/n has a positive probability to take the value $z>0$ as well as the value $-z$. The fluctuation theorem tells that the former probability is greater, which accords with the second law of thermodynamics.

Now we discuss the transient fluctuation theorem for non-stationary Markov chains. Assume that $\tilde{\xi}=\{\tilde{\xi}_n:n\in\mathbf{Z}^+\}$ is an irreducible finite Markov chain on its canonical trajectory space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbf{P}})$ with finite state space S and transition probability matrix $P=(p_{ij})_{i,j\in S}$ as ξ. Suppose that the initial distribution ν (not necessarily invariant) satisfies $\nu_i>0$ for any $i\in S$. Denote the distributions of the trajectory segments of ξ over a finite time interval $[0,n]$ and their time reversals by $\tilde{\mathbf{P}}_{[0,n]}$ and $\tilde{\mathbf{P}}^-_{[0,n]}$, respectively. Let $\tilde{W}_n=\log(d\tilde{\mathbf{P}}_{[0,n]}/d\tilde{\mathbf{P}}^-_{[0,n]})$. From the above presentation, one can easily see that $\{\tilde{\mu}_n:n\geqslant 0\}$, the family of distributions of $\{\tilde{W}_n/n:n\geqslant 0\}$, also has a large deviation property and $\{\tilde{W}_n\}$ has the same free energy function as $\{W_n\}$, which yields that $\{\tilde{\mu}_n\}$ has the same large deviation rate function as $\{\mu_n\}$ and thus the rate function has a symmetry. Moreover, for any $n>0,z\in\mathbf{R}$, it holds that

$$\begin{aligned}\tilde{\mathbf{P}}\left(\frac{\tilde{W}_n}{n}=z\right)&=\tilde{\mathbf{P}}\left(\frac{d\tilde{\mathbf{P}}_{[0,n]}}{d\tilde{\mathbf{P}}^-_{[0,n]}}=e^{nz}\right)=\tilde{\mathbf{P}}_{[0,n]}\left(\frac{d\tilde{\mathbf{P}}_{[0,n]}}{d\tilde{\mathbf{P}}^-_{[0,n]}}=e^{nz}\right)=e^{nz}\tilde{\mathbf{P}}^-_{[0,n]}\left(\frac{d\tilde{\mathbf{P}}_{[0,n]}}{d\tilde{\mathbf{P}}^-_{[0,n]}}=e^{nz}\right)\\&=e^{nz}\tilde{\mathbf{P}}_{[0,n]}\left(\frac{d\tilde{\mathbf{P}}^-_{[0,n]}}{d\tilde{\mathbf{P}}_{[0,n]}}=e^{nz}\right)=e^{nz}\tilde{\mathbf{P}}\left(\frac{\tilde{W}_n}{n}=-z\right).\end{aligned}$$

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Since S is finite, $\widetilde{W}_n/n$ only takes a finite number of values and both sides of the above equality may simultaneously be equal to zero. However, in case one can divide over, the above equality can be written as

$$\frac{\widetilde{\mathbf{P}}\left(\frac{\widetilde{W}_n}{n}=z\right)}{\widetilde{\mathbf{P}}\left(\frac{\widetilde{W}_n}{n}=-z\right)}=e^{nz}.$$

Such an equality is called the transient fluctuation theorem by Evans *et al.*[7–13,19]

III. FINITE MARKOV CHAINS WITH CONTINUOUS TIME PARAMETER

In this section, we will discuss the same problem in the case of continuous time. The emphasis is much more oriented to detailed mathematical analysis and estimates. Let $\xi=\{\xi_t:t\in\mathbf{R}\}$ be a stationary, irreducible Markov chain on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with finite state space $S=\{1,\ldots,d\}$, conservative Q-matrix $Q=(q_{ij})_{i,j\in S}$, and invariant distribution $\Pi=\{\pi_i\}_{i\in S}$. Without loss of generality, we suppose that $(\Omega,\mathcal{F},\mathbf{P})$ is the canonical trajectory space of ξ, and its trajectories are right continuous having left limits. Since the time-reversed trajectories are left continuous having right limits, we should modify them. Define the time-reversal transformation and the shift operator as

$$r:(\Omega,\mathcal{F})\to(\Omega,\mathcal{F}),\quad \xi_t(r\omega)=\lim_{s\uparrow -t}\xi_s(\omega),\quad \forall t\in\mathbf{R},$$

$$\theta^t:(\Omega,\mathcal{F})\to(\Omega,\mathcal{F}),\quad \xi_s(\theta^t\omega)=\xi_{s+t}(\omega),\quad \forall s,t\in\mathbf{R}.$$

r,θ^t are $\mathcal{F}$-measurable and invertible with $r^{-1}=r$, $r\theta^t=\theta^{-t}r$, $\theta^t\mathbf{P}=\mathbf{P}$ and $\theta^t\mathbf{P}^-=\mathbf{P}^-$, where $\mathbf{P}^-=r\mathbf{P}$. The stationary Markov chain ξ is said to be reversible if $\mathbf{P}=\mathbf{P}^-$. As is well known, ξ is reversible if and only if the entropy production rate e_p of ξ vanishes, or iff $\pi_i q_{ij}=\pi_j q_{ji}$ for any $i,j\in S$.[26,27,29] Similarly as in the discrete time case, we assume that the Q-matrix satisfies the condition

$$q_{ij}>0\Leftrightarrow q_{ji}>0,\quad \forall i,j\in S. \tag{3}$$

For each $s\in\mathbf{R}$ and $t\in\mathbf{R}^+$, denote by $\mathbf{P}_{[s,s+t)}$ and $\mathbf{P}^-_{[s,s+t)}$, respectively, the restrictions of $\mathbf{P}$ and $\mathbf{P}^-$ on $\mathcal{F}_s^{s+t}=\sigma(\xi_u:s\leqslant u<s+t)$, then we have the following proposition.

Proposition 3.1: Under the condition (3), $\mathbf{P}_{[s,s+t)}$ *and* $\mathbf{P}^-_{[s,s+t)}$ *are absolutely continuous with respect to each other, and the Radon–Nikodym derivative is given by*

$$\left.\frac{d\mathbf{P}_{[s,s+t)}}{d\mathbf{P}^-_{[s,s+t)}}\right|_{A_{i_0i_1\cdots i_n}(t)}=\frac{\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}}{\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_1i_0}},\quad \mathbf{P}-\text{a.s.},$$

where

$$A_{i_0i_1\cdots i_n}(t)=\{\omega\in\Omega:\omega \textit{ jumps n times in } [s,s+t), \textit{ and the states are } i_0,\ldots,i_n \textit{ in turn}\}.$$

Proof: Denote $T_0=s$ and the jump times in the interval $[s,s+t)$ by $T_1(\omega),T_2(\omega),\ldots$ in turn. For any $n\geqslant 0$, $i_0,i_1,\ldots,i_n\in S$, $0<s_1<\cdots<s_n<s_{n+1}=t$ and small $\delta s_1,\ldots,\delta s_n>0$ (such that $s_k+\delta s_k<s_{k+1}$, $k=1,\ldots,n$), denote

$$A=\{\omega\in A_{i_0i_1\cdots i_n}(t):s_k\leqslant T_k(\omega)<s_k+\delta s_k,\ k=1,\ldots,n\}.$$

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Then

$$\begin{aligned}\mathbf{P}(A)=&\int_{s_1}^{s_1+\delta s_1}\mathrm{d}t_1\int_{s_2}^{s_2+\delta s_2}\mathrm{d}t_2\cdots\int_{s_n}^{s_n+\delta s_n}\mathrm{d}t_n\\&\times\int_t^{+\infty}\mathrm{d}t_{n+1}\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}q_{i_n}e^{-q_{i_0}t_1}e^{-q_{i_1}(t_2-t_1)}\cdots e^{-q_{i_n}(t_{n+1}-t_n)}\\=&\int_{s_1}^{s_1+\delta s_1}\mathrm{d}t_1\int_{s_2}^{s_2+\delta s_2}\mathrm{d}t_2\cdots\int_{s_n}^{s_n+\delta s_n}\mathrm{d}t_n\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}\\&\times e^{-q_{i_0}t_1}e^{-q_{i_1}(t_2-t_1)}\cdots e^{-q_{i_{n-1}}(t_n-t_{n-1})}e^{-q_{i_n}(t-t_n)}.\end{aligned}$$

Since

$$rA=\{\omega\in A_{i_ni_{n-1}\cdots i_0}(t):t-(s_{n+1-k}+\delta s_{n+1-k})\leqslant T_k(\omega)<t-s_{n+1-k},\quad k=1,\ldots,n\},$$

it follows that

$$\begin{aligned}\mathbf{P}^-(A)=\mathbf{P}(rA)=&\int_{t-(s_n+\delta s_n)}^{t-s_n}\mathrm{d}t_1\int_{t-(s_{n-1}+\delta s_{n-1})}^{t-s_{n-1}}\mathrm{d}t_2\cdots\int_{t-(s_1+\delta s_1)}^{t-s_1}\mathrm{d}t_n\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_1i_0}\\&\times e^{-q_{i_n}t_1}e^{-q_{i_{n-1}}(t_2-t_1)}\cdots e^{-q_{i_1}(t_n-t_{n-1})}e^{-q_{i_0}(t-t_n)}\\=&\int_{s_1}^{s_1+\delta s_1}\mathrm{d}t_1\int_{s_2}^{s_2+\delta s_2}\mathrm{d}t_2\cdots\int_{s_n}^{s_n+\delta s_n}\mathrm{d}t_n\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_1i_0}\\&\times e^{-q_{i_0}t_1}e^{-q_{i_1}(t_2-t_1)}\cdots e^{-q_{i_{n-1}}(t_n-t_{n-1})}e^{-q_{i_n}(t-t_n)}\\=&\frac{\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_1i_0}}{\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}}\mathbf{P}(A).\end{aligned}$$

Notice that such A's as above generate $\mathcal{F}_{[s,s+t)}$, then one obtains the desired result immediately. □

The entropy production rate e_p of the stationary Markov chain ξ can be defined by

$$e_p\overset{\text{def}}{=}\lim_{t\to+\infty}\frac{1}{t}H(\mathbf{P}_{[0,t)},\mathbf{P}^-_{[0,t)})=\lim_{t\to+\infty}\frac{1}{t}E^{\mathbf{P}}\log\frac{\mathrm{d}\mathbf{P}_{[0,t)}}{\mathrm{d}\mathbf{P}^-_{[0,t)}},$$

or equivalently, as in Refs. 26 and 27, by

$$e_p\overset{\text{def}}{=}\lim_{t\downarrow 0+}\frac{1}{t}H(\mathbf{P}_{[s,s+t)},\mathbf{P}^-_{[s,s+t)})=\lim_{t\downarrow 0+}\frac{1}{t}E^{\mathbf{P}}\log\frac{\mathrm{d}\mathbf{P}_{[s,s+t)}}{\mathrm{d}\mathbf{P}^-_{[s,s+t)}},$$

where $s\in\mathbf{R}$. The equivalence is a direct consequence of Theorem 10.4 in Varadhan.[40] In Refs. 26, 28, and 29, an entropy production formula was given for ξ:

$$e_p=\frac{1}{2}\sum_{i,j\in S}(\pi_iq_{ij}-\pi_jq_{ji})\log\frac{\pi_iq_{ij}}{\pi_jq_{ji}}.\tag{4}$$

Employing Proposition 3.1 and (6), (7) below, it is not difficult to prove this formula strictly.

Proposition 3.2: Under the condition (3),

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$$\lim_{t\to+\infty}\frac{1}{t}\log\frac{\mathrm{d}\mathbf{P}_{[s,s+t)}}{\mathrm{d}\mathbf{P}^{-}_{[s,s+t)}}(\omega)=e_p, \quad \mathbf{P}-\text{a.s.}$$

Proof: Let $N_t(\omega)$ be the number of jumps of ω in the interval $[s,s+t)$, $N_t(i,j,\omega)$ be the number of jumps from i to j of ω in the interval $[s,s+t)$ and $\{X_k\}_{k\in\mathbf{Z}^+}$ be the embedded chain of ξ defined by $X_k=\xi_{T_k}$. By Proposition 3.1,

$$\begin{aligned}\frac{1}{t}\log\frac{\mathrm{d}\mathbf{P}_{[s,s+t)}}{\mathrm{d}\mathbf{P}^{-}_{[s,s+t)}}(\omega)&=\frac{1}{t}\sum_{n=0}^{+\infty}\left(\log\frac{\pi_{X_0(\omega)}}{\pi_{X_n(\omega)}}+\sum_{k=0}^{n-1}\log\frac{q_{X_k(\omega)X_{k+1}(\omega)}}{q_{X_{k+1}(\omega)X_k(\omega)}}\right)1_{\{N_t=n\}}(\omega)\\&=\frac{1}{t}\sum_{n=0}^{+\infty}1_{\{N_t=n\}}(\omega)\log\frac{\pi_{X_0(\omega)}}{\pi_{X_n(\omega)}}\\&\quad+\sum_{i,j\in S}\sum_{n=0}^{+\infty}\sum_{k=0}^{n-1}\frac{1}{t}1_{\{N_t=n\}}(\omega)1_{\{X_k=i,X_{k+1}=j\}}(\omega)\log\frac{q_{ij}}{q_{ji}}\\&=\frac{1}{t}\sum_{n=0}^{+\infty}1_{\{N_t=n\}}(\omega)\log\frac{\pi_{X_0(\omega)}}{\pi_{X_n(\omega)}}+\sum_{i,j\in S}\frac{1}{t}N_t(i,j,\omega)\log\frac{q_{ij}}{q_{ji}},\end{aligned}\tag{5}$$

where $1_A(\cdot)$ is the indicator function of the event $A\in\mathcal{F}$. For any $t_1,t_2\geqslant 0$ and $i,j\in S$, $N_{t_1+t_2}(i,j,\omega)=N_{t_1}(i,j,\omega)+N_{t_2}(i,j,\theta^{t_1}\omega)$, $\mathbf{P}-$a.s. By the subadditive ergodic theorem,[38] for any $\delta>0$, it holds that

$$\lim_{n\to+\infty}\frac{1}{n\delta}N_{n\delta}(i,j,\omega)=\inf_n\frac{1}{n\delta}EN_{n\delta}(i,j,\cdot)=\frac{1}{\delta}EN_{\delta}(i,j,\cdot), \quad \mathbf{P}-\text{a.s.}$$

For any t, there exist $n(t)\in\mathbf{Z}^+$ and $r(t)\in[0,\delta)$ such that $t=n(t)\delta+r(t)$, thus

$$\frac{1}{t}N_t(i,j,\omega)=\frac{1}{t}N_{n(t)\delta}(i,j,\omega)+\frac{1}{t}N_{r(t)}(i,j,\theta^{n(t)\delta}\omega), \quad \mathbf{P}-\text{a.s.}$$

Consequently,

$$\lim_{t\to+\infty}\frac{1}{t}N_{n(t)\delta}(i,j,\omega)=\frac{1}{\delta}EN_{\delta}(i,j,\cdot), \quad \mathbf{P}-\text{a.s.}$$

Now we would like to prove that $\lim_{t\to+\infty}N_{r(t)}(i,j,\theta^{n(t)\delta}\omega)/t=0$, $\mathbf{P}-$a.s. For any given $i_0,i_1,\ldots,i_n\in S$,

$$\begin{aligned}\mathbf{P}(A_{i_0i_1\cdots i_n}(t))&=\int_0^t\mathrm{d}t_n\int_{0<t_1<t_2<\cdots<t_n}\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_{n-1}\,\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}e^{-q_{i_n}t}\prod_{k=1}^{n}e^{(q_{i_k}-q_{i_{k-1}})t_k}\\&\leqslant\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}\frac{t^n}{n!}.\end{aligned}\tag{6}$$

Hence for any n,

$$\mathbf{P}(N_t(\omega)=n)\leqslant\sum_{\substack{i_0,\cdots,i_n\\ i_k\neq i_{k+1}}}\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}\frac{t^n}{n!}\leqslant\frac{(\max_i q_it)^n}{n!}.\tag{7}$$

For any $\varepsilon>0$,

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$$\sum_{n=0}^{+\infty} \mathbf{P}(N_\delta(i,j,\omega)>n\varepsilon) \leqslant \sum_{n=0}^{+\infty} \mathbf{P}(N_\delta(\omega)>n\varepsilon)$$
$$= \sum_{n=0}^{+\infty} \sum_{k>n\varepsilon} \mathbf{P}(N_\delta(\omega)=k)$$
$$= \sum_{k=1}^{+\infty} \sum_{0\leqslant n<k\varepsilon^{-1}} \mathbf{P}(N_\delta(\omega)=k)$$
$$\leqslant \sum_{k=1}^{+\infty} \mathbf{P}(N_\delta(\omega)=k)(k\varepsilon^{-1}+1)<+\infty.$$

Since the process is stationary, $\mathbf{P}(N_\delta(i,j,\theta^{n\delta}\omega)>n\varepsilon)=\mathbf{P}(N_\delta(i,j,\omega)>n\varepsilon)$ for any $n>0$. This together with the Borel–Cantelli lemma yields that

$$\mathbf{P}\left(\exists \text{ a sequence } \{t_k\geqslant 0\}_{k\in\mathbf{N}} \text{ s.t. } t_k\uparrow +\infty \text{ and } \frac{1}{t_k}N_{r(t_k)}(i,j,\theta^{n(t_k)\delta}\omega)>\varepsilon, \forall k\in\mathbf{N}\right)$$
$$\leqslant \mathbf{P}[N_\delta(i,j,\theta^{n\delta}\omega)>n\varepsilon \text{ infinitely often}]=0.$$

Hence $\mathbf{P}$−a.s., $\lim_{t\to+\infty}(1/t)\,N_{r(t)}(i,j,\theta^{n(t)\delta}\omega)=0$. So for any given $\delta>0$, it holds that

$$\lim_{t\to+\infty}\frac{1}{t}N_t(i,j,\omega)=\frac{1}{\delta}EN_\delta(i,j,\cdot), \quad \mathbf{P}-\text{a.s.}, \tag{8}$$

where $EN_\delta(i,j,\cdot)=\sum_{n=1}^{+\infty}EN_\delta(i,j,\cdot)1_{\{N_\delta=n\}}(\cdot)$. On one hand,

$$\frac{1}{\delta}EN_\delta(i,j,\cdot)1_{\{N_\delta=1\}}(\cdot)=\frac{1}{\delta}P(N_\delta(\omega)=1,N_\delta(i,j,\omega)=1)$$
$$=\frac{1}{\delta}\int_0^\delta \pi_i q_{ij} e^{-q_j\delta}e^{(q_j-q_i)\tau}\,\mathrm{d}\tau \to \pi_i q_{ij}, \quad \text{as } \delta\to 0+.$$

On the other hand, from (7)

$$\frac{1}{\delta}\sum_{n=2}^{+\infty}EN_\delta(i,j,\cdot)1_{\{N_\delta=n\}}(\cdot)\leqslant \frac{1}{\delta}\sum_{n=2}^{+\infty} nP(N_\delta(\omega)=n)\to 0, \quad \text{as } \delta\to 0+.$$

Hence $(1/\delta)\,EN_\delta(i,j,\cdot)=\pi_i q_{ij}$, which together with (5) and (8) yields that

$$\lim_{t\to+\infty}\frac{1}{t}\log\frac{\mathrm{d}\mathbf{P}_{[s,s+t)}}{\mathrm{d}\mathbf{P}^-_{[s,s+t)}}(\omega)=\sum_{i,j}\pi_i q_{ij}\log\frac{\pi_i q_{ij}}{\pi_j q_{ji}}=e_p, \quad \mathbf{P}-\text{a.s.} \qquad \square$$

Let $W_t(\omega)=\log(\mathrm{d}\mathbf{P}_{[0,t)}/\mathrm{d}\mathbf{P}^-_{[0,t)})(\omega)$, μ_t be the distribution of W_t/t and $c_t(\lambda)=(\log Ee^{\lambda W_t})/t$. According to Theorem II.6.1 in Ellis[39] (modified for continuous time parameter), in order to verify $\{\mu_t : t\in\mathbf{R}^+\}$ has a large deviation property, we only need to prove $c_t(\lambda)$ is finite and the free energy function $c(\lambda)\overset{\text{def}}{=}\lim_{t\to+\infty}c_t(\lambda)$ of $\{W_t : t\in\mathbf{R}^+\}$ exists and is differentiable.

Proposition 3.3: For any $t>0$ and $\lambda\in\mathbf{R}$, $c_t(\lambda)$ is finite.

Proof: For any $i\in S$, by Proposition 3.1, it holds that

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$$E_i(t,\lambda)\overset{\text{def}}{=}E(e^{\lambda W_t}|\xi_0=i)=\sum_{n=0}^{+\infty}\sum_{\substack{i_1,\cdots,i_n:\\ q_{ii_1}q_{i_1i_2}\cdots q_{i_{n-1}i_n}>0}}\left(\frac{\pi_i q_{ii_1}q_{i_1i_2}\cdots q_{i_{n-1}i_n}}{\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_2i_1}q_{i_1i}}\right)^{\lambda}\mathbf{P}(A_{ii_1\ldots i_n}(t)|\xi_0=i). \tag{9}$$

From (6) and (7),

$$\sum_{\substack{i_1,\cdots,i_n:\\ q_{ii_1}q_{i_1i_2}\cdots q_{i_{n-1}i_n}>0}}\left(\frac{\pi_i q_{ii_1}q_{i_1i_2}\cdots q_{i_{n-1}i_n}}{\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_2i_1}q_{i_1i}}\right)^{\lambda}\mathbf{P}(A_{ii_1\ldots i_n}(t)|\xi_0=i)\leqslant B^{\lambda}\frac{(C^{\lambda}Dt)^n}{n!},$$

where $B=\max_{i,j}\pi_i/\pi_j$, $C=\max\{q_{ij}/q_{ji}:q_{ij}>0\}$ and $D=\max_i q_i$. Hence $E_i(t,\lambda)<+\infty$ and $Ee^{\lambda W_t}=\Sigma_i\pi_iE_i(t,\lambda)<+\infty$. On the other hand, it is easy to see $Ee^{\lambda W_t}>0$. Thus $c_t(\lambda)=(1/t)\log Ee^{\lambda W_t}$ is finite. □

Theorem 3.4: *For all* $\lambda\in\mathbf{R}$, $\lim_{t\to+\infty}c_t(\lambda)$ *exists and the free energy function* $c(\lambda)$ *of* $\{W_t:t\in\mathbf{R}^+\}$ *is differentiable.*

Proof: From (6),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(A_{i_0i_1\cdots i_n}(t))=-q_{i_n}\mathbf{P}(A_{i_0i_1\cdots i_n}(t))+q_{i_{n-1}i_n}\mathbf{P}(A_{i_0i_1\cdots i_{n-1}}(t)). \tag{10}$$

By Proposition 3.1 and $\mathbf{P}^-=r\mathbf{P}$, it holds that

$$\mathbf{P}(A_{i_0i_1\cdots i_n}(t))=\frac{\pi_{i_0}q_{i_0i_1}\cdots q_{i_{n-1}i_n}}{\pi_{i_n}q_{i_ni_{n-1}}\cdots q_{i_1i_0}}\mathbf{P}(A_{i_ni_{n-1}\cdots i_0}(t)).$$

This together with (10) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(A_{i_0i_1\cdots i_n}(t))=-q_{i_0}\mathbf{P}(A_{i_0i_1\cdots i_n}(t))+\frac{\pi_{i_0}q_{i_0i_1}}{\pi_{i_1}}\mathbf{P}(A_{i_1i_2\cdots i_n}(t)),$$

and it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(A_{i_0i_1\cdots i_n}(t)|\xi_0=i_0)=-q_{i_0}\mathbf{P}(A_{i_0i_1\cdots i_n}(t)|\xi_0=i_0)+q_{i_0i_1}\mathbf{P}(A_{i_1\cdots i_n}(t)|\xi_0=i_1).$$

Taking differentials on both sides of (9) with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_i(t,\lambda)=-q_iE_i(t,\lambda)+\sum_{j:q_{ij}>0}q_{ij}\left(\frac{\pi_iq_{ij}}{\pi_jq_{ji}}\right)^{\lambda}E_j(t,\lambda). \tag{11}$$

Let

$$\mathbf{E}(t,\lambda)=\begin{pmatrix}E_1(t,\lambda)\\ \cdots\\ E_d(t,\lambda)\end{pmatrix},\quad l_{ij}(\lambda)=\begin{cases}-q_i, & \text{if } i=j,\\ q_{ij}\left(\dfrac{\pi_iq_{ij}}{\pi_jq_{ji}}\right)^{\lambda}, & \text{if } q_{ij}>0,\\ 0, & \text{if } q_{ij}=0.\end{cases}\quad \mathbf{1}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

Since $\mathbf{E}(0,\lambda)=\mathbf{1}$, (11) yields $\mathbf{E}(t,\lambda)=e^{\mathbf{L}(\lambda)t}\mathbf{1}$, where $\mathbf{L}(\lambda)=(l_{ij}(\lambda))$. Thus

$$c_t(\lambda)=\frac{1}{t}\log Ee^{\lambda W_t}=\frac{1}{t}\log\sum_i\pi_iE_i(t,\lambda)=\frac{1}{t}\log\vec{\pi}e^{\mathbf{L}(\lambda)t}\mathbf{1}, \tag{12}$$

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where $\vec{\pi}=(\pi_1,\pi_2,\dots,\pi_d)$. It is easy to see that $q_{ij}>0\Leftrightarrow l_{ij}(\lambda)>0$ and $q_{ij}=0\Leftrightarrow l_{ij}(\lambda)=0$. Since Q is irreducible, the Taylor expansion of $e^{\mathbf{L}(\lambda)t}$ at $t=0$,

$$e^{\mathbf{L}(\lambda)t}=\mathbf{I}+\mathbf{L}(\lambda)t+\frac{1}{2!}\mathbf{L}(\lambda)^2t^2+\cdots+\frac{1}{n!}\mathbf{L}(\lambda)^nt^n+\mathbf{o}(t^n),$$

tells that for any $i,j\in S$, there is a $\delta(i,j)>0$ such that $[e^{\mathbf{L}(\lambda)t}]_{ij}>0$ for any $t\in(0,\delta(i,j)]$. Hence $e^{\mathbf{L}(\lambda)\delta}>0$ for any sufficiently small δ, which yields that it holds for all $\delta>0$. For a fixed $\delta>0$, by the Perron–Frobenius theorem, the spectral radius $e(\lambda,\delta)$ of $e^{\mathbf{L}(\lambda)\delta}$ is a positive eigenvalue of $e^{\mathbf{L}(\lambda)\delta}$ with one-dimensional eigenspace. This together with (12) yields that

$$c(\lambda)=\lim_{t\to+\infty}c_t(\lambda)=\delta^{-1}\log e(\lambda,\delta).$$

$c(\lambda)$ is differentiable because $\mathbf{L}(\lambda)$ is differentiable and so are $e^{\mathbf{L}(\lambda)\delta}$ and $e(\lambda,\delta)$. $\square$

Also the innate symmetry of $\{W_t\}$ implies symmetries of its free energy function and rate function. The proof is exactly the same as that of Theorem 2.4.

Theorem 3.5: *(Fluctuation theorem) The free energy function $c(\cdot)$ and the large deviation rate function $I(\cdot)$ of $\{W_t:t\in\mathbf{R}^+\}$ satisfy*

$$c(\lambda)=c(-(1+\lambda)),\quad \forall\lambda\in\mathbf{R},\quad I(z)=I(-z)-z,\quad \forall z\in\mathbf{R}.$$

As in the discrete-time case, the transient fluctuation theorem holds for nonstationary Markov chains with continuous-time parameter. Suppose that $\tilde{\xi}=\{\tilde{\xi}_t:t\geqslant 0\}$ is a Markov chain on its canonical trajectory space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbf{P}})$ with the same state space S and the same Q-matrix Q as those of ξ. Assume that the initial distribution ν (may not invariant) satisfies $\nu_i>0$ for all $i\in S$. Denote the distributions of the trajectory segments of ξ over a finite time interval $[0,t)$ and their time reversals by $\tilde{\mathbf{P}}_{[0,t)}$ and $\tilde{\mathbf{P}}^-_{[0,t)}$, respectively. Let $\tilde{W}_t=\log(\mathrm{d}\tilde{\mathbf{P}}_{[0,t)}/\mathrm{d}\tilde{\mathbf{P}}^-_{[0,t)})$. From the above presentation, one can see that $\{\tilde{\mu}_t:t\geqslant 0\}$, the family of the distributions of $\{\tilde{W}_t/t:t\geqslant 0\}$, also has a large deviation property and $\{\tilde{W}_t\}$ has the same free energy function as $\{W_t\}$, which yields that $\{\tilde{\mu}_t\}$ has the same large deviation rate function as $\{\mu_t\}$ and thus the rate function has a symmetry. We also have

$$\tilde{\mathbf{P}}\left(\frac{\tilde{W}_t}{t}=z\right)=e^{tz}\tilde{\mathbf{P}}\left(\frac{\tilde{W}_t}{t}=-z\right),\quad \forall t>0,z\in\mathbf{R}.$$

IV. FLUCTUATIONS OF GENERAL OBSERVABLES

With the assumptions and notation of the stationary case in Sec. II, let $\varphi:S\to\mathbf{R}$ be an observable and $\Phi_n(\omega)=\sum_{k=0}^n\varphi(\xi_k(\omega))=\sum_{k=0}^n\varphi(\xi_0(\theta^k\omega))$. Clearly, Φ_n satisfies $\Phi_n(r\omega)=\Phi_n(\theta^{-n}\omega)$ for any $\omega\in\Omega$. From the Birkhoff ergodic theorem, it follows that $\lim_{n\to+\infty}\Phi_n/n=E^{\Pi}\varphi$. Use the Perron–Frobenius theorem, then one sees that

$$c(\lambda_1,\lambda_2)\overset{\text{def}}{=}\lim_{n\to+\infty}\frac{1}{n}\log Ee^{\lambda_1W_n+\lambda_2\Phi_n}$$

exists and is differentiable with respect to λ_1,λ_2. Thus $\{\mu_n:n\geqslant 0\}$, the family of the distributions of $\{(W_n/n,\Phi_n/n):n\geqslant 0\}$, has a large deviation property with rate function $I(z_1,z_2)=\sup_{\lambda_1,\lambda_2\in\mathbf{R}}\{\lambda_1z_1+\lambda_2z_2-c(\lambda_1,\lambda_2)\}$. It is not difficult to find that $c(\lambda_1,\lambda_2)=c(-(1+\lambda_1),\lambda_2)$ and $I(z_1,z_2)=I(-z_1,z_2)-z_1$. In general, let $\{\vec{\Phi}_n:n\geqslant 0\}$ and $\{\vec{\Psi}_n:n\geqslant 0\}$ be two sets of random vectors on $(\Omega,\mathcal{F},\mathbf{P})$, where $\vec{\Phi}_n$ and $\vec{\Psi}_n$ are $\mathcal{F}_0^n$ measurable. Provided that the free energy function

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$$c(\lambda,\vec{\nu},\vec{\gamma}) \stackrel{\text{def}}{=} \lim_{n\to+\infty} \frac{1}{n} \log E e^{\lambda W_n + \langle \vec{\nu},\vec{\Phi}_n\rangle + \langle \vec{\gamma},\vec{\Psi}_n\rangle}$$

exists and is differentiable, it holds that $\{\mu_n : n \geqslant 0\}$, the family of the distributions of $\{(1/n)(W_n,\vec{\Phi}_n,\vec{\Psi}_n) : n\geqslant 0\}$, has a large deviation property with rate function $I(z,\vec{u},\vec{v}) = \sup_{\lambda,\vec{\nu},\vec{\gamma}}\{\lambda z + \langle \vec{\nu},\vec{u}\rangle + \langle \vec{\gamma},\vec{v}\rangle - c(\lambda,\vec{\nu},\vec{\gamma}\}$.

Theorem 4.1: *If* $\vec{\Phi}_n(r\omega) = \vec{\Phi}_n(\theta^{-n}\omega)$ *and* $\vec{\Psi}_n(r\omega) = -\vec{\Psi}_n(\theta^{-n}\omega)$ *for any* $n\geqslant 0$ *and* $\omega \in \Omega$, *it holds that*

$$c(\lambda,\vec{\nu},\vec{\gamma}) = c(-(1+\lambda),\vec{\nu},-\vec{\gamma}), \quad I(z,\vec{u},\vec{v}) = I(-z,\vec{u},-\vec{v}) - z.$$

Proof: For any given $\lambda,\vec{\nu},\vec{\gamma}$,

$$\begin{aligned} E e^{\lambda W_n + \langle \vec{\nu},\vec{\Phi}_n\rangle + \langle \vec{\gamma},\vec{\Psi}_n\rangle} &= \int \left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(\omega)\right)^{\lambda} e^{\langle \vec{\nu},\vec{\Phi}_n(\omega)\rangle + \langle \vec{\gamma},\vec{\Psi}_n(\omega)\rangle}\, d\mathbf{P}(\omega) \\ &= \int \left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(r\omega)\right)^{\lambda} e^{\langle \vec{\nu},\vec{\Phi}_n(r\omega)\rangle + \langle \vec{\gamma},\vec{\Psi}_n(r\omega)\rangle}\, d\mathbf{P}^-(\omega) \\ &= \int \left(\frac{d\mathbf{P}_{[0,n]}}{d\mathbf{P}^-_{[0,n]}}(\theta^{-n}\omega)\right)^{-\lambda} e^{\langle \vec{\nu},\vec{\Phi}_n(\theta^{-n}\omega)\rangle + \langle \vec{\gamma},-\vec{\Psi}_n(\theta^{-n}\omega)\rangle}\, d\mathbf{P}^-(\omega) \\ &= E e^{-(1+\lambda) W_n + \langle \vec{\nu},\vec{\Phi}_n\rangle + \langle -\vec{\gamma},\vec{\Psi}_n\rangle}. \end{aligned}$$

The desired result follows immediately. □

Remark: The same result still holds for the continuous time case.

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